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Error estimates for finite difference schemes associated with Hamilton-Jacobi equations on a junction

Cyril Imbert*and Marwa Koumaiha†

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Abstract

This paper is concerned with monotone (time-explicit) finite difference schemes associated with first order Hamilton-Jacobi equations posed on a junction. They extend the schemes recently introduced by Costeseque, Lebacque and Monneau (2013) to general junction conditions. On one hand, we prove the convergence of the numerical solution towards the weak (viscosity) solution of the Hamilton-Jacobi equation as the mesh size tends to zero for general junction conditions. On the other hand, we derive error estimates of order $(\Delta x)^{\frac{1}{3}}$ in L_{loc}^{∞} for junction conditions of optimal-control type.

Keywords: Hamilton-Jacobi equations, junction conditions, viscosity solutions, flux-limited solutions, error estimates, vertex test function.

MSC: 65M06, 65M12, 49L25.

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*LAMA, UMR 8050, Univ. Paris-Est Créteil, 61 avenue du général de Gaulle, 94010 Créteil cedex, France, cyril.imbert@math.cnrs.fr

†LAMA, UMR 8050, Univ. Paris-Est Créteil, 61 avenue du général de Gaulle, 94010 Créteil cedex, France & Université Libanaise, École Doctorale des Sciences et de Technologie, Hadath, Beirut, Liban, marwa.koumaiha@math.cnrs.fr

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1 Introduction

This paper is concerned with numerical approximation of first order Hamilton-Jacobi equations posed on a junction, that is to say a network made of one node and a finite number of edges.

The theory of weak (viscosity) solutions for such equations on such domains has reached maturity by now [20, 21, 1, 18, 17]. In particular, it is now understood that general junction conditions reduce to special ones of optimal-control type [17]. Roughly speaking, it is proved in [17] that imposing a junction condition ensuring the existence of a continuous weak (viscosity) solution and a comparison principle is equivalent to imposing a junction condition obtained by “limiting the flux” at the junction point.

For the “minimal” *flux-limited* junction conditions, Costeseque, Lebacque and Monneau [10] introduced a monotone numerical scheme and proved its convergence. Their scheme can be naturally extended to general junction conditions and our first contribution is to introduce it and to prove its convergence.

Our second and main result is an error estimate à la Crandall-Lions [11] in the case of flux-limited junction conditions. It is explained in [11] that the proof of the comparison principle between sub- and super-solutions of the continuous Hamilton-Jacobi equation can be adapted in order to derive error estimates between the numerical solution associated with monotone (stable and consistent) schemes and the continuous solution. In the Euclidian case, the comparison principle is proved thanks to the technique of doubling variables; it relies on the classical penalisation term $\varepsilon^{-1}|x - y|^2$. Such a penalisation procedure is known to fail in general if the equation is posed in a junction; it is explained in [17] that it has to be replaced with a *vertex test function*.

In order to derive error estimates as in [11], it is important to study the regularity of the vertex test function. More precisely, we prove (Proposition 5.1) that it can be constructed in such a way that its gradient is locally Lipschitz continuous, at least if the flux is “strictly limited”. Such a regularity result is of independent interest.

1.1 Hamilton-Jacobi equations posed on junctions

A *junction* is a network made of one node and a finite number of infinite edges. It can be viewed as the set of N distinct copies ($N \geq 1$) of the half-line which are glued at the origin. For $\alpha = 1, \dots, N$, each branch J_α is assumed to be isometric to $[0, +\infty)$ and

$$J = \bigcup_{\alpha=1, \dots, N} J_\alpha \quad \text{with} \quad J_\alpha \cap J_\beta = \{0\} \quad \text{for} \quad \alpha \neq \beta$$

where the origin 0 is called the *junction point*. For points $x, y \in J$, $d(x, y)$ denotes the geodesic distance on J defined as

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \text{ belong to the same branch,} \\ |x| + |y| & \text{if } x, y \text{ belong to different branches.} \end{cases}$$

For a real-valued function u defined on J , $\partial_\alpha u(x)$ denotes the (spatial) derivative of u at $x \in J_\alpha \setminus \{0\}$ and the gradient of u is defined as follows,

$$u_x(x) := \begin{cases} \partial_\alpha u(x) & \text{if } x \in J_\alpha^* \\ (\partial_1 u(0), \dots, \partial_N u(0)) & \text{if } x = 0. \end{cases} \quad (1.1)$$

