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# LONG TIME BEHAVIOUR OF A HAWKES PROCESS-BASED LIMIT ORDER BOOK

FRÉDÉRIC ABERGEL AND AYMEN JEDIDI

**ABSTRACT.** Hawkes processes provide a natural framework to model dependencies between the intensities of point processes. In the context of order-driven financial markets, the relevance of such dependencies has been amply demonstrated from an empirical, as well as theoretical, standpoint. In this work, we build on previous empirical and numerical studies and introduce a mathematical model of limit order books based on Hawkes processes with exponential kernels. After proving a general stationarity result, we focus on the long-time behaviour of the limit order book and the corresponding dynamics of the suitably rescaled price. A formula for the asymptotic (in time) volatility of the price dynamics induced by that of the order book is obtained, involving the average of functions of the various order book events under the stationary distribution.

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F. Abergel, A. Jedidi, Chaire de Finance Quantitative, Laboratoire de Mathématiques Appliquées aux Systèmes, CentraleSupélec, 92290 Châtenay-Malabry, France.

## 1. INTRODUCTION

Since their introduction, Hawkes processes have been applied to a wide range of research areas, from seismology in the pioneering work [24] to credit risk, financial contagion and more recently, to the modelling of market microstructure. Among the growing literature in this latter field, [4, 3, 5] or [13, 12] introduce and study models where the joint price and order flow dynamics are driven by Hawkes processes. Several recent papers [23, 19, 28] are concerned with the stability of Hawkes processes calibrated to price dynamics, and [20] addresses the optimal execution strategies when the market orders are modelled *via* Hawkes processes. Closer in spirit to the present work, [34, 33] are empirical and numerical studies of Hawkes processes modelling limit order books, and [36] is a stylized order book model driven by Hawkes processes.

As it turns out, the relevance of Hawkes processes for limit order book modelling is amply demonstrated by several empirical properties of the order flow of market and limit orders at the microscopic level, in particular:

- time clustering: order arrivals are highly clustered in time, see e.g. [9][13];
- mutual excitation: order flow exhibit non-negligible cross-dependencies. For instance, as documented in [34][2][17], market orders excite limit orders, and limit orders that change the price excite market orders.

At the microscopic level, point process-based microstructure models capture, by construction, the intrinsic discreteness of prices and volumes. In this respect, they offer a natural framework in which to model the finer scales properties of order-driven markets. They do however exhibit a level of complexity that hides some more macroscopic properties of the markets. A question of interest in this context is then the *microscopic-to-macroscopic* transition in the price dynamics. This strand of research has attracted a lot of interest of late [1, 5, 10, 26, 6, 36], and the present work is a contribution to the domain: by casting a Hawkes process-based limit order book model into a Markovian setting and using techniques from the ergodic theory of Markov processes, we show that the order book is ergodic and leads to a

diffusive behaviour of the price at large time scales. Furthermore, we provide formulae for the asymptotic trend and variance as average of functions of the state variables under the stationary distribution.

The paper is organized as follows: Section 2 recalls some well-known facts about Hawkes processes with exponential kernels, while Section 3 describes the limit order book model we consider. In Section 4, the ergodicity of the limit order book is proven under reasonable assumptions, and the main results on the price dynamics are presented in Section 5. Finally, Section 6 contains a short summary and directions for future research.

## 2. HAWKES PROCESSES

We briefly recall in this section several classical results on multivariate Hawkes processes, and refer the interested reader to [7][14][15] for an in-depth treatment of point processes.

Let  $\mathbf{N} = (N^1, \dots, N^D)$  be a  $D$ -dimensional point process with intensity vector  $\lambda = (\lambda^1, \dots, \lambda^D)$ .

**Definition 2.1.** *We say that  $\mathbf{N} = (N^1, \dots, N^D)$  is a multivariate Hawkes process with exponential kernel if there exists  $(\lambda_0^i)_{1 \leq i \leq D} \in (\mathbb{R}_+^*)^D$ ,  $(\alpha^{ij})_{1 \leq i, j \leq D} \in (\mathbb{R}_+)^{D^2}$  and  $(\beta^{ij})_{1 \leq i, j \leq D} \in (\mathbb{R}_+^*)^{D^2}$  such that the intensities satisfy the following set of relations:*

$$\lambda_t^m = \lambda_0^m + \sum_{j=1}^D \alpha_{mj} \int_0^t e^{-\beta_{mj}(t-s)} dN_s^j$$

for  $1 \leq m \leq D$ .

The particular choice of exponential kernels is motivated by an important result that we now recall:

**Proposition 2.1.** *Define the processes  $\mu^{ij}$  as*

$$\mu_t^{ij} = \alpha_{ij} \int_0^t e^{-\beta_{ij}(t-s)} dN_s^j, \quad 1 \leq i, j \leq D,$$

and let  $\boldsymbol{\mu} = \{\mu^{ij}\}_{1 \leq i, j \leq D}$ . Then, the process  $(\mathbf{N}, \boldsymbol{\mu})$  is Markovian.

*Proof.* Lemma 6 in [30] gives a proof of this result. □

2.0.1. *Stationarity.* Extending the early stability and stationarity result in [25], Theorem 5 in [30] proves a general stability result for the multivariate Hawkes processes just introduced. In fact, under the slightly more stringent condition of *irreducibility*, one can show the existence of a Lyapunov function for such a process, that is, a nonnegative, real-valued function  $V := V(\boldsymbol{\mu})$  that goes to  $\infty$  as  $\boldsymbol{\mu} \rightarrow \infty$  and such that  $\mathcal{L}V \leq -\gamma V$  for some  $\gamma > 0$ , where  $\mathcal{L}$  is the infinitesimal generator associated to the Hawkes process.

The existence of such a function actually implies exponential convergence towards the stationary distribution, based on the concept of *V-geometric ergodicity* of [32], see e.g. [1].

