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► **To cite this version:**

Frédéric Abergel, Aymen Jedidi. LONG TIME BEHAVIOUR OF A HAWKES PROCESS-BASED LIMIT ORDER BOOK. 2015. hal-01121711v2

**HAL Id: hal-01121711**

**<https://hal.science/hal-01121711v2>**

Preprint submitted on 4 Mar 2015 (v2), last revised 17 Jul 2015 (v5)

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# LONG TIME BEHAVIOUR OF A HAWKES PROCESS-BASED LIMIT ORDER BOOK

FRÉDÉRIC ABERGEL AND AYMEN JEDIDI

**ABSTRACT.** This work is concerned with the modelling of limit order books using Hawkes processes with exponential kernels, with a specific focus on the asymptotic behaviour of the limit order book and the corresponding price dynamics.

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*Date:* March 4, 2015.

F. Abergel, A. Jedidi, Chaire de Finance Quantitative, Laboratoire de Mathématiques Appliquées aux Systèmes, CentraleSupélec, 92290 Châtenay-Malabry, France.

## 1. INTRODUCTION

Since their introduction, Hawkes processes have been applied to a wide range of research areas, from seismology in the pioneering work [17] to credit risk, financial contagion and more recently, to the modelling of market microstructure [4, 3, 5, 24, 25, 26, 12, 11].

In market microstructure, and particularly limit order book modelling, the relevance of these processes is amply demonstrated by two empirical properties of the order flow of market and limit orders at the microscopic level:

- Time clustering: order arrivals are highly clustered in time, see e.g. [9][12].
- Mutual dependence: order flow exhibit non-negligible cross-dependences. For instance, as documented in [24][2][14], market orders excite limit orders, and limit orders that change the price excite market orders.

At the microscopic level, point process-based microstructure models capture, by construction, the intrinsic discreteness of prices and volumes. In this respect, they offer a natural framework in which to model the finer scales properties of order-driven markets. They do however exhibit a level of complexity that hides some more macroscopic properties of the markets. A question of interest in this context is then the *microscopic-to-macroscopic* transition in the price dynamics. This strand of research has attracted a lot of interest of late [1, 5, 10, 19, 6, 26], and this work is a contribution to the domain. By casting a Hawkes process-based limit order book model into a markovian setting and using techniques from the ergodic theory of Markov processes, we show that the order book is ergodic and leads to a diffusive behaviour of the price at large time scales. Furthermore, we provide formulae for the asymptotic trend and variance.

The paper is organized as follows: Section 2 recalls some well-known facts about Hawkes processes with exponential kernels, while Section 3 describes the limit order book model we consider. In Section 4, the ergodicity of the limit order book is proven under reasonable assumptions, and the main results on the price dynamics are presented in Section 5.

## 2. HAWKES PROCESSES

We very briefly recall in this section several classical results on multivariate Hawkes processes. Let then  $\vec{N} = (N^1, \dots, N^D)$  be a  $D$ -dimensional point process with intensity vector  $\vec{\lambda} = (\lambda^1, \dots, \lambda^D)$ .

**Definition 2.1.** We say that  $\vec{N} = (N^1, \dots, N^D)$  is a multivariate Hawkes process when

$$\lambda_t^m = \lambda_0^m + \sum_{j=1}^D \alpha_{mj} \int_0^t e^{-\beta_{mj}(t-s)} dN_s^j$$

for  $1 \leq m \leq D$ .

**Proposition 2.1.** Define the processes  $\mu_t^{ij}$  as:  $\mu_t^{ij} = \alpha_{ij} \int_0^t e^{-\beta_{ij}(t-s)} dN_s^j$ ,  $1 \leq i, j \leq D$ , and let  $\vec{\mu} = \{\mu^{ij}\}_{1 \leq i, j \leq D}$ . Then, the process  $(\vec{N}, \vec{\mu})$  is markovian.

*Proof.* The idea of the proof is given below <sup>1</sup>. Let  $t_2 > t_1$ . Since

$$\mu_{t_2}^{mj} = e^{-\beta_{mj}(t_2-t_1)} \mu_{t_1}^{mj} + \int_{t_1}^{t_2} e^{-\beta_{mj}(t_2-s)} dN_s^j,$$

and

$$\lambda_{t_2}^m = \lambda_0^m + \sum_{j=1}^D \mu_{t_2}^{mj},$$

the law of  $(\vec{N}_{t_2}, \vec{\mu}_{t_2})$  conditional on  $\{(\vec{N}_t, \vec{\mu}_t) : 0 \leq t \leq t_1\}$  is the same as when conditioning on  $(\vec{N}_{t_1}, \vec{\mu}_{t_1})$  only, which proves the Markov property.  $\square$