With such a notation in hand, we consider the following Hamilton-Jacobi equation posed on the junction J ,

$$\begin{cases} u_t + H_\alpha(u_x) = 0 & \text{in } (0, T) \times J_\alpha \setminus \{0\}, \\ u_t + F(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}) = 0 & \text{in } (0, T) \times \{0\}, \end{cases} \quad (1.2)$$

submitted to the initial condition

$$u(0, x) = u_0(x), \quad \text{for } x \in J \quad (1.3)$$

where u_0 is globally Lipschitz in J . The second equation in (1.2) is referred to as the *junction condition*. We consider the important case of Hamiltonians H_α satisfying the following conditions:

$$\begin{cases} \textbf{(Regularity)} & H_\alpha \text{ is Lipschitz continuous} \\ \textbf{(Coercivity)} & \lim_{|p| \rightarrow +\infty} H_\alpha(p) = +\infty \\ \textbf{(Quasi-convexity)} & \{H_\alpha \leq \lambda\} \text{ is convex for all } \lambda \in \mathbb{R}. \end{cases} \quad (1.4)$$

In particular, there exists $p_0^\alpha \in \mathbb{R}$ such that H_α is non-increasing in $(-\infty, p_0^\alpha]$ and non-decreasing in $[p_0^\alpha, +\infty)$, and we set

$$H_\alpha^-(p) = \begin{cases} H_\alpha(p) & \text{for } p \leq p_0^\alpha \\ H_\alpha(p_0^\alpha) & \text{for } p \geq p_0^\alpha \end{cases} \quad \text{and} \quad H_\alpha^+(p) = \begin{cases} H_\alpha(p_0^\alpha) & \text{for } p \leq p_0^\alpha, \\ H_\alpha(p) & \text{for } p \geq p_0^\alpha \end{cases}$$

where H_α^- is non-increasing and H_α^+ is non-decreasing.

We next introduce a one-parameter family of junction conditions: given a *flux limiter* $A \in \mathbb{R} \cup \{-\infty\}$, the A -limited flux junction function is defined for $p = (p_1, \dots, p_N)$ as,

$$F_A(p) = \max \left(A, \max_{\alpha=1, \dots, N} H_\alpha^-(p_\alpha) \right) \quad (1.5)$$

for some given $A \in \mathbb{R} \cup \{-\infty\}$ where H_α^- is non-increasing part of H_α .

We now consider the following important special case of (1.2),

$$\begin{cases} u_t + H_\alpha(u_x) = 0 & \text{in } (0, T) \times J_\alpha \setminus \{0\}, \\ u_t + F_A(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}) = 0 & \text{in } (0, T) \times \{0\}. \end{cases} \quad (1.6)$$

We point out that all the junction functions F_A associated with $A \in [-\infty, A_0]$ coincide if one chooses

$$A_0 = \max_{\alpha=1, \dots, N} \min_{\mathbb{R}} H_\alpha. \quad (1.7)$$

As far as general junction conditions are concerned, we assume that the junction function $F : \mathbb{R}^n \mapsto \mathbb{R}$ satisfies

$$\begin{cases} \textbf{(Regularity)} & F \text{ is piecewise } C^1(\mathbb{R}^n) \\ \textbf{(Monotonicity)} & \forall \alpha, p_\alpha \mapsto F(p_1, \dots, p_N) \text{ is strictly decreasing} \\ \textbf{(Coercivity)} & F(p_1, \dots, p_N) \rightarrow +\infty \text{ as } (p_1)_- + \dots + (p_N)_- \rightarrow +\infty. \end{cases} \quad (1.8)$$

1.2 Presentation of the scheme

The domain $(0, +\infty) \times J$ is discretized with respect to time and space. We choose a regular grid in order to simplify the presentation but it is clear that more general meshes could be used here.