We summarize these results in the

**Proposition 2.2.** *Let the matrix  $\mathbf{A}$  be defined by*

$$\mathbf{A}_{ij} = \frac{\alpha_{ji}}{\beta_{ji}}, \quad 1 \leq i, j \leq D.$$

*Assume furthermore that  $\mathbf{A}$  is irreducible and that its spectral radius  $\rho(\mathbf{A})$  satisfies the condition*

$$\rho(\mathbf{A}) < 1. \tag{1}$$

*Then, there exists a (unique) multivariate point process  $\mathbf{N} = (N^1, \dots, N^m)$  whose intensity is specified as in Definition 2.1. Moreover, this process is stable, and converges exponentially fast in the total variation norm towards its unique stationary distribution.*

Note that a sufficient condition for irreducibility is that  $\forall i, j, \alpha_{ij} > 0$ . Appendix A provides an explicit construction of Lyapunov functions of arbitrary high polynomial growth at infinity for Hawkes processes.

### 3. LIMIT ORDER BOOK DRIVEN BY HAWKES PROCESSES

The limit order book model under scrutiny is presented in details in this section. After describing the model setup, in particular the processes driving the arrivals of orders of various types, the dynamics of the order book is written down and the infinitesimal generator of the associated Markov process is worked out.

### 3.1. Model setup.

3.1.1. *Order book representation.* Each side of the order book is supposed to be fully described by a *finite* number of limits  $K$ , ranging from 1 to  $K$  ticks away from the best available opposite quote. We use the notation

$$(\mathbf{a}(t); \mathbf{b}(t)) := (a_1(t), \dots, a_K(t); b_1(t), \dots, b_K(t)),$$

where  $\mathbf{a} := (a_1, \dots, a_K)$  represents the ask side of the order book,  $a_i$  being the number of shares available  $i$  ticks away from the best opposite quote; and similarly for  $\mathbf{b} := (b_1, \dots, b_K)$  on the bid side. Note that, contrarily to the representations used in [11] or [35], see also [21] for an interesting discussion, we adopt as in [1] a *finite moving frame*, since it reflects more faithfully the limit order books seen by traders on their screens. For this reason,  $\mathbf{a}$ ,  $\mathbf{b}$  will sometimes be referred to as the *visible* limits.

The quantities  $a_i, b_i$ 's are supposed as in [1] to live in the discrete space  $q\mathbb{Z}$ , where  $q \in \mathbb{N}^*$  is the minimum order size on each specific market (the *lot size*), but the results presented in this work are valid in the more general case of real-valued  $a_i, b_i$ 's. Some extra-care is necessary when writing down the order book dynamics (Equations (5)(6) below), but no essential change is required. In particular, the results concerning ergodicity and long-time behaviour, obtained *via* the theory developed in [31], are valid in a locally compact state space.

The cumulative depths up to level  $i$  are defined by

$$A(i) := \sum_{k=1}^i a_k \tag{2}$$

$$B(i) := \sum_{k=1}^i |b_k|, \tag{3}$$

and their generalized inverse functions are also introduced:

$$A^{-1}(x) := \inf\{p : \sum_{j=1}^p a_j > x\}$$

$$B^{-1}(x) := \inf\{p : \sum_{j=1}^p |b_j| > x\}.$$

In particular, the (common) index  $i_S$  corresponding to the spread measured in numbers of ticks is given by

$$i_S := A^{-1}(0) = B^{-1}(0). \quad (4)$$

The boundary conditions described below will ensure that  $i_S < \infty$ .

**3.1.2. Boundary conditions.** Constant boundary conditions are imposed outside the moving frame of size  $2K$ : every time the moving frame leaves a price level, the number of shares at that level is set to  $a_\infty$  or  $b_\infty$ , depending on the side of the book. Our choice of a finite moving frame and constant boundary conditions has three motivations: firstly, it assures that the order book does not become empty and that the best ask (resp. best bid) price  $P^A$  (resp.  $P^B$ ) is always defined. Secondly, it keeps the spread  $S$  and the increments of  $P^A, P^B$  bounded - this will be important when addressing the scaling limit of the price. Thirdly, it helps make the order book model Markovian, as we do not keep track of the price levels that have been visited, and then left, by the moving frame at some prior time.

Figure 1 is a representation of the order book using the above notations.

**3.1.3. Arrival of orders.** A Markovian Hawkes process as in Definition 2.1 drives the arrivals of new market and limit orders:

- $M^\pm(t)$ : arrival of new buy or sell market order, with intensity  $\lambda^{M^+}$  and  $\lambda^{M^-}$ ;
- $L_i^\pm(t)$ : arrival of a new sell or buy limit order at level  $i$ , with intensity  $\lambda_i^{L^\pm}$ .

Note that buy limit orders  $L_i^-(t)$  arrive below the ask price  $P^A(t)$ , and sell limit orders  $L_i^+(t)$  arrive above the bid price  $P^B(t)$ .

As explained in the introduction, the motivation for this model comes from the empirically observed fact that there exists a definite interplay between liquidity taking and providing on order-driven markets. Hawkes processes offer a very natural, analytically tractable and rather intuitive framework to model such an interplay.

The case of cancellation orders is a little different. Depending on the data that are made available by exchanges or data providers, empirical studies

