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Discrete Hammersley's Lines with sources and sinks

A-L. Basdevant*, N. Enriquez†, L. Gerin‡, J-B. Gouéré§

March 2, 2015

Abstract

We introduce two stationary versions of two discrete variants of Hammersley's process in a finite box, this allows us to recover in a unified and simple way the laws of large numbers proved by T. Seppäläinen for two generalized Ulam's problems. As a by-product we obtain an elementary solution for the original Ulam problem.

We also prove that for the first process defined on \mathbb{Z} , Bernoulli product measures are the only extremal and translation-invariant stationary measures.

MSC 2010 Classification: 60K35, 60F15.

Keywords: Hammersley's process, Ulam's problem, longest increasing subsequences.

1 Introduction

In a celebrated paper, J.M.Hammersley used Poissonization to attack the so-called *Ulam problem* of the typical length $\ell(n)$ of the longest increasing subsequence of a uniform permutation of size n . Namely, he reduced this problem to finding the greatest number of points of a Poisson point process inside a square, an increasing path can go through. He proved ([4], Theorem 4) that $\ell(n)/\sqrt{n}$ converges in probability to some constant c , sometimes referred to as the *Ulam constant*, and conjectured that $c = 2$.

The proof of $c = 2$ was achieved independently by Logan and Shepp and by Vershik and Kerov in 1977, using algebraic methods. Various authors were then interested in finding a more probabilistic proof of this result. First, Aldous and Diaconis [1] gave one, using the properties of what they called Hammersley's process, which was implicitly introduced in [4] (p.358 and following). Hammersley's process is continuous in time and space and Aldous and Diaconis studied its properties on the infinite line, in particular its stationary distributions. A few years later, Groeneboom [3] and Cator and Groeneboom [2] studied Hammersley's process on a quarter plane. By adding what they called Poisson *sinks* and *sources* on the x and y -axis, they also found a stationary version of this process on the quarter plane. Using this point of view, they were able to recover again the value of c .

In this paper, we study two discrete variants of Ulam's problem. Namely, for all p in $[0, 1]$, we replace the original Poisson point process by the following random set ξ of integer points: each integer point of the square $[1, n]^2$ is chosen independently with probability p . We are interested in the two following quantities.

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- The length of the longest increasing subsequence through points of ξ :

$$L_n^{(1)} = \max \{L; (i_1, j_1), \dots, (i_L, j_L) \in \xi, i_1 < \dots < i_L \text{ and } j_1 < j_2 < \dots < j_L\}.$$

- The length of the longest non-decreasing subsequence through points of ξ :

$$L_n^{(2)} = \max \{L; (i_1, j_1), \dots, (i_L, j_L) \in \xi, i_1 < \dots < i_L \text{ and } j_1 \leq j_2 \leq \dots \leq j_L\}.$$

One aim of the present work is to recover by simple and unified probabilistic arguments the first order asymptotics of $L_n^{(1)}$ and $L_n^{(2)}$ already obtained by T. Seppäläinen in two independent papers, for $L_n^{(1)}$ in [9] and for $L_n^{(2)}$ in [10].

Theorem 1. • *The length $L_n^{(1)}$ of the longest increasing subsequence satisfies*

$$\frac{L_n^{(1)}}{n} \xrightarrow[n \rightarrow +\infty]{a.s.} \frac{2\sqrt{p}}{\sqrt{p} + 1}. \quad (1)$$

- *The length $L_n^{(2)}$ of the longest non-decreasing subsequence satisfies*

$$\frac{L_n^{(2)}}{n} \xrightarrow[n \rightarrow +\infty]{a.s.} \begin{cases} 2\sqrt{p(1-p)} & \text{if } p < 1/2, \\ 1 & \text{if } p \geq 1/2. \end{cases} \quad (2)$$

To prove this, Seppäläinen associates to each problem a particle system on the infinite line \mathbb{Z} and checks that Bernoulli product measures are stationary. The position of particles at a given time is then characterized as the solution of a discrete optimization problem, and the a.s. limit of $L_n^{(i)}/n$ is identified using convex analysis arguments. Note also that for the second question, Johansson used later ([6], Th.5.3) another description of $L_n^{(2)}/n$ and proved that the rescaled fluctuations converge to the Tracy-Widom distribution.

In [8], Rolla *et al.* considered the closely related model of last-passage percolation with geometric weights. Here is a quick description of the model. Let $(\tilde{\xi}(i, j))_{1 \leq i, j \leq n}$ be a family of independent geometric random variables with parameter p . Set:

$$L_n^{(3)} = \max \left\{ \sum_t \tilde{\xi}(\gamma(t)) \right\}$$

where the maximum is taken over all paths γ from $(1, 1)$ to (n, n) with $(1, 0)$ or $(0, 1)$ steps. With respect to $L_n^{(1)}$ or $L_n^{(2)}$, we have simply changed the weight of each point (from Bernoulli to geometric) and the set of paths considered. Rolla *et al.* gave a new proof of the following results:

$$\frac{L_n^{(3)}}{n} \xrightarrow[n \rightarrow +\infty]{a.s.} \frac{2\sqrt{p}}{1 - \sqrt{p}}.$$

Our proof of the lower bound of Theorem 1 is inspired by the proof of Rolla *et al.* of the above result.

The strategy is to consider a particle system on a *bounded domain* which turns out to coincide with the restriction of Seppäläinen's particle system on \mathbb{Z} . This simplifies the definition of the process (especially, in the case of the second problem). Moreover, it turns out that a *local* balance property around a single site, see Lemma 1 and 2 below, is enough to check the stationarity of the process on the whole square. Our proofs are essentially the same for both models. This kind of remarkable local balance property also occurs in last-passage percolation with geometric

weights (see Rolla *et al.*, Section 3.1 in [8] or Seppäläinen, Lemma 2.3 in [11]). The particle systems studied here provide two other examples where such a property holds. Theorem 1 is then proven by investigating the behaviour of the models under the different stationary measures which have been exhibited. Again, the strategy is the same as in Rolla *et al.* [8].

A nice by-product of our proof is that we obtain non-asymptotic estimates which provide an elementary proof of $c = 2$ for the original Ulam problem (see the discussion at the end of Section 4).

In the last section, which is of independent interest, we study the particle system defined on the whole line \mathbb{Z} associated to the first problem and prove that the only extremal translation-invariant stationary measures of this process are the independent product of Bernoulli random variables.

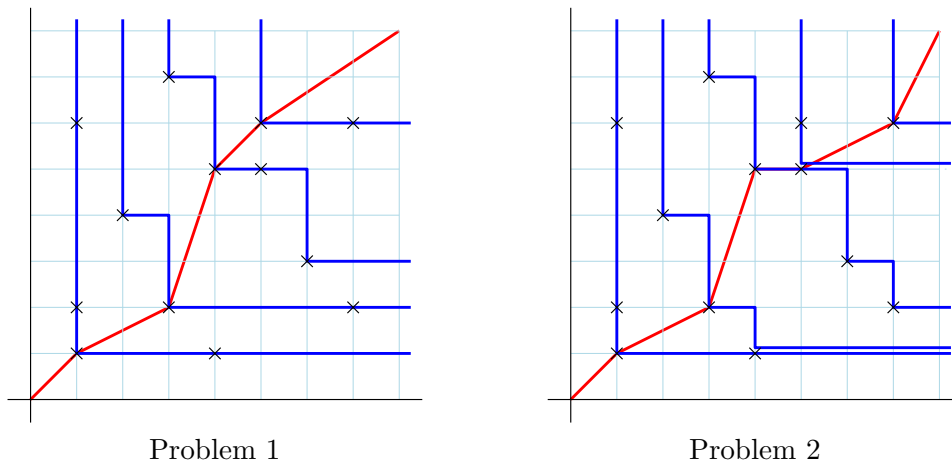
2 Discrete Hammersley's processes

Like Hammersley did, we construct two sets of broken lines whose cardinality is respectively the variable $L_n^{(1)}$ and $L_n^{(2)}$. For each case, we first introduce a partial order:

$$\begin{array}{ll} \text{Case 1:} & \text{Case 2:} \\ (x, y) \prec (x', y') \text{ iff} & (x, y) \prec (x', y') \text{ iff} \\ x < x' \text{ and } y < y' & x < x' \text{ and } y \leq y' \end{array}$$

Now, Hammersley's lines are paths starting from the top side of the square, ending at its right side and making only south or east steps. They are constructed recursively. The first line is the highest non-increasing path connecting the minimal points of ξ for \prec . We withdraw these points from ξ and connect the new minima to get the second line, and so on.

