Transcendental Morse Inequality and Generalized Okounkov Bodies

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Abstract

The main goal of this article is to construct "arithmetic Okounkov bodies" for an arbitrary pseudo-effective (1,1)-class α on a Kähler manifold. Firstly, using Boucksom's divisorial Zariski decompositions for pseudo-effective (1,1)-classes on compact Kähler manifolds, we prove the differentiability of volumes of big classes for Kähler manifolds on which modified nef cones and nef cones coincide; this includes Kähler surfaces. We then apply our differentiability results to prove Demailly's transcendental Morse inequality for these particular classes of Kähler manifolds. In the second part, we construct the convex body $\Delta(\alpha)$ for any big class α with respect to a fixed flag by using positive currents, and prove that this newly defined convex body coincides with the Okounkov body when $\alpha \in \mathrm{NS}_{\mathbb{R}}(X)$; such convex sets $\Delta(\alpha)$ will be called generalized Okounkov bodies. As an application we prove that any rational point in the interior of Okounkov bodies is "valuative". Next we give a complete characterisation of generalized Okounkov bodies on surfaces, and show that the generalized Okounkov bodies behave very similarly to original Okounkov bodies. By the differentiability formula, we can relate the standard Euclidean volume of $\Delta(\alpha)$ in $\mathbb{R}^2$ to the volume of a big class α, as defined by Boucksom; this solves a problem raised by Lazarsfeld in the case of surfaces. Finally, we study the behavior of the generalized Okounkov bodies on the boundary of the big cone, which are characterized by numerical dimension.

Keywords. Generalized Okounkov body, positive current, Siu decomposition, divisorial Zariski decomposition, Lelong number, transcendental Morse inequality, numerical dimension, differentiability formula

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1 Introduction

In [Oko96] Okounkov introduced a natural procedure to associate a convex body $\Delta(D)$ in $\mathbb{R}^n$ to any ample divisor D on an n-dimensional projective variety. Relying on the work of Okounkov, Lazarsfeld and Mustaţă [LM09], and Kaveh and Khovanskii [KK09, KK10], have systematically studied Okounkov's construction, and associated to any big divisor and any fixed flag of subvarieties a convex body which is now called the Okounkov body.

We now briefly recall the construction of the Okounkov body. We start with a complex projective variety X of dimension n. Fix a flag

$$Y_\bullet : X = Y_0 \supset Y_1 \supset Y_2 \supset \ldots \supset Y_{n-1} \supset Y_n = \{p\}$$

where Y_i is a smooth irreducible subvariety of codimension i in X. For a given big divisor D, one defines a valuation-like function

$$\mu = \mu_{Y_\bullet, D} : (H^0(X, \mathcal{O}_X(D)) - \{0\}) \to \mathbb{Z}^n.$$

as follows. First set $\mu_1 = \mu_1(s) = \mathrm{ord}_{Y_1}(s)$. Dividing s by a local equation of Y_1, we obtain a section

$$\widetilde{s}_1 \in H^0(X, \mathcal{O}_X(D - \mu_1 Y_1))$$

that does not vanish identically along Y_1. We restrict $\widetilde{s}_1$ on Y_1 to get a non-zero section

$$s_1 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \mu_1 Y_1)),$$

then we write $\mu_2(s) = \mathrm{ord}_{Y_2}(s_1)$, and continue in this fashion to define the remaining integers $\mu_i(s)$. The image of the function μ in $\mathbb{Z}^n$ is denoted by $\mu(D)$. With this in hand, we define the *Okounkov body of D with respect to the fixed flag* $Y_\bullet$ to be

$$\Delta(D) = \Delta_{Y_\bullet}(D) = \text{closed convex hull} \left(\bigcup_{m \geq 1} \frac{1}{m} \cdot \mu(mD) \right) \subseteq \mathbb{R}^n.$$

According to the open question raised in the final part of [LM09], it is quite natural to wonder whether one can construct "arithmetic Okounkov bodies" for an arbitrary pseudo-effective (1,1)-class α on a Kähler manifold, and realize the volumes of these classes by convex bodies as well. In our paper, using positive currents in a natural way, we give a construction of a convex body $\Delta(\alpha)$ associated to such a class α, and show that this newly defined convex body coincides with the Okounkov body when $\alpha \in \mathrm{NS}_{\mathbb{R}}(X)$.

Theorem 1.1. *Let X be a smooth projective variety of dimension n, L be a big line bundle on X and $Y_\bullet$ be a fixed admissible flag. Then we have*

$$\Delta(c_1(L)) = \Delta(L) = \overline{\bigcup_{m=1}^{\infty} \frac{1}{m} \nu(mL)}.$$

Moreover, in the definition of Okounkov body $\Delta(L)$, it suffices to take the closure of the set of normalized valuation vectors instead of the closure of the convex hull.

By Theorem 1.1, we know that our definition of the Okounkov body for any pseudo-efffective class could be treated as a generalization of the original Okounkov body. A very interesting problem is to find out exactly which points in the Okounkov body $\Delta(L)$ are given by valuations of sections. This is expressed by saying that a rational point of $\Delta(L)$ is "valuative". By Theorem 1.1 we can give some partial answers to this question which have been given in [KL14] in the case of surfaces.

Corollary 1.2. *Let X be a projective variety of dimension n and $Y_\bullet$ be an admissible flag. If L is a big line bundle, then any rational point in $\mathrm{int}(\Delta(L))$ is a valuative point.*

It is quite natural to wonder whether our newly defined convex body for big classes behaves similarly as the original Okounkov body. In the situation of complex surfaces, we give an affirmative answer to the question raised in [LM09], as follows:

Theorem 1.3. *Let X be a compact Kähler surface, $\alpha \in H^{1,1}(X,\mathbb{R})$ be a big class. If C is an irreducible divisor of X, there are piecewise linear continuous functions*

$$f\ ,\ g\ :\ [a,s] \mapsto \mathbb{R}_+$$

with f convex, g concave, and $f \leq g$, such that $\Delta(\alpha) \subset \mathbb{R}^2$ is the region bounded by the graphs of f and g:

$$\Delta(\alpha) = \{(t,y) \in \mathbb{R}^2 \mid a \leq t \leq s, \textit{and } f(t) \leq y \leq g(t)\}.$$

Here $\Delta(\alpha)$ is the generalized Okounkov body with respect to the fixed flag

$$X \supseteq C \supseteq \{x\},$$

and $s = \sup\{t > 0 \mid \alpha - tC \text{ is big}\}$. If C is nef, $a = 0$ and $f(t)$ is increasing; otherwise, $a = \sup\{t > 0 \mid C \subseteq E_{nK}(\alpha - tC)\}$, where $E_{nK} := \bigcap_T E_+(T)$ for T ranging among the Kähler currents in α, which is the non-Kähler locus. Moreover, $\Delta(\alpha)$ is a finite polygon whose number of vertices is bounded by $2\rho(X) + 2$, where $\rho(X)$ is the Picard number of X, and

$$\mathrm{vol}_X(\alpha) = 2\,\mathrm{vol}_{\mathbb{R}^2}(\Delta(\alpha)).$$

In [LM09], it was asked whether the Okounkov body of a divisor on a complex surface could be an infinite polygon. In [KLM10], it was shown that the Okounkov body is always a finite polygon. Here we give an explicit description for the "finiteness" of the polygons appearing as generalized Okounkov bodies of big classes, and conclude that it also holds for the original Okounkov bodies by Theorem 1.1.

As one might suspect from the construction of Okounkov bodies, the Euclidean volume of $\Delta(D)$ has a strong connection with the growth of the groups $H^0(X, \mathcal{O}_X(mD))$. In [LM09], the following precise relations were shown:

$$n! \cdot \mathrm{vol}_{\mathbb{R}^n}(\Delta(D)) = \mathrm{vol}_X(D) := \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, \mathcal{O}_X(kD)). \tag{1.1}$$

The proof of (1.1) relies on properties of sub-semigroups of $\mathbb{N}^{n+1}$ constructed from the graded linear series $\{H^0(X, \mathcal{O}_X(mD))\}_{m\geq 0}$. However, when α is a big class which does not belong to $\mathrm{NS}_{\mathbb{R}}(X)$, there are no such algebraic objects which correspond to $\mathrm{vol}_X(\alpha)$, and we only have the following analytic definition due to Boucksom:

$$\mathrm{vol}_X(\alpha) := \sup_T \int_X T_{ac}^n,$$

where T ranges among all positive $(1,1)$-currents. Therefore, it is quite natural to propose the following conjecture:

Conjecture 1.4. *Let X be a compact Kähler manifold of dimension n. For any big class $\alpha \in H^{1,1}(X, \mathbb{R})$, we have*

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta(\alpha)) = \frac{1}{n!} \cdot \mathrm{vol}_X(\alpha).$$

In Theorem 1.3, we prove this conjecture in dimension 2. Our method is to relate the Euclidean volume of the slice of the generalized Okounkov body to the differential of the volume of the big class. We prove the following differentiability formula for volumes of big classses.

Theorem 1.5 (Differentiability of volumes)**.** *Let X be a compact Kähler surface and α be a big class. If β is a nef class or $\beta = \{C\}$ where C is an irreducible curve, we have*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}_X(\alpha + t\beta) = 2Z(\alpha) \cdot \beta,$$

where $Z(\alpha)$ is the divisorial Zariski decomposition of α defined in Section 2.6.

A direct corollary of this formula is the *transcendental Morse inequality*:

Theorem 1.6. *Let X be a compact Kähler surface. If α and β are nef classes satisfying the inequality $\alpha^2 - 2\alpha \cdot \beta > 0$, then $\alpha - \beta$ is big and $\mathrm{vol}_X(\alpha - \beta) \geq \alpha^2 - 2\alpha \cdot \beta$.*

In higher dimension, we also have a differentiability formula for big classes on some special Kähler manifolds.

Theorem 1.7. *Let X be a compact Kähler manifold of dimension n on which the modified nef cone $\mathcal{MN}$ coincides with the nef cone $\mathcal{N}$. If $\alpha \in H^{1,1}(X,\mathbb{R})$ is a big class, $\beta \in H^{1,1}(X,\mathbb{R})$ is a nef class, then*

$$\mathrm{vol}_X(\alpha+\beta) = \mathrm{vol}_X(\alpha) + n\int_0^1 Z(\alpha+t\beta)^{n-1}\cdot\beta\,\mathrm{d}t. \tag{1.2}$$

As a consequence, $\mathrm{vol}_X(\alpha+t\beta)$ is $\mathcal{C}^1$ for $t\in\mathbb{R}^+$ and we have

$$\frac{d}{dt}\Big|_{t=t_0}\mathrm{vol}_X(\alpha+t\beta) = nZ(\alpha+t_0\beta)^{n-1}\cdot\beta \tag{1.3}$$

for $t_0\geq 0$.

Finally, we study the generalized Okounkov bodies for pseudo-effective classes in Kähler surfaces. We summerize our results as follows

Theorem 1.8. *Let X be a Kähler surface, α be any pseudo-effective but not big class,*

(i) *if the numerical dimension $n(\alpha)=0$, then for any irreducible curve C which is not contained in the negative part $N(\alpha)$, we have the generalized Okounkov body*

$$\Delta_{(C,x)}(\alpha) = 0\times\nu_x(N(\alpha)|_C),$$

where $\nu_x(N(\alpha)|_C)=\nu(N(\alpha)|_C,x)$ is the Lelong number of $N(\alpha)$ at x;

(ii) *if $n(\alpha)=1$, then for any irreducible curve C satisfying $Z(\alpha)\cdot C>0$, we have*

$$\Delta_{(C,x)}(\alpha) = 0\times[\nu_x(N(\alpha)|_C),\nu_x(N(\alpha)|_C)+Z(\alpha)\cdot C].$$

In particular, the numerical dimension determines the dimension of the generalized Okounkov body.