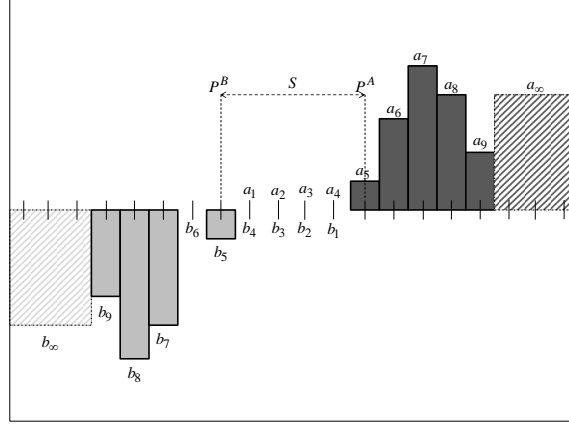


FIGURE 1. Order book dynamics: in this example,  $K = 9$ ,  $q = 1$ ,  $a_\infty = 4$ ,  $b_\infty = -4$ . The shape of the order book is such that  $\mathbf{a} = (0, 0, 0, 0, 1, 3, 5, 4, 2)$  and  $\mathbf{b} = (0, 0, 0, 0, -1, 0, -4, -5, -3)$ . The spread in ticks is given by  $i_S = 5$ . Assume that a sell market order arrives, then  $\mathbf{a}, \mathbf{b}, i_S$  become  $\mathbf{a}' = (0, 0, 0, 0, 0, 0, 1, 3, 5)$ ,  $\mathbf{b}' = (0, 0, 0, 0, 0, 0, -4, -5, -3)$  and  $i'_S = 7$ . Assume instead that a new buy limit order arrives one tick away from the best ask price, then  $\mathbf{a}' = (1, 3, 5, 4, 2, 4, 4, 4, 4)$ ,  $\mathbf{b}' = (-1, 0, 0, 0, -1, 0, -4, -5, -3)$  and  $i'_S = 1$ .

on the arrival of cancellations are more difficult to perform than those corresponding to market and limit orders. The minimal hypothesis of an exponential lifetime for limit orders that are not executed therefore leads us to model as in [1][35] the arrival of cancellations by a doubly stochastic Poisson process with proportional intensity:

- $C_i^\pm(t)$ : cancellation of a limit order at level  $i$ , with intensity  $\lambda_i^{C^+} a_i$  and  $\lambda_i^{C^-} |b_i|$ ,

where the superscripts “+” and “-” respectively refer to the ask and bid side of the book.

Actually, it could be interesting and relevant to allow all kinds of orders, including cancellations, to interact with one another<sup>1</sup>. This extension and some of its consequences are discussed in Remark 4.1.

<sup>1</sup>We thank one of the anonymous referees for this suggestion.



**3.2. Dynamics of the limit order book.** The dynamics of the limit order book is governed by the following set of SDE's

$$\begin{aligned}
da_i(t) &= -\mathbf{1}_{\{a_i(t) \neq 0\}} (q - A(i-1))_+ dM^+(t) + qdL_i^+(t) - qdC_i^+(t) \\
&+ \left( J^{M^-}(\mathbf{a}) - \mathbf{a} \right)_i dM^-(t) + \sum_{i=1}^K \left( J^{L_i^-}(\mathbf{a}) - \mathbf{a} \right)_i dL_i^-(t) \\
&+ \sum_{i=1}^K \left( J^{C_i^-}(\mathbf{a}) - \mathbf{a} \right)_i dC_i^-(t)
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
db_i(t) &= \mathbf{1}_{\{b_i(t) \neq 0\}} (q - B(i-1))_+ dM^-(t) - qdL_i^-(t) + qdC_i^-(t) \\
&+ \left( J^{M^+}(\mathbf{b}) - \mathbf{b} \right)_i dM^+(t) + \sum_{i=1}^K \left( J^{L_i^+}(\mathbf{b}) - \mathbf{b} \right)_i dL_i^+(t) \\
&+ \sum_{i=1}^K \left( J^{C_i^+}(\mathbf{b}) - \mathbf{b} \right)_i dC_i^+(t)
\end{aligned} \tag{6}$$

(remember that, by convention, the  $b_i$ 's are non-positive).

In Equations (5)(6), the first three terms on the right-hand side describe the evolution of the quantity available at a given limit  $i$  under the influence of the three type of events that can directly affect it:

- a buy market order decreasing by an amount  $q$  the first non-zero limit on the ask side, possibly hitting the liquidity reservoir if all visible limits are empty (and similarly on the bid side);
- a new limit order increasing by an amount  $q$  the available quantity;
- a cancellation order decreasing by an amount  $q$  the available quantity.

By assumption, the intensity of the point process triggering a cancellation is 0 when the corresponding quantity is 0, avoiding all inconsistencies. However, for market orders, no such assumption is made, hence the use of the indicator function. Note that this formulation holds without change in the case of varying order sizes.

As for the  $J$ 's, they are *shift operators* corresponding to the renumbering of the ask side following an event affecting the bid side of the book and vice-versa. For instance, the shift operator corresponding to the arrival of a

sell market order  $dM^-(t)$  of size  $q$  is

$$J^{M^-}(\mathbf{a}) = \left( \underbrace{0, 0, \dots, 0}_{k \text{ times}}, a_1, a_2, \dots, a_{K-k} \right) \quad (7)$$

with

$$\begin{aligned} k &= \inf\{p : \sum_{j=1}^p |b_j| > q\} - \inf\{p : |b_p| > 0\} \\ &= A^{-1}(q) - i_S. \end{aligned}$$

Equation (7) is the mathematical formulation of the fact that the limit order book always has exactly  $K$  visible limits, and that the reference price for the ask side of the book possibly changes if a sell market order eats up all the available liquidity at the best bid price. Similarly, a new buy limit order *within the spread*  $k$  ticks above the previous best bid price - so that  $k := i_S - i > 0$  - will shift the ask side according to

$$J^{L_i^-}(\mathbf{a}) = \left( a_{1+k}, a_{2+k}, \dots, a_K, \underbrace{a_\infty, \dots, a_\infty}_{k \text{ times}} \right),$$

see Figure 1 for a graphical representation. Similar expressions are readily derived for the other events affecting the order book.

**3.3. The infinitesimal generator.** The limit order book model just introduced is naturally cast under the form of a  $D$ -dimensional Markov process  $(\mathbf{a}; \mathbf{b}; \boldsymbol{\mu})$  using the decomposition of the intensities of the Hawkes processes presented in Section 2. Here,  $D = (2K + 2)^2 + 2K$  is the dimension of the state space.

