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REAL RATIONAL SURFACES

by

Frédéric Mangolte

1. Introduction

During the last decade⁽¹⁾, there were many progresses in the understanding of the topology of real algebraic manifolds, above all in dimensions 2 and 3. Results on real algebraic threefolds were addressed in the survey [Man14] with a particular emphasis on Kollár's results and conjectures concerning real uniruled and real rationally connected threefolds, see [Kol01], [HM05b, HM05a], [CM08, CM09], [MW12]. In the present paper, we will focus on real rational surfaces and especially on their birational geometry. Thus the three next sections are devoted to real rational surfaces which are presented in a most elementary way. We state Comessatti's and Nash-Tognoli's famous theorems (Theorem 7 and Theorem 20). Among other things, we give a sketch of proof of the statement: "Up to isomorphism, there is exactly one single real rational model of each nonorientable surface" (Theorem 10); a sketch of proof of the statement: "the groups of birational diffeomorphisms are infinitely transitive" (Theorem 11); a sketch of proof of the statement: "the groups of birational diffeomorphisms is dense in the group of diffeomorphisms" (Theorem 22).

We conclude the paper with Section 5 devoted to a new line of research: the theory of *regulous functions* and the geometry we are able to define with them.

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Key words and phrases. — rational real algebraic surface, topological surface, rational model, birational diffeomorphism, automorphism group, regulous map, continuous rational map.

⁽¹⁾With the exception of some classical references, only references over the past years from the preceding "RAAG conference in Rennes", which took place in 2001, are included.

Besides the progresses in the theory of real rational surfaces, the classification of other real algebraic surfaces has considerably advanced during the last decade (see [Kha06] for a survey): topological types and deformation types of real Enriques surfaces [DIK00], deformation types of geometrically⁽²⁾ rational surfaces [DK02], deformation types of real ruled surfaces [Wel03], topological types and deformation types of real bielliptic surfaces [CF03], topological types and deformation types of real elliptic surfaces [AM08, BM07, DIK08].

The present survey is an expansion of the preprint written by Johannes Huisman [Hui11] from which we have borrowed several parts.

Convention. — In this paper, a *real algebraic surface* (resp. *real algebraic curve*) is a projective complex algebraic manifold of complex dimension 2 (resp. 1) endowed with an anti-holomorphic involution whose set of fixed points is called *the real locus* and denoted by $X(\mathbf{R})$. A *real map* is a complex map commuting with the involutions. A *topological surface* is a real 2-dimensional \mathcal{C}^∞ -manifold. If nonempty, the real locus $X(\mathbf{R})$ of a real algebraic surface⁽³⁾ gets a natural structure of a topological surface when endowed with the euclidean topology. Furthermore $X(\mathbf{R})$ is compact since X is projective.

Acknowledgments. — Thanks to Daniel Naie for sharing his picture of the real locus of a blow-up, see Figure 1.

2. Real rational surfaces

A real algebraic surface X is *rational* if it contains a Zariski-dense subset real isomorphic to the affine plane \mathbf{A}^2 or, equivalently, if its function field is isomorphic to the field of rational functions $\mathbf{R}(x, y)$. In the sequel, a rational real algebraic surface will be called a *real rational surface* for short and by our general convention, always assumed to be projective and nonsingular.

Example 1. — 1. The real projective plane $\mathbf{P}_{x:y:z}^2$ is rational. Indeed, each of the coordinate open charts $U_0 = \{x \neq 0\}$, $U_1 = \{y \neq 0\}$, $U_2 = \{z \neq 0\}$ is isomorphic to \mathbf{A}^2 . The real locus $\mathbf{P}^2(\mathbf{R})$ endowed with the euclidean topology is the topological real projective plane.

2. The product surface $\mathbf{P}_{x:y}^1 \times \mathbf{P}_{u:v}^1$ is rational. Indeed, the product open subset $\{x \neq 0\} \times \{u \neq 0\}$ is isomorphic to \mathbf{A}^2 . The set of real points

⁽²⁾See p. 11 before Theorem 12.

⁽³⁾nonsingular by our convention

$(\mathbf{P}^1 \times \mathbf{P}^1)(\mathbf{R}) = \mathbf{P}^1(\mathbf{R}) \times \mathbf{P}^1(\mathbf{R})$ is diffeomorphic to the 2-dimensional torus $\mathbf{S}^1 \times \mathbf{S}^1$ where \mathbf{S}^1 denotes the unit circle in \mathbf{R}^2 .

3. The quadric $Q_{3,1}$ in the projective space $\mathbf{P}_{w:x:y:z}^3$ given by the affine equation $x^2 + y^2 + z^2 = 1$ is rational. Indeed, for a real point P of $Q_{3,1}$, let's denote by $T_P Q_{3,1}$ the real projective plane in \mathbf{P}^3 tangent to $Q_{3,1}$ at P . Then the stereographic projection $Q_{3,1} \setminus T_P Q_{3,1} \rightarrow \mathbf{A}^2$ is an isomorphism of real algebraic surfaces. For example in the case P is the North pole $N = (1 : 0 : 0 : 1)$, let $\pi_N : Q_{3,1} \rightarrow \mathbf{P}_{U:V:W}^2$ be the rational map given by

$$\pi_N : (w : x : y : z) \dashrightarrow (x : y : w - z) .$$

Then π_N restricts to the stereographic projection from $Q_{3,1} \setminus T_N Q_{3,1}$ onto its image $\pi_N(Q_{3,1} \setminus T_N Q_{3,1}) = \{w \neq 0\} \simeq \mathbf{A}^2$.

(The inverse rational map $\pi_N^{-1} : \mathbf{P}^2 \dashrightarrow Q_{3,1}$ is given by

$$\pi_N^{-1} : (x : y : z) \dashrightarrow (x^2 + y^2 + z^2 : 2xz : 2yz : x^2 + y^2 - z^2) .$$

The real locus $Q_{3,1}(\mathbf{R})$ is the unit sphere \mathbf{S}^2 in \mathbf{R}^3 .

To produce more examples, we will recall the construction of the blow-up which is especially simple in the context of rational surfaces.