In the below picture $L_n^{(1)} = 4$, $L_n^{(2)} = 5$ (crosses denote points of ξ , Hammersley's lines are in blue, one of the longest subsequences is in red):



In order to analyze the number of Hammersley's lines, we introduce two discrete time Markovian particle systems in which the particle trajectories precisely follow these lines.

These processes take their values in $\{0, 1\}^n$, we denote them by $(X_t^1)_{t \geq 0} := (X_t^1(x), x \in \llbracket 1, n \rrbracket)_{t \geq 0}$ and $(X_t^2)_{t \geq 0} := (X_t^2(x), x \in \llbracket 1, n \rrbracket)_{t \geq 0}$. We will say that there is a *particle* at time t at position x in Model i if $X_t^i(x) = 1$. For both models we start with the initial condition $X_0^i \equiv 0$.

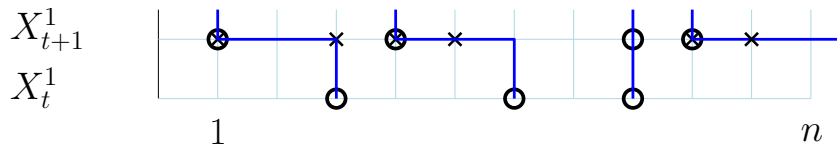
In order to define the dynamic of these processes, we introduce a family of i.i.d. random variables $\xi = \{\xi_t(x), x \in \llbracket 1, n \rrbracket, t \in \mathbb{N}\}$ with law Bernoulli(p). We say that there is a *cross* at

time t located at x if $\xi_t(x) = 1$. A cross will *act* on a given configuration of particles in the following way : if a cross is located at x , then the left-most particle in the interval $\llbracket x, n \rrbracket$ (if any) moves to x ; if there is no such particle then a particle is created at x .

With this definition, we can now construct the value of X_{t+1}^i as a function of X_t^i and the crosses at time $t + 1$:

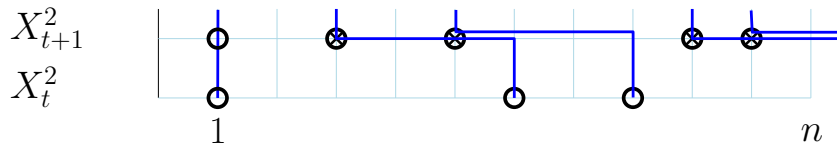
Model 1. We define X_{t+1}^1 as the result of the *successive* actions on X_t^1 of all crosses at time $t + 1$ *from the right to the left*.

An example is drawn in this figure where the circles represent the particles and the crosses the locations where ξ_{t+1} equals 1. Here, we have $X_{t+1}^1 = (1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0)$ (as it is usual in the literature on Hammersley's processes, time goes from bottom to top in our pictures):



Let us note that there is an alternative way to construct the model 1 : At time t , if there is a particle at x and if the particle immediately on its left is at y , then the particle at x jumps at time $t + 1$ at the rightmost cross of ξ_{t+1} in interval (y, x) (if any, otherwise it stays still). If there are some points of ξ_{t+1} at the right of the rightmost particle of X_t^1 then a new particle appears at the leftmost point among them.

Model 2. We define X_{t+1}^2 as the result of the *successive* actions on X_t^2 of all crosses at time $t + 1$ *from the left to the right*.



We let the reader convince themselves that the trajectories of the particles coincide indeed with the lines previously defined. In particular, and this is actually the important point, the number of particles at time n is, for both model, equal to $L_n^{(i)}$.

Although the definition of both models seems, at first glance, very close, the nature of the two processes is in fact quite different. Indeed, in the first model, to find the location at time $t + 1$ of a particle located at time t at x , one just need to know the location y of the particle immediately on his left at time t and the position x of the crosses of ξ_{t+1} in the interval (y, x) . In particular, with no difficulty, one can define a process with similar transitions on the whole line \mathbb{Z} as it is done in [9]. For Model 2, the dependances are more intricate. Indeed, to determine the location at time $t + 1$ of a particle located at time t at x , one need to know the whole configuration of X_t^2 and ξ_{t+1} on the interval $\llbracket 1, x \rrbracket$. In particular, the definition on the whole line \mathbb{Z} of a similar process is more delicate and requires a condition between the density of crosses and particles (see [10]).

3 Sinks and sources

Since, in both models, the number of particles can only increase as time goes on, both processes converge a.s. to $X_\infty^i := 1$. In this section, we are going to modify a little bit the rules of evolution of the processes such that less trivial stationary measures exist. To do so, we define two families of random variables : $(\xi_t(0))_{t \geq 1}$ on the y -axis and $(\xi_0(x))_{x \in \llbracket 1, n \rrbracket}$ on the x -axis. Using the same

terminology as in [2], we call these two families respectively the *sinks* and the *sources* of our processes. The sources will take values in $\{0, 1\}^n$ whereas the sinks will take values in $\{0, 1\}^{\mathbb{N}}$ for Model 1 and in $\mathbb{Z}_+^{\mathbb{N}}$ for Model 2. The two processes with *sources* and *sinks* evolve with the following rules:

Model i with sources and sinks

- the process starts with the initial configuration given by the sources *i.e.* $X_0^i \equiv \xi_0$.
- a sink acts on a configuration of particles in the following way: if the value of a sink equals k , then the k leftmost particles jump to this sink and disappear.
- we define the value of X_{t+1}^1 with sources and sinks as the result of the successive actions on X_t^1 of
 - all crosses $(\xi_{t+1}(x), x \in \llbracket 1, n \rrbracket)$ at time $t + 1$ from the right to the left
 - the sink $\xi_{t+1}(0)$.
- we define the value of X_{t+1}^2 with sources and sinks as the result of the successive actions on X_t^2 of
 - the sink $\xi_{t+1}(0)$
 - all crosses $(\xi_{t+1}(x), x \in \llbracket 1, n \rrbracket)$ at time $t + 1$ from the left to the right.

Theorem 2 (Stationarity of Hammersley's processes in a finite box, with sources and sinks).