Its infinitesimal generator is now worked out - in fact, the following result holds true:

**Lemma 3.1.** *The infinitesimal generator associated to the limit order book process is the operator  $\mathcal{L}$  defined for functions  $F : (\mathbf{a}; \mathbf{b}; \boldsymbol{\mu}) \rightarrow F(\mathbf{a}; \mathbf{b}; \boldsymbol{\mu})$*

that are of class  $C^1$  in their last argument, by

$$\begin{aligned}
\mathcal{L}F(\mathbf{a}; \mathbf{b}; \boldsymbol{\mu}) &= \lambda^{M^+} \left( F \left( [a_i - (q - A(i-1))_+]_+; J^{M^+}(\mathbf{b}); \boldsymbol{\mu} + \Delta^{M^+}(\boldsymbol{\mu}) \right) - F \right) \\
&+ \sum_{i=1}^K \lambda_i^{L^+} \left( F \left( a_i + q; J^{L^+}(\mathbf{b}); \boldsymbol{\mu} + \Delta^{L^+}(\boldsymbol{\mu}) \right) - F \right) \\
&+ \sum_{i=1}^K \lambda_i^{C^+} a_i \left( F \left( a_i - q; J^{C^+}(\mathbf{b}); \boldsymbol{\mu} \right) - F \right) \\
&+ \lambda^{M^-} \left( F \left( J^{M^-}(\mathbf{a}); [b_i + (q - B(i-1))_+]_-; \boldsymbol{\mu} + \Delta^{M^-}(\boldsymbol{\mu}) \right) - F \right) \\
&+ \sum_{i=1}^K \lambda_i^{L^-} \left( F \left( J^{L^-}(\mathbf{a}); b_i - q; \boldsymbol{\mu} + \Delta^{L^-}(\boldsymbol{\mu}) \right) - F \right) \\
&+ \sum_{i=1}^K \lambda_i^{C^-} |b_i| \left( F \left( J^{C^-}(\mathbf{a}); b_i + q; \boldsymbol{\mu} \right) - F \right) \\
&- \sum_{i,j=1}^D \beta_{ij} \mu_{ij} \frac{\partial F}{\partial \mu_{ij}}. \tag{8}
\end{aligned}$$

*Proof.* By direct computations as in [1]. □

In order to ease the already cumbersome notations, we have written  $F(a_i; \mathbf{b}; \boldsymbol{\mu})$  instead of  $F(a_1, \dots, a_i, \dots, a_K; \mathbf{b}; \boldsymbol{\mu})$ . The notation  $\Delta^{(\cdot)}(\boldsymbol{\mu})$  stands for the jump of the intensity vector  $\boldsymbol{\mu}$  corresponding to a jump of the process  $N^{(\cdot)}$ , see Definition 2.1.

The operator  $\mathcal{L}$  is a combination of

- standard difference operators corresponding to the arrival and cancellation of orders at each limit;
- shift operators expressing the moves of the origins of the reference frames;
- drift terms coming from the mean-reverting behaviour of the intensities of the Hawkes processes between jumps.

Note that, as already mentioned, the operator in (8) corresponds to the case of the discrete state-space for the available quantities; some trivial but notationally cumbersome modifications are necessary in order to account for the case of general real-valued quantities  $a_i, b_i$ 's and variable order sizes  $q$ .

## 4. STABILITY OF THE ORDER BOOK

In this section, we study the long-time behaviour of the limit order book. Our main result is stated in the

**Proposition 4.1.** *Under the standing assumptions, in particular those of Proposition 2.2 and the proportional cancellation rate assumption, the limit order book  $(\mathbf{a}, \mathbf{b}, \boldsymbol{\mu})$  is an ergodic process. It converges exponentially fast towards its unique stationary distribution.*

*Proof.* Given the existence of the Lyapunov function, see Lemma 4.1 below, and the *geometric drift condition* (10), the result is proven exactly as in [1] using Theorem 7.1 in [31].  $\square$

The main technical result of this section is the following

**Lemma 4.1.** *For  $\eta > 0$  small enough, the function  $V$  defined by*

$$V(\mathbf{a}; \mathbf{b}; \boldsymbol{\mu}) = \sum_{i=1}^K a_i + \sum_{i=1}^K |b_i| + \frac{1}{\eta} \sum_{i,j=1}^{(2K+2)^2} \delta_{ij} \mu_{ij} \equiv V_1 + \frac{1}{\eta} V_2 \quad (9)$$

with  $V_1 := V(\mathbf{a}; \mathbf{b}; \mathbf{0})$  and  $V_2 := V(\mathbf{0}; \mathbf{0}; \boldsymbol{\mu})$ , is a Lyapunov function satisfying a *geometric drift condition*

$$\mathcal{L}V \leq -\zeta V + C, \quad (10)$$

for some  $\zeta > 0$  and  $C \in \mathbb{R}$ . The coefficients  $\delta_{ij}$ 's are defined in (22) in Appendix A.

*Proof.* First specialize  $V_2$  to be identical - up to a change in the indices - to the function defined by (23) in Appendix A. Regarding the "small" parameter  $\eta > 0$ , it will become handy as a penalization parameter, as we shall see below.

Thanks to the linearity of  $\mathcal{L}$ , there holds

$$\mathcal{L}V = \mathcal{L}V_1 + \frac{1}{\eta} \mathcal{L}V_2.$$

The first term  $\mathcal{L}V_1$  is dealt with as in [1]:

$$\begin{aligned} \mathcal{L}V_1 &\leq -(\lambda^{M^+} + \lambda^{M^-})q + \sum_{i=1}^K (\lambda_i^{L^+} + \lambda_i^{L^-})q - \sum_{i=1}^K (\lambda_i^{C^+} a_i + \lambda_i^{C^-} |b_i|)q \\ &\quad + \sum_{i=1}^K \lambda_i^{L^+} (i_S - i)_+ a_\infty + \sum_{i=1}^K \lambda_i^{L^+} (i_S - i)_+ |b_\infty| \end{aligned} \quad (11)$$

$$\begin{aligned} &\leq -(\lambda^{M^+} + \lambda^{M^-})q + (\Lambda^{L^-} + \Lambda^{L^+})q - \underline{\lambda}^C q V_1(\mathbf{x}) \\ &\quad + K(\Lambda^{L^-} a_\infty + \Lambda^{L^+} |b_\infty|), \end{aligned} \quad (12)$$

where

$$\Lambda^{L^\pm} := \sum_{i=1}^K \lambda_i^{L^\pm} \text{ and } \underline{\lambda}^C := \min_{1 \leq i \leq K} \{\lambda_i^{C^\pm}\} > 0.$$