The space step is denoted by Δx and the time step by Δt . If ε denotes $(\Delta t, \Delta x)$, the mesh (or grid) \mathcal{G}_ε is chosen as

$$\mathcal{G}_\varepsilon = \{n\Delta t : n \in \mathbb{N}\} \times J^{\Delta x}$$

where

$$J^{\Delta x} = \bigcup_{\alpha=1, \dots, N} J_\alpha^{\Delta x} \quad \text{with} \quad J_\alpha \supset J_\alpha^{\Delta x} \simeq \{i\Delta x : i \in \mathbb{N}\}.$$

It is convenient to write x_i^α for $i\Delta x \in J_\alpha$.

A numerical approximation u^ε of the solution u of the Hamilton-Jacobi equation is defined in \mathcal{G}_ε ; the quantity $u^\varepsilon(n\Delta t, x_i^\alpha)$ is simply denoted by $U_i^{\alpha, n}$. We want it to be an approximation of $u(n\Delta t, x_i^\alpha)$ for $n \in \mathbb{N}$, $i \in \mathbb{N}$, where α stands for the index of the branch.

We consider the following time-explicit scheme: for $n \geq 0$,

$$\begin{cases} \frac{U_i^{\alpha, n+1} - U_i^{\alpha, n}}{\Delta t} + \max\{H_\alpha^+(p_{i,+}^{\alpha, n}), H_\alpha^-(p_{i,+}^{\alpha, n})\} = 0, & i \geq 1, \quad \alpha = 1, \dots, N \\ U_0^{\beta, n} := U_0^\beta, & i = 0, \quad \beta = 1, \dots, N \\ \frac{U_0^{\alpha, n+1} - U_0^{\alpha, n}}{\Delta t} + F(p_{0,+}^{1, n}, \dots, p_{0,+}^{N, n}) = 0, \end{cases} \quad (1.9)$$

where $p_{i,\pm}^{\alpha, n}$ are the discrete (space) gradients defined by

$$p_{i,+}^{\alpha, n} := \frac{U_{i+1}^{\alpha, n} - U_i^{\alpha, n}}{\Delta x}, \quad p_{i,-}^{\alpha, n} := \frac{U_i^{\alpha, n} - U_{i-1}^{\alpha, n}}{\Delta x} \quad (1.10)$$

with the initial condition

$$U_i^{\alpha, 0} = u_0(x_i^\alpha), \quad i \geq 0, \quad \alpha = 1, \dots, N. \quad (1.11)$$

The following Courant-Friedrichs-Lewy (CFL) condition ensures that the explicit scheme is monotone,

$$\frac{\Delta x}{\Delta t} \geq \max \left\{ \max_{\substack{i \geq 0, \\ \alpha=1, \dots, N, \\ 0 \leq n \leq n_T}} |H'_\alpha(p_{i,+}^{\alpha, n})|; \max_{0 \leq n \leq n_T} \left\{ (-\nabla \cdot F)(p_{0,+}^{1, n}, \dots, p_{0,+}^{N, n}) \right\} \right\} \quad (1.12)$$

where the integer n_T is assumed to be defined as $n_T = \lfloor \frac{T}{\Delta t} \rfloor$ for a given $T > 0$.

1.3 Main results

As previously noticed in [10] in the special case $F = F_{A_0}$, it is not obvious that the CFL condition (1.12) can be satisfied; the reason is that, for α, i and n given, the discrete gradients $p_{i,+}^{\alpha, n}$ depend itself on Δt and Δx through the numerical scheme. For this reason we will consider the following more restrictive CFL condition that can be checked from initial data,

$$\frac{\Delta x}{\Delta t} \geq \max \left\{ \max_{\substack{\alpha=1, \dots, N, \\ \underline{p}_\alpha \leq p \leq \bar{p}_\alpha}} |H'_\alpha(p)|; \max_{\underline{p}_\alpha \leq p_\alpha \leq \bar{p}_\alpha} \left\{ (-\nabla \cdot F)(p_1, \dots, p_N) \right\} \right\} \quad (1.13)$$

for some $\underline{p}_\alpha, \bar{p}_\alpha, p_\alpha^0 \in \mathbb{R}$ to be fixed. We can argue as in [10] and prove that $\underline{p}_\alpha, \bar{p}_\alpha, p_\alpha^0 \in \mathbb{R}$ can be chosen such that the CFL condition (1.12) is satisfied and, in turn, the scheme is monotone (Lemma 4.1 in Section 4). We will also see that it is stable (Lemma 4.4) and consistent (Lemma 4.5). It is thus known that it converges [11, 4].