The blow-up $B_{(0,0)} \mathbf{A}^2$ of \mathbf{A}^2 at $(0, 0)$ is the quadric hypersurface defined in $\mathbf{A}^2 \times \mathbf{P}^1$ by

$$B_{(0,0)} \mathbf{A}^2 = \{((x, y), [u : v]) \in \mathbf{A}_{x,y}^2 \times \mathbf{P}_{u:v}^1 : uy = vx\} .$$

The blow-up $B_{(0:0:1)} \mathbf{P}^2$ of \mathbf{P}^2 at $P = (0 : 0 : 1)$ is the algebraic surface

$$B_{(0:0:1)} \mathbf{P}^2 = \{([x : y : z], [u : v]) \in \mathbf{P}_{x:y:z}^2 \times \mathbf{P}_{u:v}^1 : uy - vx = 0\} .$$

The open subset $V_0 = \{((x, y), [u : v]) \in B_{(0,0)} \mathbf{A}^2 : u \neq 0\}$ is Zariski-dense in $B_{(0,0)} \mathbf{A}^2$ and the map $\varphi : V_0 \rightarrow \mathbf{A}^2$, $((x, y), [u : v]) \mapsto (x, \frac{y}{u})$ is an isomorphism. Similarly, the open subset

$$\widetilde{U}_2 = \{([x : y : z], [u : v]) \in B_{(0:0:1)} \mathbf{P}^2 : z \neq 0, u \neq 0\}$$

is Zariski-dense in $B_{(0:0:1)} \mathbf{P}^2$ and the map $\widetilde{U}_2 \rightarrow U_2 \simeq \mathbf{A}^2$,

$$([x : y : z], [u : v]) \mapsto [ux : v : uz]$$

is an isomorphism. Thus $B_{(0:0:1)} \mathbf{P}^2$ is rational. Now remark that the map $\varphi : V_1 = \{v \neq 0\} \rightarrow \mathbf{A}^2$, $((x, y), [u : v]) \mapsto (x, \frac{y}{v})$ is also an isomorphism and the surface $B_{(0,0)} \mathbf{A}^2$ is thus covered by two open subsets, both isomorphic to \mathbf{A}^2 . We deduce that the surface $B_{(0:0:1)} \mathbf{P}^2$ is covered by the three open subsets $U_0, U_1, \widetilde{U}_2 = B_{(0:0:1)} U_2 \simeq B_{(0,0)} \mathbf{A}^2$ hence covered by four open subsets, both isomorphic to \mathbf{A}^2 . Up to affine transformation, we can define $B_P \mathbf{P}^2$ for

any $P \in \mathbf{P}^2$ and it is now clear that the surface $B_P\mathbf{P}^2$ is covered by a finite number of open subsets, each isomorphic to \mathbf{A}^2 . The same is clearly true for $\mathbf{P}^1 \times \mathbf{P}^1$. It is also true for $Q_{3,1}$. Indeed, choose 3 distinct real points P_1, P_2, P_3 of $Q_{3,1}$, and denote the open set $Q_{3,1} \setminus T_{P_i}Q_{3,1}$ by U_i , for $i = 1, 2, 3$. Since the common intersection of the three projective tangent planes is a single point, that, moreover does not belong to $Q_{3,1}$, the subsets U_1, U_2, U_3 constitute an open affine covering of $Q_{3,1}$.

Let X be an algebraic surface and P a real point of X . Assume that P admits a neighborhood U isomorphic to \mathbf{A}^2 which is dense in $X^{(4)}$, and define the blow-up of X at P to be the real algebraic surface obtained from $X \setminus \{P\}$ and $B_P U$ by gluing them along their common open subset $U \setminus \{P\}$. Then $B_P U \simeq B_P U_0$ is dense in $B_P X$ and contains a dense open subset isomorphic to $U_0 \simeq \mathbf{A}^2$. We have to admit at this point that this construction does neither depend of the choice of U , nor on the choice of the isomorphism between U and \mathbf{A}^2 . See e.g. [Sha94, §II.4.1] or [Man14, Appendice A] for a detailed exposition.

We get:

Proposition 2. — *Let*

$$X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_1} X_0 = \mathbf{P}^2, \mathbf{P}^1 \times \mathbf{P}^1 \text{ or } Q_{3,1}$$

be a sequence of blows-up at real points. Then X_n is a real rational surface.

Proof. — Indeed, from Example 1 and comments above, any point $P \in X_i$ admits a neighborhood U isomorphic to \mathbf{A}^2 which is dense in X_i . \square

Let $\pi: B_P X \rightarrow X$ be the blow-up of X at P . The curve $E_P = \pi^{-1}\{P\}$ is the *exceptional curve* of the blow-up. We say that $B_P X$ is the blow-up of X at P and that X is obtained from $B_P X$ by the *contraction* of the curve E_P .

Example 3. — Notice that if P is a real point of X , the resulting blown-up surface gets an anti-holomorphic involution lifting the one of X . If P is not real, we can obtain a real surface anyway by blowing-up both P and \bar{P} : let U be an open neighborhood of P complex isomorphic to $\mathbf{A}^2(\mathbf{C})$ and define $B_{P,\bar{P}} X$ to be the result of the glueing of $X \setminus \{P, \bar{P}\}$ with both $B_P U$ and $B_{\bar{P}} \bar{U}$.

⁽⁴⁾As a corollary of Theorem 9 below we can see that if X is rational, any real point of X has this property. Otherwise said: any (nonsingular) real rational surface is covered by a finite number of open subsets, each isomorphic to \mathbf{A}^2 .

Remark 4. — In Example 1.3, the rational map π_N decomposes into the blow-up of $Q_{3,1}$ at N , followed by the contraction of the strict transform of the curve $z = w$ (intersection of $Q_{3,1}$ with the tangent plane $T_N Q_{3,1}$), which is the union of two non-real conjugate lines. The rational map π_N^{-1} decomposes into the blow-up of the two non-real points $(1 : \pm i : 0)$, followed by the contraction of the strict transform of the line $z = 0$.

The exceptional curve is a real rational curve isomorphic to \mathbf{P}^1 whose real locus $E_P(\mathbf{R})$ is diffeomorphic to the circle \mathbf{S}^1 . Furthermore, the normal bundle of the smooth curve $E_P(\mathbf{R})$ in the smooth surface $B_P X(\mathbf{R})$ is nonorientable, thus $E_P(\mathbf{R})$ possesses a neighborhood diffeomorphic to the Möbius band in $B_P X(\mathbf{R})$. Hence, topologically speaking, $B_P X(\mathbf{R})$ is obtained from $X(\mathbf{R})$ through the following surgery (see Figure 1): from $X(\mathbf{R})$, remove a disk D centered at P (the boundary ∂D is diffeomorphic to the circle \mathbf{S}^1) and paste a Möbius band M (the boundary ∂M is also diffeomorphic to the circle \mathbf{S}^1) to get $B_P X(\mathbf{R})$ which is then diffeomorphic to the connected sum:

$$(1) \quad B_P X(\mathbf{R}) \approx X(\mathbf{R}) \# \mathbf{P}^2(\mathbf{R}) .$$

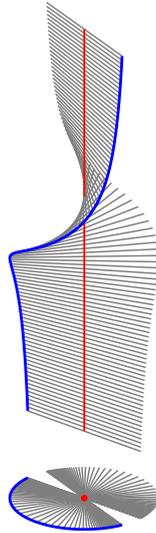


FIGURE 1. The real locus of the exceptional curve is depicted by the vertical line.