For all n ,

- **Model 1.** For all $p, \alpha \in (0, 1)$, the process $(X_t^1(x), x \in \llbracket 1, n \rrbracket)_{t \geq 0}$ is stationary if sources are *i.i.d.* $\text{Ber}(\alpha)$ and sinks are *i.i.d.* $\text{Ber}(\alpha^*)$ (and sinks independent from sources) with

$$\alpha^* = \frac{p(1 - \alpha)}{\alpha + p(1 - \alpha)}. \quad (3)$$

- **Model 2.** For all $p, \alpha \in (0, 1)$ such that $\alpha > p$, the process $(X_t^2(x), x \in \llbracket 1, n \rrbracket)_{t \geq 0}$ is stationary if sources are *i.i.d.* $\text{Ber}(\alpha)$ and sinks are *i.i.d.* $(\text{Geo}(\alpha^*) - 1)$ (*i.e.* $\mathbb{P}(\xi_t(0) = k) = \alpha^*(1 - \alpha^*)^k$ for $k = 0, 1, \dots$) (and sinks independent from sources) with

$$\alpha^* = \frac{\alpha - p}{\alpha(1 - p)}. \quad (4)$$

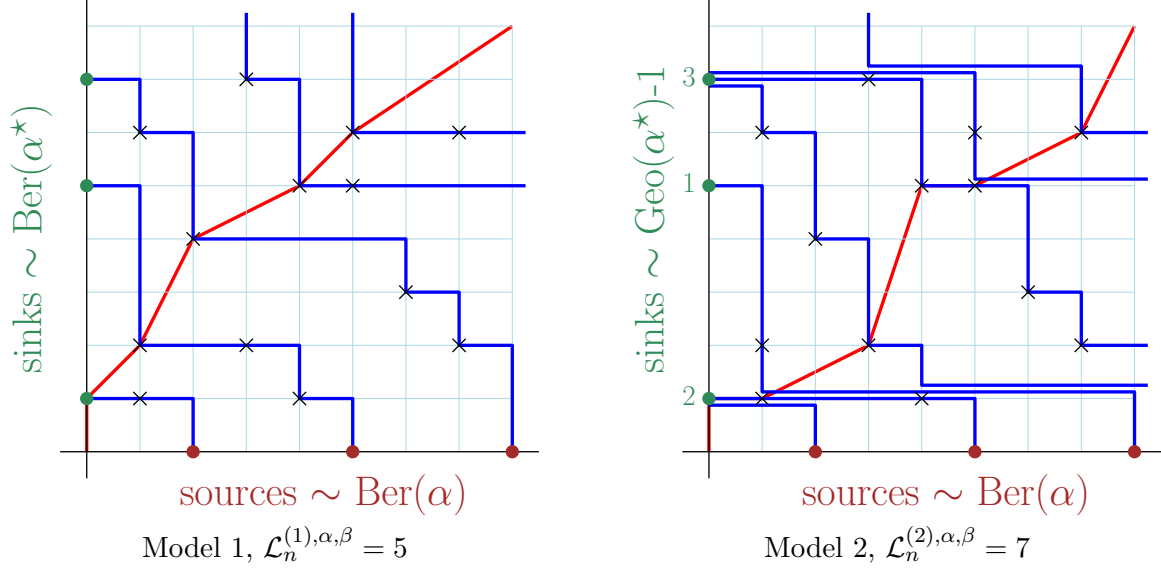
(Note that $\alpha^* \in (0, 1)$ if $\alpha > p$.)

Remark 1. • For Model 1 (resp. Model 2), Theorem 2 should be compared to Lemma 2.1 in [9] (resp. Proposition 1 in [10]) which states that Bernoulli product measures are stationary for the analogous model on the infinite line.

- The condition $\alpha > p$ in Model 2 reminds the condition on the density of the initial configuration needed to define the process on \mathbb{Z} in [10].
- The theorem states stationarity in time (*i.e.* from bottom to top in our figures). In fact, the proof also shows stationarity from left to right.

Let us collect a few consequences for further use. Let $\mathcal{L}_{(n,m)}^{(i),\alpha,\beta}$ be the number of Hammersley's lines in Model i with parameters α (sources), β (sinks) in the box $\{1, \dots, n\} \times \{1, \dots, m\}$ and let $\mathcal{T}_{(n,m)}^{(i),\alpha,\beta}$ be the number of Hammersley's lines that leave the same box from the top. Then, for $i = 1, 2$ and every $\alpha, \beta \in [0, 1]$

$$\begin{aligned} \mathcal{L}_{(n,m)}^{(i),0,0} &= \mathcal{T}_{(n,m)}^{(i),0,0} \stackrel{d}{=} L_{(n,m)}^{(i)}, \\ \mathcal{L}_{(n,m)}^{(i),\alpha,\beta} &= \mathcal{T}_{(n,m)}^{(i),\alpha,\beta} + \# \{ \text{sinks between 1 and } m \}, \end{aligned} \quad (5)$$



note that in the right-hand side the two terms are not independent. Moreover, $\mathcal{L}_{(n,m)}^{(i),\alpha,\beta}$ can be interpreted as the longest weakly increasing or non-decreasing path. Let us give a precise definition in the case, for example, of Model 1. The quantity $\mathcal{L}_{(n,m)}^{(1),\alpha,\beta}$ is the maximum of

$$\max\{L; (i_1, 0), \dots, (i_k, 0), (i_{k+1}, j_{k+1}), \dots, (i_L, j_L) \in \xi, i_1 < \dots < i_L \text{ and } 0 < j_{k+1} < \dots < j_L\}$$

and

$$\max\{L; (0, j_1), \dots, (0, j_k), (i_{k+1}, j_{k+1}), \dots, (i_L, j_L) \in \xi, 0 < i_{k+1} < \dots < i_L \text{ and } j_1 < \dots < j_L\}.$$

In other words, the path can pick either sources either sinks before going into $\mathbb{N} \times \mathbb{N}$. This property can be proven directly. It can also be seen as a consequence of the similar property without sinks or sources by seeing sinks and sources as crosses placed at a convenient location outside \mathbb{N}^2 and by considering the longest increasing path from a conveniently chosen point in $(-\mathbb{N})^2$. A similar property holds $\mathcal{L}_{(n,m)}^{(2),\alpha,\beta}$. Note that, in this case, the path can pick several sinks at the same point.

Thanks to these properties, we obtain the following inequality

$$\bar{L}_{(n,m)}^{(i)} \leq \mathcal{L}_{(n,m)}^{(i),\alpha,\beta}. \quad (6)$$

Moreover, Theorem 2 implies that

$$\mathcal{T}_{(n,m)}^{(i),\alpha,\alpha^*} \stackrel{d}{=} \text{Binom}(n, \alpha). \quad (7)$$

Proof. We first prove for both models a local balance property, in Lemmas 1 and 2. These lemmas are elementary but do not seem to be written elsewhere, and form the heart of our proof.

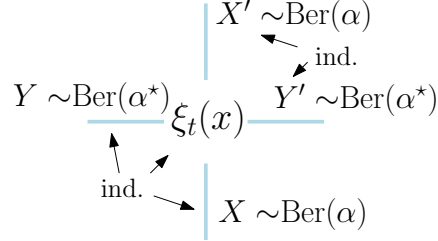
Model 1.

We first focus on what happens around a single point (x, t) . Recall that $\xi_t(x)$ is a Bernoulli(p). Denote by X (resp. Y, X', Y') be the indicator that a Hammersley's line hits (x, t) from the bottom (resp. left, top, right).

Lemma 1 (Local balance for Model 1). *The random variables (X', Y') are measurable with respect to $(X, Y, \xi_t(x))$ and we have*

$$\left((X, Y, \xi_t(x)) \stackrel{(d)}{=} \text{Ber}(\alpha) \otimes \text{Ber}(\alpha^*) \otimes \text{Ber}(p) \right) \Rightarrow \left((X', Y') \stackrel{(d)}{=} \text{Ber}(\alpha) \otimes \text{Ber}(\alpha^*) \right)$$

where α^* is as in (3).