2.0.1. *Stationarity.* Extending the early stability and stationarity result in [18], [7] proves a general result for the multivariate Hawkes processes just introduced:

**Proposition 2.2.** Let the matrix  $\mathbf{A}$  be defined by

$$\mathbf{A}_{ij} = \frac{\alpha_{ji}}{\beta_{ji}}, \quad 1 \leq i, j \leq D.$$

If its spectral radius  $\rho(\mathbf{A})$  satisfies the condition

$$\rho(\mathbf{A}) < 1, \tag{1}$$

then there exists a (unique) multivariate point process  $\vec{N} = (N^1, \dots, N^m)$  whose intensity is specified as in Definition 2.1. Moreover, this process is stable, and reaches its unique stationary distribution exponentially fast.

The interested reader is referred to the original paper [7] for the proof of this result. Note that, in Appendix A, a straightforward, explicit construction of a Lyapunov function for such a process is given. This function actually provides an alternate proof of stability, together with the exponential convergence towards the stationary distribution, based on the concept of *V-geometric ergodicity* of [23].

<sup>1</sup>a rigorous proof can be found e.g. in [26]

## 3. LIMIT ORDER BOOK DRIVEN BY HAWKES PROCESSES

**3.1. Model setup.** A stylized order book model whose dynamics is governed by Hawkes processes is now introduced. Since a similar limit order book model with Poissonian arrivals has been discussed at length in [1], the description provided here is brief.

Assume therefore that each side of the order book is fully characterized by a finite number of limits  $K$ , ranging from 1 to  $K$  ticks away from the best available opposite quote. We use the notation

$$(\vec{a}; \vec{b}) = (a_1, \dots, a_K; b_1, \dots, b_K),$$

where  $\vec{a} = (a_1, \dots, a_K)$  designates the ask side of the order book and  $a_i$  the number of shares available  $i$  ticks away from the best opposite quote, and  $\vec{b} = (b_1, \dots, b_K)$  designates the bid side of the book. By convention,  $a_i \geq 0$  and  $b_i \leq 0$ .

It may be natural to consider as in [1] that the  $a_i, b_i$ 's live in a discrete space, but all the results presented in this work are valid in the case of a more general, locally compact state space, allowing in particular for real-valued  $a_i, b_i$ 's.

Three types of events can modify the limit order book:

- arrival of a new limit order;
- arrival of a new market order;
- cancellation of an already existing limit order.

The arrival of market and limit orders are described by *mutually exciting* Hawkes processes:

- $M^\pm(t)$ : arrival of new buy or sell market order, with intensity  $\lambda^{M^+}$  and  $\lambda^{M^-}$ ;
- $L_i^\pm(t)$ : arrival of a limit order at level  $i$ , with intensity  $\lambda_i^{L^\pm}$ ,

whereas the arrival of a cancellation order is modelled as in [1] by a doubly stochastic Poisson process:

- $C_i^\pm(t)$ : cancellation of a limit order at level  $i$ , with intensity  $\lambda_i^{C^+} a_i$  and  $\lambda_i^{C^-} |b_i|$ .

Here,  $q$  denotes the size of an incoming order, and the superscript “+” (respectively “-”) refers to the ask (respectively bid) side of the book. Buy limit orders  $L_i^-(t)$  arrive below the ask price  $P^A(t)$ , and sell limit orders  $L_i^+(t)$  arrive above the bid price  $P^B(t)$ . Furthermore, we impose constant boundary conditions outside the moving frame of size  $2K$ : every time the moving frame leaves a price level, the number of shares at that level is set to  $a_\infty$  or  $b_\infty$ , depending on the side of the book. The quantities  $a_\infty$  and  $b_\infty$  represent two “reservoirs” of liquidity.