In particular $(B_P \mathbf{P}^2)(\mathbf{R}) \approx \mathbf{P}^2(\mathbf{R}) \# \mathbf{P}^2(\mathbf{R})$ is the *Klein bottle*. From the classification of compact connected topological surfaces, we know that any

nonorientable compact connected topological surface S is diffeomorphic to the connected sum of g copies of the real projective plane $\mathbf{P}^2(\mathbf{R})$:

$$S \approx \mathbf{P}^2(\mathbf{R}) \# \mathbf{P}^2(\mathbf{R}) \# \dots \# \mathbf{P}^2(\mathbf{R}) .$$

The nonnegative integer g is uniquely determined by S and is called the *genus* of S . Hence the genus of $\mathbf{P}^2(\mathbf{R})$ is 1 and the genus of the Klein bottle is 2.

Definition 5. — Let S be a compact connected topological surface. A real rational surface X is a *real rational model* of S if the real locus is diffeomorphic to S :

$$X(\mathbf{R}) \approx S .$$

The preceding observations and Examples 1.2 and 1.3 above lead to the following consequence:

Corollary 6. — *Let S be a compact connected topological surface. If S is nonorientable, or orientable of genus 0 or 1, then S admits a real rational model.*

A deep result of Comessatti [Com14, p. 257] states that the other topological surfaces do not have any real rational model:

Theorem 7 (Comessatti). — *Let X be a nonsingular projective real rational surface. Then, if orientable, the real locus $X(\mathbf{R})$ is diffeomorphic to the sphere \mathbf{S}^2 or to the torus $\mathbf{S}^1 \times \mathbf{S}^1$.*

Otherwise said: the real locus of a real rational surface is diffeomorphic to a sphere, a torus, or a nonorientable compact connected topological surface.

A modern proof uses the Minimal Model Program for real algebraic surfaces as developed by Kollár [Kol01, p. 206, Theorem. 30] (see also [Sil89, Prop. 4.3] for an alternative proof). In fact that approach gives us an even more precise statement.

Let X and Y be two real rational models of a given topological surface S . We will say that X and Y are *isomorphic as real rational models* if their real loci $X(\mathbf{R})$ and $Y(\mathbf{R})$ have isomorphic Zariski open neighborhoods in X and Y , respectively. Equivalently, the surfaces $X(\mathbf{R})$ and $Y(\mathbf{R})$ are *birationally diffeomorphic*, that is: there is a diffeomorphism $f: X(\mathbf{R}) \rightarrow Y(\mathbf{R})$ whose coordinate functions are rational functions on $X(\mathbf{R})$ without poles on $X(\mathbf{R})$, and f^{-1} has also coordinate rational functions on $Y(\mathbf{R})$ without poles on $Y(\mathbf{R})$.

Example 8. — Let P be a real point of the sphere $\mathbf{S}^2 = Q_{3,1}(\mathbf{R})$. Then the blow-up $B_P Q_{3,1}$ at P is a real rational model of the topological real projective plane $\mathbf{P}^2(\mathbf{R})$. The projective plane \mathbf{P}^2 is also a real rational model of $\mathbf{P}^2(\mathbf{R})$ as well. Although the real algebraic surfaces $B_P Q_{3,1}$ and \mathbf{P}^2 are not isomorphic, the stereographic projection induces a birational diffeomorphism from $B_P Q_{3,1}(\mathbf{R})$ onto $\mathbf{P}^2(\mathbf{R})$ sending the exceptional curve to the line at infinity. The real rational surfaces $B_P Q_{3,1}$ and \mathbf{P}^2 are therefore isomorphic real rational models of the topological surface $\mathbf{P}^2(\mathbf{R})$.

Collecting preceding observations: $\mathbf{P}^1 \times \mathbf{P}^1$ is a real rational model of the torus $\mathbf{S}^1 \times \mathbf{S}^1$, $Q_{3,1}$ is a real rational model of the sphere \mathbf{S}^2 and if S is a nonorientable topological surface of genus g , the blow-up $B_{P_1, \dots, P_g} Q_{3,1}$, where P_1, \dots, P_g are g distinct real points, is a real rational model of S :

$$B_{P_1, \dots, P_g} Q_{3,1}(\mathbf{R}) \approx B_{P_1, \dots, P_g} \mathbf{S}^2 \approx \mathbf{P}^2(\mathbf{R}) \# \dots \# \mathbf{P}^2(\mathbf{R}) \quad (g \text{ terms}).$$

Using Kollár’s Minimal Model Program [Kol01, *loc. cit.*], one can prove the following statement (compare [BH07, Thm. 3.1]):

Theorem 9. — *Let S be a compact connected topological surface and X be a real rational model of S .*

1. *If S is nonorientable then X is isomorphic to a real rational model of S obtained from $Q_{3,1}$ by successively blowing up at reals points only.*
2. *If S is orientable then X is isomorphic to $Q_{3,1}$ or $\mathbf{P}^1 \times \mathbf{P}^1$ as a real rational model.*

This clearly implies Comessatti’s Theorem above, but it also highlight the importance of classifying real rational models of a given topological surface (compare [Man06, Theorem 1.3 and comments following it]). Surprisingly enough, all real rational models of a given topological surface turn out to be isomorphic as real rational models. This has been proved by Biswas and Huisman [BH07, Thm. 1.2]:

Theorem 10. — *Let S be a compact connected topological surface. Then any two real rational models of S are isomorphic.*

Proof of Theorem 9. — Apply the Minimal Model Program to X in order to obtain a sequence of blows-up

$$X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \dots \xrightarrow{\pi_1} X_0$$

analogous to the one of Proposition 2 except that we allow also blows-up at pairs of nonreal points as in Example 3 and that X_0 is now one of the following (see [Kol01, p. 206, Theorem. 30]):

1. a surface with nef canonical bundle;
2. a conic bundle $X_0 \rightarrow B$ over a nonsingular real algebraic curve with an even number of real singular fibers, each of them being real isomorphic to $x^2 + y^2 = 0$;
3. a "del Pezzo" surface: \mathbf{P}^2 , $Q_{3,1}$ or a del Pezzo surface with non connected real locus;

Since X is rational, X_0 is rational and we proceed through a case by case analysis:

1. The only thing we need to know about the condition "nef canonical bundle" is that a rational surface cannot satisfy such a condition.

2. Since X_0 is rational, the base curve B of the conic bundle is rational, that is B is isomorphic to \mathbf{P}^1 . Since $X_0(\mathbf{R})$ is connected and nonempty, the number of real singular fibers of the conic bundle is 0 or 2. If it is 2, $X_0(\mathbf{R})$ is then diffeomorphic to \mathbf{S}^2 . In fact X_0 is isomorphic to $Q_{3,1}$ blown-up at a pair of nonreal points (see [BM14, Example 2.13(3)] for details). This reduces to the case when X_0 is isomorphic to $Q_{3,1}$. If there is no real singular fibers, X_0 is isomorphic to a \mathbf{P}^1 -bundle over \mathbf{P}^1 . By [Man06, Theorem 1.3], $X_0(\mathbf{R})$ is then birationally diffeomorphic to the Klein bottle $(B_P \mathbf{P}^2)(\mathbf{R})$, see (1), or to the torus $(\mathbf{P}^1 \times \mathbf{P}^1)(\mathbf{R})$. If S is orientable we are done, since $X(\mathbf{R})$ is orientable too, and X is obtained from X_0 by blowing-up at nonreal points only. If S is nonorientable, then $X(\mathbf{R})$ is nonorientable either, and X is obtained from X_0 by blowing-up, at least, one real point. Since $X_0 = \mathbf{P}^1 \times \mathbf{P}^1$, a blow-up of X_0 at one real point is isomorphic to a blow-up of \mathbf{P}^2 at two real points and then is isomorphic as a real rational model to some blow-up of $Q_{3,1}$.