Proof of Lemma 1. Recalling that, in Model 1, Hammersley's lines are non increasing lines which do not touch each other and noticing that $X' = Y' = 1$ iff $X = Y = 0$ and $\xi_t(x) = 1$, we get that

$$(X', Y') = \begin{cases} (X, Y) & \text{if } X \neq Y \\ (1, 1) & \text{if } X = Y = 0 \text{ and } \xi_t(x) = 1. \\ (0, 0) & \text{otherwise.} \end{cases}$$

Hence, we see that $(X', Y') \in \sigma(X, Y, \xi_t(x))$ and one can easily check that if $(X', Y') \sim \text{Ber}(\alpha) \otimes \text{Ber}(\alpha^*)$ with α^* is as in (3), then $(X', Y') \sim \text{Ber}(\alpha) \otimes \text{Ber}(\alpha^*)$. \square

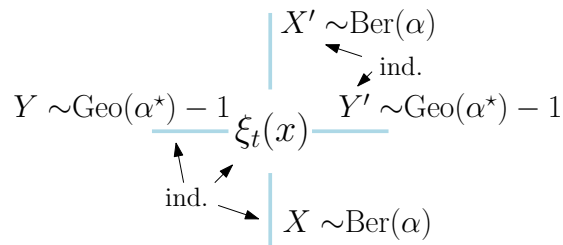
We now explain how Lemma 1 proves Theorem 2 for Model 1. For every $(x, 1)$ define the corresponding $(X'(x), Y'(x))$ be the *output* of $(x, 1)$. Independence of sources and sinks and Lemma 1 ensure that the output of $(1, 1)$ is independent of sources at $(2, 0), (3, 0), \dots$ and sinks at $(0, 2), (0, 3), \dots$. Then a simple induction on x proves that the random variables $(X'(x), x \in \llbracket 1, n \rrbracket)$ are distributed as $\text{Ber}(\alpha)$.

Model 2. The strategy is similar:

Lemma 2 (Local balance for Model 2). *The random variables (X', Y') are measurable with respect to $(X, Y, \xi_t(x))$ and we have*

$$\left((X, Y, \xi_t(x)) \stackrel{(d)}{=} \text{Ber}(\alpha) \otimes (\text{Geo}(\alpha^*) - 1) \otimes \text{Ber}(p) \right) \Rightarrow \left((X', Y') \stackrel{(d)}{=} \text{Ber}(\alpha) \otimes (\text{Geo}(\alpha^*) - 1) \right)$$

where α^* is as in (4).



Proof of Lemma 2. Recall that $X, X' \in \{0, 1\}$ whereas $Y, Y' \in \mathbb{Z}^+$. Besides, using that Hammersley's lines are non-increasing, the balance between incoming and outgoing lines at a site yields $X' + Y = X + Y'$. Moreover, $X' = 1$ iff $(\xi_t(x) = 1 \text{ or } (X, Y) = (1, 0))$.

We observe that the different cases can be summed up in

$$(X', Y') = \begin{cases} (1, 0) & \text{if } (X, Y) = (1, 0) \\ (\xi_t(x), Y - X + \xi_t(x)) & \text{otherwise.} \end{cases}$$

One can easily check that this equality implies Lemma 2. For instance, for $k \geq 1$, $\mathbb{P}(X = 1, Y = k) = \alpha\alpha^*(1 - \alpha^*)^k$ should be equal to

$$\begin{aligned} \mathbb{P}(X' = 1, Y' = k) &= \mathbb{P}(\xi_t(x) = 1, X = 0, Y = k - 1) + \mathbb{P}(\xi_t(x) = 1, X = 1, Y = k) \\ &= p(1 - \alpha)\alpha^*(1 - \alpha^*)^{k-1} + p\alpha\alpha^*(1 - \alpha^*)^k \\ &= p(1 - \alpha^*)^{k-1}\alpha^* [(1 - \alpha) + \alpha(1 - \alpha^*)], \end{aligned}$$

which requires (4). □

Theorem 2 for Model 2 then follows from Lemma 2 in the same way as before Theorem 2 for Model 1 follows from Lemma 1. □

4 Law of Large Numbers in Hammersley's processes

In this section, we explain how Theorem 2 on the stationary measure of the processes with sources and sinks implies Theorem 1. In fact, we are going to prove the following proposition.

Proposition 1. *For $a, b > 0$, let $L_{(an, bn)}^{(i)}$ denote the number of Hammersley's lines in the rectangle $\{1, \dots, \lfloor an \rfloor\} \times \{1, \dots, \lfloor bn \rfloor\}$ for Model i . Then, when $n \rightarrow +\infty$,*

$$\frac{L_{(an, bn)}^{(1)}}{n} \xrightarrow{a.s.} \begin{cases} \frac{\sqrt{p}(2\sqrt{ab} - (a+b)\sqrt{p})}{1-p} & \text{if } p < \min\{a/b, b/a\}, \\ \min\{a, b\} & \text{otherwise,} \end{cases} \quad (8)$$

$$\frac{L_{(an, bn)}^{(2)}}{n} \xrightarrow{a.s.} \begin{cases} 2\sqrt{abp(1-p)} + (a-b)p & \text{if } p < a/(a+b), \\ a & \text{otherwise.} \end{cases} \quad (9)$$

Proof. Trivial case.

We first consider Model 1 in the case $p \geq \min\{a/b, b/a\}$. The upper bound is straightforward. The lower bound can be proven as follows. Let us consider for example the case $p \geq ab^{-1}$. We have to prove:

$$\liminf_{n \rightarrow \infty} \frac{L_{(an, bn)}^{(1)}}{n} \geq a. \quad (10)$$

One can build an increasing path as follows. The first point $(1, y_1)$ of the path is the lowest point of ξ having first coordinate equal to 1. The second point $(2, y_2)$ of the path is the lowest point of ξ having first coordinate equal to 2 and second coordinate strictly larger than y_1 . The other points are defined in a similar fashion. As $pb \geq a$, a study of this path easily provides (10). The case $p \geq ba^{-1}$ can be handled in a similar way.

The same strategy treats the case $p \geq a/(a+b)$ in Model 2.

Non trivial case

We only need to compute $\lim \frac{1}{n} \mathbb{E}[L_{(an,bn)}^{(i)}]$ since almost sure convergence follows from superadditivity.

Model 1. Upper bound. We assume that $p \geq \min\{a/b, b/a\}$. For any $\alpha \in (0, 1)$, taking α^* as in (3) and using (5), (6) and (7), we get

$$\begin{aligned} \frac{1}{n} \mathbb{E}[L_{(an,bn)}^{(1)}] &\leq \frac{1}{n} \mathbb{E}[\mathcal{L}_{(an,bn)}^{(1),\alpha,\alpha^*}] \leq \frac{1}{n} \mathbb{E}[\mathcal{T}_{(an,bn)}^{(1),\alpha,\alpha^*}] + \frac{1}{n} \mathbb{E}[\#\{\text{sinks between 1 and } bn\}] \\ &= a\alpha + b\alpha^* = a\alpha + b \frac{p(1-\alpha)}{\alpha + p(1-\alpha)} =: \phi^{(1)}(a, b, \alpha). \end{aligned} \quad (11)$$

The latter is minimized for

$$\alpha(a, b) := \frac{\sqrt{p}\sqrt{\frac{b}{a}} - p}{1-p}, \quad \alpha^*(a, b) := \frac{\sqrt{p}\sqrt{\frac{a}{b}} - p}{1-p}, \quad (12)$$

which both are in $(0, 1)$ if $p < \min\{a/b, b/a\}$. This yields

$$\frac{1}{n} \mathbb{E}[L_{(an,bn)}^{(1)}] \leq \phi^{(1)}(a, b, \alpha(a, b)) = \frac{\sqrt{p}(2\sqrt{ab} - (a+b)\sqrt{p})}{1-p}.$$

Model 2. Upper bound. We assume that $p \geq a/(a+b)$. Using (7) with $\alpha^* = \frac{\alpha-p}{\alpha(1-p)}$ and that $\mathbb{E}[\text{Geo}(\alpha^*) - 1] = \frac{1}{\alpha^*} - 1$, we get

$$\begin{aligned} \frac{1}{n} \mathbb{E}[L_{(an,bn)}^{(2)}] &\leq \frac{1}{n} \mathbb{E}[\mathcal{T}_{(an,bn)}^{(i),\alpha,\alpha^*}] + \frac{1}{n} \mathbb{E}[\#\{\text{sinks between 1 and } bn\}] \\ &= a\alpha + b\left(\frac{1}{\alpha^*} - 1\right) = a\alpha + b \frac{p(1-\alpha)}{\alpha-p}. \end{aligned}$$

We minimize the latter by taking

$$\alpha = p + \sqrt{\frac{b}{a}p(1-p)}.$$

Note that this choice is allowed since for $p \in (0, a/(a+b))$ then $0 < p < \alpha < 1$.