2 Technical preliminaries

2.1 Siu decomposition

Let T be a closed positive current of bidegree (p,p) on a complex manifold X. We denote by $\nu(T,x)$ its Lelong number at a point $x\in X$. For any $c>0$, the Lelong upperlevel sets are defined by

$$E_c(T) := \{x\in X, \nu(T,x)\geq c\}.$$

In [Siu74], Siu proved that $E_c(T)$ is an analytic subset of X, of codimension at least p. Moreover, T can be written as a convergent series of closed positive currents

$$T = \sum_{k=1}^{+\infty}\nu(T,Z_k)[Z_k]+R$$

where $[Z_k]$ is a current of integration over an irreducible analytic set of dimension p, and R is a residual current with the property that $\dim E_c(R) < p$ for every $c > 0$. This decomposition is locally and globally unique: the sets Z_k are precisely the p-dimensional components occurring in the upperlevel sets $E_c(T)$, and $\nu(T, Z_k) := \inf\{\nu(T, x)|x \in Z_k\}$ is the generic Lelong number of T along Z_k.

2.2 Currents with analytic singularities

A closed positive (1,1) current T on a compact complex manifold X is said to have analytic (resp. algebraic) singularities along a subscheme $V(\mathcal{I})$ defined by an ideal $\mathcal{I}$ if there exists some $c \in \mathbb{R}_{>0}$ (resp. $\mathbb{Q}_{>0}$) such that locally we have

$$T = \frac{c}{2} dd^c \log(|f_1|^2 + \ldots + |f_k|^2) + dd^c v$$

where $f_1, \ldots, f_k$ are local generators of $\mathcal{I}$ and $v \in L^\infty_{\text{loc}}$ (resp. and additionally, X and $V(\mathcal{I})$ are algebraic). Moreover, if v is smooth, T will be said to have mild analytic singularities. In these situations, we call the sum $\sum \nu(T, D)D$ which appears in the Siu decomposition of T the divisorial part of T. Using the Lelong-Poincaré formula, it is straightforward to check that the divisorial part $\sum \nu(T, D)D$ of a closed (1,1)-current T with analytic singularities along the subscheme $V(\mathcal{I})$ is just the divisorial part of $V(\mathcal{I})$, times the constant $c > 0$ appearing in the definition of analytic singularities. The residual part R has analytic singularities in codimension at least 2. If we denote $E_+(T) := \{x \in X|\nu(T, x) > 0\}$, then $E_+(T)$ is exactly the support of $V(\mathcal{I})$. Moreover, if $V \not\subseteq E_+(T)$ for some smooth variety V, $T|_V := \frac{c}{2} dd^c log(|f_1|^2 + \ldots + |f_k|^2)|_V + dd^c v|_V$ is well defined, for $|f_1|^2 + \ldots + |f_k|^2$ and v are not identically equal to $-\infty$ on V. It is easy to check that this definition does not depend on the choice of the local potential of T.

Definition 2.1 (Non-Kähler locus)**.** If $\alpha \in H^{1,1}_{\partial\overline{\partial}}(X, \mathbb{R})$ is a big class, we define its *non-Kähler locus* as $E_{nK} := \bigcap_T E_+(T)$ for T ranging among the Kähler currents in α.

We will usually use the following theorem due to Collins and Tosatti.

Theorem 2.2 ([CT13])**.** *Let X be a compact Kähler manifold of dimension n. Given a nef and big class α, we define a subset of X which measures its non-Kählerianity, namely the null locus*

$$\text{Null}(\alpha) := \bigcup_{\int_V \alpha^{dim V} = 0} V,$$

where the union is taken over all positive dimensional irreducible analytic subvarieties of X. Then we have

$$\text{Null}(\alpha) = E_{nK}(\alpha).$$

2.3 Regularization of currents

We will need Demailly's regularization theorem for closed (1,1)-currents, which enables us to approximate a given current by currents with analytic singularities, with a loss of positivity that is arbitrary small. In particular, we could approximate a Kähler current T inside its cohomology class by Kähler currents T_k with algebraic singularities, with a good control of the singularities. A big class therefore contains plenty of Kähler currents with analytic singularities.

Theorem 2.3. *Let T be a closed almost positive (1,1)-current on a compact complex manifold X, and fix a Hermitian form ω. Suppose that $T \geq \gamma$ for some real (1,1)-form γ on X. Then there exists a sequence T_k of currents with algebraic singularities in the cohomology class $\{T\}$ which converges weakly to T, such that $T_k \geq \gamma - \epsilon_k\omega$ for some sequence $\epsilon_k > 0$ decreasing to 0, and $\nu(T_k, x)$ increases to $\nu(T, x)$ uniformly with respect to $x \in X$.*

2.4 Currents with minimal singularities

Let $T_1 = \theta_1 + dd^c\varphi_1$ and $T_2 = \theta_2 + dd^c\varphi_2$ be two closed almost positive (1,1)-currents on X, where θ_i are smooth forms and φ_i are almost pluri-subharmonic functions, we say that T_1 is less singular than T_2 (write $T_1 \preceq T_2$) if we have $\varphi_2 \leq \varphi_1 + C$ for some constant C.

Let α be a class in $H^{1,1}_{\partial\overline{\partial}}(X, \mathbb{R})$ and γ be a smooth real (1,1)-form, we denote by $\alpha[\gamma]$ the set of closed almost positive (1,1)-currents $T \in \alpha$ with $T \geq \gamma$. Since the set of potentials of such currents is stable by taking a supremum, we conclude by standard pluripotential theory that there exists a closed almost positive (1,1)-current $T_{\min,\gamma} \in \alpha[\gamma]$ which has minimal singularities in $\alpha[\gamma]$. $T_{\min,\gamma}$ is well defined modulo dd^cL^∞. For each $\epsilon > 0$, denote by $T_{\min,\epsilon} = T_{\min,\epsilon}(\alpha)$ a current with minimal singularities in $\alpha[-\omega]$, where ω is some reference Hermitian form. The minimal multiplicity at $x \in X$ of the pseudo-effective class $\alpha \in H^{1,1}_{\partial\overline{\partial}}(X, \mathbb{R})$ is defined as

$$\nu(\alpha, x) := \sup_{\epsilon>0} \nu(T_{\min,\epsilon}, x).$$

For a prime divisor D, we define the generic minimal multiplicity of α along D as

$$\nu(\alpha, D) := \inf\{\nu(\alpha, x) | x \in D)\}.$$

We then have $\nu(\alpha, D) = \sup_{\epsilon>0} \nu(T_{\min,\epsilon}, D)$.

2.5 Lebesgue decomposition

A current T can be locally seen as a form with distribution coefficients. When T is positive, the distributions are positive measures which admit a Lebesgue decomposition into

an absolutely continuous part (with respect to the Lebesgue measure on X) and a singular part. Therefore we obtain the decomposition $T = T_{\mathrm{ac}} + T_{\mathrm{sing}}$, with T_{ac} (resp. T_{sing}) globally determined thanks to the uniqueness of the Lebesgue decomposition.

Now we assume that T is a (1,1)-current. The absolutely continuous part T_{ac} is considered as a (1,1)-form with L^1_{loc} coefficients, and more generally we have $T_{\mathrm{ac}} \geq \gamma$ whenever $T \geq \gamma$ for some real smooth real form γ. Thus we can define the product T^k_{ac} of T_{ac} almost everywhere. This yields a positive Borel (k,k)-form.

2.6 Modified nef cone and divisorial Zariski decomposition

In this subsection, we collect some definitions and properties of the modified nef cone and divisorial Zariski decomposition. See [Bou04] for more details.

Definition 2.4. Let X be compact complex manifold, and ω be some reference Hermitian form. Let α be a class in $H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{R})$.

(i) α is said to be a *modified Kähler class* iff it contains a Kähler current T with $\nu(T,D) = 0$ for all prime divisors D in X.

(ii) α is said to be a *modified nef class* iff, for every $\epsilon > 0$, there exists a closed (1,1)-current $T_\epsilon \geq -\epsilon\omega$ and $\nu(T_\epsilon, D) = 0$ for every prime D.

Remark 2.5. The modified nef cone $\mathcal{MN}$ is a closed convex cone which contains the nef cone $\mathcal{N}$. When X is a Kähler manifold, $\mathcal{MN}$ is just the interior of the modified Kähler cone $\mathcal{MK}$.

Remark 2.6. For a complex surface, the Kähler (nef) cone and the modified Kähler (modified nef) cone coincide. Indeed, analytic singularities in codimension 2 of a Kähler current T are just isolated points. Therefore the class $\{T\}$ is a Kähler class.

Definition 2.7 (Divisorial Zariski decomposition)**.** The *negative part* of a pseudo-effective class $\alpha \in H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{R})$ is defined as $N(\alpha) := \sum \nu(\alpha, D)D$. The *Zariski projection* of α is $Z(\alpha) := \alpha - \{N(\alpha)\}$. We call the decomposition $\alpha = Z(\alpha) + \{N(\alpha)\}$ the *divisorial Zariski decomposition of* α.

Remark 2.8. We claim that the volume of $Z(\alpha)$ is equal to the volume of α. Indeed, if T is a positive current in α, then we have $T \geq N(\alpha)$ since $T \in \alpha[-\epsilon\omega]$ for each $\epsilon > 0$ and we conclude that $T \mapsto T - N(\alpha)$ is a bijection between the positive currents in α and those in $Z(\alpha)$. Furthermore, we notice that $(T - N(\alpha))_{\mathrm{ac}} = T_{\mathrm{ac}}$, and thus by the definition of volume of the pseudo-effective classes we conclude that $\mathrm{vol}_X(\alpha) = \mathrm{vol}_X(Z(\alpha))$.

Definition 2.9 (Exceptional divisors)**.** (i) A family $D_1, \ldots, D_q$ of prime divisors is said to be an *exceptional family* iff the convex cone generated by their cohomology classes meets the modified nef cone at 0 only.

(ii) An effective $\mathbb{R}$-divisor E is said to be *exceptional* iff its prime components constitute an exceptional family.

We have the following properties of exceptional divisors:

Theorem 2.10. (i) *An effective $\mathbb{R}$-divisor E is exceptional iff $Z(E) = 0$.*

(ii) *If E is an exceptional effective $\mathbb{R}$-divisor, we have $E = N(\{E\})$.*

(iii) *If $D_1, \ldots, D_q$ is an exceptional family of primes, then their classes $\{D_1\}, \ldots, \{D_q\}$ are linearly independent in $\mathrm{NS}_{\mathbb{R}}(X) \subset H^{1,1}(X, \mathbb{R})$. In particular, the length of the exceptional families of primes is uniformly bounded by the Picard number $\rho(X)$.*

(iv) *Let X be a surface, a family $D_1, \ldots, D_r$ of prime divisors is exceptional iff its intersection matrix $(D_i \cdot D_j)$ is negative definite.*

In this paper, we need the following properties of the modified nef cone $\mathcal{MN}$ and the divisorial Zariski decomposition due to Boucksom (ref. [Bou04]). We state these properties without proofs.

Theorem 2.11. *Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudo-effective class. Then we have:*

(i) *Its Zariski projection $Z(\alpha)$ is a modified nef class.*

(ii) *$Z(\alpha) = \alpha$ iff α is modified nef.*

(iii) *$Z(\alpha)$ is big iff α is.*

Remark 2.12. Let X be a complex Kähler surface. For a big class $\alpha \in H^{1,1}(X, \mathbb{R})$, $Z(\alpha)$ is a big and modified nef class. By Remark 2.5, any modified nef class is nef, it follows that $Z(\alpha)$ is big and nef.