3. The real locus of a real rational surface being connected, this rules out del Pezzo surfaces with non connected real locus.

It remains to show that the statement of the theorem holds if X_0 is isomorphic to \mathbf{P}^2 or to $Q_{3,1}$. If X_0 is isomorphic to \mathbf{P}^2 , then by Example 8, the stereographic projection reduces to the case X_0 is isomorphic to $Q_{3,1}$ as a real rational model. Now if S is orientable, then $X(\mathbf{R})$ is orientable too, and like in the torus case, X is obtained from X_0 by blowing-up at nonreal points only. It follows that X is isomorphic to $Q_{3,1}$ as a real rational model. If S is nonorientable, then $X(\mathbf{R})$ is nonorientable either, and it is obtained from $Q_{3,1}$ by blowing-up at real points. \square

Proof of Theorem 10. — The proof which is given below is quite different from the one in [BH07]; it is built on the fact that the group of self-birational diffeomorphisms of the sphere is *infinitely transitive*, see Theorem 11 in next section, this is the approach followed in [HM09].

A crucial ingredient of the proof of Theorem 10 is the following. Let S be a nonorientable surface, According to Theorem 9, any real rational model X of S is isomorphic to a real rational model Y of S obtained from the sphere $\mathbf{S}^2 = Q_{3,1}(\mathbf{R})$ by successively blowing up real points. This means that there is a sequence of blow-ups at real points

$$Y = Y_n \xrightarrow{\pi_n} \dots \longrightarrow Y_2 \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0 = Q_{3,1} .$$

If, for example, $Y_2 = B_P(B_Q Q_{3,1})$ is the blow-up of $Y_1 = B_Q Q_{3,1}$ at a real point P of the exceptional curve $E_Q \subset Y_1$ of π_1 , it is not *a priori* clear that we can reduce to the case where Y_2 is the blow-up of $Q_{3,1}$ at two distinct points of $Q_{3,1}(\mathbf{R}) = \mathbf{S}^2$. One gets rid of this difficulty by using Example 8. For simplicity, we explain this in the case $n = 2$.

The algebraic surface Y_1 is a real rational model of $\mathbf{P}^2(\mathbf{R})$ isomorphic to \mathbf{P}^2 , i.e. there is a birational diffeomorphism $f_Q: Y_1(\mathbf{R}) \rightarrow \mathbf{P}^2(\mathbf{R})$. Up to projectivities, we get moreover that for any real projective line D of \mathbf{P}^2 , there is a birational diffeomorphism that maps the set of real points $E_Q(\mathbf{R})$ of the exceptional curve E_Q to the real locus $D(\mathbf{R})$. Choose a real projective line $D(\mathbf{R})$ of $\mathbf{P}^2(\mathbf{R})$ that does not contain the real point $f_Q(P)$ of \mathbf{P}^2 .

There is a blow-up $Y'_1 = B_{Q'} Q_{3,1}$ of the sphere at a real point, and a birational diffeomorphism $f_{Q'}: Y'_1(\mathbf{R}) \rightarrow \mathbf{P}^2(\mathbf{R})$ mapping the real locus of the exceptional curve $E_{Q'}$ onto $D(\mathbf{R})$. Let $f = f_{Q'}^{-1} \circ f_Q$ and P' be the real point of Y'_1 corresponding to P via the birational diffeomorphism $f: Y_1(\mathbf{R}) \rightarrow Y'_1(\mathbf{R})$. Then the point P' is not a point of the exceptional curve of the blow-up $\pi': Y'_1 = B_{Q'} Q_{3,1} \rightarrow Q_{3,1}$; which means that π' maps isomorphically some affine neighborhood of P' to an affine neighborhood of $\pi'(P')$.

Since there is a birational diffeomorphism from $Y_1(\mathbf{R})$ into $Y'_1(\mathbf{R})$ that maps P to P' , there is also a birational diffeomorphism from $Y_2(\mathbf{R})$ into $Y'_2(\mathbf{R})$, the real locus of the blow-up Y'_2 of Y'_1 at P' . Now, $Y'_2 = B_{\pi'(P'), Q'} Q_{3,1}$ is the blow-up of $Q_{3,1}$ at 2 distinct real points, and is isomorphic as a real rational model to $Y_2 = B_P(B_Q Q_{3,1})$.

By an induction argument, one shows more generally that any real rational model X of a nonorientable compact connected topological surface of genus g is isomorphic to the blow-up $B_{P_1, \dots, P_g} Q_{3,1}$ where P_1, \dots, P_g are g distinct real points of the sphere.

The second main ingredient of the proof is the fact that for any two g -tuples (P_1, \dots, P_g) and (Q_1, \dots, Q_g) of distinct elements of \mathbf{S}^2 , there is a birational diffeomorphism $f: \mathbf{S}^2 \rightarrow \mathbf{S}^2$ such that $f(P_i) = Q_i$ for all i (see Theorem 11 below). Hence the blow-up $B_{P_1, \dots, P_g} Q_{3,1}$ is birationally diffeomorphic to the blow-up $B_{Q_1, \dots, Q_g} Q_{3,1}$. \square

3. Automorphism groups of real loci

The group of automorphisms of a complex algebraic variety is small: indeed, it is finite in general. Moreover, the group of automorphisms is 3-transitive only if the variety is $\mathbf{P}^1(\mathbf{C})$. On the other hand, the group $\text{Aut}(X(\mathbf{R}))$ of birational self-diffeomorphisms (also called automorphisms of $X(\mathbf{R})$) of a real rational surface X is quite big as the next result shows.

Recall that a group G , acting on a set M , acts *n-transitively* on M if for any two n -tuples (P_1, \dots, P_n) and (Q_1, \dots, Q_n) of distinct elements of M , there is an element g of G such that $g \cdot P_i = Q_i$ for all i . The group G acts *infinitely transitively* ⁽⁵⁾ on M if for every positive integer n , its action is n -transitive on M . The next result is proved in [HM09, Thm.1.4].