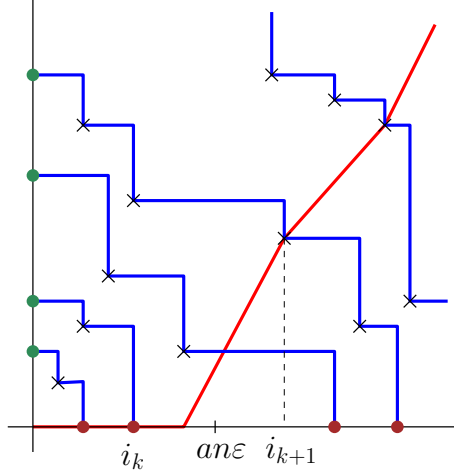
Model 1 and 2. Lower bound.

In [9, 10] the lower bound was obtained with a convexity argument on the scaling limit. Instead, we adapt a more probabilist argument due to Rolla *et al.* (proof of Theorem 4.1 in [8]) for the closely related model of last passage percolation with geometric weights. Here, we only give the details for Model 1 but the same argument applies for Model 2.

Let us consider Model 1 on the rectangle $[0, an] \times [0, bn]$ with sinks and sources with optimal source intensity $\alpha = \alpha(a, b)$ and $\alpha^* = \alpha^*(a, b)$. For $\varepsilon \in [0, 1]$, denote by $L_{(an,bn)}^{(1)}(\varepsilon)$ the length of the largest non-decreasing subsequence defined by

$$\begin{aligned} L_{(an,bn)}^{(1)}(\varepsilon) &= \max \{L; (i_1, 0), \dots, (i_k, 0), (i_{k+1}, j_{k+1}), \dots, (i_L, j_L) \in \xi, \\ &\quad i_1 < \dots < i_k \leq an\varepsilon < i_{k+1} < \dots < i_L \leq an \text{ and } 0 < j_{k+1} < \dots < j_L \leq bn\}. \end{aligned}$$

Here is an example where $L_{(an,bn)}^{(1)}(\varepsilon) = 4$. Note that because of the constraint $L_{(an,bn)}^{(1)}(\varepsilon)$ is smaller than the number of Hammersley's lines $\mathcal{L}_{(an,bn)}^{(1),\alpha,\alpha^*}$.



Lemma 3 (Largest subsequence using some sources). *There exist positive and non-decreasing functions f, g on $(0, 1]$, which depend on a, b , such that*

$$\mathbb{P}\left(L_{(an, bn)}^{(1)}(\varepsilon) \geq n\left(\phi^{(1)}(a, b, \alpha(a, b)) - f(\varepsilon)\right)\right) \leq \exp(-ng(\varepsilon)).$$

Proof of Lemma 3. We write $L_{(an, bn)}^{(1)}(\varepsilon) = I_1 + I_2$ with

$$I_1 := \max\{k; (i_1, 0), \dots, (i_k, 0) \in \xi, i_1 < \dots < i_k \leq an\varepsilon\}$$

$$I_2 := \max\{k; (i_1, j_1), \dots, (i_k, j_k) \in \xi, an\varepsilon < i_1 < \dots < i_k \leq an \text{ and } 0 < j_1 < \dots < j_k \leq bn\}.$$

The random variables I_1 and I_2 are independent, I_1 is binomially distributed with parameters $(\lfloor an\varepsilon \rfloor, \alpha(a, b))$. The random variable I_2 has the law of $L_{(an(1-\varepsilon), bn)}^{(1)}$, thus is dominated by a random variable I_3 with law $\mathcal{L}_{(an(1-\varepsilon), bn)}^{(1), \alpha(a(1-\varepsilon), b), \alpha^*(a(1-\varepsilon), b))}$. Using (11), we get:

$$\begin{aligned} \frac{1}{n}\mathbb{E}\left[L_{(an, bn)}^{(1)}(\varepsilon)\right] &\leq \frac{1}{n}\mathbb{E}[I_1 + I_3] \\ &\leq a\varepsilon\alpha(a, b) + \phi^{(1)}\left(a(1-\varepsilon), b, \alpha(a(1-\varepsilon), b)\right), \end{aligned}$$

where, by convention, $\alpha(a', b') = 1$ when $p \geq a'/b'$ and $\alpha(a', b') = 0$ when $p < a'/b'$. One can check that for $p < \min\{a/b, b/a\}$ and $a', b' > 0$ such that $a/b \neq a'/b'$ we have

$$\phi^{(1)}(a', b', \alpha(a, b)) > \phi^{(1)}(a', b', \alpha(a', b')).$$

This is due to the fact that $\alpha(a, b) \neq \alpha(a', b')$ and that there is a unique $\alpha \in [0, 1]$ which minimizes $\phi(a', b', \alpha)$. Thus,

$$\begin{aligned} a\varepsilon\alpha(a, b) + \phi^{(1)}\left(a(1-\varepsilon), b, \alpha(a(1-\varepsilon), b)\right) &< a\varepsilon\alpha(a, b) + \phi^{(1)}\left(a(1-\varepsilon), b, \alpha(a, b)\right) \\ &= a\varepsilon\alpha(a, b) + a(1-\varepsilon)\alpha(a, b) + b\alpha^*(a, b) \\ &= a\alpha(a, b) + b\alpha^*(a, b) \\ &= \phi^{(1)}(a, b, \alpha(a, b)). \end{aligned}$$

Denote by $d(\varepsilon)$ the difference between the right and left hand side of the inequality. The function d is positive on $(0, 1]$ and continuous. We set $f(\varepsilon) = \frac{1}{2} \min\{d(\delta), \delta \in [\varepsilon, 1]\}$ which is non-decreasing and positive. Then, we have

$$\begin{aligned} \mathbb{P}\left(L_{(an, bn)}^{(1)}(\varepsilon) \geq n\left(\phi^{(1)}(a, b, \alpha(a, b)) - f(\varepsilon)\right)\right) &\leq \mathbb{P}(I_1 + I_3 \geq \mathbb{E}(I_1 + I_3) + n(d(\varepsilon) - f(\varepsilon))) \\ &\leq \mathbb{P}(I_1 + I_3 \geq \mathbb{E}(I_1 + I_3) + nf(\varepsilon)). \end{aligned}$$

Recalling that I_1 is binomially distributed and I_3 is the sum of two binomial random variables, function g is obtained by applying Hoeffding's inequality to I_1 and I_3 . □

We still consider Model 1 on the rectangle $[0, an] \times [0, bn]$ with sinks and sources with optimal source intensity $\alpha = \alpha(a, b)$ and $\alpha^* = \alpha^*(a, b)$. Let π_n be an optimal path for $\mathcal{L}_{(an, bn)}^{(1), \alpha, \alpha^*}$. If there are several optimal paths, we choose one of them in an arbitrary way. Define D_n as the number of points $(i, j) \in \pi_n$ with $j = 0$.