Theorem 1.1 (Convergence for general junction conditions). *Let $T > 0$ and u_0 be Lipschitz continuous. There exist $\underline{p}_\alpha, \bar{p}_\alpha, p_\alpha^0 \in \mathbb{R}$, $\alpha = 1, \dots, N$, depending only on the initial data, the Hamiltonians and the junction function F , such that, if ε satisfies the CFL condition (1.13), then the numerical solution u^ε defined by (1.9)-(1.11) converges locally uniformly as ε goes to zero to the unique weak (relaxed viscosity) solution u of (1.2)-(1.3), on any compact set $\mathcal{K} \subset [0, T] \times J$, i.e.*

$$\limsup_{|\varepsilon| \rightarrow 0} \sup_{(t, x) \in \mathcal{K} \cap \mathcal{G}_\varepsilon} |u^\varepsilon(t, x) - u(t, x)| = 0. \quad (1.14)$$

The main result of this paper lies in getting error estimates in the case of flux-limited junction conditions.

Theorem 1.2 (Error estimates for flux-limited junction conditions). *Let u_0 be Lipschitz continuous, u^ε be the solution of the associated numerical scheme (1.9)-(1.11) and u be the weak (viscosity) solution of (1.6)-(1.3) for some $A \in \mathbb{R}$. If the CFL condition (1.13) is satisfied, then there exists $C > 0$ such that*

$$\sup_{[0,T) \times J \cap \mathcal{G}_\varepsilon} |u^\varepsilon(t, x) - u(t, x)| \leq C(\Delta x)^{1/3}. \quad (1.15)$$

1.4 Related results

Numerical schemes for Hamilton-Jacobi equations on networks. The discretization of weak (viscosity) solutions of Hamilton-Jacobi equations posed on networks has been studied in a few papers only. Apart from [10] mentioned above, we are only aware of two other works. A convergent semi-Lagrangian scheme is introduced in [6] for equations of eikonal type. In [15], an adapted Lax-Friedrichs scheme is used to solve a traffic model; it is worth mentioning that this discretization implies to pass from the scalar conservation law to the associated Hamilton-Jacobi equation at each time step.

Link with monotone schemes for scalar conservation laws. We first follow [10] by emphasizing that the convergence result, Theorem 1.1, implies the convergence of a monotone scheme for scalar conservation laws (in the sense of distributions).

In order to introduce the scheme, it is useful to introduce a notation for the numerical Hamiltonian \mathcal{H}_α ,

$$\mathcal{H}_\alpha(p^+, p^-) = \max\{H_\alpha^-(p^+), H_\alpha^+(p^-)\}.$$

The discrete solution (V^n) of the scalar conservation law is defined as follows,

$$V_{i+\frac{1}{2}}^{\alpha,n} = \begin{cases} \frac{U_{i+1}^{\alpha,n} - U_i^{\alpha,n}}{\Delta x} & \text{if } i \geq 1 \\ \frac{U_1^{\alpha,n} - U_0^n}{\Delta x} & \text{if } i = 0. \end{cases}$$

In view of (1.9), it satisfies for all $\alpha = 1, \dots, N$,

$$\begin{cases} \frac{V_{i+\frac{1}{2}}^{\alpha,n+1} - V_{i+\frac{1}{2}}^{\alpha,n}}{\Delta t} + (\Delta x)^{-1} \left(\mathcal{H}_\alpha(V_{i+\frac{3}{2}}^{\alpha,n}, V_{i+\frac{1}{2}}^{\alpha,n}) - \mathcal{H}_\alpha(V_{i+\frac{1}{2}}^{\alpha,n}, V_{i-\frac{1}{2}}^{\alpha,n}) \right) = 0, & i \geq 1, \\ \frac{V_{\frac{1}{2}}^{\alpha,n+1} - V_{\frac{1}{2}}^{\alpha,n}}{\Delta t} + (\Delta x)^{-1} \left(\mathcal{H}_\alpha(V_{\frac{3}{2}}^{\alpha,n}, V_{\frac{1}{2}}^{\alpha,n}) - F(V_{\frac{1}{2}}^{1,n}, \dots, V_{\frac{1}{2}}^{N,n}) \right) = 0. \end{cases}$$

submitted to the initial condition

$$V_{i+\frac{1}{2}}^{\alpha,0} = \frac{u_0(x_i^\alpha) - u_0(0)}{\Delta x}, \quad i \geq 0, \quad \alpha = 1, \dots, N.$$

In view of Theorem 1.1, we thus can conclude that the discrete solution v^ε constructed from (V^n) converges towards u_x in the sense of distributions, at least far from the junction point.