Theorem 2.13. (i) *The map $\alpha \mapsto N(\alpha)$ is convex and homogeneous on pseudo-effective class cone $\mathcal{E}$. It is continuous on the interior of $\mathcal{E}$.*

(ii) *The Zariski projection $Z : \mathcal{E} \to \mathcal{MN}$ is concave and homogeneous. It is continuous on the interior of $\mathcal{E}$.*

Theorem 2.14. *Let p be a big and modified nef class. Then the primes $D_1, \ldots, D_q$ contained in the non-Kähler locus $E_{nK}(p)$ form an exceptional family A, and the fiber of Z over p is the simplicial cone $Z^{-1}(p) = p + V_+(A)$, where $V_+(A) := \sum_{D \in A} \mathbb{R}_+\{D\}$.*

Theorem 2.15. *Let X be a compact surface. If $\alpha \in H^{1,1}(X, \mathbb{R})$ is a pseudo-effective class, its divisorial Zariski decomposition $\alpha = Z(\alpha) + \{N(\alpha)\}$ is the unique orthogonal decomposition of α with respect to the non-degenerate quadratic form $q(\alpha) := \int \alpha^2$ into the sum of a modified nef class and the class of an exceptional effective $\mathbb{R}$-divisor.*

Remark 2.16. Let X be a surface, α is the class of an effective $\mathbb{Q}$-divisor D on a projective surface, the divisorial Zariski decomposition of α is just the original Zariski decomposition of D.

3 Transcendental Morse inequality

3.1 Proof of the transcendental Morse inequality for complex surfaces

The main goal of this section is to prove the differentiability of the volume function and the transcendental Morse inequality for complex surfaces. In fact, in the next subsection we will give a more general method to prove the transcendental Morse inequality for Kähler manifolds on which modified nef cones $\mathcal{MN}$ coincide with the nef cones; this includes complex surfaces. However, since the methods and results here are very special in studying generalized Okounkov bodies, we will treat complex surface and higher dimensional Kähler manifolds separately. Throughout this subsection, if not specially mentioned, X will stand for a complex Kähler surface. We denote by $q(\alpha) := \int \alpha^2$ the quadratic form on $H^{1,1}(X,\mathbb{R})$. By the Hodge index theorem, $(H^{1,1}(X,\mathbb{R}), q)$ has signature $(1, h^{1,1}(X)-1)$. The open cone $\{\alpha \in H^{1,1}(X,\mathbb{R}) | q(\alpha) > 0\}$ has thus two connected components which are convex cones, and we denote by $\mathcal{P}$ the component containing the Kähler cone $\mathcal{K}$.

Lemma 3.1. *Let X be a compact Kähler manifold of dimension n. If $\alpha \in H^{1,1}(X,\mathbb{R})$ is a big class, $\beta \in H^{1,1}(X,\mathbb{R})$ is a nef class, then $N(\alpha+t\beta) \leq N(\alpha)$ as effective $\mathbb{R}$-divisors for $t \geq 0$. Furthermore, when t is small enough, the prime components of $N(\alpha+t\beta)$ will be the same as those of $N(\alpha)$.*

Proof. Since β is nef, by Theorem 2.13, we have

$$N(\alpha+t\beta) \leq N(\alpha) + tN(\beta) = N(\alpha).$$

Since the map $\alpha \mapsto N(\alpha)$ is convex on pseudo-effective class cone $\mathcal{E}$, it is continuous on the interior of $\mathcal{E}$, and thus the theorem follows. □

Theorem 3.2. *If $\alpha \in H^{1,1}(X,\mathbb{R})$ is a big class and $\beta \in H^{1,1}(X,\mathbb{R})$ is a nef class, then*

$$\frac{d}{dt}\Big|_{t=0} \mathrm{vol}_X(\alpha+t\beta) = 2Z(\alpha)\cdot\beta \tag{3.1}$$

Proof. By Lemma 3.1, there exists an $\epsilon > 0$ such that when $0 \leq t < \epsilon$, we can write $N(\alpha+t\beta) = \sum_{i=1}^{r} a_i(t)N_i$, where $0 < a_i(t) \leq a_i(0) =: a_i$, and each $a_i(t)$ is a continuous and decreasing function with respect to t. According to the orthogonal property of

divisorial Zariski decomposition (ref. Theorem 2.15), $Z(\alpha+t\beta)\cdot N(\alpha+t\beta)=0$ for $t\geq 0$. Since $Z(\alpha+t\beta)$ is modified nef and thus nef (by Remark 2.6), we have $Z(\alpha+t\beta)\cdot N_i\geq 0$ for every i. When $0\leq t<\epsilon$, we have $a_i(t)>0$ for $i=1,\ldots,r$, therefore, $Z(\alpha+t\beta)$ is orthogonal to each $\{N_i\}$ with respect to q. We denote by $V\subset H^{1,1}(X,\mathbb{R})$ the finite vector space spanned by $\{N_1\},\ldots,\{N_r\}$, by $V^\perp$ the orthogonal space of V with respect to q. Thus $\alpha+t\beta=Z(\alpha+t\beta)+\sum_{i=1}^r a_i(t)\{N_i\}$ is the decomposition in the direct sum $V^\perp\oplus V$. We decompose $\beta=\beta^\perp+\beta_0$ in the direct sum $V^\perp\oplus V$, and we have

$$Z(\alpha+t\beta)=Z(\alpha)+t\beta^\perp,$$
$$\sum_{i=1}^r a_i(t)\{N_i\}=\sum_{i=1}^r a_i\{N_i\}+t\beta_0.$$

Since $\mathrm{vol}_X(\alpha+t\beta)=\mathrm{vol}_X(Z(\alpha+t\beta))=Z(\alpha+t\beta)^2$ (by Remark 2.8), it is easy to deduce that

$$\left.\frac{d}{dt}\right|_{t=0}\mathrm{vol}_X(\alpha+t\beta)=2Z(\alpha)\cdot\beta^\perp=2Z(\alpha)\cdot\beta.$$

The last equality follows from $\beta_0\in V$ and $Z(\alpha)\in V^\perp$. We get the first half of Theorem 1.5. □

To prove the transcendental Morse inequality for complex surfaces, we will need a criterion for bigness of a class:

Theorem 3.3. *Let α and β be two nef classes such that $\alpha^2-2\alpha\cdot\beta>0$, then $\alpha-\beta$ is a big class.*

Proof. We denote by $\mathcal{P}$ the connected component of the open cone $\{\alpha\in H^{1,1}(X,\mathbb{R})\mid q(\alpha)>0\}$ containing the Kähler cone $\mathcal{K}$, then $\mathcal{P}\subset\mathcal{E}^0$. As a consequence of the Nakai-Moishezon criterion for surfaces (ref. [Lam99]), we know that, if γ is a real (1,1)-class with $\gamma^2>0$, then γ or $-\gamma$ is big. Since α and β are both nef, we have that $(\alpha-t\beta)^2>0$ for $0\leq t\leq 1$. This means that $\alpha-t\beta$ is contained in some component of the open cone $\{\alpha\in H^{1,1}(X,\mathbb{R})|q(\alpha)>0\}$. But since α is big, $\alpha-t\beta$ is contained in $\mathcal{P}\subset\mathcal{B}$, and *a fortiori* $\alpha-\beta$ is. □

Now we are ready to prove the transcendental Morse inequality for complex surfaces.

Proof of Theorem 1.6. By Theorem 3.3, when $\alpha^2-2\alpha\cdot\beta>0$, the cohomology class $\alpha-\beta$ is big. By the differentiability formula (3.1), we have

$$\mathrm{vol}_X(\alpha-\beta)=\alpha^2-2\int_0^1 Z(\alpha-t\beta)\cdot\beta\,dt.$$

Since the Zariski projection $Z:\mathcal{E}\to\mathcal{MN}$ is concave and homogeneous by Theorem 2.13, we have

$$\alpha=Z(\alpha)\geq Z(\alpha-t\beta)+Z(t\beta)\geq Z(\alpha-t\beta).$$

Since β is nef, we have

$$\alpha \cdot \beta \geq Z(\alpha - t\beta) \cdot \beta,$$

and thus

$$\mathrm{vol}_X(\alpha - \beta) \geq \alpha^2 - 2\alpha \cdot \beta.$$

□

In the last part of this subsection, we prove the second half of Theorem 1.5.

Theorem 3.4. *Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class and C be an irreducible divisor, then*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}_X(\alpha + tC) = 2Z(\alpha) \cdot C. \tag{3.2}$$

Proof. It suffices to prove the theorem for C not nef. Thus we have $C^2 < 0$. Write $N(\alpha) = \sum_{i=1}^r a_i N_i$, where each N_i is prime divisor. If $C \subseteq E_{nK}(Z(\alpha))$, we deduce that $Z(\alpha) \cdot C = 0$ by Theorem 2.2, and $\{C, N_1, \ldots, N_r\}$ forms an exceptional family by Theorem 2.14. Thus we have

$$Z(\alpha + tC) = Z(\alpha),$$

and

$$N(\alpha + tC) = N(\alpha) + tC$$

for $t \geq 0$. The theorem is thus proved in this case.

From now on we assume $C \not\subseteq E_{nK}(Z(\alpha))$, thus we have $Z(\alpha) \cdot C > 0$ and $C \not\subseteq \mathrm{Supp}(N(\alpha))$. We define

$$\begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} = -S^{-1} \cdot \begin{pmatrix} C \cdot N_1 \\ \vdots \\ C \cdot N_r \end{pmatrix},$$

where $S = (s_{ij})$ denotes the intersection matrix of $\{N_1, \ldots, N_r\}$. By Theorem 2.15 we know that S is negative definite satisfying $s_{ij} \geq 0$ for all $i \neq j$. We claim that $Z(\alpha) + t(\{C\} + \sum_{i=1}^r b_i\{N_i\})$ is big and nef if $0 \leq t < -\frac{Z(\alpha)\cdot C}{C^2}$. We need the following lemma from [BKS03] to prove our claim.

Lemma 3.5. *Let A be a negative definite $r \times r$-matrix over the reals such that $a_{ij} \geq 0$ for all $i \neq j$. Then all entries of the inverse matrix A^{-1} are ≤ 0.*

By Lemma 3.5 we know that all entries of S^{-1} are ≤ 0, thus $b_j \geq 0$ for all $1 \leq j \leq r$ and we get the bigness of $Z(\alpha) + t(\{C\} + \sum_{i=1}^r b_i\{N_i\})$. By the construction of b_j, we have

$$(Z(\alpha) + t(\{C\} + \sum_{i=1}^r b_i\{N_i\})) \cdot N_j = 0$$

for $1 \leq j \leq r$, and

$$(Z(\alpha) + t(\{C\} + \sum_{i=1}^{r} b_i\{N_i\})) \cdot C > 0$$

for $0 \leq t < -\frac{Z(\alpha)\cdot C}{C^2}$. Thus we have the nefness and our claim follows. Since the divisorial Zariski decomposition is orthogonal and unique (see Theorem 2.15), we conclude that

$$N(\alpha + t\{C\}) = \sum_{i=1}^{r}(a_i - tb_i)N_i, \tag{3.3}$$

$$Z(\alpha + t\{C\}) = Z(\alpha) + t\{C\} + \sum_{i=1}^{r} tb_i\{N_i\}, \tag{3.4}$$

for t small enough. Since $\mathrm{vol}_X(\alpha + tC) = Z(\alpha + t\{C\})^2$, we have thus also obtained formula (3.2) in this case.

□

3.2 Transcendental Morse inequality for some special Kähler manifolds

One can modify the proof of Theorem 1.6 a little bit, to extend the transcendental Morse inequality to Kähler manifolds whose modified nef cone $\mathcal{MN}$ coincides with the nef cone $\mathcal{N}$. In this subsection, we assume X to be a compact Kähler manifold of dimension n which satisfies this condition.

Lemma 3.6. *If $\alpha \in \mathcal{E}^\circ$, then the divisorial Zariski decomposition $\alpha = Z(\alpha) + N(\alpha)$ is such that*

$$Z(\alpha)^{n-1} \cdot N(\alpha) = 0.$$

Remark 3.7. Lemma 3.6 is very similar to the Corollary 4.5 in [BDPP13]: If $\alpha \in \mathcal{E}_{\mathrm{NS}}$, then the divisorial Zariski decomposition $\alpha = Z(\alpha) + N(\alpha)$ is such that $\langle Z(\alpha)^{n-1}\rangle \cdot N(\alpha) = 0$. However, the proof of [BDPP13] is based on the orthogonal estimate for divisorial Zariski decomposition of $\mathcal{E}_{\mathrm{NS}}$, which is still a conjecture for $\alpha \in \mathcal{E}$. Here we will use Theorem 2.2 to prove this lemma directly.