Computing  $\mathcal{L}V_2$  yields an expression identical to that obtained in Appendix A:

$$\mathcal{L}V_2 = \sum_{i,j} \lambda_0^j \delta_{ij} \alpha_{ij} + (\kappa - 1) \sum_{j,k} \epsilon_k \mu_{jk},$$

so that there holds

$$\mathcal{L}V = \mathcal{L}V_1 + \frac{1}{\eta} \mathcal{L}V_2 \leq -\underline{\lambda}^C q V_1 - \frac{\gamma}{\eta} V_2 - \mathbf{G} \cdot \boldsymbol{\mu} + C,$$

where  $\gamma$  is as in Equation (24),  $\mathbf{G} \cdot \boldsymbol{\mu}$  is a compact notation for the linear form in the  $\mu_{ij}$ 's obtained in (12), and  $C$  is some constant. Now, thanks to the positivity of the coefficients in  $V_2$  and of the  $\mu_{ij}$ 's, one can choose  $\eta$  small enough that there holds

$$\forall \boldsymbol{\mu}, |\mathbf{G} \cdot \boldsymbol{\mu}| \leq \frac{\gamma}{2\eta} V_2(\boldsymbol{\mu}),$$

which yields

$$\mathcal{L}V \equiv \mathcal{L}V_1 + \frac{1}{\eta} \mathcal{L}V_2 \leq -\underline{\lambda}^C q V_1 - \frac{\gamma}{2\eta} V_2 + C, \quad (13)$$

and finally

$$\mathcal{L}V \leq -\zeta V + C,$$

with  $\zeta = \min\left(\underline{\lambda}^C q, \frac{\gamma}{2\eta}\right)$  and  $C$  is some constant.  $\square$

**Remark 4.1.** *The proportionality of the cancellation rates clearly plays an important role in the stability of the order book. A careful analysis of the proof of Lemma 4.1 shows that one can easily add an excitation from*

*the market and limit orders towards the cancellation orders. This feature would account for instance for the fact that the cancellation rate at the second limit is likely to increase if a market order starts emptying the first limit. However, should the arrival of a cancellation order excite other types of orders, a new term will appear in the computation of  $\mathcal{L}V$ , possibly with the wrong sign. Hence, the proof of stability will not work out in a similar fashion. From a physical standpoint, it is actually rather clear that, should cancellations increase the arrival of new limit orders, the order book may become quite fat and ergodicity, harder to prove - or even false !*

## 5. LARGE SCALE LIMIT OF THE PRICE PROCESS

This section is devoted to the asymptotics in time of the price process. The ergodic theory of Markov processes, see [8], is combined with the martingale convergence theorem [18], to obtain the results. This approach is extremely general and flexible, and prone to many generalizations for Markovian models of limit order books. It is somewhat similar to that used in [26][6], where various long-time, large-scale behaviour of limit order books are studied. However, the stochastic behaviour of the intensities of the point processes triggering the order book events makes the situation we consider slightly different.

**5.1. Price dynamics and the Ergodic Theorem.** We first write down the expression for the price dynamics. Consider for instance the best ask and bid prices, denoted by  $P^A(t)$  and  $P^B(t)$ . One can easily see that they satisfy the following SDE's:

$$\begin{aligned} dP^A(t) &= \Delta P \left[ \left( A^{-1}(q) - i_S \right) dM^+(t) \right. \\ &\quad \left. - \sum_{i=1}^K (i_S - i)_+ dL_i^+(t) + \left( A^{-1}(q) - i_S \right) dC_{i_S}^+(t) \right] \end{aligned}$$

and

$$\begin{aligned} dP^B(t) &= -\Delta P \left[ \left( B^{-1}(q) - i_S \right) dM^-(t) \right. \\ &\quad \left. - \sum_{i=1}^K (i_S - i)_+ dL_i^-(t) + \left( B^{-1}(q) - i_S \right) dC_{i_S}^-(t) \right], \end{aligned}$$

describing the various events that affect them: change due to a market order, change due to a new limit order inside the spread, and change due to the cancellation of a limit order at the best limit (recall that  $i_s$  is defined in (4)). Let us recast these equations under a general form as follows:

$$P_t = P_0 + \int_0^t \sum_i F_i(X_u) dN_u^i, \quad (14)$$

where the  $N^i$  are the point processes driving the limit order book,  $X$  is the Markovian process describing its state, and  $P$  is one of the price processes we are interested in. For instance, in the Poisson case dealt with in [1],  $X = (\mathbf{a}, \mathbf{b})$  and the  $N^i$  are the Poisson processes driving the arrival of market, limit and cancellation orders. In the context of Hawkes processes considered in this work,  $X = (\mathbf{a}, \mathbf{b}, \boldsymbol{\mu})$  and the  $N^i$  are the Poisson and Hawkes processes driving the limit order book. Moreover, the  $F_i$  are bounded functions, thanks to the non-zero boundary conditions  $a_\infty, b_\infty$ .