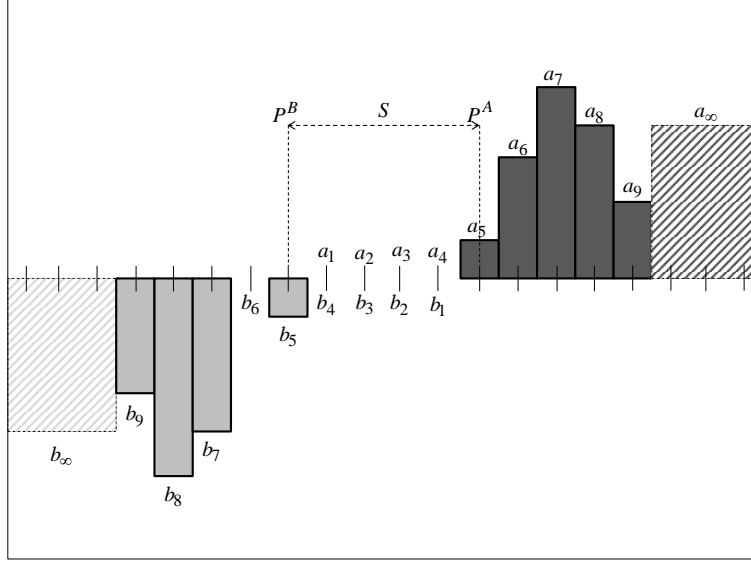


FIGURE 1. Order book dynamics: in this example,  $K = 9$ ,  $q = 1$ ,  $a_\infty = 4$ ,  $b_\infty = -4$ . The shape of the order book is such that  $a(t) = (0, 0, 0, 0, 1, 3, 5, 4, 2)$  and  $b(t) = (0, 0, 0, 0, -1, 0, -4, -5, -3)$ . The spread  $S(t) = 5$  ticks. Assume that at time  $t' > t$  a sell market order  $dM^-(t')$  arrives, then  $a(t') = (0, 0, 0, 0, 0, 0, 1, 3, 5)$ ,  $b(t') = (0, 0, 0, 0, 0, 0, -4, -5, -3)$  and  $S(t') = 7$ . Assume instead that at  $t' > t$  a buy limit order  $dL_1^-(t')$  arrives one tick away from the best opposite quote, then  $a(t') = (1, 3, 5, 4, 2, 4, 4, 4, 4)$ ,  $b(t') = (-1, 0, 0, 0, -1, 0, -4, -5, -3)$  and  $S(t') = 1$ .

Our choice of a finite moving frame and constant boundary conditions has three motivations: firstly, it assures that the order book does not become empty and that  $P^A$ ,  $P^B$  are always well defined. Secondly, it keeps the spread  $S = P^A - P^B$  and the increments of  $P^A$ ,  $P^B$  and of the mid-price  $P_{mid} = (P^A + P^B)/2$  bounded - this will be important when addressing the scaling limit of the price. Thirdly, it makes the model markovian, as we do not need to keep track of the price levels that have been visited by the moving frame at some prior time.

Figure 1 is a schematic representation of the order book.

**3.2. The infinitesimal generator.** A markovian  $(2K + 2)$ -dimensional Hawkes process now models the intensities of the arrivals of market and limit orders. With the same notations as in [1], one can work out the infinitesimal generator associated with the process describing the joint evolution of the limit order book. It is now characterized by the  $D$ -dimensional process  $(\vec{a}; \vec{b}; \vec{\mu})$  of the available quantities and the intensities of the Hawkes processes decomposed as in Section 2, where

$D = (2K + 2)^2 + 2K$  is the dimension of the state space.

In fact, one can write that

$$\begin{aligned}
\mathcal{L}F(\vec{a}; \vec{b}; \vec{\mu}) &= \lambda^{M^+} \left( F\left([a_i - (q - A(i-1))_+]_+; J^{M^+}(\vec{b}); \vec{\mu} + \Delta^{M^+}(\vec{\mu})\right) - F \right) \\
&+ \sum_{i=1}^K \lambda_i^{L^+} \left( F\left(a_i + q; J^{L^+}(\vec{b}); \vec{\mu} + \Delta^{L^+}(\vec{\mu})\right) - F \right) \\
&+ \sum_{i=1}^K \lambda_i^{C^+} a_i \left( F\left(a_i - q; J^{C^+}(\vec{b})\right) - F \right) \\
&+ \lambda^{M^-} \left( F\left(J^{M^-}(\vec{a}); [b_i + (q - B(i-1))_-]_-; \vec{\mu} + \Delta^{M^-}(\vec{\mu})\right) - F \right) \\
&+ \sum_{i=1}^K \lambda_i^{L^-} \left( F\left(J^{L^-}(\vec{a}); b_i - q; \vec{\mu} + \Delta^{L^-}(\vec{\mu})\right) - F \right) \\
&+ \sum_{i=1}^K \lambda_i^{C^-} |b_i| \left( f\left(J^{C^-}(\vec{a}); b_i + q\right) - f \right) \\
&- \sum_{i,j=1}^D \beta_{ij} \mu_{ij} \frac{\partial F}{\partial \mu_{ij}}. \tag{2}
\end{aligned}$$