Proof of Lemma 3.6. By Theorem 2.11, if α is big, then $Z(\alpha)$ is big and modified nef, thus nef by the assumption for X. By Theorem 2.14, the primes $D_1, \ldots, D_q$ contained in the non-Kähler locus $E_{nK}(Z(\alpha))$ form an exceptional family, and $\alpha = Z(\alpha) + \sum_{i=1}^{r} a_i D_i$ for $a_i \geq 0$. Since $\mathrm{Null}(Z(\alpha)) = E_{nK}(Z(\alpha))$ by Theorem 2.2, we have $Z(\alpha)^{n-1} \cdot D_i = 0$ for each i, and thus $Z(\alpha)^{n-1} \cdot N(\alpha) = 0$. The lemma is proved. □

Proof of Theorem 1.7. By Lemma 3.1, there exists $\epsilon > 0$ such that the prime components of $N(\alpha + t\beta)$ will be the same when $0 \leq t \leq \epsilon$. Moreover if we denote $N(\alpha + t\beta) =$

$\sum_{i=1}^{r} a_i(t)N_i$, then each $a_i(t)$ is continuous and decreasing satisfying $a_i(t) > 0$. By Lemma 3.6, we have

$$Z(\alpha+t\beta)^{n-1}\cdot N(\alpha+t\beta)=\sum_{i=1}^{r} a_i(t)Z(\alpha+t\beta)^{n-1}\cdot N_i=0.$$

Since $Z(\alpha+t\beta)$ is modified nef thus nef, we deduce that $Z(\alpha+t\beta)^{n-1}\cdot N_i = 0$ for $0\leq t\leq\epsilon$ and $i=1,\ldots,r$.

Since $a_i(t)$ is continuous and decreasing, it is almost everywhere differentiable. Thus $Z(\alpha+t\beta)=\alpha+t\beta-\sum_{i=1}^{r} a_i(t)N_i$ is an a.e. differentiable and continuous curves in the finite dimensional space $H^{1,1}(X,\mathbb{R})$ parametrized by t. Meanwhile, since $\alpha\mapsto\alpha^n$ is a quadratic form (possibly degenerate) in $H^{1,1}(X,\mathbb{R})$, we thus deduce that $\mathrm{vol}_X(\alpha+t\beta)=Z(\alpha+t\beta)^n$ is an a.e. differentiable function with respect to t. Therefore, if $\mathrm{vol}_X(\alpha+t\beta)$ and $a_i(t)$ are both differentiable at $t=t_0$, we have

$$\frac{d}{dt}\Big|_{t=t_0}\mathrm{vol}_X(\alpha+t\beta)=nZ(\alpha+t_0\beta)^{n-1}\cdot(\beta-\sum_{i=1}^{r} a_i'(t_0)N_i)=nZ(\alpha+t_0\beta)^{n-1}\cdot\beta.$$

Since $\mathrm{vol}_X(\alpha+t\beta)$ is increasing and continuous, it is also a.e. differentiable and thus we have

$$\begin{aligned}\mathrm{vol}_X(\alpha+s\beta) &= \mathrm{vol}_X(\alpha)+\int_0^s \frac{d}{dt}\mathrm{vol}_X(\alpha+t\beta)\mathrm{d}t\\ &= \mathrm{vol}_X(\alpha)+n\int_0^s Z(\alpha+t\beta)^{n-1}\cdot\beta\,\mathrm{d}t. \qquad (3.5)\end{aligned}$$

for $0\leq s\leq\epsilon$. Since $Z(\alpha+t\beta)$ is continuous (by Theorem 2.13), by (3.5) we deduce that $\mathrm{vol}_X(\alpha+t\beta)$ is differentiable with respect to t and its derivative

$$\frac{d}{dt}\Big|_{t=t_0}\mathrm{vol}_X(\alpha+t\beta)=nZ(\alpha+t_0\beta)^{n-1}\cdot\beta. \quad \square$$

In order to prove transcendental Morse inequality, we will need the following bigness criterion obtained in [Xia13] and [Popo14].

Theorem 3.8. *Let X be an n-dimensional compact Kähler manifold. Assume α and β are two nef classes on X satisfying $\alpha^n-n\alpha^{n-1}\cdot\beta>0$, then $\alpha-\beta$ is a big class.*

The proof of the next theorem is similar to that of Theorem 1.6 and is therefore omitted.

Theorem 3.9. *Let X be a compact Kähler manifold on which the modified nef cone $\mathcal{MN}$ and the nef cone $\mathcal{N}$ coincide. If α and β are nef cohomology classes of type (1,1) on X satisfying the inequality $\alpha^n-n\alpha^{n-1}\cdot\beta>0$. Then $\alpha-\beta$ contains a Kähler current and $\mathrm{vol}_X(\alpha-\beta)\geq\alpha^{n-1}-n\alpha^{n-1}\cdot\beta$.*

Remark 3.10. In [BCJ09], the authors proved the following differentiability theorem:

$$\frac{d}{dt}\Big|_{t=t_0} \mathrm{vol}_X(L+tD) = n\langle L^{n-1}\rangle \cdot D, \tag{3.6}$$

where L is a big line bundle on the smooth projective variety X and D is a prime divisor. The right-hand side of the equation above involves the *positive intersection product* $\langle L^{n-1}\rangle \in H^{n-1,n-1}_{\geq 0}(X,\mathbb{R})$, first introduced in the analytic context in [BDPP13]. Theorem 1.7 could be seen as a transcendental version of (3.6) for some special Kähler manifolds. In the general Kähler situation, we propose the following conjecture:

Conjecture 3.11. *Let X be a Kähler manifold of dimensional n, α be a big class. If β is a pseudo-effective class, then we have*

$$\frac{d}{dt}\Big|_{t=0} \mathrm{vol}_X(\alpha+t\beta) = n\langle \alpha^{n-1}\rangle \cdot \beta.$$

4 Generalized Okounkov bodies on Kähler manifolds

4.1 Definition and relation with the algebraic case

Throughout this subsection, X will stand for a Kähler manifold of dimensional n. Our main goal in this subsection is to generalize the definition of Okounkov body to any pseudo-effective class $\alpha \in H^{1,1}(X,\mathbb{R})$. First of all, we define a valuation-like function. For any positive current $T \in \alpha$ with analytic singularites, we define the valuation-like function

$$T \to \nu(T) = \nu_{Y_\bullet}(T) = (\nu_1(T), \dots \nu_n(T))$$

as follows. First, set

$$\nu_1(T) = \sup\{\lambda \mid T - \lambda[Y_1] \geq 0\},$$

where $[Y_1]$ is the current of integration over Y_1. By Section 2.1 we know that $\nu_1(T)$ is the coefficient $\nu(T,Y_1)$ of the positive current $[Y_1]$ appearing in the Siu decomposition of T. Since T has analytic singularities, by the arguments in Section 2.2, $T_1 := (T-\nu_1[Y_1])|_{Y_1}$ is a well-defined positive current in the pseudo-effective class $(\alpha - \nu_1\{Y_1\})|_{Y_1}$ and it also has analytic singularities. Then take

$$\nu_2(T) = \sup\{\lambda \mid T_1 - \lambda[Y_2] \geq 0\},$$

and continue in this manner to define the remaining values $\nu_i(T) \in \mathbb{R}^+$.

Remark 4.1. If one assumes $\alpha \in \mathrm{NS}_{\mathbb{Z}}(X)$, there exists a holomorphic line bundle L such that $\alpha = c_1(L)$. If D is the divisor of some holomorphic section $s_D \in H^0(X,\mathcal{O}_X(L))$, then we have

$$\nu([D]) = \mu(s_D),$$

where μ is the valuation-like function appeared in the definition of the original Okounkov body. Roughly speaking our definition of valuation-like function has a bigger domain of definition and thus the image of our valuation-like function contains $\bigcup_{m=1}^{\infty} \frac{1}{m}\mu(mL)$.

For any big class α, we define a $\mathbb{Q}$-*convex body* $\Delta_{\mathbb{Q}}(\alpha)$ (resp. $\mathbb{R}$-*convex body* $\Delta_{\mathbb{R}}(\alpha)$) to be the set of valuation vectors $\nu(T)$, where T ranges among all the Kähler (resp. positive) currents with algebraic (resp. analytic) singularities. Then $\Delta_{\mathbb{Q}}(\alpha) \subseteq \Delta_{\mathbb{R}}(\alpha)$. It is easy to check that this is a convex set in $\mathbb{Q}^n$ (resp. $\mathbb{R}^n$). Indeed, for any two positive currents T_0 and T_1 with algebraic (resp. analytic) singularities, we have $\nu(\epsilon T_0 + (1-\epsilon)T_1) = \epsilon\nu(T_0) + (1-\epsilon)\nu(T_1)$ for $0 \leq \epsilon \leq 1$ rational (resp. real). It is also obvious to see the homogeneous property of $\Delta_{\mathbb{Q}}(\alpha)$, that is, for all $c \in \mathbb{Q}^+$, we have

$$\Delta_{\mathbb{Q}}(c\alpha) = c\Delta_{\mathbb{Q}}(\alpha).$$

Indeed, since we have $\nu(cT) = c\nu(T)$ for all $c \in \mathbb{R}^+$, the claim follows directly.

Example 4.2. Let L be a line bundle of degree $c > 0$ on a smooth curve C of genus g. Then we have

$$\Delta_{\mathbb{Q}}(c_1(L)) = \mathbb{Q} \cap [0, c).$$

Since $\mathrm{NS}_{\mathbb{R}}(C) = H^{1,1}(C, \mathbb{R})$, for any ample class α on C we have

$$\Delta_{\mathbb{Q}}(\alpha) = \mathbb{Q} \cap [0, \alpha \cdot C).$$

Lemma 4.3. *Let α be a big class, then the $\mathbb{R}$-convex body $\Delta_{\mathbb{R}}(\alpha)$ lies in a bounded subset of $\mathbb{R}^n$.*

Proof. It suffices to show that there exists a $b > 0$ large enough such that $\nu_i(T) < b$ for any positive current T with analytic singularities. We fix a Kähler class ω. Choose first of all $b_1 > 0$ such that

$$(\alpha - b_1 Y_1) \cdot \omega^{n-1} < 0.$$

This guarantees that $\nu_1(T) < b_1$ since $\alpha - b_1 Y_1 \notin \mathcal{E}$. Next choose b_2 large enough so that

$$((\alpha - aY_1)|_{Y_1} - b_2 Y_2) \cdot \omega^{n-2} < 0$$

for all real numbers $0 \leq a \leq b_1$. Then $\nu_2(T) \leq b_2$ for any positive current T with analytic singularities. Continuing in this manner we construct $b_i > 0$ for $i = 1, \ldots, n$ such that $\nu_i(T) \leq b_i$ for any positive current T with analytic singularities. We take $b = \max\{b_i\}$. □

Lemma 4.4. *For any big class α, $\Delta_{\mathbb{Q}}(\alpha)$ is dense in $\Delta_{\mathbb{R}}(\alpha)$. Thus we have $\overline{\Delta_{\mathbb{Q}}(\alpha)} = \overline{\Delta_{\mathbb{R}}(\alpha)}$.*

Proof. It is easy to verify that if T is a Kähler current with analytic singularities, then for any $\epsilon > 0$, there exists a Kähler current S_ϵ with algebraic singularities such that $\|\nu(S_\epsilon) - \nu(T)\| < \epsilon$ with respect to the standard norm in $\mathbb{R}^n$. For the general case, We fix a Kähler current $T_0 \in i\Theta(L)$ with algebraic singularities. Then for any positive current T with analytic singularities, $T_\epsilon := (1-\epsilon)T + \epsilon T_0$ is still a Kähler current. By Lemma 4.3, $\|\nu(T_\epsilon) - \nu(T)\| = \epsilon \|(\nu(T_0) - \nu(T))\|$ will tend to 0 since $\nu(T)$ is uniformly bounded for any positive current T with analytic singularities. Thus $\Delta_{\mathbb{Q}}(\alpha)$ is dense in $\Delta_{\mathbb{R}}(\alpha)$. □

Now we study the relations between $\Delta_{\mathbb{Q}}(c_1(L))$ and $\Delta(L)$ for L a big line bundle on X. First we begin with the following two lemmas.