Denote by  $\Pi$  the stationary distribution of  $X$  as provided by Proposition 4.1. Then, the Ergodic Theorem for Markov processes states the following:

**Theorem 5.1** ([8][29]). *Let  $G$  be in  $L^1(\Pi(dX))$ . Then,*

$$\lim_{t \rightarrow +\infty} \frac{a.s.}{t} \int_0^t G(X_s) ds = \int G(X) \Pi(dX).$$

Using this classical result, together with the deep Theorem 7.1.4 of [18] on the convergence of martingales, one can prove the

**Proposition 5.1.** *Consider the price process described by Equation (14) above, and introduce the sequence of martingales  $\hat{P}^n$  formed by the centered, rescaled price*

$$\hat{P}_t^n \equiv \frac{P_{nt} - Q_{nt}}{\sqrt{n}}$$

where  $Q$  is the compensator of  $P$

$$Q_t = \int_0^t \sum_i F_i(X_s) \lambda_s^i ds.$$

Then,  $\hat{P}^n$  converges in distribution in the space  $D([0, \infty); \mathbb{R})$  endowed with the Skorohod topology, to a Wiener process  $\hat{\sigma}W$ , where the volatility  $\hat{\sigma}$  is

given by

$$\hat{\sigma}^2 = \lim_{t \rightarrow +\infty} \frac{a.s.}{t} \int_0^t \sum_i (F_i(X_s))^2 \lambda_s^i ds = \int \sum_i (F_i(X))^2 \lambda_i \Pi(dX). \quad (15)$$

*Proof.* Proposition 5.1 follows from the convergence of the predictable quadratic variation of  $\hat{P}^n$ . By construction, there holds

$$\langle \hat{P}^n, \hat{P}^n \rangle_t = \frac{1}{n} \int_0^{nt} \sum_i (F_i(X_s))^2 \lambda_s^i ds,$$

or else

$$\langle \hat{P}^n, \hat{P}^n \rangle_t = t \left( \frac{1}{nt} \int_0^{nt} \sum_i (F_i(X_s))^2 \lambda_s^i ds \right),$$

and Theorem 5.1 ensures that

$$\lim_{t \rightarrow +\infty} \frac{a.s.}{nt} \int_0^{nt} \sum_i (F_i(X_s))^2 \lambda_s^i ds = \int \sum_i (F_i(X))^2 \lambda_i \Pi(dX)$$

whenever the integrability conditions of Theorem 5.1 are satisfied. Now, those are easily seen to hold true, since the Lyapunov function  $V$  itself is in  $L^1(\Pi(dX))$ , see e.g [22], and the integrand in the predictable quadratic variation, being linear in the  $\lambda$ 's and bounded as a function of the  $a_i, b_i$ 's, is bounded by a multiple of  $V$ .

The other condition for the martingale convergence theorem to apply is trivially satisfied, since the size of the jumps of  $\hat{P}^n$  is bounded by  $\frac{C}{\sqrt{n}}$ ,  $C$  being some constant.  $\square$

**5.2. The dynamics of the rescaled price process.** It is tempting to use Equation (15) as a characterization of the volatility of the price process at the larger time scales - as indeed one of our main motivations for this work was to establish the connection between microstructural models and diffusive behaviour in the long run. As it turns out, Proposition 5.1 is not completely satisfactory: in order to give a more precise characterization of the dynamics of the rescaled price process, it is necessary to understand thoroughly the behaviour of its compensator  $Q_{nt}$ . As a matter of fact,  $Q_{nt}$  itself satisfies an ergodic theorem, and if its asymptotic variance is not negligible w.r. to  $nt$ , one cannot use directly Proposition 5.1 to assess the volatility of the rescaled, deterministically centered price process.



The next result provides a more accurate answer, valid under general ergodicity conditions.

**Theorem 5.2.** *Write as above the price*

$$P_t = P_0 + \int_0^t \sum_i F_i(X_s) dN_s^i$$

*and its compensator*

$$Q_t = \int_0^t \sum_i F_i(X_s) \lambda_s^i ds.$$

Set

$$h = \sum_i F_i(X) \lambda_i$$

*and let  $\alpha \in \mathbb{R}$  be defined by*

$$\alpha := \lim_{t \rightarrow +\infty} \frac{a.s.}{t} \int_0^t \sum_i (F_i(X_s)) \lambda_s^i ds = \int h(X) \Pi(dX).$$

*Finally, introduce the solution  $g$  to the Poisson equation*

$$\mathcal{L}g = h - \alpha \tag{16}$$

*and the associated martingale*

$$Z_t = g(X_t) - g(X_0) - \int_0^t \mathcal{L}g(X_s) ds \equiv g(X_t) - g(X_0) - Q_t + \alpha t.$$

*Then, the deterministically centered, rescaled price*

$$\bar{P}_t^n \equiv \frac{P_{nt} - \alpha nt}{\sqrt{n}}$$

*converges in distribution in the space  $D([0, \infty); \mathbb{R})$  endowed with the Skorohod topology, to a Wiener process  $\bar{\sigma}W$ . The asymptotic volatility  $\bar{\sigma}$  satisfies the identity*

$$\begin{aligned} \bar{\sigma}^2 &= \lim_{t \rightarrow +\infty} \frac{a.s.}{t} \int_0^t \sum_i ((F_i - \Delta^i(g))(X_s))^2 \lambda_s^i ds \\ &\equiv \int \sum_i ((F_i - \Delta^i(g))(X))^2 \lambda_i \Pi(dX), \end{aligned} \tag{17}$$

*where  $\Delta^i(g)(X)$  denotes the jump of the process  $g(X)$  when the process  $N^i$  jumps and the limit order book is in the state  $X$ .*

*Proof.* The martingale method, see e.g. [22][16][27], consists in rewriting the price process under the form

$$P_t = (P_t - Q_t) - Z_t + g(X_t) - g(X_0) + \alpha t = (M_t - Z_t) + g(X_t) - g(X_0) + \alpha t,$$

so that

$$\bar{P}_t^n = \frac{V_{nt} + g(X_t) - g(X_0)}{\sqrt{n}},$$

where  $V = M - Z$  is a martingale. Therefore, the theorem is proven *iff* one can show that  $\frac{g(X_t) - g(X_0)}{\sqrt{n}}$  converges to 0 in  $L^2(\Pi(dX))$ , or simply, that  $g \in L^2(\Pi(dX))$ .