Scalar conservation laws with Dirichlet boundary conditions and constrained fluxes.

We would like next to explain why our result can be seen as the Hamilton-Jacobi counterpart of the error estimates obtained by Ohlberger and Vovelle [19] for scalar conservation laws submitted to Dirichlet boundary conditions.

On one hand, it is known since [3] that Dirichlet boundary conditions imposed to scalar conservation laws should be understood in a generalized sense. This can be seen by studying the parabolic regularization of the problem. A boundary layer analysis can be performed for systems

if the solution of the conservation law is smooth; see for instance [14, 16]. Depending on the fact that the boundary is characteristic or not, the error is $\varepsilon^{\frac{1}{2}}$ or ε . In the scalar case, it is proved in [13] that the error between the solution of the regularized equation with a vanishing viscosity coefficient equal to ε and the entropy solution of the conservation law (which is merely of bounded variation in space) is of order $\varepsilon^{1/3}$ (in $L_t^\infty L_x^1$ norm). In [19], the authors derive error estimates for finite volume schemes associated with such boundary value problems and prove that it is of order $(\Delta x)^{1/6}$ (in $L_{t,x}^1$ norm). More recently, scalar conservation laws with flux constraints were studied [9, 8] and some finite volume schemes were built [2]. In [7], assuming that the flux is bell-shaped, that is to say the opposite is quasi-convex, it is proved that the error between the finite volume scheme and the entropy solution is of order $(\Delta x)^{\frac{1}{3}}$ and that it can be improved to $(\Delta x)^{\frac{1}{2}}$ under an additional condition on the traces of the BV entropy solution.

On the other hand, the derivative of a weak (viscosity) solution of a Hamilton-Jacobi equation posed on the real line is known to coincide with the entropy solution of the corresponding scalar conservation law. It is therefore reasonable to expect that the error between the viscosity solution of the Hamilton-Jacobi equation and its approximation is as good as the one obtained between the entropy solution of the scalar conservation law and its approximation.

Moreover, it is explained in [18] that the junction conditions of optimal-control type are related to the BLN condition mentioned above; such a correspondance is recalled in Appendix A. It is therefore interesting to get an error estimate of order $(\Delta x)^{1/3}$ for the Hamilton-Jacobi problem.

1.5 Open problems

Let us first mention that it is not known if the error estimate between the (entropy) solution of the scalar conservation law with Dirichlet boundary condition and the solution of the parabolic approximation [13] or with the numerical scheme [19] is optimal or not. Similarly, we do not know if our error estimate is optimal or not.

Deriving error estimates for general junction conditions seems difficult to us. The main difficulty is the singular geometry of the domain. The vertex test function, used in deducing the error estimates with flux limited solutions, is designed to compare flux limited solutions. Consequently, when applying the reasoning of Section 6, the discrete viscosity inequality cannot be combined with the continuous one. We expect that a layer develops between the continuous solution and the discrete scheme at the junction point.

Organization of the article. The remaining of the paper is organized as follows. In Section 2, we recall definitions and results from [17] about viscosity solutions for (1.2)-(1.3) and the so-called vertex test function. Section 3 is dedicated to the derivation of discrete gradient estimates for the numerical scheme. In Section 4, the convergence result, Theorem 1.1 is proved. In Section 5, it is proved that the vertex test function constructed in [17] can be chosen so that the gradient is locally Lipschitz continuous (at least if the flux is strictly limited). The final section, Section 6, is dedicated to the proof of the error estimates.

2 Preliminaries

2.1 Viscosity solutions

We introduce the main definitions related to viscosity solutions for Hamilton-Jacobi equations that are used in the remaining. For a more general introduction to viscosity solutions, the reader could refer to Barles [5] and to Crandall, Ishii, Lions [12].