In order to ease the already cumbersome notations, we have written  $F(a_i; \vec{b}; \vec{\mu})$  instead of  $F(a_1, \dots, a_i, \dots, a_K; \vec{b}; \vec{\mu})$ . Moreover, the notation  $\Delta^{(\dots)}(\vec{\mu})$  stands for the jump of the intensity vector  $\vec{\mu}$  corresponding to a jump of the process  $N^{(\dots)}$  (see Appendix A). As for the  $J$ 's, they are *shift operators* corresponding to the renumbering on one side of the limit order book following an event affecting the other side. For instance the shift operator corresponding to the arrival of a sell market order ( $\Delta M^-(t) = 1$ ) of size  $q$  is

$$J^{M^-}(\mathbf{a}) = \left( \underbrace{0, 0, \dots, 0}_{k \text{ times}}, a_1, a_2, \dots, a_{K-k} \right),$$

with

$$k := \inf\{p : \sum_{j=1}^p |b_j| > q\} - \inf\{p : |b_p| > 0\}.$$

Similar expressions are derived for the other events affecting the order book. We refer to [1] for a detailed analysis of the operator just introduced, and simply note that it is a combination of

- standard difference operators corresponding to the arrival or cancellation of orders at each limit;

- shift operators expressing the moves in the best limits and therefore, in the origins of the frames for the two sides of the order book. These shifts depend on the profile of the order book on the opposite side, namely the cumulative depth up to level  $i$  defined by

$$A(i) := \sum_{k=1}^i a_k \quad (3)$$

and

$$B(i) := \sum_{k=1}^i |b_k|, \quad (4)$$

and their generalized inverse functions

$$A^{-1}(q') := \inf\{p : \sum_{j=1}^p a_j > q'\} \quad (5)$$

and

$$B^{-1}(q') := \inf\{p : \sum_{j=1}^p |b_j| > q'\}; \quad (6)$$

- drift terms coming from the mean-reverting behaviour of the intensities of the Hawkes processes between jumps.

Note that, as mentioned above, the infinitesimal generator is fully worked out in (2) in the case of the discrete state-space; some trivial but notationally cumbersome modifications would be necessary in order to account for the case of real-valued quantities  $a_i, b_i$ 's and order size  $q$ .

#### 4. STABILITY OF THE ORDER BOOK

In this section, we study the long-time behaviour of the limit order book. Combining the approach in [1] and the results in Appendix A, a Lyapunov function can be built, ensuring the ergodicity of the limit order book under the natural assumption (1). More precisely, there holds the

**Proposition 4.1.** *Under the standing assumptions, in particular (1), the limit order book process  $X$  is ergodic. It converges exponentially fast towards its unique stationary distribution  $\Pi$ .*

*Proof.* Given the existence of the Lyapunov function provided in Lemma 4.1 below, and the *geometric drift condition* (8), the result is proven exactly as in [1] using Theorem 7.1 in [22].  $\square$



**Lemma 4.1.** *For  $\eta > 0$  small enough, the function  $V$  defined by*

$$V(\vec{a}; \vec{b}; \vec{\mu}) = \sum_{i=1}^K a_i + \sum_{i=1}^K |b_i| + \frac{1}{\eta} \sum_{i,j=1}^{(2K+2)^2} \delta_{ij} \mu_{ij} \equiv V_1 + \frac{1}{\eta} V_2 \quad (7)$$

where  $V_1$  (resp.  $V_2$ ) corresponds to the part that depends only on  $(\vec{a}; \vec{b})$  (resp.  $\vec{\mu}$ ), is a Lyapunov function satisfying a geometric drift condition

$$\mathcal{L}V \leq -\zeta V + C, \quad (8)$$

for some  $\zeta > 0$  and  $C \in \mathbb{R}$ . The coefficients  $\delta_{ij}$ 's are defined in (17) in Appendix A.

*Proof.* First specialize  $V_2$  to be identical - up to a change in the indices - to the function defined by (4.1) in Appendix A. Regarding the "small" parameter  $\eta > 0$ , it will become handy as a penalization parameter, as we shall see below.

Thanks to the linearity of  $\mathcal{L}$ , there holds

$$\mathcal{L}V = \mathcal{L}V_1 + \frac{1}{\eta} \mathcal{L}V_2.$$

The first term  $\mathcal{L}V_1$  is dealt with exactly as in [1]:

$$\begin{aligned} \mathcal{L}V_1(\mathbf{x}) &\leq -(\lambda^{M^+} + \lambda^{M^-})q + \sum_{i=1}^K (\lambda_i^{L^+} + \lambda_i^{L^-})q - \sum_{i=1}^K (\lambda_i^{C^+} a_i + \lambda_i^{C^-} |b_i|)q \\ &\quad + \sum_{i=1}^K \lambda_i^{L^+} (i_S - i)_+ a_\infty + \sum_{i=1}^K \lambda_i^{L^+} (i_S - i)_+ |b_\infty| \end{aligned} \quad (9)$$

$$\begin{aligned} &\leq -(\lambda^{M^+} + \lambda^{M^-})q + (\Lambda^{L^-} + \Lambda^{L^+})q - \underline{\lambda}^C q V_1(\mathbf{x}) \\ &\quad + K(\Lambda^{L^-} a_\infty + \Lambda^{L^+} |b_\infty|), \end{aligned} \quad (10)$$

where

$$\Lambda^{L^\pm} := \sum_{i=1}^K \lambda_i^{L^\pm} \text{ and } \underline{\lambda}^C := \min_{1 \leq i \leq K} \{\lambda_i^{C^\pm}\} > 0.$$