Lemma 4.5 (Extension property)**.** *Let L be a big line bundle on the projective variety X of dimension n, with a singular Hermitian metric $h = e^{-\varphi}$ satisfying*

$$i\Theta_{L,h} = dd^c\varphi \geq \epsilon\omega$$

for some $\epsilon > 0$ and a given Kähler form ω. If the restriction of φ on a smooth hypersurface Y is not identically equal to $-\infty$, then there exists a positive integer m_0 which depends only on Y so that any holomorphic section $s_m \in H^0(Y, \mathcal{O}_Y(mL) \otimes \mathcal{I}(m\varphi|_Y))$ can be extended to $S_m \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(m\varphi))$ for any $m \geq m_0$.

We need the following Ohsawa-Takegoshi extension theorem to prove Lemma 4.5.

Theorem 4.6 (Ohsawa-Takegoshi)**.** *Let X be a smooth projective variety. Let Y be a smooth divisor defined by a holomorphic section of the line bunle H with a smooth metric $h_0 = e^{-\psi}$. Let L be a holomorphic line bunle with a singular metric $h = e^{-\phi}$, satisfying the curvature assumptions*

$$dd^c\phi \geq 0$$

and

$$dd^c\phi \geq \delta dd^c\psi$$

with $\delta > 0$. Then for any holomorphic section $s \in H^0(Y, \mathcal{O}_Y(K_Y + L) \otimes \mathcal{I}(h|_Y))$, there exists a global holomorphic section $S \in H^0(X, \mathcal{O}_X(K_X + L + Y) \otimes \mathcal{I}(h))$ such that $S|_Y = s$.

Proof of Lemma 4.5. Taking a smooth metric $e^{-\psi}$ and $e^{-\eta}$ on Y and K_X, we can choose m_0 large enough satisfying the curvature assumptions

$$dd^c(m\phi - \eta - \psi) \geq 0$$

and

$$dd^c(m\phi - \eta - \psi) \geq dd^c\psi$$

for any $m \geq m_0$.

By Theorem 4.6, any holomorphic section $s \in H^0(Y, \mathcal{O}_Y(K_Y + (mL - K_X - Y)|_Y) \otimes \mathcal{I}(h^m|_Y))$ can be extended to a global holomorphic section $S \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m))$ such that $S|_Y = s$. By the adjunction theorem, we have $(K_X + Y)|_Y = K_Y$, thus the lemma is proved.

□

Lemma 4.7. *Let L be a big line bundle on the Riemann surface C with a singular Hermitian metric $h = e^{-\varphi}$ such that φ has algebraic singularities and*

$$i\Theta_{L,h} = dd^c\varphi \geq \epsilon\omega$$

for some $\epsilon > 0$. Then for a fixed point p, there exists an integer $k > 0$ such that we have a holomorphic section $s_k \in H^0(C, \mathcal{O}_C(kL) \otimes \mathcal{I}(h^k))$ satisfying $\mathrm{ord}_p(s_k) = k\nu(i\Theta_{L,h}, p)$.

Proof. Since φ has algebraic singularities, we have the following Lebsegue decomposition

$$i\Theta_{L,h} = (i\Theta_{L,h})_{\mathrm{ac}} + \sum_{i=1}^{r} c_i x_i,$$

where each $c_i > 0$ is rational and $x_1, \ldots, x_r$ are the log poles of $i\Theta_{L,h}$ (possibly p is among them). Since we have

$$\int_C i(\Theta_{L,h})_{\mathrm{ac}} + \sum_{i=1}^{r} c_i = \deg(L),$$

thus

$$\sum_{i=1}^{r} c_i < \deg(L).$$

By Riemann-Roch theorem there exists an integer $k > 0$ satisfying

(i) kc_i is integer,

(ii) there is a holomorphic section $s_k \in H^0(C, \mathcal{O}_C(kL))$ such that $\mathrm{ord}_{x_i}(s_k) \geq kc_i$ and $\mathrm{ord}_p(s_k) = k\nu(i\Theta_{L,h}, p)$.

Thus s_k is locally integrable with respect to the weight $e^{-k\varphi}$. The theorem is proved.

□

Theorem 4.8. *Let X be a smooth projective manifold of dimension n. For any Kähler current $T \in c_1(L)$ with algebraic singularities, there exists a holomorphic section $s \in H^0(X, \mathcal{O}_X(kL))$ such that $\mu(s) = k\nu(T)$, i.e., we have*

$$\nu(T) \in \bigcup_{m=1}^{\infty} \frac{1}{m}\mu(mL).$$

In particular,

$$\Delta_{\mathbb{Q}}(c_1(L)) \subseteq \bigcup_{m=1}^{\infty} \frac{1}{m}\mu(mL) \subseteq \Delta(L).$$

Proof. First, set $\nu_i = \nu_i(T)$ and define

$$T_0 := T,\ T_1 := (T_0 - \nu_1[Y_1])|_{Y_1},\ \ldots\ , T_{n-1} := (T_{n-2} - \nu_{n-1}[Y_{n-1}])|_{Y_{n-1}};$$

$$L_0 := L - \nu_1 Y_1,\ L_1 := L_0|_{Y_1} - \nu_2 Y_2,\ \ldots\ , L_{n-2} := L_{n-3}|_{Y_{n-2}} - \nu_{n-1}Y_{n-1}.$$

Since $T_0 \geq \epsilon\omega$, we have $T_1 \geq \epsilon\omega|_{Y_1}, \ldots, T_{n-1} \geq \epsilon\omega|_{Y_{n-1}}$. Since each ν_i is rational, we could find an integer m to make each $m\nu_i$ be integer so that each mL_i is a big line bundle on Y_i. If we could prove

$$\nu(mT) \in \bigcup_{k=1}^{\infty} \frac{1}{k}\mu(kmL),$$

then we will have

$$\nu(T) \in \bigcup_{m=1}^{\infty} \frac{1}{m}\mu(mL),$$

by the homogeneous property $\frac{1}{m}\nu(mT) = \nu(T)$. Thus we can assume that each $\nu_i(T)$ is an integer after we replace L by mL and T by mT.

Firstly, since $T_0 \in c_1(L)$ is a Kähler current with algebraic singularities, there exists a singular metric $h = e^{-\varphi_0}$ on L whose curvature current is T_0 and φ has algebraic singularities; on the other hand, there is a canonical metric $e^{-\eta_0}$ on $\mathcal{O}_{Y_0}(Y_1)$ such that $dd^c\eta_0 = [Y_1]$ in the sense of currents, thus by the definition of ν_1 we deduce that $h_0 := e^{-\varphi_0+\nu_1\eta_0}$ is a singular metric of L_0 such that $-\varphi_0 + \nu_1\eta_0$ does not vanish identically on Y_1, and $h_0|_{Y_1}$ is a singular metric of $L_0|_{Y_1}$ with algebraic singularities whose curvature current is $T_1 \geq \epsilon\omega|_{Y_1}$.

Secondly, there is a canonical singular metric $e^{-\eta_1}$ of $\mathcal{O}_{Y_1}(Y_2)$ on Y_1 with the curvature current $[Y_2]$. Thus the singular metric $h_1 := h_0|_{Y_1} + e^{\nu_2\eta_1}$ of the big line bundle L_1 gives a curvature current $T_1 - \nu_2[Y_2] \geq \epsilon\omega|_{Y_1}$. We continue in this manner to define the remaining singular metrics $h_i := h_{i-1}|_{Y_i} + e^{\nu_{i+1}\eta_i}$ of the big line bundle L_i on Y_i with curvature current $T_i - \nu_{i+1}[Y_{i+1}] \geq \epsilon\omega|_{Y_i}$ for $i = 0, \ldots, n-1$. It is easy to see that $h_i|_{Y_{i+1}}$ is well-defined.

By Lemma 4.5, there exists a k_0 such that for each $k \geq k_0$, the following short sequence is exact

$$H^0(Y_{i-1}, \mathcal{O}_{Y_{i-1}}(kL_{i-1}) \otimes \mathcal{I}(h_{i-1}^k)) \longrightarrow H^0(Y_i, \mathcal{O}_{Y_i}(kL_{i-1}) \otimes \mathcal{I}(h_{i-1}^k|_{Y_i})) \longrightarrow 0 \quad (4.1)$$

for $i = 1, \ldots, n-1$.

Now we begin our construction. T_{n-1} is the curvature current of the singular metric $h_{n-2}|_{Y_{n-1}}$ of $L_{n-2}|_{Y_{n-1}}$ over the Riemann surface Y_{n-1}. By Lemma 4.7, there exists a

$k \geq k_0$ and a holomorphic section $s_{n-1} \in H^0(Y_{n-1}, \mathcal{O}_{Y_{n-1}}(kL_{n-2}) \otimes \mathcal{I}(h^k_{n-2}|_{Y_{n-1}}))$, such that $\mathrm{ord}_p(s_{n-1}) = k\nu(T_{n-1}, p) = k\nu_n$.

By the exact sequence (4.1), s_{n-1} could be extend to

$$\widetilde{s}_{n-2} \in H^0(Y_{n-2}, \mathcal{O}_{Y_{n-2}}(kL_{n-2}) \otimes \mathcal{I}(h^k_{n-2})).$$

Now we choose a canonical section t_{n-2} of $H^0(Y_{n-2}, \mathcal{O}_{Y_{n-2}}(Y_{n-1}))$ such that the divisor of t_{n-2} is Y_{n-1}. We define $s_{n-2} := \widetilde{s}_{n-2} t_{n-2}^{\otimes \nu_{n-1}}$, by the construction of $h_{n-2} := h_{n-3}|_{Y_{n-2}} + e^{\nu_{n-1}\eta_{n-2}}$, we obtain that

$$s_{n-2} \in H^0(Y_{n-2}, \mathcal{O}_{Y_{n-2}}(kL_{n-3}) \otimes \mathcal{I}(h^k_{n-3}|_{Y_{n-2}}).$$

We can continue in this manner to construct a section $s_0 \in H^0(X, \mathcal{O}_X(kL))$ and by our construction we have

$$\mu(s_0) = (k\nu_1, \ldots, k\nu_n) = k\nu(T),$$

this concludes the theorem.

□

Proposition 4.9. *For any big line bundle L and any admissible flag $Y_\bullet$, one has $\overline{\Delta_{\mathbb{Q}}(c_1(L))} = \Delta(L)$. In particular,*

$$\Delta(L) = \overline{\bigcup_{m=1}^{\infty} \frac{1}{m}\nu(mL)}.$$

Proof. Firstly, since $\Delta_{\mathbb{Q}}(c_1(L))$ is a convex set in Q^n, its closure $\overline{\Delta_{\mathbb{Q}}(c_1(L))}$ is also a closed convex set in $\mathbb{R}^n$. By Proposition 4.8, we have

$$\Delta_{\mathbb{Q}}(c_1(L)) \subset \bigcup_{m=1}^{\infty} \frac{1}{m} \cdot \nu(mL),$$

thus

$$\overline{\Delta_{\mathbb{Q}}(c_1(L))} \subseteq \Delta(L).$$

By Remark 4.1, we have $\bigcup_{m=1}^{\infty} \frac{1}{m}\nu(mL) \subseteq \Delta_{\mathbb{R}}(c_1(L))$, thus by the definition of Okounkov body $\Delta(L)$, we deduce that

$$\Delta(L) \subseteq \overline{\Delta_{\mathbb{R}}(c_1(L))}.$$

By Lemma 4.4, we have $\overline{\Delta_{\mathbb{Q}}(c_1(L))} = \overline{\Delta_{\mathbb{R}}(c_1(L))}$, thus the theorem is proved.

□

Remark 4.10. By Proposition 4.9, in the definition of the Okounkov body $\Delta(L)$, it suffices to close up the set of normalized valuation vectors instead of the closure of the convex hull of this set.