Lemma 4 (Optimal paths do not take many sources). *There exists a positive function h on $(0, 1]$ such that for any $\delta \in (0, 1]$ and for any n large enough*

$$\mathbb{P}(D_n > an\delta) \leq \exp(-nh(\delta)).$$

In particular D_n/n converges to 0 in L^1 .

Proof of Lemma 4. By definition, if $D_n > an\delta$ then for some $\varepsilon \geq \delta$ such that $\varepsilon n \in \mathbb{N}$ and $\varepsilon \leq 1$

- there are more than $an\delta$ sources in $\{1, \dots, \varepsilon n\}$;
- $L_{(an, bn)}^{(1)}(\varepsilon) = \mathcal{L}_{(an, bn)}^{(1), \alpha, \alpha^*} = \mathcal{T}_{(an, bn)}^{(1), \alpha, \alpha^*} + \#\{\text{sinks between 1 and } bn\}$.

Therefore

$$\begin{aligned} \mathbb{P}\{D_n > an\delta\} &\leq \mathbb{P}\left(\mathcal{T}_{(an, bn)}^{(1), \alpha, \alpha^*} \leq n(a\alpha - \frac{1}{2}f(\delta))\right) \\ &\quad + \mathbb{P}\left(\#\{\text{sinks}\} \leq n(b\alpha^* - \frac{1}{2}f(\delta))\right) \\ &\quad + \mathbb{P}\left(L_n^{(1)}(\varepsilon) > n(a\alpha + b\alpha^* - f(\delta)) \text{ for some } \varepsilon \geq \delta\right) \\ &\leq \exp(-n\tilde{g}(\delta)) + \sum_{1 \geq \varepsilon \geq \delta; \varepsilon n \in \mathbb{N}} \mathbb{P}\left(L_n^{(1)}(\varepsilon) > n(a\alpha + b\alpha^* - f(\delta))\right) \\ &\leq \exp(-n\tilde{g}(\delta)) + \sum_{1 \geq \varepsilon \geq \delta; \varepsilon n \in \mathbb{N}} \mathbb{P}\left(L_n^{(1)}(\varepsilon) > n(a\alpha + b\alpha^* - f(\varepsilon))\right) \\ &\leq \exp(-n\tilde{g}(\delta)) + \sum_{1 \geq \varepsilon \geq \delta; \varepsilon n \in \mathbb{N}} \exp(-ng(\varepsilon)) \\ &\leq \exp(-n\tilde{g}(\delta)) + n \exp(-ng(\delta)) \end{aligned}$$

for large n , where we used Lemma 3 and where \tilde{g} is some positive function. This implies the first part of the lemma. The convergence of D_n/n to 0 in probability and in L^1 follows, since the sequence is bounded. □

We now conclude the proof of the lower bound noticing that we have

$$\mathcal{L}_{(an, nb)}^{(1), \alpha, \alpha^*} - D_n - D'_n \leq L_{(an, bn)}^{(1)}$$

where D'_n is the number of points $(i, j) \in \pi_n$ with $i = 0$. By the previous lemma we know that D_n/n tends to 0 in L^1 . By symmetry, the same result holds for D'_n/n . Taking expectations of both sides in the previous inequality we get

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E}[L_{(an, bn)}^{(1)}] \geq a\alpha + b\alpha^* = \phi^{(1)}(a, b, \alpha(a, b)).$$

□

Back to Ulam's constant

We observe that if we take $p = 1/n$, Theorem 1 for Model 1 suggests $L_n^{(1)} \approx 2\sqrt{n}$, which is consistent with the asymptotics of Ulam's problem, since ξ is then close, after renormalization, to a Poisson point process with intensity n .

In fact, one can rigorously recover that $c = 2$ using our proof of Theorem 1. To do so, consider a Poisson point process Ξ with intensity n in the unit square and denote by $\ell(n)$ the greatest number of points of Ξ an increasing path can go through. To get a lower bound and an upper bound of $\ell(n)$, we divide the square $[0, 1]^2$ in two different ways.

First, we fix some $k \geq 1$ and we divide it into small squares of length side $1/(k\sqrt{n})$. Say that $\xi_j(i) = 1$ if at least one point of Ξ is in the square with top-right corner $(i/(k\sqrt{n}), j/(k\sqrt{n}))$ and consider the quantity $L_{k\sqrt{n}}^{(1)} = L_{(k\sqrt{n}, k\sqrt{n})}^{(1)}$ associated to the family $(\xi_j(i))_{i,j \leq k\sqrt{n}}$. It is clear that

$$\ell(n) \geq L_{k\sqrt{n}}^{(1)}. \quad (13)$$

Denoting

$$p_k = \mathbb{P}(\xi_j(i) = 1) = \mathbb{P}(\text{Poiss}(1/k^2) \geq 1) = 1 - e^{-1/k^2},$$

Theorem 1 implies

$$\frac{L_{k\sqrt{n}}^{(1)}}{\sqrt{n}} \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} \frac{2k\sqrt{p_k}}{\sqrt{p_k} + 1}.$$

Using (13) and letting k tend to infinity, we get $c := \lim \ell(n)/\sqrt{n} \geq 2$.

To prove the upper bound, we divide now the square into small squares of length side $1/n^4$. Say that $\xi_j(i) = 1$ if at least one point of Ξ is in the square with top-right corner $(i/n^4, j/n^4)$ and consider the quantity $L_{n^4}^{(1)}$ associated to the family $(\xi_j(i))_{i,j \leq n^4}$. The parameter of these Bernoulli random variables is now

$$\tilde{p}_n = \mathbb{P}(\text{Poiss}(1/n^7) \geq 1) = 1 - e^{-1/n^7}.$$

With probability higher than $1 - n^{-2}$, all the columns and lines of width $1/n^4$ contain at most one point of Ξ . On this event that we denote F_n , $L_{n^4}^{(1)}$ coincides with $\ell(n)$. We now use some intermediate results of the proof of the upper bound for $a = b = 1$. Inequality (6) still holds despite the dependence of \tilde{p}_n on n *i.e.*

$$L_{n^4}^{(1)} \leq \mathcal{L}_{n^4}^{(1), \alpha_n, \alpha_n^*},$$

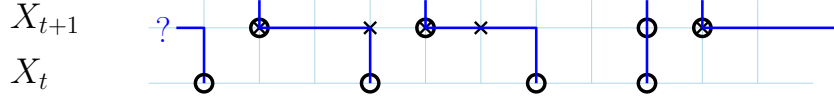
with $\alpha_n = \alpha_n^* = \frac{\sqrt{\tilde{p}_n - p_n}}{1 - p_n} \sim n^{-7/2}$. Using (11), we get

$$\begin{aligned} \mathbb{E}[\ell(n)] &\leq \mathbb{E}[\Xi | \mathbf{1}_{F_n}] + \mathbb{E}[\mathcal{L}_{n^4}^{(1), \alpha_n, \alpha_n^*}] \\ &\leq \sqrt{\mathbb{E}[\text{Poiss}(n)^2]/n^2} + n^4(\alpha_n + \alpha_n^*), \end{aligned}$$

where we used the Cauchy-Schwarz inequality. Dividing by n and letting n go to infinity, we obtain $c \leq 2$.

5 Stationary measures of Hammersley's process on \mathbb{Z}

In this section, which can be read independently from the rest of the article, we study the analogous of Model 1 of the previous sections on the whole line \mathbb{Z} . As before, $(\xi_t(i))_{i \in \mathbb{Z}, t \in \mathbb{N}}$ denotes i.i.d. Bernoulli(p) random variables. As already noticed, the process $X_t = (X_t(i))_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ on the whole line \mathbb{Z} is well defined using the same construction as explained in the introduction for Model 1 as soon as it starts with an initial configuration $X_0 \sim \mu$ such that μ -almost surely, for all i there is $j < i$ such that $X_0(j) = 1$.