Computing  $\mathcal{L}V_2$  yields an expression identical to that obtained in Appendix A:

$$\mathcal{L}(V_2) = \sum_{i,j} \lambda_0^j \delta_{ij} \alpha_{ij} + (\kappa - 1) \sum_{j,k} \epsilon_k \mu_{jk},$$

so that there holds

$$\mathcal{L}V = \mathcal{L}V_1 + \frac{1}{\eta} \mathcal{L}V_2 \leq -\underline{\lambda}^C q V_1 - \frac{\gamma}{\eta} V_2 - \vec{G} \cdot \vec{\mu} + C,$$

where  $\gamma$  is as in Equation (19),  $\vec{G} \cdot \vec{\mu}$  is a compact notation for the linear form in the  $\mu_{ij}$ 's obtained in (10), and  $C$  is some constant. Now, thanks to the positivity of the

coefficients in  $V_2$  and of the  $\mu_{ij}$ 's, one can choose  $\eta$  small enough that there holds

$$\sqrt{\vec{\mu}}, |\vec{G} \cdot \vec{\mu}| \leq \frac{\gamma}{2\eta} V_2(\vec{\mu}),$$

which yields

$$\mathcal{L}V \equiv \mathcal{L}V_1 + \frac{1}{\eta} \mathcal{L}V_2 \leq -\underline{\lambda}^C q V_1 - \frac{\gamma}{2\eta} V_2 + C, \quad (11)$$

and finally

$$\mathcal{L}V \leq -\zeta V + C,$$

with  $\zeta = \text{Min}(\underline{\lambda}^C q, \frac{\gamma}{2\eta})$  and  $C$  is some constant.  $\square$

## 5. LARGE SCALE LIMIT OF THE PRICE PROCESS

This section is devoted to a study of the asymptotics of the price process. The ergodic theory of Markov processes, see [8], is combined with the martingale convergence theorem [15], to obtain the results. This approach is extremely general and flexible, and prone to many generalizations for markovian models of limit order books. It is somewhat similar to that used in [19][6], where various long-time, large scale behaviour of limit order books are studied. However, the stochastic behaviour of the intensities of the point processes triggering the order book events makes the situation we consider slightly different.

We first derive the expression of the price dynamics in our limit order book model. Consider for instance the best ask and bid prices, denoted by  $P^A(t)$  and  $P^B(t)$ . One can easily see that they satisfy the following SDE's:

$$\begin{aligned} dP^A(t) &= \Delta P[(A^{-1}(q) - A^{-1}(0))dM^+(t) \\ &\quad - \sum_{i=1}^K (A^{-1}(0) - i)_+ dL_i^+(t) + (A^{-1}(q) - A^{-1}(0))dC_{i_A}^+(t)] \end{aligned}$$

and

$$\begin{aligned} dP^B(t) &= -\Delta P[(B^{-1}(q) - B^{-1}(0))dM^-(t) \\ &\quad - \sum_{i=1}^K (B^{-1}(0) - i)_+ dL_i^-(t) + (B^{-1}(q) - B^{-1}(0))dC_{i_B}^-(t)] \end{aligned}$$

describing the various events that affect them: change due to a market order, change due to limit orders inside the spread, and change due to the cancellation of a limit order at the best limit.

Let us recast these equations under a general form as follows:

$$P_t = \int_0^t \sum_{i=1}^{2K+2} F_i(X_u) dN_u^i,$$

where the  $N^i$  are the point processes driving the limit order book,  $X$  is the markovian process describing its state, and  $P$  is one of the price processes we are interested in. For instance, in the Poisson case dealt with in [1],  $X = (\vec{a}, \vec{b})$  and the  $N^i$  are the Poisson processes driving the arrival of market, limit and cancellation orders. In the context of Hawkes processes considered in this work,  $X = (\vec{a}, \vec{b}, \vec{\mu})$  and the  $N^i$  are the Poisson and Hawkes processes driving the limit order book. Also, recall that, as mentinoed earlier, the  $F_i$  are bounded functions, as the price changes are bounded by the total number of limits in the book, thanks to the non-zero boundary conditions  $a_\infty, b_\infty$ .