Remark 4.11. It is easy to reprove that the Okounkov body $\Delta(L)$ depends only on the numerical equivalence class of the big line bundle L. Indeed, if L_1 and L_2 are numerically equivalent, we have $c_1(L_1) = c_1(L_2)$ thus

$$\Delta_{\mathbb{Q}}(c_1(L_1)) = \Delta_{\mathbb{Q}}(c_1(L_2)).$$

By Proposition 4.9, we have

$$\Delta(L_1) = \Delta(L_2).$$

Now we are ready to find some valuative points in the Okounkov bodies.

Proof of Corollary 1.2. In [LM09] we know that $\mathrm{vol}_{\mathbb{R}^n}(\Delta(L)) = \mathrm{vol}_X(L) > 0$ by the bigness of L. Since we have $\Delta(L) = \overline{\Delta_{\mathbb{Q}}(c_1(L))}$ by Proposition 4.9, then for any $p \in \mathrm{int}(\Delta(L)) \cap \mathbb{Q}^n$, there exists an n-simplex Δ_n containing p with all the vertices lying in $\Delta_{\mathbb{Q}}(c_1(L))$. Since $\Delta_{\mathbb{Q}}(c_1(L))$ is a convex set in $\mathbb{Q}^n$, we have $\Delta_n \cap \mathbb{Q}^n \subseteq \Delta_{\mathbb{Q}}(c_1(L))$, and thus

$$\Delta_{\mathbb{Q}}(c_1(L)) \supseteq \mathrm{int}(\Delta(L)) \cap \mathbb{Q}^n.$$

From Theorem 4.8 we have $\Delta_{\mathbb{Q}}(c_1(L)) \subseteq \bigcup_{m=1}^{\infty} \frac{1}{m}\mu(mL)$, thus we get the inclusion

$$\mathrm{int}(\Delta(L)) \cap \mathbb{Q}^n \subseteq \bigcup_{m=1}^{\infty} \frac{1}{m}\mu(mL),$$

which means that all rational interior points of $\Delta(L)$ are valuative. □

Pursuing the same philosophy as in Proposition 4.9, it is natual to extend results related to Okounkov bodies for big line bundles, to the more general case of an arbitrary big class $\alpha \in H^{1,1}(X, \mathbb{R})$. We propose the following definition.

Definition 4.12 (Generalized Okounkov body)**.** Let X be a Kähler manifold of dimension n. We define the *generalized Okounkov body* of a big class $\alpha \in H^{1,1}(X, \mathbb{R})$ with respect to the fixed flag $Y_\bullet$ by

$$\Delta(\alpha) = \overline{\Delta_{\mathbb{R}}(\alpha)} = \overline{\Delta_{\mathbb{Q}}(\alpha)}.$$

We have the following properties for generalized Okounkov bodies:

Proposition 4.13. *Let α and β be big classes, ω be any Kähler class. Then:*

(i) $\Delta(\alpha) + \Delta(\beta) \subseteq \Delta(\alpha + \beta)$.

(ii) $\mathrm{vol}_{\mathbb{R}^n}(\Delta(\omega)) > 0$.

(iii) $\Delta(\alpha) = \bigcap_{\epsilon>0} \Delta(\alpha + \epsilon\omega)$.

Proof. (i) is obvious from the definition of generalized Okounkov body. To prove (ii), we use induction for dimension. The result is obvious if $n = 1$, assume now that (ii) is true for $n-1$. We choose $t > 0$ small enough such that $\omega - tY_1$ is still a Kähler class. By the main theorem of [CT14], any Kähler current $T \in (\omega - tY_1)|_{Y_1}$ with analytic singularities can be extended to a Kähler current $\widetilde{T} \in \omega - tY_1$, thus we have

$$\Delta(\omega) \bigcap t \times \mathbb{R}^{n-1} = t \times \Delta((\omega - tY_1)|_{Y_1}),$$

where $\Delta((\omega - tY_1)|_{Y_1})$ is the generalized Okounkov body of $(\omega - tY_1)|_{Y_1}$ with respect to the flag

$$Y_1 \supset Y_2 \supset \ldots \supset Y_n = \{p\}.$$

By the induction, we have $\mathrm{vol}_{\mathbb{R}^{n-1}}(\Delta((\omega - tY_1)|_{Y_1})) > 0$. Since $\Delta(\omega)$ contains the origin, we have $\mathrm{vol}_{\mathbb{R}^n}(\Delta(\omega)) > 0$.

Now we are ready to prove (iii). By the concavity we have

$$\Delta(\alpha + \epsilon_1\omega) + \Delta((\epsilon_2 - \epsilon_1)\omega) \subseteq \Delta(\alpha + \epsilon_2\omega)$$

if $0 \leq \epsilon_1 < \epsilon_2$. Since $\Delta(\omega)$ contains the origin, we have

$$\Delta(\alpha) \subseteq \bigcap_{\epsilon > 0} \Delta(\alpha + \epsilon\omega),$$

and

$$\Delta(\alpha + \epsilon_1\omega) \subseteq \Delta(\alpha + \epsilon_2\omega).$$

From the concavity property, we conclude that $\mathrm{vol}_{\mathbb{R}^n}(\Delta(\alpha + t\omega))$ is a concave function for $t \geq 0$, thus continuous. Then we have

$$\mathrm{vol}_{\mathbb{R}^n}(\bigcap_{\epsilon > 0} \Delta(\alpha + \epsilon\omega)) = \mathrm{vol}_{\mathbb{R}^n}(\Delta(\alpha)) > 0.$$

Since they are all closed and convex, we have

$$\Delta(\alpha) = \bigcap_{\epsilon > 0} \Delta(\alpha + \epsilon\omega).$$

□

Remark 4.14. We don't know whether $\mathrm{vol}_{\mathbb{R}^n}(\Delta(\alpha))$ is independent of the choice of the admissible flag. However, in the next subsection we will prove that in the case of surfaces we have

$$\mathrm{vol}_X(\alpha) = 2\,\mathrm{vol}_{\mathbb{R}^2}(\Delta((\alpha)),$$

in particular the Euclidean volume of the generalized Okounkov body is independent of the choice of the flag. We conjecture that

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta(\alpha)) = \frac{1}{n!} \cdot \mathrm{vol}_X(\alpha),$$

as we proposed in the introduction.

4.2 Generalized Okounkov bodies on complex surfaces

Now we will mainly focus on generalized Okounkov bodies of compact Kähler surfaces. In this section, X denotes a compact Kähler surface. We fix henceforth an admissible flag

$$X \supseteq C \supseteq \{x\},$$

on X, where $C \subset X$ is an irreducible curve and $x \in C$ is a smooth point.

Definition 4.15. For any big class $\alpha \in H^{1,1}(X, \mathbb{R})$, we denote the *restricted* $\mathbb{R}$*-convex body* of α along C by $\Delta_{\mathbb{R},X|C}(\alpha)$, which is defined to be the set of Lelong numbers $\nu(T|_C, x)$, where $T \in \alpha$ ranges among all the positive currents with analytic singularities such that $C \not\subseteq E_+(T)$. The *restricted Okounkov body* of α along C is defined as

$$\Delta_{X|C}(\alpha) := \overline{\Delta_{\mathbb{R},X|C}(\alpha)}.$$

When $\alpha = c_1(L)$ for some big line bunle L on X, it is noticeable that $\Delta_{X|C}(\alpha) = \Delta_{X|C}(L)$, where $\Delta_{X|C}(L)$ is defined in [LM09]. When L is ample, we have $\Delta_{X|C}(L) = \Delta(L|_C)$. Indeed, it is suffice to show that for any section $s \in H^0(C, \mathcal{O}_C(L))$, there exists an integer m such that $s^{\otimes m}$ can be extended to a section $S_m \in H^0(X, \mathcal{O}_X(mL))$. This can be garanteed by Kodaira vanishing theorem. When α is any ample class, there is a very similar theorem which has appeared in the proof of Proposition 4.13. However, the proof there relies on the difficult extension theorem in [CT14]. Here we give a simple and direct proof when X is a complex surface. Anyway, the idea of proof here is borrowed from [CT14].

Proposition 4.16. *If α is an ample class, then we have*

$$\Delta_{X|C}(\alpha) = \Delta(\alpha|_C) = [0, \alpha \cdot C].$$

Proof. From Definition 4.15, we have $\Delta_{X|C}(\alpha) \subseteq \Delta(\alpha|_C)$. It suffices to prove that for any Kähler current $T \in \alpha|_C$ with mild analytic singularities, we have a positive current $\widetilde{T} \in \alpha$ with analytic singularites such that $\widetilde{T}|_C = T$. First we choose a Kähler form $\omega \in \alpha$. By assumption, we can write $T = \omega|_V + dd^c\varphi$ for some quasi-plurisubharmonic function φ on C which has mild analytic singularities. Our goal is to extend φ to a function Φ on X such that $\omega + dd^c\Phi$ is a Kähler current with analytic singularities.

Choose $\epsilon > 0$ small enough so that

$$T = \omega|_C + idd^c\varphi \geq 3\epsilon\omega,$$

holds as currents on C. We can cover C by finitely many charts $\{W_j\}_{1 \leq j \leq N}$ satisfying the following properties:

(i) On each $W_j (j \leq k)$ there are local coordinates $(z_1^{(j)}, z_2^{(j)})$ such that $C \bigcap W_j = \{z_2^{(j)} = 0\}$ and
$$\varphi = \frac{c_j}{2} \log |z_1^{(j)}|^2 + g_j(z_1^{(j)})$$
where $g_j(z_1^{(j)})$ is smooth and bounded on $W_j \bigcap C$. We denote the single pole of T in $W_j (j \leq k)$ by x_j;

(ii) On each $W_j (j > k)$ the local potential φ is smooth and bounded on $W_j \bigcap C$;

(iii) $x_i \notin \overline{W_j}$ for $i = 1, \ldots, k$ and $j \neq i$.

Define a function φ_j on W_j (with analytic singularities) by
$$\varphi_j(z_1^{(j)}, z_2^{(j)}) = \begin{cases} \varphi(z_1^{(j)}) + A|z_2^{(j)}|^2 & \text{if} \quad j > k, \\ \frac{c_j}{2} \log(|z_1^{(j)}|^2 + |z_2^{(j)}|^2) + g_j(z_1^{(j)}) + A|z_2^{(j)}|^2 & \text{if} \quad j \leq k, \end{cases}$$
where $A > 0$ is a constant. If we shrink the W_j's slightly, still preserving the property that $C \subseteq \bigcup W_j$, we can choose A sufficiently large so that
$$\omega + dd^c \varphi_j \geq 2\epsilon\omega,$$
holds on W_j for all j. We also need to construct slightly smaller open sets $W_j' \subset\subset U_j \subset\subset W_j$ such that $\bigcup W_j'$ is still a covering of C.

By construction φ_j is smooth when $j > k$, and φ_j is smooth outside the log pole x_j when $j \leq k$. By property (iii) above, we can glue the functions φ_j together to produce a Kähler current
$$\widetilde{T} = \omega|_U + dd^c \widetilde{\varphi} \geq \epsilon\omega$$
defined in a neighborhood U of C in X, thanks to Richberg's gluing procedure. Indeed, φ_i is smooth on $W_i \bigcap W_j$ for any $j \neq i$, which is a sufficient condition in using the Richberg technique. From the construction of $\widetilde{\varphi}$, we know that $\widetilde{\varphi}|_C = \varphi$, $\widetilde{\varphi}$ has log poles in every x_i and is continuous outside $x_1, \ldots, x_k$.

On the other hand, since α is an ample class, there exists a rational number $\delta > 0$ such that $\alpha - \delta\{C\}$ is still ample, thus we have a Kähler form $\omega_1 \in \alpha - \delta\{C\}$. We can write $\omega_1 + \delta[C] = \omega + dd^c\phi$, where ϕ is smooth outside C, and for any point $x \in C$, we have
$$\phi = \frac{\delta}{2} \log |z_2|^2 + O(1),$$
where z_2 is the local equation of C.