Theorem 3. *The only extremal translation-invariant stationary measures of Hammersley's process on \mathbb{Z} are measures $\text{Ber}(\alpha)^{\otimes \mathbb{Z}}$ for all $\alpha \in (0, 1]$.*

This theorem is precisely the discrete counterpart of ([1], Lemma 7). The fact that Bernoulli product measures are stationary was proved by T.Seppäläinen ([9] Lemma 2.1). The reason for which we only focus on Model 1 on the infinite line is that the analogous of Model 2 is much more complicated to analyze. Indeed, the evolution of a particle in Model 2 depends on the whole configuration on its left.

In order to prove that there are no other extremal measures we adapt a classical strategy used to deal with exclusion processes (see for instance [5] p.17-20 and [7] Ch.VIII).

Proof of uniqueness. Let $\alpha \in (0, 1]$ and μ be an extremal translation-invariant stationary measure on $\{0, 1\}^{\mathbb{Z}}$ with marginals $\mu(X(i) = 1) = \alpha$, we want to prove that $\mu = \text{Ber}(\alpha)^{\otimes \mathbb{Z}}$.

The proof is made of the following steps:

1. we introduce a measure $\underline{\pi}$ on $\{0, 1\}^{\mathbb{Z}} \times \{0, 1\}^{\mathbb{Z}}$ which is a *minimal* coupling between $\text{Ber}(\alpha)^{\otimes \mathbb{Z}}$ and μ ;
2. we prove (Lemma 6) that some patterns that would decrease $\text{card}\{i; x_i \neq y_i\}$, have $\underline{\pi}$ -probability zero ;
3. we conclude: $\mu = \text{Ber}(\alpha)^{\otimes \mathbb{Z}}$.

Let \mathcal{M}_* be the set of translation-invariant measures on $\{0, 1\}^{\mathbb{Z}} \times \{0, 1\}^{\mathbb{Z}}$ that are a coupling of $\text{Ber}(\alpha)^{\otimes \mathbb{Z}}$ and μ . The set \mathcal{M}_* is non-empty (it contains $\text{Ber}(\alpha)^{\otimes \mathbb{Z}} \otimes \mu$) and compact for the weak topology.

We will prove that in \mathcal{M}_* there is a coupling $((X(i))_{i \in \mathbb{Z}}, (Y(i))_{i \in \mathbb{Z}})$ such that $X = Y$ a.s. Set

$$D : \mathcal{M}_* \rightarrow [0, 1]$$

$$\pi \mapsto \pi(X(0) \neq Y(0)).$$

The function D is continuous on the compact set \mathcal{M}_* and thus attains its minimum $\delta \geq 0$ at some coupling $\underline{\pi}$. If $\delta = 0$, then the theorem is proved: by translation-invariance $X = Y$ $\underline{\pi}$ -a.s.

Let $(X_0, Y_0) \sim \underline{\pi}$, we use the same random variables $(\xi_t(i))_{i \in \mathbb{Z}, t \in \mathbb{N}}$ to define the dynamics of X and Y . We denote by (X_t, Y_t) the pair of configurations at time t , its joint distribution is denoted by $\underline{\pi}_t$. By construction, for every t , $\underline{\pi}_t$ is translation-invariant. For every $k \geq 1$ we denote by

$$\Delta_k(X_t, Y_t)$$

the number of $i \in \{1, \dots, k\}$ such that $X_t(i) \neq Y_t(i)$.

Let $n \geq 2$ and $t \geq 0$. We say that there is a n -forbidden pattern at location $x \in \mathbb{Z}$ at time t if the configuration (X_t, Y_t) between locations x and $x + n - 1$ is either

$$\left(\begin{array}{c} x \qquad \qquad \qquad x + n - 1 \\ X_t = (\begin{array}{cccc} 1 & 0 & \dots & 0 \\ Y_t = (\begin{array}{cccc} 0 & 0 & \dots & 1 \end{array} \end{array} \right)$$

or

$$\left(\begin{array}{c} x \qquad \qquad \qquad x + n - 1 \\ X_t = (\begin{array}{cccc} 0 & 0 & \dots & 0 \\ Y_t = (\begin{array}{cccc} 1 & 0 & \dots & 0 \end{array} \end{array} \right).$$

Lemma 5. For $n \geq 2$, let $E_{x,n}$ be the event $\xi_1(x) = 1, \xi_1(x+1) = \xi_1(x+2) = \dots = \xi_1(x+n-1) = 0$.

Let $j \geq 1$, denote by $F(n, j)$ the subset of locations of $\{0, n, 2n, \dots, (j-1)n\}$ at which is located a forbidden pattern at time 0. Let

$$A(n, j) = \text{card}\{x \in F(n, j), E_{x,n} \text{ occurs}\}.$$

Then:

$$\Delta_{jn}(X_1, Y_1) \leq \Delta_{jn}(X_0, Y_0) + 1 - 2A(n, j).$$

Proof of Lemma 5. We let the crosses at time 1 act one by one from the right to the left. We consider the impact of each cross action on the discrepancy Δ_{jn} between the two configurations. Let us note that, when a cross located at i acts on some configuration of particles, it changes the value of at most two sites of the configuration: i and the leftmost 1 of the configuration in the interval $\llbracket i, \infty \rrbracket$. Thus, we deduce the following facts:

- The crosses strictly to the right of jn have no impact on the discrepancy.
- A cross located at a point x in $\{1, \dots, jn\}$ cannot increase the discrepancy. To check this fact, one can study all the cases. They are all shown below, up to symmetries between X and Y .

$$\begin{aligned} & \left(\begin{array}{c} x \\ (1) \\ (1) \end{array} \right), \left(\begin{array}{c} x \\ (1 \ ? \ \dots \ ? \ 1) \\ (0 \ 0 \ \dots \ 0 \ 1) \end{array} \right), \\ & \left(\begin{array}{c} x \\ (1 \ ? \ \dots \ ? \ 0) \\ (0 \ 0 \ \dots \ 0 \ 1) \end{array} \right), \left(\begin{array}{c} x \\ (0 \ \dots \ 0 \ 1) \\ (0 \ \dots \ 0 \ 1) \end{array} \right), \\ & \left(\begin{array}{c} x \\ (0 \ \dots \ 0 \ 0 \ 0 \ 0 \ 0 \ 1) \\ (0 \ \dots \ 0 \ 1 \ ? \ \dots \ ? \ 1) \end{array} \right), \left(\begin{array}{c} x \\ (0 \ \dots \ 0 \ 0 \ 0 \ 0 \ 0 \ 1) \\ (0 \ \dots \ 0 \ 1 \ ? \ \dots \ ? \ 0) \end{array} \right). \end{aligned}$$

- For a forbidden pattern at some in , if $E_{in,n}$ occurs, the cross at in decreases the discrepancy by 2.
- In the worst case, the right-most cross which is strictly to the left of 1 increases the discrepancy by 1.
- The crosses to the left of the previous one have no impact on the discrepancy.

The result follows. □

Lemma 6 (Forbidden patterns). Let $n \geq 2$. The π_0 probability of a n -forbidden pattern at any location is 0.