Denote now by  $\Pi$  the stationary distribution of  $X$  as provided by Proposition 4.1. Then, the ergodic theorem for Markov processes states the following:

**Theorem 5.1** ([8][21]). *Let  $H$  be in  $L^1(\Pi(dX))$ . Then,*

$$\lim_{t \rightarrow +\infty} \frac{a.s.}{t} \int_0^t G(X_t) dt = \int G(X) \Pi(dX).$$

Using this classical result, together with the deep Theorem 7.1.4 of [15] regarding the convergence of martingales, one can show the

**Proposition 5.1.** *Consider the price process described by Equation (12) above, and introduce the sequence of martingales  $\hat{P}^n$  formed by the centered, rescaled price*

$$\hat{P}_t^n \equiv \frac{P_{nt} - Q_{nt}}{\sqrt{n}},$$

where  $Q$  is the compensator of  $P$

$$Q_t = \int_0^t \sum_i F_i(X_t) \lambda_t^i dt.$$

Then,  $\hat{P}^n$  converges in distribution to a Wiener process  $\hat{\sigma}W$ , where the volatility  $\hat{\sigma}$  is given by

$$\hat{\sigma}^2 = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \sum_i (F_i(X_t))^2 \lambda_t^i dt = \int \sum_i (F_i(X))^2 \lambda_i \Pi(dX).$$

*Proof.* Proposition 5.1 will follow from the convergence of the predictable quadratic variation of  $\hat{P}^n$ . By construction, there holds

$$\langle \hat{P}^n, \hat{P}^n \rangle_t = \frac{1}{n} \int_0^{nt} \sum_i (F_i(X_t))^2 \lambda_t^i dt,$$

or else

$$\langle \hat{P}^n, \hat{P}^n \rangle_t = t \left( \frac{1}{nt} \int_0^{nt} \sum_i (F_i(X_t))^2 \lambda_t^i dt \right),$$

and Theorem 5.1 ensures that

$$\lim_{t \rightarrow +\infty} \frac{a.s.}{nt} \int_0^{nt} \sum_i (F_i(X_t))^2 \lambda_t^i dt = \int \sum_i (F_i(X))^2 \lambda_i \Pi(dX)$$

whenever the integrability conditions (see Theorem 5.1) are satisfied. Now, those are easily seen to hold true, since the Lyapunov function  $V$  itself is in  $L^1(\Pi(dx))$ , see e.g. [16], and the integrand in the predictable quadratic variation, being linear in the  $\lambda$ 's and bounded as a function of the  $a_i, b_i$ 's, is bounded by a multiple of  $V$ . The other condition for the martingale convergence theorem to apply is trivially satisfied, since the size of the jumps of  $\hat{P}^n$  is bounded by  $\frac{C}{\sqrt{n}}$ ,  $C$  being some constant.  $\square$

Appealing as it first seems, Proposition 5.1 is not satisfactory: in order to give a more precise characterization of the dynamics of the rescaled price process, it is necessary to understand thoroughly the behaviour of its compensator  $Q_{nt}$ . As a matter of fact,  $Q_{nt}$  itself satisfies an ergodic theorem, and if its asymptotic variance is not negligible w.r. to  $nt$ , one cannot conclude directly from Proposition 5.1 that the rescaled price process  $\frac{P_{nt}}{\sqrt{n}}$  behaves like a Wiener process with a deterministic drift. In the case of a Poisson-driven limit order book as in [1], Proposition 5.1 is actually sufficient, but in the case of Hawkes processes as in the present paper, or for more general Markovian limit order books, the situation becomes more intricate.

The next result provides a more accurate answer, valid under general ergodicity conditions.

**Theorem 5.2.** *Write as above the price*

$$P_t = \int_0^t \sum_i F_i(X_t) dN_t^i$$

*and its compensator*

$$Q_t = \int_0^t \sum_i F_i(X_t) \lambda_t^i dt.$$

*Define*

$$h = \sum_i F_i(X) \lambda_i$$

*and let*

$$\alpha = \lim_{t \rightarrow +\infty} \frac{a.s.}{t} \int_0^t \sum_i (F_i(X_s)) \lambda_s^i dt = \int h(X) \Pi(dX).$$

*Finally, introduce the solution  $g$  to the Poisson equation*

$$\mathcal{L}g = h - \alpha$$

and the associated martingale

$$Z_t = g(X_t) - g(X_0) - \int_0^t \mathcal{L}g(X_s) ds \equiv g(X_t) - g(X_0) - Q_t + \alpha t.$$

Then, the deterministically centered, rescaled price

$$\bar{P}_t^n \equiv \frac{P_{nt} - \alpha nt}{\sqrt{n}}$$

converges in distribution to a Wiener process  $\bar{\sigma}W$ . The asymptotic volatility  $\bar{\sigma}$  satisfies the identity

$$\bar{\sigma}^2 = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \sum_i ((F_i - \Delta^i(g))(X_s))^2 \lambda_s^i ds \equiv \int \sum_i ((F_i - \Delta^i(g))(X))^2 \lambda_i \Pi(dx),$$

where  $\Delta^i(g)(X)$  denotes the jump of the process  $g(X)$  when the process  $N^i$  jumps and the limit order book is in the state  $X$ .