Since ϕ is continuous outside C, we can choose a large constant $B > 0$ such that $\phi > \widetilde{\varphi} - B$ in a neighborhood of ∂U. Therefore we define

$$\Phi = \begin{cases} \max\{\widetilde{\varphi}, \phi + B\} & \text{on } U \\ \phi + B & \text{on } X - U, \end{cases}$$

which is well defined on the whole of X, and satisfies $\omega + dd^c\Phi \geq \epsilon'\omega$ for some $\epsilon' > 0$. Since $\phi = -\infty$ on C, while $\widetilde{\varphi}|_C = \varphi$, it follows that $\Phi|_C = \varphi$.

We claim that Φ also has analytic singularities. Since around x_j, we have

$$\widetilde{\varphi}(z_1, z_2) = \frac{c_j}{2}\log(|z_1|^2 + |z_2|^2) + O(1),$$

and

$$\phi(z_1, z_2) = \frac{\delta}{2}\log|z_2|^2 + O(1),$$

for some local coordinates (z_1, z_2) of x_j. Thus locally we have

$$\max\{\widetilde{\varphi}, \phi + A\} = \frac{1}{2}\log(|z_1|^{2c_j} + |z_2|^{2c_j} + |z_2|^{2\delta}) + O(1).$$

Since Φ is continuous outside $x_1, \ldots, x_k$, our claim is proved.

□

Lemma 4.17. *Let α be a big and nef class on X, then for any $\epsilon > 0$, there exsists a Kähler current $T_\epsilon \in \alpha$ with analytic singularities such that the Lelong number $\nu(T_\epsilon, x) < \epsilon$ for any point in X. Moreover, T_ϵ also satisfies*

$$E_+(T) = E_{nK}(\alpha).$$

Proof. Since α is big, there exists a Kähler current with analytic singularities such that $E_+(T_0) = E_{nK}(\alpha)$ and $T_0 > \omega$ for some Kähler form ω. Since α is also a nef class, for any $\delta > 0$, there exists a smooth form θ_δ such that $\theta_\delta \geq -\delta\omega$. Thus $T_\delta := \delta T_0 + (1-\delta)\theta_\delta \geq \delta^2\omega$ is a Kähler current with analytic singularities satisfying that

$$E_+(T_\delta) = E_+(T_0) = E_{nK}(\alpha),$$

and

$$\nu(T_\delta, x) = \delta\nu(T_0, x)$$

for $x \in X$. Since the Lelong number $\nu(T_0, x)$ is an upper continuous function (thus bounded from above), $\nu(T_\delta, x)$ converges uniformly to zero as δ tends to 0. The lemma is proved. □

Proposition 4.18. *Let α be a big and nef class, $C \not\subseteq E_{nK}(\alpha)$. Then we have*

$$\Delta_{X|C}(\alpha) = \Delta(\alpha|_C) = [0, \alpha \cdot C].$$

Proof. Asumme $E_{nK}(\alpha) = \bigcup_{i=1}^{r} C_i$, where each C_i is an irreducible curve. By Lemma 4.17, for any $\epsilon > 0$ there exists a Kähler current $T_\epsilon \in \alpha$ with analytic singularities such that

$$E_+(T_\epsilon) = E_{nK}(\alpha) = \text{Null}(\alpha) = \bigcup_{i=1}^{r} C_i$$

and $\nu(T_\epsilon, x) < \epsilon$ for all $x \in X$. Thus the Siu decomposition

$$T_\epsilon = R_\epsilon + \sum_{i=1}^{r} a_{i,\epsilon} C_i$$

satisfies $0 \leq a_{i,\epsilon} < \epsilon$, and R_ϵ is a Kähler current whose analytic singularities are isolated points. By Remark 2.5, the cohomology class $\{R_\epsilon\}$ is a Kähler class and converges to α as $\epsilon \to 0$. In particular, $|\{R_\epsilon\} \cdot C - \alpha \cdot C| < A\epsilon$, where A is a constant.

By Proposition 4.16, there exists a Kähler current $S_\epsilon \in \{R_\epsilon\}$ with analytic singularities such that $C \not\subseteq E_+(S_\epsilon)$ and $-\epsilon < \nu(S_\epsilon|_C, x) - \{R_\epsilon\} \cdot C < 0$. Thus $T'_\epsilon := S_\epsilon + \sum_{i=1}^{r} a_{i,\epsilon} C_i$ is a Kähler current in α with analytic singularities, and $-(1+A)\epsilon < \nu(T'_\epsilon|_C, x) - \alpha \cdot C$. Since α is big and nef, there exists a Kähler current P_ϵ in α with analytic singularities such that $\nu(P_\epsilon|_C, x) < \epsilon$. Therefore, by the definition of $\Delta_{X|C}(\alpha)$ and the convexity property we deduce that $[0, \alpha \cdot C] \subseteq \Delta_{X|C}(\alpha)$. On the other hand, $\Delta_{X|C}(\alpha) \subseteq \Delta(\alpha|_C) = [0, \alpha \cdot C]$ by definition. The proposition is proved.

□

Lemma 4.19. *Let α be a big class on X with divisorial Zariski decomposition $\alpha = Z(\alpha) + N(\alpha)$. Assume that $C \not\subseteq E_{nK}(Z(\alpha))$, so that $C \not\subseteq \text{Supp}(N(\alpha))$ by Theorem 2.14. Moreover, set*

$$f(\alpha) = \nu_x(N(\alpha)|_C), \quad g(\alpha) = \nu_x(N(\alpha)|_C) + Z(\alpha) \cdot C,$$

where $\nu_x(N(\alpha)|_C) = \nu(N(\alpha)|_C, x)$. Then the restricted Okounkov body of α along C is the interval

$$\Delta_{X|C}(\alpha) = [f(\alpha), g(\alpha)]$$

Proof. First, by Remark 2.8 we conclude that $T \mapsto T - N(\alpha)$ is a bijection between the positive currents in α and those in $Z(\alpha)$, thus we have

$$E_{nK}(\alpha) = E_{nK}(Z(\alpha)) \bigcup \text{supp}(N(\alpha)),$$

and

$$C \not\subseteq E_{nK}(Z(\alpha)) \iff C \not\subseteq E_{nK}(\alpha). \tag{4.2}$$

By the assumption of theorem, $N(\alpha)|_C$ is a well-defined positive current with analytic singularites on C. By the definition of $\Delta_{\mathbb{R},X|C}(\alpha)$, we have

$$\Delta_{\mathbb{R},X|C}(\alpha) = \Delta_{\mathbb{R},X|C}(Z(\alpha)) + \nu_x(N(\alpha)|_C).$$

We take the closure of the sets to get

$$\Delta_{X|C}(\alpha) = \Delta_{X|C}(Z(\alpha)) + \nu_x(N(\alpha)|_C).$$

Since α is big, thus $Z(\alpha)$ is big and nef, and by Proposition 4.18 we have $\Delta_{X|C}(Z(\alpha)) = [0, Z(\alpha)\cdot C]$. We have proved the lemma.

□

Definition 4.20. If α is big and β is pseudo-effective, then the slope of β with respect to α is defined as

$$s = s(\alpha, \beta) = \sup\{t > 0 \mid \alpha - t\beta \text{ is big}\}.$$

Remark 4.21. Since the big cone is open, we know that $\{t > 0 \mid \alpha > t\beta\}$ is an open set in $\mathbb{R}^+$. Thus $\alpha - s\beta$ belongs to the boundary of the big cone $\mathcal{E}$, and $\mathrm{vol}_X(\alpha - s\beta) = 0$.

Proof of Theorem 1.3. For $t \in [0, s)$, we put $\alpha_t = \alpha - t\{C\}$, and let $Z_t := Z(\alpha_t)$ and $N_t := N(\alpha_t)$ be the positive and negative part of the divisorial Zariski decomposition of α_t.

(i) First we assume C is nef. By Theorem 2.14, the prime divisors in $E_{nK}(Z(\alpha_t))$ form an exceptional family, thus $C \not\subseteq E_{nK}(Z(\alpha_t))$, thus $C \not\subseteq E_{nK}(\alpha_t)$ by (4.2). By Lemma 4.19 we have $\Delta_{X|C}(\alpha_t) = [\nu_x(N_t|_C), Z_t \cdot C + \nu_x(N_t|_C)]$.

By the definition of $\mathbb{R}$-convex body and restrict $\mathbb{R}$-convex body, we have

$$\Delta_{\mathbb{R}}(\alpha)\bigcap t \times \mathbb{R} = t \times \Delta_{\mathbb{R},X|C}(\alpha_t).$$

Thus

$$t \times \overline{\Delta_{\mathbb{R},X|C}(\alpha_t)} \subseteq \overline{\Delta_{\mathbb{R}}(\alpha)}\bigcap t \times \mathbb{R}.$$

However, since both $\Delta_{\mathbb{R},X}(\alpha)$ and $\Delta_{\mathbb{R},X|C}(\alpha_t)$ are closed convex sets in $\mathbb{R}^2$ and $\mathbb{R}$, we have

$$t \times \overline{\Delta_{\mathbb{R},X|C}(\alpha_t)} = \overline{\Delta_{\mathbb{R}}(\alpha)}\bigcap t \times \mathbb{R},$$

therefore

$$t \times \Delta_{X|C}(\alpha_t) = \Delta(\alpha)\bigcap t \times \mathbb{R}. \tag{4.3}$$

Let

$$f(t) = \nu_x(N_t|_C)\ ,\ g(t) = Z_t \cdot C + \nu_x(N_t|_C),$$

then $\Delta(\alpha)\bigcap[0, s) \times \mathbb{R}$ is the region bounded by the graphs of $f(t)$ and $g(t)$.

Now we prove the piecewise linear property of $f(t)$ and $g(t)$. By Lemma 3.1, we have $N_{t_1} \le N_{t_2}$ if $0 \le t_1 \le t_2 < s$, thus $f(t)$ is increasing. Since N_t is an exceptional divisor by Theorem 2.15, the number of the prime components of N_t is uniformly bounded by the Picard number $\rho(X)$. Thus we can denote $N_t = \sum_{i=1}^{r} a_i(t)N_i$, where $a_i(t) \ge 0$ is an increasing and continuous function. Moreover, there exsists $0 = t_0 < t_1 < \ldots < t_k = s$

such that the prime components of N_t are the same when t lies in the interval (t_i, t_{i+1}) for $i = 0, \ldots, k-1$, and the number of prime components of N_t will increase at every t_i for $i = 1, \ldots, k-1$. We write $s_i = \frac{t_{i-1}+t_i}{2}$ for $i = 1, \ldots, k$.

We denote the linear subspace of $H^{1,1}(X, \mathbb{R})$ spanned by the prime components of N_{s_i} by V_i, and let $V_i^{\perp}$ be the orthogonal space of V_i with respect to q. By the proof of Lemma 3.1, for $t \in (t_{i-1}, t_i)$ we have

$$Z_t = Z_{s_i} + (s_i - t)\{C\}_i^{\perp} \tag{4.4}$$

$$N_t = N_{s_i} + (t_i - t)C_i^{\|}, \tag{4.5}$$

where $\{C\}_i^{\perp}$ is the projection of $\{C\}$ to $V_i^{\perp}$, and $C_i^{\|}$ is a linear combination of the prime components of N_{s_i} satisfying that the cohomology class $\{C_i^{\|}\}$ is equal to the projection of $\{C\}$ to V_i. By Theorem 2.14, the prime components of N_{s_i} are independent, thus $C_i^{\|}$ is uniquely defined. The piecewise linearity property of $f(t)$ and $g(t)$ follows directly from (4.4) and (4.5), and thus $f(t)$ and $g(t)$ can be continuously extended to s. Therefore we conclude that $\Delta(\alpha)$ is the region bounded by the graphs of $f(t)$ and $g(t)$ for $t \in [0, s]$. Thus the vertices of $\Delta(\alpha)$ are contained in the set $\{(t_i, f(t_i)), (t_j, g(t_j)) \in \mathbb{R}^2 \mid i, j = 0, \ldots, k\}$. This means that a vertex of $\Delta(\alpha)$ may only occur for those $t \in [0, s]$, where a new curve appears in N_t. Since $r \leq \rho(X)$, the number of vertices is bounded by $2\rho(X)+2$. The fact that $f(t)$ is convex and $g(t)$ concave is a consequence of the convexity of $\Delta(\alpha)$.