Theorem 11. — *Let X be a nonsingular projective real rational surface. Then the group of birational diffeomorphisms $\text{Aut}(X(\mathbf{R}))$ acts infinitely transitively on $X(\mathbf{R})$.*

Proof. — In order to give an idea of the proof of the above theorem, let us show how one can construct many birational diffeomorphisms of the sphere $Q_{3,1}(\mathbf{R}) \approx \mathbf{S}^2$. Let I be the interval $[-1, 1]$ in \mathbf{R} . Choose any smooth rational map $f: I \rightarrow \mathbf{S}^1$. This simply means that the two coordinate functions of f are rational functions in one variable without poles in I . Define $\phi_f: \mathbf{S}^2 \rightarrow \mathbf{S}^2$ (ϕ_f is called the *twisting map* associated to f) by

$$\phi_f(x, y, z) = (f(z) \cdot (x, y), z)$$

where \cdot denotes complex multiplication in $\mathbf{R}^2 = \mathbf{C}$. Then ϕ_f is a birational self-diffeomorphism of \mathbf{S}^2 . Indeed, its inverse is ϕ_g where $g: I \rightarrow \mathbf{S}^1$ maps z to the multiplicative inverse $(f(z))^{-1}$ of $f(z)$. Now let x_1, \dots, x_n be n distinct points of I and ρ_1, \dots, ρ_n be elements of \mathbf{S}^1 . Then from Lagrange polynomial interpolation, there is a smooth rational map $f: I \rightarrow \mathbf{S}^1$ such that $f(x_j) = \rho_j$ for $j = 1, \dots, n$. The isomorphism $\mathbf{S}^1 \simeq \mathbf{SO}(2, \mathbf{R})$ makes multiplication by ρ_i

⁽⁵⁾In the literature, an *infinitely transitive* group action is sometimes called a *very transitive* action.

a rotation, hence there exists a twisting map ϕ_f which moves n given distinct points P_1, \dots, P_n on the sphere to n another given points R_1, \dots, R_n provided that each pair P_i, R_i (same i) belong to a vertical plane ($z = cst$). To get a birational self-diffeomorphism mapping each P_i to each Q_i from the original n -tuples, it suffices to consider two transversal families of parallel planes in order to get n intersection points R_i , then up to linear changes of coordinates, apply twice the preceding construction to get 2 twisting maps, see Figure 2. The composition of these twisting maps gives the desired birational self-diffeomorphism. ⁽⁶⁾

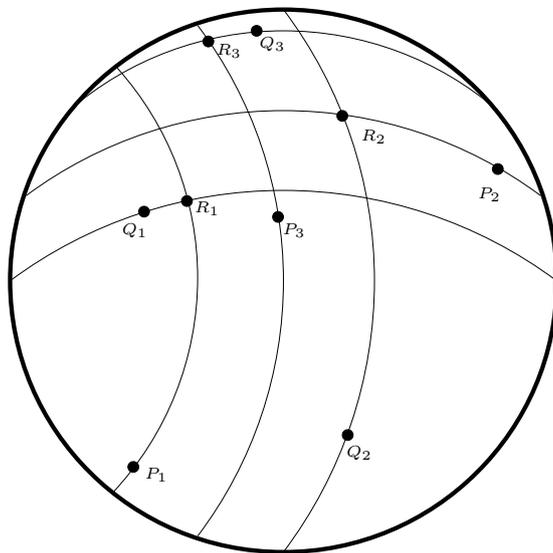


FIGURE 2. The sphere \mathbf{S}^2 with two sets of parallels.

□

Theorem 11 deals with real algebraic surfaces which are rational. More generally, a real algebraic surface is *geometrically rational* if the complex surface (that is the real surface without the anti-holomorphic involution) contains a dense open subset complex isomorphic to $\mathbf{A}^2(\mathbf{C})$. Clearly, a real rational surface is geometrically rational but the converse is not true. In the paper [BM11,

⁽⁶⁾By induction on the dimension, we can prove with this construction that in fact the group $\text{Aut}(\mathbf{S}^n)$ acts infinitely transitively on \mathbf{S}^n for $n > 1$.

Thm. 1], the question of infinite transitivity of the automorphism group is settled for geometrically rational surfaces and in fact for all real algebraic surfaces. Below is one result of *ibid.*

Theorem 12. — *Let X be a real algebraic surface⁽⁷⁾. The group $\text{Aut}(X(\mathbf{R}))$ is then infinitely transitive on each connected component if and only if X is geometrically rational and $\#X(\mathbf{R}) \leq 3$.*

In the statement above, the action of $\text{Aut}(X(\mathbf{R}))$ on $X(\mathbf{R})$ is said to be infinitely transitive *on each connected component* if for any pair of n -tuples of distinct points (P_1, \dots, P_n) and (Q_1, \dots, Q_n) of $X(\mathbf{R})$ such that for each i , P_i and Q_i belong to the same connected component of $X(\mathbf{R})$, there exists a birational diffeomorphism $f: X(\mathbf{R}) \rightarrow X(\mathbf{R})$ such that $f(P_i) = Q_i$ for all i .

Remark 13. — The infinite transitivity of the automorphism groups of real algebraic varieties has been proved also for rational surfaces with mild singularities in [HM10]; and the question of infinite transitivity in the context of affine varieties is studied in [KM12].

A closely related line of research studies generators of $\text{Aut}(X(\mathbf{R}))$ for various real rational surfaces X . The classical Noether-Castelnuovo Theorem [Cas01] (see also [AC02, Chapter 8] for a modern exposition of the proof) gives generators of the group $\text{Bir}_{\mathbf{C}}(\mathbf{P}^2)$ of birational transformations of the complex projective plane. The group is generated by the biregular automorphisms, which form the group $\text{Aut}_{\mathbf{C}}(\mathbf{P}^2) = \mathbf{PGL}(3, \mathbf{C})$ of projectivities, and by the standard quadratic transformation

$$\sigma_0: (x : y : z) \dashrightarrow (yz : xz : xy).$$

This result does not work over the real numbers. Indeed, recall that a *base point* of a birational transformation is a (possibly infinitely near) point of indeterminacy; and note that two of the base points of the quadratic involution

$$\sigma_1: (x : y : z) \dashrightarrow (y^2 + z^2 : xy : xz)$$

are not real. Thus σ_1 cannot be generated by projectivities and σ_0 . More generally, we cannot generate this way maps having nonreal base-points. Hence the group $\text{Bir}_{\mathbf{R}}(\mathbf{P}^2)$ of birational transformations of the real projective plane is not generated by $\text{Aut}_{\mathbf{R}}(\mathbf{P}^2) = \mathbf{PGL}(3, \mathbf{R})$ and σ_0 .