*Proof.* The martingale method, see e.g. [16][13][20], consists in rewriting the price process under the form

$$P_t = (P_t - Q_t) - Z_t + g(X_t) - g(X_0) + \alpha t = (M_t - Z_t) + g(X_t) - g(X_0) + \alpha t,$$

so that

$$\bar{P}_t^n = \frac{V_{nt} + g(X_t) - g(X_0)}{\sqrt{n}},$$

where  $V = M - Z$  is a martingale. Therefore, the theorem is proven *iff* one can show that  $\frac{g(X_t) - g(X_0)}{\sqrt{n}}$  converges to 0 in  $L^2(\Pi(dx))$ , or simply, that  $g \in L^2(\Pi(dx))$ .

Theorem 4.4 of [16] states that the condition

$$h^2 \leq V \tag{12}$$

(where  $V$  is a Lyapunov function for the process) is sufficient for  $g$  to be in  $L^2(\Pi(dx))$ . The linear Lyapunov function  $V$  introduced in (7) does not yield the result<sup>2</sup>, because  $h$  is linearly increasing in the  $\lambda$ 's. However, see Lemma A.1 in Appendix A, one can design a Lyapunov function having a polynomial growth of arbitrary high order in the  $\lambda$ 's at infinity, thereby ensuring that Condition (12) holds.  $\square$

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<sup>2</sup>It is suitable for a Poisson-driven limit order book, since in that case  $h$  is bounded

## APPENDIX A. LYAPUNOV FUNCTIONS FOR HAWKES PROCESSES

For the sake of completeness, an explicit construction of a Lyapunov function for a multi-dimensional Hawkes processes  $\vec{N} = (N^i)$  with intensities

$$\lambda_t^i = \lambda_0^i + \sum_j \int_0^t \alpha_{ij} e^{-\beta_{ij}(t-s)} dN_s^j$$

is provided here.

Denote as in Section 2

$$\mu_t^{ij} = \int_0^t \alpha_{ij} e^{-\beta_{ij}(t-s)} dN_s^j,$$

so that there holds

$$\lambda_t^i = \lambda_0^i + \sum_j \mu_t^{ij}. \quad (13)$$

We assume for simplicity <sup>3</sup> the following

$$\forall i, j, \alpha_{ij} > 0, \beta_{ij} > 0, \quad (14)$$

as well as the spectral condition (1). The infinitesimal generator associated to the markovian process  $(\mu^{ij})$ ,  $1 \leq i, j \leq D$ , is the operator

$$\mathcal{L}_H F(\vec{\mu}) = \sum_j \lambda^j (F(\vec{\mu} + \Delta^j(\vec{\mu})) - F(\vec{\mu})) - \sum_{i,j} \beta_{ij} \mu_{ij} \frac{\partial F}{\partial \mu_{ij}},$$

where  $\vec{\mu}$  is the vector with components  $\mu_{ij}$  and  $\lambda_j$  is as in (13). The notation  $\Delta^j(\vec{\mu})$  characterizes the jumps in those of the entries in  $\vec{\mu}$  that are affected by a jump of the process  $N^j$ . For a **fixed** index  $j$ , it is given by the vector with entries  $\alpha_{ij}$  at the relevant spots, and zero entries elsewhere.

A Lyapunov function for the associated semi-group is sought under the form

$$V(\vec{\mu}) = \sum_{i,j} \delta_{ij} \mu_{ij} \quad (15)$$

(since the intensities are always positive, a linear function will be coercive). Assuming (15), there holds

$$\mathcal{L}_H V = \sum_j \lambda^j \left( \sum_i \delta_{ij} \alpha_{ij} \right) - \sum_{i,j} \beta_{ij} \mu_{ij} \delta_{ij}$$

or

$$\mathcal{L}_H V = \sum_{i,j} (\lambda_0^j + \sum_k \mu_{jk}) \delta_{ij} \alpha_{ij} - \beta_{ij} \mu_{ij} \delta_{ij}. \quad (16)$$

<sup>3</sup>The classical assumption of irreducibility would be sufficient, as long as all the  $\beta$ 's are strictly positive

At this stage, it is convenient to introduce  $\vec{\epsilon}$  the maximal eigenvector of the matrix  $\mathbf{A}$  (introduced in Section 2) with entries

$$A_{ij} = \frac{\alpha_{ji}}{\beta_{ji}}.$$

Denote by  $\kappa$  the associated maximal eigenvalue. By Assumption (1), one has that  $0 < \kappa < 1$  and furthermore, by Perron-Frobenius theorem, there holds:  $\forall i, \epsilon_i > 0$ .