By (4.3), we have

$$\begin{aligned} 2\operatorname{vol}_{\mathbb{R}^2}(\Delta(\alpha)) &= 2\int_0^s \operatorname{vol}_{\mathbb{R}}(\Delta_{X|C}(\alpha_t))\mathrm{d}t \\ &= 2\int_0^s Z_t \cdot C\mathrm{d}t \\ &= \operatorname{vol}_X(\alpha) - \operatorname{vol}_X(\alpha - sC) \\ &= \operatorname{vol}_X(\alpha). \end{aligned}$$

where the second equality follows by Proposition 4.18, the third one by Theorem 3.2 and the last one by Remark 4.21. We have proved the theorem under the assumption that C is nef.

(ii) Now we prove the theorem when C is not nef, i.e., $C^2 < 0$. Recall that $a := \sup\{t > 0 \mid C \subseteq E_{nK}(\alpha)\}$. By (4.2), if $C \subseteq E_{nK}(\alpha_t)$ for some $t \in [0, s)$, we have $C \subseteq E_{nK}(Z(\alpha_t))$. By the proof in Theorem 3.4 we have

$$\begin{aligned} Z(\alpha_s) \cdot C = 0, \\ Z(\alpha_s) = Z(\alpha_t), \end{aligned}$$

for $0 \leq s \leq t$. Thus we have

$$\{0 \leq t < s \mid C \not\subseteq E_{nK}(\alpha_t)\} = (a, s),$$

and $\Delta(\alpha)$ is contained in $[a, s] \times \mathbb{R}$. By Theorem 3.4 we also have

$$\begin{aligned} 2\,\mathrm{vol}_{\mathbb{R}^2}(\Delta(\alpha)) &= 2\int_a^s \mathrm{vol}_{\mathbb{R}}(\Delta_{X|C}(\alpha_t))\mathrm{d}t \\ &= 2\int_a^s Z_t \cdot C\mathrm{d}t \\ &= \mathrm{vol}_X(\alpha_a) - \mathrm{vol}_X(\alpha_s) \\ &= \mathrm{vol}_X(\alpha). \end{aligned}$$

Since the prime components of N_{t_1} is contained in that of N_{t_2} if $a < t_1 \leq t_2 < s$, using the same arguments above, we obtain the piecewise linear property of $f(t)$ and $g(t)$ which can also be extended to s. The theorem is proved completely. □

Remark 4.22. If X is a projective surface, by the main result in [BKS03], the cone of big divisors of X admits a locally finite decomposition into locally polyhedral subcones such that the support of the negative part in the Zariski decomposition is constant on each subcone. It is noticeable that if we only assume X to be Kähler, this decomposition still holds if we replace the cone of big divisors by the cone of big classes and use divisorial Zariski decomposition instead. This property ensures that the generalized Okounkov bodies should also be polygons.

4.3 Generalized Okounkov bodies for pseudo-effective classes

Throughout this subsection, X will stand for a Kähler surface if not specially mentioned. Our main goal in this subsection is to study the behavior of generalized Okounkov bodies on the boundary of the big cone.

Definition 4.23. Let X be any Kähler manifold, if $\alpha \in H^{1,1}(X, \mathbb{R})$ is any pseudo-effective class. We define the *generalized Okounkov body* $\Delta(\alpha)$ with respect to the fixed flag by

$$\Delta(\alpha) := \bigcap_{\epsilon>0} \Delta(\alpha + \epsilon\omega),$$

where ω is any Kähler class.

It is easy to check that our definition does not depend on the choice of ω, and if α is big, by Proposition 4.13, the definition is consistent with Definition 4.12. Now we recall the definition of numerical dimension for any real (1,1)-class.

Definition 4.24 (numerical dimension)**.** Let X be a compact Kähler manifold. For a class $\alpha \in H^{1,1}(X, \mathbb{R})$, the *numerical dimension* $n(\alpha)$ is defined to be $-\infty$ if α is not pseudo-effective, and

$$n(\alpha) = \max\{p \in \mathbb{N}, \langle \alpha^p \rangle \neq 0\},$$

if α is pseudo-effective.

We recall that the right-hand side of the equation above involves the *positive intersection product* $\langle \alpha^p \rangle \in H^{p,p}_{\geq 0}(X, \mathbb{R})$ defined in [BDPP13]. When X is a Kähler surface, we simply have

$$n(\alpha) = \max\{p \in \mathbb{N}, Z(\alpha)^p \neq 0\}, \;\; p \in \{0, 1, 2\}.$$

If $n(\alpha) = 2$, α is big and the situation is studied in the last subsection. Throughout this subsection, we assume $\alpha \in \partial\mathcal{E}$.

Lemma 4.25. *Let $\{N_1, \ldots, N_r\}$ be an exceptional family of prime divisors, ω be any Kähler class. Then there exists unique positive numbers $b_1, \ldots, b_r$ such that $\omega + \sum_{i=1}^r b_i N_i$ is big and nef satisfying* $\mathrm{Null}(\omega + \sum_{i=1}^r b_i N_i) = \bigcup_{i=1}^r N_i$.

Proof. If we set

$$\begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} = -S^{-1} \cdot \begin{pmatrix} \omega \cdot N_1 \\ \vdots \\ \omega \cdot N_r \end{pmatrix},$$

where S denotes the intersection matrix of $\{N_1, \ldots, N_r\}$, we have $(\omega + \sum_{i=1}^r b_i N_i) \cdot N_j = 0$ for $j = 1, \ldots, r$. By Lemma 3.5, we conclude that all b_i are positive and thus $\omega + \sum_{i=1}^r b_i N_i$ is big and nef. □

Proposition 4.26. *Let α be any pseudo-effective class with $N(\alpha) = \sum_{i=1}^r a_i N_i$, ω be a Kähler class. Then for $\epsilon > 0$ small enough, we have the divisorial Zariski decomposition*

$$Z(\alpha + \epsilon\omega) = Z(\alpha) + \epsilon(\omega + \sum_{i=1}^r b_i N_i),$$

$$N(\alpha + \epsilon\omega) = \sum_{i=1}^r (a_i - \epsilon b_i) N_i,$$

where b_i is the positive number defined in Lemma 4.25.

Proof. Since $Z(\alpha) + \epsilon(\omega + \sum_{i=1}^r b_i N_i)$ is nef and orthogonal to all N_i by Lemma 4.25, by Theorem 2.15, if ϵ satisfies that $a_i - \epsilon b_i > 0$ for all i, the divisorial decomposition in the proposition holds. □

If $n(\alpha) = 0$, we have $Z(\alpha) = 0$ and thus $\alpha = \sum_{i=1}^r a_i N_i$ is an exceptional effective $\mathbb{R}$-divisor. We fix a flag

$$X \supseteq C \supseteq \{x\},$$

where $C \neq N_i$ for all i. Then we have

Theorem 4.27. *For any pseudo-effective class α whose numerical dimension $n(\alpha) = 0$, we have*

$$\Delta_{(C,x)}(\alpha) = 0 \times \nu_x(N(\alpha)|_C).$$

Proof. We asumme $N(\alpha) = \sum_{i=1}^{r} a_i N_i$. Fix a Kähler class ω, by Proposition 4.26, we have

$$Z(\alpha + \epsilon\omega) = \epsilon(\omega + \sum_{i=1}^{r} b_i N_i), \tag{4.6}$$

$$N(\alpha + \epsilon\omega) = \sum_{i=1}^{r} (a_i - \epsilon b_i) N_i, \tag{4.7}$$

where b_i is the positive number defined in Lemma 4.25. Since $T \mapsto T - N(\alpha + \epsilon\omega)$ is a bijection between the positive currents in $\alpha + \epsilon\omega$ and those in $Z(\alpha + \epsilon\omega)$, we have

$$\Delta(\alpha + \epsilon\omega) = \epsilon\Delta(\omega + \sum_{i=1}^{r} b_i N_i) + \nu(\sum_{i=1}^{r} (a_i - \epsilon b_i) N_i),$$

where $\nu(\sum_{i=1}^{r} (a_i - \epsilon b_i) N_i) = \nu_{(C,x)}(\sum_{i=1}^{r} (a_i - \epsilon b_i) N_i)$ is the valuation-like function defined in Section 4.1. Thus the diameter of $\Delta(\alpha + \epsilon\omega)$ converges to 0 when ϵ tends to 0, and we conclude that $\Delta(\alpha)$ is a single point in $\mathbb{R}^2$. Since

$$\begin{aligned} \Delta(\alpha + \epsilon\omega) \bigcap 0 \times \mathbb{R} &= 0 \times \Delta_{X|C}(\alpha + \epsilon\omega) \\ &= 0 \times [\nu_x(N(\alpha + \epsilon\omega)|_C), \nu_x(N(\alpha + \epsilon\omega)|_C) + Z(\alpha + \epsilon\omega) \cdot C], \end{aligned}$$

by (4.6) and (4.7) we have

$$\Delta(\alpha) \bigcap 0 \times \mathbb{R} = 0 \times \nu_x(\sum_{i=1}^{r} a_i N_i|_C),$$

and we prove the first part of Theorem 1.8.. □

If $n(\alpha) = 1$, $Z(\alpha)$ is nef but not big. If there exists one irreducible curve C such that $Z(\alpha) \cdot C > 0$, we fix the flag

$$X \supseteq C \supseteq \{x\},$$

then we have

Theorem 4.28. *For any pseudo-effective class α whose numerical dimension $n(\alpha) = 1$, we have*

$$\Delta(\alpha) = 0 \times [\nu_x(N(\alpha)|_C), \nu_x(N(\alpha)|_C) + Z(\alpha) \cdot C].$$

Proof. By the assumption $Z(\alpha) \cdot C > 0$ we know that $C \nsubseteq \mathrm{Supp}(N(\alpha))$. By Proposition 4.26, when ϵ small enough, the divisorial Zariski decomposition for $\alpha + \epsilon\omega$ is

$$Z(\alpha + \epsilon\omega) = Z(\alpha) + \epsilon(\omega + \sum_{i=1}^{r} b_i N_i), \tag{4.8}$$

$$N(\alpha + \epsilon\omega) = \sum_{i=1}^{r} (a_i - \epsilon b_i) N_i, \tag{4.9}$$

where b_i is the positive number defined in Lemma 4.25. Combine (4.8) and (4.9), we have

$$\begin{aligned}\Delta(\alpha)\bigcap 0\times\mathbb{R} &= \bigcap_{\epsilon>0}\Delta(\alpha+\epsilon\omega)\bigcap 0\times\mathbb{R}\\ &= \bigcap_{\epsilon>0} 0\times[\nu_x(N(\alpha+\epsilon\omega)|_C),\nu_x(N(\alpha+\epsilon\omega)|_C)+Z(\alpha+\epsilon\omega)\cdot C]\\ &= 0\times[\nu_x(\sum_{i=1}^{r}a_iN_i|_C),\nu_x(\sum_{i=1}^{r}a_iN_i|_C)+Z(\alpha)\cdot C].\end{aligned}$$

Since we have

$$\mathrm{vol}_{\mathbb{R}^2}(\Delta(\alpha))=\lim_{\epsilon\to 0}\mathrm{vol}_{\mathbb{R}^2}(\Delta(\alpha+\epsilon\omega))=\lim_{\epsilon\to 0}Z(\alpha+\epsilon\omega)^2=0,$$

and $\Delta(\alpha)$ is a closed convex set, we conclude that there are no points of $\Delta(\alpha)$ which lie outside $0\times\mathbb{R}$ as $\mathrm{vol}_{\mathbb{R}}(\Delta(\alpha)\bigcap 0\times\mathbb{R})=Z(\alpha)\cdot C>0$. We finish the proof of Theorem 1.8. □

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