Theorem 4.4 of [22] states that the condition

$$h^2 \leq V \tag{18}$$

(where  $V$  is a Lyapunov function for the process) is sufficient for  $g$  to be in  $L^2(\Pi(dX))$ . As opposed to the case of Poisson intensities, the linear Lyapunov function  $V$  introduced in (9) does not yield the desired result, because  $h$  is linearly increasing in the  $\lambda$ 's. However, see Lemma A.1 in Appendix A, one can design a Lyapunov function having a polynomial growth of arbitrary high order in the  $\lambda$ 's at infinity, thereby ensuring that (18) holds.  $\square$

**5.3. Interpreting the asymptotic volatility.** A general formula for the low frequency volatility of the price process is provided in (17); it is related to the frequency of events that cause a price change, and to the size of price jumps when a change occurs. Formula (17) can easily be implemented numerically by using its formulation as a time average, but its analytical computation would require the knowledge of the stationary distribution of the order book. However, some simplifying hypotheses help shed some light on its interpretation and qualitative dependency on the model parameters. Assume for instance that one is interested in modelling *large tick* assets, for which the size of price changes is *always* equal to 1 tick. In our framework, this is made possible by choosing  $K = 1$ : only one limit on each side of the order book is modelled. In this case, all the  $F_i$ 's introduced in Section 5 are equal to 1 or 0, and the asymptotic variance can be rewritten by separating the events that change the price from those that do not.

Let us introduce such a decomposition of market, limit and cancellation

orders depending on whether an event change the price or not - and use a 1 (resp. 0) superscript to indicate that the event changes (resp. does not change) the price:

$$\begin{aligned} M^\pm &= M^{\pm,1} + M^{\pm,0}, \\ L_i^\pm &= L_i^{\pm,1} + L_i^{\pm,0}, \\ C_i^\pm &= C_i^{\pm,1} + C_i^{\pm,0}. \end{aligned}$$

Now, should all these processes be independent Poisson processes, the asymptotic variance would be given using (15) or (17) (see comment below) by

$$\bar{\sigma}^2 = (\Delta P)^2 \left( \lambda^{M^+,1} + \lambda^{M^-,1} + \sum_i (\lambda_i^{L^+,1} + \lambda_i^{L^-,1}) + \lambda_{is}^{C^+,1} + \lambda_{is}^{C^-,1} \right),$$

where all the quantities involved are easily interpreted, and can be measured empirically from the data. Obviously the Poisson hypothesis is violated in the framework of a limit order book driven by Hawkes processes, but we think that this rewriting makes the formula more intuitive.

Another interesting question concerning Formula (17) is the role played by the correcting term coming from the solution  $g$  to the Poisson equation (16). In the case of Poisson arrival for the price-changing processes and deterministic price changes, the right-hand-side of (16) is 0, so that the correcting terms are also 0: Formulae (15) and (17) coincide. In general this is not the case, and one should find an estimate of the correcting terms - essentially, a control of the variance of  $h = \sum_i F_i(X)\lambda_i$  when the  $\lambda_i$ 's are now random. Some analytic computations may be performed as in the Poisson case under the simplifying assumptions of deterministic price changes and Hawkes processes driving the events that change the prices, but the general case is more intricate, although easily attainable *via* numerical simulations.

## 6. CONCLUDING REMARKS

In this work, a model for a limit order book driven by Markovian Hawkes processes has been studied. The model is motivated by empirical observations on the interplay between liquidity taking and providing, and captures this phenomenon at the high frequency level. Under standard stability

conditions for the Hawkes processes driving the arrival of orders, stability and exponential convergence towards the stationary state have been proven. Then, the long time asymptotics of the price has been studied, and a formula for the volatility at large time scales has been given.

Several further research directions naturally come to mind. First, there is the issue of the relevance of the model: does a multivariate exponential Hawkes process satisfactorily describe the interplay one is interested in, and how does it compare to empirical measurements ? Recent works on Hawkes processes in the context of price and trade arrivals offer diverging, sometimes contradictory opinions on that subject; what seems to be clear is that a model such as ours, where the base intensity and the kernel parameters are constant, should be used in the context of stable market conditions. Another question of interest is related to cancellations: it would be natural to allow for mutual excitations between cancellations and other types of orders. As already noted in Remark 4.1, very minor modifications to this work should be made if one only allows cancellations to be excited by orders of other types; but the reverse situation is not as simple. It is easy to obtain some smallness conditions on the size of the jumps so as to ensure ergodicity, but whether such conditions are necessary is not clear at the moment.

#### APPENDIX A. LYAPUNOV FUNCTIONS FOR HAWKES PROCESSES

For the sake of completeness, some explicit constructions of Lyapunov functions for a multi-dimensional Hawkes processes  $\mathbf{N} = (N^i)$  with intensities

$$\lambda_t^i = \lambda_0^i + \sum_j \int_0^t \alpha_{ij} e^{-\beta_{ij}(t-s)} dN_s^j$$

are now given. Note that [37] provides a somewhat similar construct of a linearly growing Lyapunov function<sup>2</sup>.

Denote as in Section 2

$$\mu_t^{ij} = \int_0^t \alpha_{ij} e^{-\beta_{ij}(t-s)} dN_s^j,$$

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<sup>2</sup>The authors thank one of the anonymous referees for drawing their attention to this paper.

so that there holds

$$\lambda_t^i = \lambda_0^i + \sum_j \mu_t^{ij}. \quad (19)$$

The infinitesimal generator associated to the Markovian process  $(\mu^{ij})$ ,  $1 \leq i, j \leq D$ , is the operator

$$\mathcal{L}_H F(\boldsymbol{\mu}) = \sum_j \lambda^j \left( F(\boldsymbol{\mu} + \Delta^j(\boldsymbol{\mu})) - F(\boldsymbol{\mu}) \right) - \sum_{i,j} \beta_{ij} \mu_{ij} \frac{\partial F}{\partial \mu_{ij}},$$

where  $\boldsymbol{\mu}$  is the vector with components  $\mu_{ij}$  and  $\lambda_j$  is as in (19). The notation  $\Delta^j(\boldsymbol{\mu})$  characterizes the jumps in those of the entries in  $\boldsymbol{\mu}$  that are affected by a jump of the process  $N^j$ . For a **fixed** index  $j$ , it is given by the vector with entries  $\alpha_{ij}$  at the relevant spots, and zero entries elsewhere.