Assuming that

$$\delta_{ij} \equiv \frac{\epsilon_i}{\beta_{ij}}, \quad (17)$$

the expression for  $V$  becomes

$$V(\vec{\mu}) = \sum_{i,j} \epsilon_i \frac{\mu_{ij}}{\beta_{ij}}. \quad (18)$$

Plugging (18) in (16) yields

$$\begin{aligned} \mathcal{L}_H V &= \sum_{i,j} \lambda_0^j \delta_{ij} \alpha_{ij} + \sum_{i,j,k} \mu_{jk} \epsilon_i \frac{\alpha_{ij}}{\beta_{ij}} - \sum_{j,k} \beta_{jk} \mu_{jk} \delta_{jk} \\ &= \sum_{i,j} \lambda_0^j \delta_{ij} \alpha_{ij} + (\kappa - 1) \sum_{j,k} \epsilon_k \mu_{jk}, \end{aligned}$$

using the identity  $\sum_j \mathbf{A}_{ji} \epsilon_i = \kappa \epsilon_j$ . A comparison with (18) easily yields the upper bound

$$\mathcal{L}_H V \leq -\gamma V + C, \quad (19)$$

with  $\gamma = (1 - \kappa) \beta_{min}$ ,  $\beta_{min} \equiv \text{Inf}_{i,j}(\beta_{ij}) > 0$  by assumption, and  $C = \sum_{i,j} \lambda_0^j \delta_{ij} \alpha_{ij} \equiv \kappa \vec{\epsilon} \cdot \vec{\lambda}_0$ .

Sufficient as it is to prove Theorem 5.1 using the Lyapunov function for the limit order book detailed in Proposition 4.1, a linearly growing Lyapunov function is too weak to prove Theorem 5.2: as already noted, Theorem 4.4 in [16] requires a Lyapunov function with *quadratic* growth, in order that the Poisson equation with a linearly growing RHS have a solution in  $L^2(\Pi(dx))$ . A useful extension of Proposition 4.1 is given in the following

**Lemma A.1.** *Under the standing assumptions (1) and (14), one can construct a Lyapunov function of arbitrary high polynomial growth at infinity.*

*Proof.* Let  $n \in \mathbb{N}^*$ , and  $V$  be the function defined in (18). The function  $V^n$  satisfies

$$\mathcal{L}_H V^n(\vec{\mu}) = \sum_j \lambda^j (V^n(\vec{\mu} + \Delta^j(\vec{\mu})) - V^n(\vec{\mu})) - n V^{n-1} \left( \sum_{i,j} \beta_{ij} \mu_{ij} \frac{\partial V}{\partial \mu_{ij}} \right). \quad (20)$$

Upon factoring  $V^n(\vec{\mu} + \Delta^j(\vec{\mu})) - V^n(\vec{\mu})$ :

$$V^n(\vec{\mu} + \Delta^j(\vec{\mu})) - V^n(\vec{\mu}) = (V(\vec{\mu} + \Delta^j(\vec{\mu})) - V(\vec{\mu})) \left( \sum_{k=0}^{n-1} V^{n-1-k}(\vec{\mu} + \Delta^j(\vec{\mu})) V^k(\vec{\mu}) \right),$$

the linearity of  $V$  yields the following expression

$$V^n(\vec{\mu} + \Delta^j(\vec{\mu})) - V^n(\vec{\mu}) = nV^{n-1}(\vec{\mu})(V(\vec{\mu} + \Delta^j(\vec{\mu})) - V(\vec{\mu})) + \mathcal{M}_j V(\vec{\mu}),$$

where  $\mathcal{M}_j V(\vec{\mu})$  can be bounded by a polynomial function of order  $n - 1$  at infinity in  $\vec{\mu}$ . Therefore, one can rewrite (20) as follows

$$\mathcal{L}_H V^n(\vec{\mu}) = (nV^{n-1} \mathcal{L}_H V)(\vec{\mu}) + \mathcal{M} V(\vec{\mu}), \quad (21)$$

where  $\mathcal{M}(V)(\vec{\mu})$  is a polynomial of order  $n - 1$  in  $\vec{\mu}$ . Combining (18) with (21) shows that  $V^n$  is also a Lyapunov function for the Hawkes process.  $\square$

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