Proof of Lemma 6. Let $j \geq 1$. By Lemma 5 we have:

$$\Delta_{jn}(X_1, Y_1) \leq \Delta_{jn}(X_0, Y_0) + 1 - 2A(n, j).$$

Note that the probability that, at some location in , there is a n -forbidden pattern and that $E_{in,n}$ occurs is $\zeta p(1-p)^{n-1}$ where ζ is the probability of a n -forbidden pattern at a given location. We aim to prove $\zeta = 0$. Taking expectation in the previous display we get

$$jnD(\underline{\pi}_1) \leq jnD(\underline{\pi}_0) + 1 - 2j\zeta p(1-p)^{n-1}.$$

Dividing by nj and letting $j \rightarrow \infty$ we get:

$$D(\underline{\pi}_1) \leq D(\underline{\pi}_0) - \frac{2\zeta p(1-p)^{n-1}}{n}.$$

But the minimality of $\underline{\pi}$ yields $D(\underline{\pi}_1) \geq D(\underline{\pi}_0)$ and we get $\zeta = 0$. \square

Lemma 7 (There is a forbidden pattern somewhere). *Assume that $\delta > 0$, there exists n such that*

$$\underline{\pi} \left(\begin{array}{c} 1 \qquad \qquad n \\ X = (1 \ ? \ \dots \ ? \ 0 \) \\ Y = (0 \ ? \ \dots \ ? \ 1 \) \end{array} \right) > 0 \quad \text{or} \quad \underline{\pi} \left(\begin{array}{c} 1 \qquad \qquad n \\ X = (0 \ ? \ \dots \ ? \ 1 \) \\ Y = (1 \ ? \ \dots \ ? \ 0 \) \end{array} \right) > 0$$

Proof. Let $(X, Y) \sim \underline{\pi}$,

$$\delta = \underline{\pi}(X(0) \neq Y(0)) = \underline{\pi}(X(0) = 1, Y(0) = 0) + \underline{\pi}(X(0) = 0, Y(0) = 1).$$

The two terms on the right side are equal:

$$\begin{aligned} \underline{\pi}(X(0) = 1, Y(0) = 0) &= \underline{\pi}(X(0) = 1) - \underline{\pi}(X(0) = 1, Y(0) = 1) \\ &= \underline{\pi}(Y(0) = 1) - \underline{\pi}(X(0) = 1, Y(0) = 1) \quad (X, Y \text{ have same marginals}) \\ &= \underline{\pi}(X(0) = 0, Y(0) = 1). \end{aligned}$$

Then $0 < \delta/2 = \underline{\pi}(X(0) = 1, Y(0) = 0)$. Assume that the lemma is false (for every n), we have $\underline{\pi}(A \cup B) = 1$, where

$$\begin{aligned} A &= \{(x, y), x_i \leq y_i \text{ for all } i \in \mathbb{Z}\}, \\ B &= \{(x, y), x_i \geq y_i \text{ for all } i \in \mathbb{Z}\}. \end{aligned}$$

As seen above, $\delta > 0$ implies then that $\underline{\pi}(A), \underline{\pi}(B \setminus A) > 0$. Besides, it is easy to check that A and $B \setminus A$ are preserved by the dynamics. Hence, considering the restriction of $\underline{\pi}$ on these two subsets, $\underline{\pi}_A := (\nu_A, \mu_A)$ and $\underline{\pi}_{B \setminus A} := (\nu_{B \setminus A}, \mu_{B \setminus A})$, we see that ν_A (resp. μ_A) and $\nu_{B \setminus A}$ (resp. $\mu_{B \setminus A}$) define two translation-invariant stationary measures such that

$$\begin{aligned} \text{Ber}(\alpha)^{\otimes \mathbb{Z}} &= \underline{\pi}(A)\nu_A + \underline{\pi}(B \setminus A)\nu_{B \setminus A}. \\ \mu &= \underline{\pi}(A)\mu_A + \underline{\pi}(B \setminus A)\mu_{B \setminus A}. \end{aligned}$$

The extremality of $\text{Ber}(\alpha)^{\otimes \mathbb{Z}}$ and μ implies that $\text{Ber}(\alpha)^{\otimes \mathbb{Z}} = \nu_A = \nu_{B \setminus A}$ and $\mu = \mu_A = \mu_{B \setminus A}$. But, by definition of A and B , $\nu_A \preceq \mu_A$ and $\mu_{B \setminus A} \preceq \nu_{B \setminus A}$. Thus, necessarily, $\mu = \text{Ber}(\alpha)^{\otimes \mathbb{Z}}$. \square

Now we can obtain our contradiction. Indeed, there exists an integer n such that

$$\underline{\pi} \left(\begin{array}{c} 1 \qquad \qquad n \\ X = (1 \ ? \ \dots \ ? \ 0 \) \\ Y = (0 \ ? \ \dots \ ? \ 1 \) \end{array} \right) > 0 \quad \text{or} \quad \underline{\pi} \left(\begin{array}{c} 1 \qquad \qquad n \\ X = (0 \ ? \ \dots \ ? \ 1 \) \\ Y = (1 \ ? \ \dots \ ? \ 0 \) \end{array} \right) > 0.$$

Without loss of generality we do the first case. We can assume that X, Y coincide between 2 and $n-1$, otherwise we can decrease n . Let $1 < k < n$ be the leftmost position at which $\binom{X(k)}{Y(k)} = \binom{1}{1}$.

With positive probability we can turn this $\binom{1}{1}$ into $\binom{1}{0}$:

$$\left(\begin{array}{c} \times \quad - \quad - \quad - \quad - \quad - \\ X_t = (1 \ 0 \ \dots \ 0 \ 1 \ ? \ 0 \) \\ Y_t = (0 \ 0 \ \dots \ 0 \ 1 \ ? \ 1 \) \end{array} \right) \xrightarrow{t+1} \left(\begin{array}{c} 1 \qquad \qquad k \qquad \qquad n \\ X_{t+1} = (? \ 0 \ \dots \ 0 \ 1 \ ? \ 0 \) \\ Y_{t+1} = (? \ 0 \ \dots \ 0 \ 0 \ ? \ 1 \) \end{array} \right)$$

We repeat this process between positions k and n and finally for some t, i, j we have

$$\pi \left(\begin{array}{l} X_t = (\begin{array}{cccc} & i & & j \\ 1 & 0 & \dots & 0 & 0 \end{array}) \\ Y_t = (\begin{array}{cccc} & & & \\ 0 & 0 & \dots & 0 & 1 \end{array}) \end{array} \right) > 0,$$

which contradicts Lemma 6. □

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References

- [1] D.Aldous, P.Diaconis. Hammersley's interacting particle process and longest increasing subsequences. *Probability Theory and Related Fields* 103 (1995) n.2 p.199-213.
- [2] E. Cator, P. Groeneboom. Hammersley's process with sources and sinks. *Annals of Probability* vol.33 (2005), n.3, p.879-903.
- [3] P. Groeneboom. Hydrodynamical methods for analyzing longest increasing subsequences. *J. Comput. Appl. Math.* vol. 142 (2002), p.83-105
- [4] J.M.Hammersley. A few seedlings of research. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, vol.1 (1972), p.345-394.
- [5] W.Jockusch, J.Propp, P.Shor. Random domino tilings and the arctic circle theorem (1995). [arXiv:math/9801068](https://arxiv.org/abs/math/9801068).
- [6] K.Johansson. Discrete orthogonal polynomial ensembles and the Plancherel measure. *Annals of Mathematics* (2001), vol.153 n.1, p.259-296.
- [7] T.M.Liggett. *Interacting particle systems*. Springer (1985).
- [8] L.Rolla, V.Sidoravicius, D.Surgailis, M.E.Vares. The discrete and continuum broken line process. *Markov Process. Related Fields* 16 (2010), n.1, p.79-116.
- [9] T.Seppäläinen. Increasing sequences of independent points on the planar lattice. *Annals of Applied Probability* vol.7 (1997) n.4 p.886-898.
- [10] T.Seppäläinen. Exact limiting shape for a simplified model of first-passage percolation on the plane. *Annals of Probability* vol.26 (1998), n.3, p.1232-1250.
- [11] T.Seppäläinen. Lecture notes on the corner growth model. Available at <http://www.math.wisc.edu/~seppalai/>