A Lyapunov function for the associated semi-group is sought under the form

$$V(\boldsymbol{\mu}) = \sum_{i,j} \delta_{ij} \mu_{ij} \quad (20)$$

(since the intensities are always positive, a linear function will be coercive).

Assuming (20), there holds

$$\mathcal{L}_H V = \sum_j \lambda^j \left( \sum_i \delta_{ij} \alpha_{ij} \right) - \sum_{i,j} \beta_{ij} \mu_{ij} \delta_{ij}$$

or

$$\mathcal{L}_H V = \sum_{i,j} \left( \lambda_0^j + \sum_k \mu_{jk} \right) \delta_{ij} \alpha_{ij} - \beta_{ij} \mu_{ij} \delta_{ij}. \quad (21)$$

Recall the matrix  $\mathbf{A}$  defined in Section 2 with entries

$$A_{ij} = \frac{\alpha_{ji}}{\beta_{ji}}.$$

Under the assumptions of Proposition 2.2,  $\mathbf{A}$  is irreducible, and the spectral condition (1) holds. Let  $\boldsymbol{\epsilon}$  be the maximal eigenvector of  $\mathbf{A}$  and denote by  $\kappa$  the associated maximal eigenvalue. By Assumption (1), one has that  $0 < \kappa < 1$  and furthermore, by Perron-Frobenius theorem, there holds:  $\forall i, \epsilon_i > 0$ .

Assuming that

$$\delta_{ij} \equiv \frac{\epsilon_i}{\beta_{ij}}, \quad (22)$$

the expression for  $V$  becomes

$$V(\boldsymbol{\mu}) = \sum_{i,j} \epsilon_i \frac{\mu_{ij}}{\beta_{ij}}. \quad (23)$$

Plugging (23) in (21) yields

$$\begin{aligned} \mathcal{L}_H V &= \sum_{i,j} \lambda_0^j \delta_{ij} \alpha_{ij} + \sum_{i,j,k} \mu_{jk} \epsilon_i \frac{\alpha_{ij}}{\beta_{ij}} - \sum_{j,k} \beta_{jk} \mu_{jk} \delta_{jk} \\ &= \sum_{i,j} \lambda_0^j \delta_{ij} \alpha_{ij} + (\kappa - 1) \sum_{j,k} \epsilon_k \mu_{jk}, \end{aligned}$$

where we have used the identity  $\sum_j \mathbf{A}_{ji} \epsilon_i = \kappa \epsilon_j$ . A comparison with (23) easily yields the upper bound

$$\mathcal{L}_H V \leq -\gamma V + C, \quad (24)$$

with  $\gamma = (1 - \kappa)\beta_{\min}$ ,  $\beta_{\min} \equiv \inf_{i,j}(\beta_{ij}) > 0$  by assumption, and  $C = \sum_{i,j} \lambda_0^j \delta_{ij} \alpha_{ij} \equiv \kappa \epsilon \cdot \lambda_0$ .

Sufficient as it is to prove Proposition 5.1 using the Lyapunov function for the limit order book provided by Proposition 4.1, a linearly growing Lyapunov function is too weak to prove Theorem 5.2: as already noted, Theorem 4.4 in [22] requires a Lyapunov function with *quadratic* growth, in order that the Poisson equation with a linearly growing RHS have a solution in  $L^2(\Pi(dX))$ . To this aim, a useful extension of Lemma 4.1 is given in the following

**Lemma A.1.** *Under the standing assumptions, one can construct a Lyapunov function of arbitrary high polynomial growth at infinity.*

*Proof.* Let  $n \in \mathbb{N}^*$ , and  $V$  be defined in (23). The function  $V^n$  satisfies

$$\mathcal{L}_H V^n(\boldsymbol{\mu}) = \sum_j \lambda^j \left( V^n(\boldsymbol{\mu} + \Delta^j(\boldsymbol{\mu})) - V^n(\boldsymbol{\mu}) \right) - n V^{n-1} \left( \sum_{i,j} \beta_{ij} \mu_{ij} \frac{\partial V}{\partial \mu_{ij}} \right). \quad (25)$$

Upon factoring  $V^n(\boldsymbol{\mu} + \Delta^j(\boldsymbol{\mu})) - V^n(\boldsymbol{\mu})$ :

$$V^n(\boldsymbol{\mu} + \Delta^j(\boldsymbol{\mu})) - V^n(\boldsymbol{\mu}) = \left( V(\boldsymbol{\mu} + \Delta^j(\boldsymbol{\mu})) - V(\boldsymbol{\mu}) \right) \left( \sum_{k=0}^{n-1} V^{n-1-k}(\boldsymbol{\mu} + \Delta^j(\boldsymbol{\mu})) V^k(\boldsymbol{\mu}) \right),$$

the linearity of  $V$  yields the following expression

$$V^n(\boldsymbol{\mu} + \Delta^j(\boldsymbol{\mu})) - V^n(\boldsymbol{\mu}) = nV^{n-1}(\boldsymbol{\mu})(V(\boldsymbol{\mu} + \Delta^j(\boldsymbol{\mu})) - V(\boldsymbol{\mu})) + \mathcal{M}_j(V)(\boldsymbol{\mu}),$$

where  $\mathcal{M}_j(V)(\boldsymbol{\mu})$  can be bounded by a polynomial function of degree  $n - 1$  at infinity in  $\boldsymbol{\mu}$ . Therefore, one can rewrite (25) as follows

$$\mathcal{L}_H V^n(\boldsymbol{\mu}) = (nV^{n-1} \mathcal{L}_H V)(\boldsymbol{\mu}) + \mathcal{M}(V)(\boldsymbol{\mu}), \quad (26)$$

where  $\mathcal{M}(V)(\boldsymbol{\mu})$  is a polynomial of degree  $n - 1$  in  $\boldsymbol{\mu}$ . Combining (23) with (26) shows that  $V^n$  is also a Lyapunov function for the Hawkes process.  $\square$

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