In [17], the following assumption on F is imposed, which is weaker than (1.8) above (no coercivity is needed in the theory developed in [17]).

$$\begin{cases} \text{(Continuity)} & F \in C(\mathbb{R}^n) \\ \text{(Monotonicity)} & \forall \alpha, p_\alpha \mapsto F(p_1, \dots, p_N) \text{ is non-increasing.} \end{cases} \quad (2.1)$$

Space of test functions For a smooth real valued function u defined on J , we denote by u^α the restriction of u to $(0, T) \times J_\alpha$.

Then we define the natural space of functions on the junction:

$$C^1(J_T) = \{u \in C(J_T) : \forall \alpha = 1, \dots, N, u \in C^1((0, T) \times J_\alpha)\}.$$

Viscosity solutions In order to define classical viscosity solutions, we recall the definition of upper and lower semi-continuous envelopes u^* and u_* of a (locally bounded) function u defined on $[0, T) \times J$:

$$u^*(t, x) = \limsup_{(s, y) \rightarrow (t, x)} u(s, y) \quad u_*(t, x) = \liminf_{(s, y) \rightarrow (t, x)} u(s, y).$$

It is convenient to introduce the following shorthand notation

$$H(x, p) = \begin{cases} H_\alpha(p) & \text{for } p = p_\alpha & \text{if } x \in J_\alpha^*, \\ F(p) & \text{for } p = (p_1, \dots, p_N) & \text{if } x = 0. \end{cases} \quad (2.2)$$

Definition 1 (Viscosity solution). Assume that the Hamiltonians satisfy (1.4) and that F satisfies (2.1) and let $u: (0, T) \times J \rightarrow \mathbb{R}$.

(i) We say that u is a *sub-solution* (resp. *super-solution*) of (1.2) in $(0, T) \times J$ if for all test function $\varphi \in C^1(J_T)$ such that

$$u^* \leq \varphi \quad (\text{resp. } u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in J_T$$

with equality at (t_0, x_0) for some $t_0 > 0$, we have

$$\varphi_t + H(x, \varphi_x) \leq 0 \quad (\text{resp. } \geq 0) \quad \text{at } (t_0, x_0)$$

(ii) We say that u is a *sub-solution* (resp. *super-solution*) of (1.2)-(1.3) on $[0, T) \times J$ if additionally

$$u^*(0, x) \leq u_0(x) \quad (\text{resp. } u_*(0, x) \geq u_0(x)) \quad \text{for all } x \in J.$$

(iii) We say that u is a (viscosity) solution if u is both a sub-solution and a super-solution.

As explained in [17], it is difficult to construct viscosity solutions in the sense of Definition 1 because of the junction condition. It is possible in the case of the flux-limited junction conditions F_A . For general junction conditions, the Perron process generates a viscosity solution in the following relaxed sense [17].

Definition 2 (Relaxed viscosity solution). Assume that the Hamiltonians satisfy (1.4) and that F satisfies (2.1) and let $u: (0, T) \times J \rightarrow \mathbb{R}$.

(i) We say that u is a *relaxed sub-solution* (resp. *relaxed super-solution*) of (1.2) in $(0, T) \times J$ if for all test function $\varphi \in C^1(J_T)$ such that

$$u^* \leq \varphi \quad (\text{resp. } u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in J_T$$

with equality at (t_0, x_0) for some $t_0 > 0$, we have

$$\varphi_t + H_\alpha(\varphi_x) \leq 0 \quad (\text{resp. } \geq 0) \quad \text{at } (t_0, x_0)$$

if $x_0 \neq 0$ and,

$$\begin{cases} \text{either } \varphi_t + F(\varphi_x) \leq 0 & (\text{resp. } \geq 0) & \text{at } (t_0, x_0) \\ \text{or } \varphi_t + H_\alpha(\partial_\alpha \varphi) \leq 0 & (\text{resp. } \geq 0) & \text{at } (t_0, x_0) \quad \text{for some } \alpha. \end{cases}$$

(ii) We say that u is a *relaxed (viscosity) solution* of (1.2) if u is both a sub-solution and a super-solution.