⁽⁷⁾As always, smooth and projective

The main result of [BM14, Thm. 1.1] is that $\text{Bir}_{\mathbf{R}}(\mathbf{P}^2)$ is generated by $\text{Aut}_{\mathbf{R}}(\mathbf{P}^2)$, σ_0 , σ_1 , and a family of birational maps of degree 5 having only nonreal base-points:

Example 14. — Let $p_1, \bar{p}_1, p_2, \bar{p}_2, p_3, \bar{p}_3 \in \mathbf{P}^2$ be three pairs of non-real points of \mathbf{P}^2 , not lying on the same conic. Denote by $\pi: X \rightarrow \mathbf{P}^2$ the blow-up of the six points, which induces a birational diffeomorphism $X(\mathbf{R}) \rightarrow \mathbf{P}^2(\mathbf{R})$. Note that X is isomorphic to a smooth cubic surface in \mathbf{P}^3 . The set of strict transforms of the conics passing through five of the six points corresponds to three pairs of non-real lines on the cubic, and the six lines are disjoint. The contraction of the six lines gives a birational morphism $\eta: X \rightarrow \mathbf{P}^2$, inducing an isomorphism $X(\mathbf{R}) \rightarrow \mathbf{P}^2(\mathbf{R})$, which contracts the curves onto three pairs of non-real points $q_1, \bar{q}_1, q_2, \bar{q}_2, q_3, \bar{q}_3 \in \mathbf{P}^2$; we choose the order so that q_i is the image of the conic not passing through p_i . The map $\psi = \eta\pi^{-1}$ is a birational map $\mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ inducing a birational diffeomorphism $\mathbf{P}^2(\mathbf{R}) \rightarrow \mathbf{P}^2(\mathbf{R})$.

Theorem 15. — *The group $\text{Bir}_{\mathbf{R}}(\mathbf{P}^2)$ is generated by $\text{Aut}_{\mathbf{R}}(\mathbf{P}^2)$, σ_0 , σ_1 , and by the quintic transformations of \mathbf{P}^2 defined in Example 14.*

The proof is based on an extensive study of Sarkisov links. As a consequence, [BM14] recover the set of generators given in [RV05, Teorema II]:

Theorem 16. — *The group $\text{Aut}(\mathbf{P}^2(\mathbf{R}))$ is generated by*

$$\text{Aut}_{\mathbf{R}}(\mathbf{P}^2) = \mathbf{PGL}(3, \mathbf{R})$$

and by the quintic transformations of \mathbf{P}^2 defined in Example 14.

And also the set of generators given in [KM09, Thm. 1]:

Theorem 17. — *The group $\text{Aut}(Q_{3,1}(\mathbf{R}))$ is generated by*

$$\text{Aut}_{\mathbf{R}}(Q_{3,1}) = \mathbf{PO}(3, 1)$$

and by the cubic transformations defined in [BM14, Example 5.1].

As remarked in [BM14, Proposition 5.6], the twisting maps defined in the proof of Theorem 11 are compositions of twisting maps of degree 1 and 3. And in the latter case the twisting maps belong to the set of cubic transformations used in the above theorem.

A new set of generators, completing the list for "minimal" real rational surfaces is also given [BM14, Thm. 1.4]:

Theorem 18. — *The group $\text{Aut}((\mathbf{P}^1 \times \mathbf{P}^1)(\mathbf{R}))$ is generated by*

$$\text{Aut}_{\mathbf{R}}(\mathbf{P}^1 \times \mathbf{P}^1) \cong \text{PGL}(2, \mathbf{R})^2 \rtimes \mathbf{Z}/2\mathbf{Z}$$

and by the involution

$$\tau_0: ((x_0 : x_1), (y_0 : y_1)) \dashrightarrow ((x_0 : x_1), (x_0y_0 + x_1y_1 : x_1y_0 - x_0y_1)).$$

Remark 19. — For the interested reader, we put the stress on recent "real" results on Cremona groups: a rather complete classification of real structures on del Pezzo surfaces [Rus02]; the study of the structure of some real subgroups of the Cremona group [Rob14] and [Zim14].

4. Approximation of differentiable maps by algebraic maps

We have defined *real rational models* of topological surfaces in Section 2. More generally, let M be a compact \mathcal{C}^∞ -manifold without boundary; a real algebraic manifold X is a *real algebraic model* of M if the real locus is diffeomorphic to M :

$$X(\mathbf{R}) \approx M.$$

Clearly, a topological surface admitting a real rational model admits also a real algebraic model but the converse is not true by Comessatti's Theorem 7 and the fact that one of the two real algebraic surfaces given by the affine equations $z^2 = \pm f(x, y)$, where f is the product of equations of 3 well chosen circles, is a real algebraic model⁽⁸⁾ of a genus 2 orientable surface. A striking theorem of Nash [Nas52] improved by Tognoli [Tog73] is the following:

Theorem 20 (Nash 1952, Tognoli 1973). — *Let M be compact \mathcal{C}^∞ -manifold without boundary, then there exists a nonsingular projective real algebraic variety X whose real locus is diffeomorphic to M :*

$$M \approx X(\mathbf{R}).$$

One of the most famous application of the Nash Theorem is the Theorem of Artin-Mazur [AM65] below. For any self-map $f: M \rightarrow M$, denote by $N_\nu(f)$ the number of *isolated* periodic points of f , of period ν (*i.e.*, the number of isolated fixed points of f^ν).

⁽⁸⁾In fact, such a surface is not a manifold since it has nonreal singular points; but it is easy to get a manifold by "resolution" of these singular points or by a small deformation of the plane curve $f(x, y) = 0$.

Theorem 21. — *Let M be a compact C^∞ -manifold⁽⁹⁾ without boundary, and let $F(M)$ be the space of C^∞ -self maps of M endowed with the C^∞ -topology. There is a dense subset of $\mathcal{E} \subset F(M)$ such that if $f \in \mathcal{E}$, then $N_\nu(f)$ grows at most exponentially (as ν varies through the positive integers).*

The question has been raised whether the group $\text{Aut}(X(\mathbf{R}))$ is dense in the group $\text{Diff}(X(\mathbf{R}))$ of all self-diffeomorphisms of $X(\mathbf{R})$, for a real rational surface X . This turns out to be true and has been proved in [KM09, Theorem 4], see Theorem 22 below. Before stating the whole result, we want to stress here a big gap between the Nash diffeomorphisms used to prove Artin-Mazur’s Theorem and the birational diffeomorphisms. A diffeomorphism which is also a rational map without poles on the real locus is a Nash diffeomorphism but not necessarily a birational diffeomorphism. Indeed, the converse diffeomorphism is not always rational. For instance the map $x \mapsto x + x^3$ is a Nash self-diffeomorphism of \mathbf{R} but it is not birational since the converse map has radicals. This is a consequence of the fact that Implicit function Theorem holds in analytic setting but does not hold in the algebraic setting.