Theorem 2.1 (Comparison principle on a junction). *Let $A \in \mathbb{R} \cup \{-\infty\}$. Assume that the Hamiltonians satisfy (1.4) and the initial datum u_0 is uniformly continuous. Then for all sub-solution u and super-solution v of (1.6)-(1.3) satisfying for some $T > 0$ and $C_T > 0$*

$$u(t, x) \leq C_T(1 + d(0, x)), \quad v(t, x) \geq -C_T(1 + d(0, x)), \quad \text{for all } (t, x) \in [0, T) \times J,$$

we have

$$u \leq v \quad \text{in } [0, T) \times J.$$

Theorem 2.2 (General junction conditions reduce to flux-limited ones). *Assume that the Hamiltonians satisfy (1.4) and that F satisfies (2.1). Then there exists $A_F \in \mathbb{R}$ such that any relaxed viscosity (sub-/super-)solution of (1.2) is in fact a viscosity (sub-/super-)solution of (1.6) with $A = A_F$.*

Theorem 2.3 (Existence and uniqueness on a junction). *Assume that the Hamiltonians satisfy (1.4) and that F satisfies (2.1) and that the initial datum u_0 is Lipschitz continuous. Then there exists a unique relaxed viscosity solution u of (1.2)-(1.3), such that*

$$|u(t, x) - u_0(x)| \leq Ct \quad \text{for all } (t, x) \in [0, T) \times J$$

for some constant C only depending on H and u_0 . Moreover, it is Lipschitz continuous with respect to time and space, in particular,

$$\|\nabla u\|_\infty \leq C.$$

2.2 Inverse functions of Hamiltonians and junction functions

In the proofs of discrete gradient estimates, as well as in the construction of the vertex test functions, “generalized” inverse functions of H_α^\pm are needed; they are defined as follows:

$$\begin{cases} \pi_\alpha^+(a) := \sup\{p : H_\alpha^+(p) = \max(a, A_\alpha)\} \\ \pi_\alpha^-(a) := \inf\{p : H_\alpha^-(p) = \max(a, A_\alpha)\} \end{cases} \quad (2.3)$$

with the additional convention that $(H_\alpha^\pm)^{-1}(+\infty) = \pm\infty$, where $A_\alpha := \min_{\mathbb{R}} H_\alpha$.

In order to define a “generalized” inverse function of F , we remark that (1.8) implies that

$$\text{for all } K \in \mathbb{R}, \text{ there exists } \underline{p}(K) \in \mathbb{R}^N \text{ such that } F(p_1, \dots, p_N) \leq K \Rightarrow p_\alpha \geq \underline{p}_\alpha(K).$$

Remark that the functions \underline{p}_α can be chosen non-increasing.

2.3 Vertex test function

In this subsection, we recall what is a *vertex test function*. It is introduced in [17] in order to prove a comparison principle for (1.2). This function G plays the role of $|x - y|^2$ in the classical “doubling variables” method [12].

Theorem 2.4 (Vertex test function – [17]). *Let $A \in \mathbb{R} \cup \{-\infty\}$ and $\gamma > 0$. Assume the Hamiltonians satisfy (1.4) and $p_0^\alpha = 0$, that is to say,*

$$\min H_\alpha = H_\alpha(0). \quad (2.4)$$

Then there exists a function $G : J^2 \rightarrow \mathbb{R}$ enjoying the following properties.

(i) (Regularity)

$$G \in C(J^2) \quad \text{and} \quad \begin{cases} G(x, \cdot) \in C^1(J) & \text{for all } x \in J, \\ G(\cdot, y) \in C^1(J) & \text{for all } y \in J. \end{cases}$$

(ii) (Bound from below) $G \geq 0 = G(0, 0)$.

(iii) (Compatibility condition on the diagonal) For all $x \in J$

$$0 \leq G(x, x) \leq \gamma.$$

(iv) (Compatibility condition on the gradients) For all $(x, y) \in J^2$

$$H(y, -G_y(x, y)) - H(x, G_x(x, y)) \leq \gamma.$$

(v) (Superlinearity) There exists $g: [0, +\infty) \rightarrow \mathbb{R}$ non-decreasing and s.t. for $(x, y) \in J^2$

$$g(d(x, y)) \leq G(x, y) \quad \text{and} \quad \lim_{a \rightarrow +\infty} \frac{g(a)}{a} = +\infty.$$