Theorem 22. — [KM09, Theorem 4]

Let S be a compact connected topological surface and $\text{Diff}(S)$ its group of self-diffeomorphisms. Then

1. *If S is nonorientable or of genus $g(S) \leq 1$, then there exists a real algebraic model X of S such that $\overline{\text{Aut}(X(\mathbf{R}))} = \text{Diff}(X(\mathbf{R}))$;*
2. *If S is orientable of genus $g(S) \geq 2$, then for any real algebraic model X of S , we have $\overline{\text{Aut}(X(\mathbf{R}))} \neq \text{Diff}(X(\mathbf{R}))$.*

Sketch of proof. — We start by proving the second part of the theorem. Let X be a real algebraic surface with orientable real locus. Then following up the classification of surfaces (see e.g. [BHPVdV04, Sil89]): if X is geometrically rational or ruled, then $X(\mathbf{R}) \approx \mathbf{S}^2$ or $X(\mathbf{R}) \approx \mathbf{S}^1 \times \mathbf{S}^1$; if X is K3 or abelian, then $\text{Aut}(X(\mathbf{R}))$ preserves a volume form, hence density does not hold; if X is Enriques or bi-elliptic, it admits a finite cover by one surface in the former case, hence density does not hold; if X is properly elliptic, then $\text{Aut}(X(\mathbf{R}))$ preserves a fibration, hence density does not hold; if X is of general type, then $\text{Aut}(X(\mathbf{R}))$ is finite, hence density does not hold. Summing up, if $g(S) > 1$, then for any real algebraic model, density does not hold.

⁽⁹⁾In fact, the following results are valid for any C^k -regularity, $k = 1, \dots, \infty$.

Proof of part 1. Any such topological surface admits a real rational model which is $\mathbf{P}^1 \times \mathbf{P}^1$, $Q_{3,1}$ or the blow-up $B_{P_1, \dots, P_g} Q_{3,1}$ where P_1, \dots, P_g are g distinct real points of the sphere. Leaving aside the torus case for simplicity, we start with a theorem of Lukackiĭ to the effect that the density holds for the sphere. The paper [Luk77, Thm. 2] proves indeed that for any integer $n > 1$, the topological group $\mathbf{SO}(n+1, 1)$ is a maximal closed subgroup of the neutral component $\text{Diff}_0(\mathbf{S}^n)$ of $\text{Diff}(\mathbf{S}^n)$, meaning that any topological subgroup of the topological group $\text{Diff}_0(\mathbf{S}^n)$ containing $\mathbf{SO}(n+1, 1)$ is dense in $\text{Diff}_0(\mathbf{S}^n)$. In particular, the group $\mathbf{O}(n+1, 1)$ and anything else generate a dense subgroup of $\text{Diff}(\mathbf{S}^n)$. Thanks to this argument, we prove that $\overline{\text{Aut}(\mathbf{S}^n)} = \text{Diff}(\mathbf{S}^n)$ for $X(\mathbf{R}) \approx \mathbf{S}^n$, $n > 1$. For the case we are concerned with, the group $\mathbf{O}(3, 1)$ together with any nontrivial twisting map⁽¹⁰⁾ of \mathbf{S}^2 generate a dense subgroup of $\text{Diff}(\mathbf{S}^2)$.

The remaining cases are the nonorientable surfaces $B_{P_1, \dots, P_g} Q_{3,1}(\mathbf{R})$. Let $X = B_{P_1, \dots, P_g} Q_{3,1}$. The proof is in three steps:

1. (Marked points). Let f be a self-diffeomorphism of \mathbf{S}^2 . Let g be a birational self-diffeomorphism of \mathbf{S}^2 close to f given by density. Then the point $Q_i = g(P_i)$ is just as close to P_i for $i = 1, \dots, g$. By Theorem 11, we get a birational self-diffeomorphism h such that $P_i = h(Q_i)$ for $i = 1, \dots, g$. Moreover, the construction of such a h shows that h is just as close to identity. Thus, starting with a map g closer to f if needed, we get that the group $\text{Aut}(\mathbf{S}^2, P_1, \dots, P_g)$ of birational self-diffeomorphisms of \mathbf{S}^2 fixing each P_i is dense in the group $\text{Diff}(\mathbf{S}^2, P_1, \dots, P_g)$ of self-diffeomorphisms of \mathbf{S}^2 fixing each P_i .
2. (Identity component). By a partition of unity argument (Fragmentation Lemma), we deduce the density of the identity component $\text{Aut}_0(X(\mathbf{R}))$ in $\text{Diff}_0(X(\mathbf{R}))$, see [KM09, Proposition 20] for details.
3. (Mapping class group). The conclusion follows from the fact that the *modular group* $\text{Mod}(X(\mathbf{R})) = \text{Diff}(X(\mathbf{R}))/\text{Diff}_0(X(\mathbf{R}))$ (also called the *mapping class group*) is generated by birational self-diffeomorphisms of $X(\mathbf{R})$, see Theorem 23 below.

□

Let X be a real algebraic model of a topological surface S , then for any diffeomorphism $g: S \rightarrow X(\mathbf{R})$, the map $\text{Diff}(S) \rightarrow \text{Diff}(X(\mathbf{R}))$, $f \mapsto g^{-1} \circ f \circ g$ induces a group isomorphism $\text{Mod}(S) \rightarrow \text{Mod}(X(\mathbf{R}))$.

⁽¹⁰⁾See p. 10

Theorem 23. — [KM09, Theorem 27]

Let S be a nonorientable compact connected topological surface and $\text{Mod}(S)$ its modular group. Then there exists a real algebraic model X of S such that the group homomorphism

$$\pi: \begin{cases} \text{Aut}(X(\mathbf{R})) & \longrightarrow & \text{Mod}(X(\mathbf{R})) \\ f & \longmapsto & [f] \end{cases}$$

is surjective.

In the above statement, the model X is rational; it is straightforward to see that this statement is also true for the sphere \mathbf{S}^2 , whose modular group is generated by the antipodal map, and the torus $\mathbf{S}^1 \times \mathbf{S}^1$, whose modular group is $\mathbf{SL}(2, \mathbf{Z})$. Thus any surface S admitting a real rational model satisfies the statement.

A byproduct of the proof of Theorem 22 is that $\text{Aut}(X(\mathbf{R}))$ is dense in $\text{Diff}(X(\mathbf{R}))$ when X is a geometrically rational surface with $\#X(\mathbf{R}) = 1$ (or equivalently when X is rational, see [Sil89, Corollary VI.6.5]). In [KM09], it is said that $\#X(\mathbf{R}) = 2$ is probably the only other case where the density holds. This case remains open nowadays. The following sums up the known results in this direction.

Theorem 24. — [KM09, BM11]

Let X be a smooth real projective surface.

– If X is not a geometrically rational surface, then

$$\overline{\text{Aut}(X(\mathbf{R}))} \neq \text{Diff}(X(\mathbf{R})) ;$$

– If X is a geometrically rational surface, then

- If $\#X(\mathbf{R}) \geq 5$, then $\overline{\text{Aut}(X(\mathbf{R}))} \neq \text{Diff}(X(\mathbf{R}))$;
- if $\#X(\mathbf{R}) = 1$, then $\overline{\text{Aut}(X(\mathbf{R}))} = \text{Diff}(X(\mathbf{R}))$.

For $i = 3, 4$, there exists smooth real projective surfaces X with $\#X(\mathbf{R}) = i$ such that $\overline{\text{Aut}(X(\mathbf{R}))} \neq \text{Diff}(X(\mathbf{R}))$.

Note that the study of automorphism groups of other real algebraic surfaces than the rational ones has been developed from the point of view of topological entropy of automorphisms by several authors. In particular, Moncet [Mon12] defines the *concordance* $\alpha(X)$ for a real algebraic surface X which is a number between 0 and 1 with the property that $\overline{\text{Aut}_{\mathbf{R}}(X)} \neq \text{Diff}(X(\mathbf{R}))$ as soon as $\alpha(X) > 0$. (Notice that $\text{Aut}_{\mathbf{R}}(X)$ is the subgroup of $\text{Aut}(X(\mathbf{R}))$ of real automorphisms of the real algebraic surface X .)

An important application of the Density Theorem 22 is the following, see [KM14]:

Theorem 25. — *An embedded circle in a nonsingular real rational surface admits a C^∞ -approximation by smooth rational curves if and only if it is not diffeomorphic to a null-homotopic circle on a torus.*

Corollary 26. — *Let X be a nonsingular real rational variety, then an embedded circle is approximated by smooth rational curves if and only if it is not diffeomorphic to a null-homotopic circle on a 2-dimensional torus.*

5. Regulous maps

In general the problem of approximation of differentiable maps by algebraic maps is still open. For instance, the existence of algebraic representatives of homotopy classes of continuous maps between spheres of different dimension does not have a complete solution nowadays. In [Kuc09], Kucharz introduces the notion of *continuous rational maps* generalizing algebraic maps between real algebraic varieties. The particular case of continuous rational functions has also been studied by Kollár very recently, see Kollár-Nowak [KN14]. Continuous rational maps between nonsingular⁽¹¹⁾ real algebraic varieties are now often called *regulous maps* following [FHMM15].

Let X and Y be irreducible nonsingular real algebraic varieties whose sets of real points are dense. A *regulous map* from $X(\mathbf{R})$ to $Y(\mathbf{R})$ is a rational map $f: X \dashrightarrow Y$ with the following property. Let $U \subset X$ be the domain of the rational map f . The restriction of f to $U(\mathbf{R})$ extends to a continuous map from $X(\mathbf{R})$ to $Y(\mathbf{R})$ for the euclidean topology. Kucharz shows that all homotopy classes can be represented by regulous maps [Kuc09, Thm. 1.1].

Theorem 27. — *Let n and p be nonzero natural integers. Any continuous map from \mathbf{S}^n to \mathbf{S}^p is homotopic to a regulous map.*

In fact the statement is more precise: Let n, p and k be natural integers, n and p being nonzero. Any continuous map from \mathbf{S}^n to \mathbf{S}^p is homotopic to a k -regulous map. see below.

The paper [FHMM15] sets up foundations of a regulous geometry: algebra of regulous functions and regulous topologies. Here is a short account. Recall that a rational function f on \mathbf{R}^n is called a *regular function* on \mathbf{R}^n if f has no

⁽¹¹⁾In the singular case, the two notions may differ, see [KN14].

pole on \mathbf{R}^n . For instance, the rational function $f(x) = 1/(x^2 + 1)$ is regular on \mathbf{R} . The set of regular functions on \mathbf{R}^n is a subring of the field $\mathcal{K}(\mathbf{R}^n)$ of rational function on \mathbf{R}^n . A *regulous function* on \mathbf{R}^n is a real valued function defined at any point of \mathbf{R}^n , which is continuous for the euclidean topology and whose restriction to a nonempty Zariski open set is regular. A typical example is the function

$$f(x, y) = \frac{x^3}{x^2 + y^2}$$

which is regular on $\mathbf{R}^2 \setminus \{0\}$ and regulous on the whole \mathbf{R}^2 . Its graph is the canopy of the famous Cartan umbrella, see Figure 3. The set of regulous

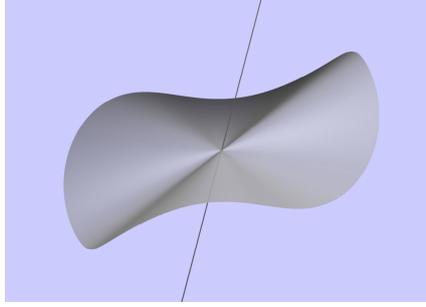


FIGURE 3. The Cartan umbrella: $z(x^2 + y^2) = x^3$.

functions on \mathbf{R}^n is a subring $\mathcal{R}^0(\mathbf{R}^n)$ of the field $\mathcal{K}(\mathbf{R}^n)$. More generally, a function defined on \mathbf{R}^n is *k-regulous*, if it is at the same time, regular on a nonempty Zariski open set, and of class \mathcal{C}^k on \mathbf{R}^n . Here, $k \in \mathbf{N} \cup \{\infty\}$. For instance, the function

$$f(x, y) = \frac{x^{3+k}}{x^2 + y^2}$$

is *k-regulous* on \mathbf{R}^2 for any natural integer k . We can prove that an ∞ -regulous function on \mathbf{R}^n is in fact regular (the converse statement is straightforward) and we get an infinite chain of subrings:

$$\mathcal{R}^\infty(\mathbf{R}^n) \subseteq \dots \subseteq \mathcal{R}^2(\mathbf{R}^n) \subseteq \mathcal{R}^1(\mathbf{R}^n) \subseteq \mathcal{R}^0(\mathbf{R}^n) \subseteq \mathcal{K}(\mathbf{R}^n).$$

where $\mathcal{R}^k(\mathbf{R}^n)$ denotes the subring of $\mathcal{K}(\mathbf{R}^n)$ consisting of *k-regulous* functions.

The *k-regulous* topology is the topology whose closed sets are zero sets of *k-regulous* functions. Figure 4 represents a "horned umbrella" which is an algebraic subset of \mathbf{R}^3 irreducible for the ∞ -regulous topology, but reducible for the *k-regulous* topology for any natural integer k .

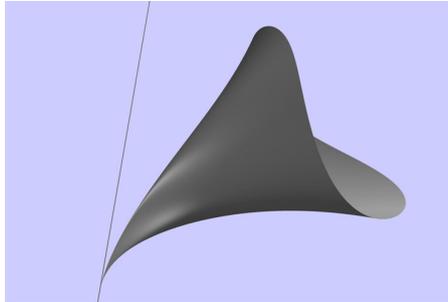


FIGURE 4. A horned umbrella: $x^2 + y^2((y - z^2)^2 + yz^3) = 0$.

In the paper [FHMM15], several properties of the rings $\mathcal{R}^k(\mathbf{R}^n)$ are established. In particular, a strong Nullstellensatz is proved. The scheme theoretic properties are studied and regulous versions of Theorems A and B of Cartan are proved. There is also a geometrical characterization of prime ideals of $\mathcal{R}^k(\mathbf{R}^n)$ in terms of the zero-locus of regulous functions and a relation between k -regulous topology and the topology generated by euclidean closed Zariski-constructible sets. Many papers are related to this new line of research and among them see: [Kuc13, BKVV13, KK13, Kuc14a, Kuc14b, Now14] [FMQ14].

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