(vi) (Gradient bounds) For all $K \geq 0$, there exists $C_K > 0$ such that for all $(x, y) \in J^2$,

$$d(x, y) \leq K \quad \Rightarrow \quad |G_x(x, y)| + |G_y(x, y)| \leq C_K.$$

Remark 1. Following [17], the vertex test function G is obtained as a regularized version of $A + G^0$ where G^0 is defined as follows: for $(x, y) \in J_\alpha \times J_\beta$:

$$G^0 := \sup_{(p, \lambda) \in \mathcal{G}(A)} (p_\alpha x - p_\beta y - \lambda) \quad (2.5)$$

where $\mathcal{G}(A)$ is referred to as the germ and is defined as follows

$$\mathcal{G}(A) = \begin{cases} \{(p, \lambda) \in \mathbb{R}^N \times \mathbb{R}, H_\alpha(p_\alpha) = F_A(p) = \lambda \text{ for } \alpha = 1, \dots, N\} & \text{if } N \geq 2, \\ \{(p_1, \lambda) \in \mathbb{R} \times \mathbb{R}, H_1(p_1) = \lambda \geq A\} & \text{if } N = 1. \end{cases}$$

3 Gradient estimates for the scheme

This section is devoted to the proofs of the discrete (time and space) gradient estimates. These estimates ensure the monotonicity of the scheme and, in turn, its convergence.

Theorem 3.1 (Discrete gradient estimates). *If $u^\varepsilon = (U_i^{\alpha, n})$ is the numerical solution of (1.9)-(1.11) and if the CFL condition (1.13) is satisfied with m^0 finite, then the following two properties hold true for any $n \geq 0$.*

(i) (Gradient estimate) There exist $\underline{p}_\alpha, \bar{p}^\alpha, \underline{p}_\alpha^0$ (only depending on H_α, u_0 and F) such that

$$\begin{cases} \underline{p}_\alpha \leq p_{i,+}^{\alpha, n} \leq \bar{p}^\alpha & i \geq 1, \alpha = 1, \dots, N, \\ \underline{p}_\alpha^0 \leq p_{0,+}^{\alpha, n} \leq \bar{p}^\alpha & i = 0, \alpha = 1, \dots, N. \end{cases} \quad (3.1)$$

(ii) (Time derivative estimate) The discrete time derivative defined as

$$W_i^{\alpha, n} := \frac{U_i^{\alpha, n+1} - U_i^{\alpha, n}}{\Delta t}$$

satisfies

$$m^0 \leq m^n \leq m^{n+1} \leq M^{n+1} \leq M^n \leq M^0$$

where

$$m^n := \inf_{\alpha, i} W_i^{\alpha, n}, \quad M^n := \sup_{\alpha, i} W_i^{\alpha, n}.$$

Remark 2. The quantities $\underline{p}_\alpha, \bar{p}_\alpha, p_\alpha^0$ are defined as follows

$$\begin{cases} \underline{p}_\alpha = \begin{cases} \pi_\alpha^-(-m^0) & \text{if } -m^0 > A_\alpha \\ \pi_\alpha^-(-m^0 + 1) & \text{if } -m^0 = A_\alpha \end{cases} \\ \bar{p}_\alpha = \begin{cases} \pi_\alpha^+(-m^0) & \text{if } -m^0 > A_\alpha \\ \pi_\alpha^+(-m^0 + 1) & \text{if } -m^0 = A_\alpha \end{cases} \\ p_\alpha^0 = \begin{cases} \underline{p}(-m^0) & \text{if } \underline{p}(-m^0) < \bar{p}_\alpha \\ \underline{p}(-m^0 + 1) & \text{if } \underline{p}(-m^0) = \bar{p}_\alpha \end{cases} \end{cases} \quad (3.2)$$

where π_α^\pm and \underline{p} are the ‘‘generalized’’ inverse functions of H_α and F respectively, and where

$$m^0 = \inf_{\substack{\beta=1, \dots, N, \\ i \in \mathbb{N}}} W_i^{\beta, 0}. \quad (3.3)$$

In order to establish Theorem 3.1, we first prove two auxiliary results. In order to state them, some notation should be introduced.

3.1 Discrete time derivative estimates

In order to state the first one, Proposition 3.2 below, we introduce some notation. For $\sigma \in \{+, -\}$, we set

$$I_{i, \sigma}^{\alpha, n} := [\min(p_{i, \sigma}^{\alpha, n}, p_{i, \sigma}^{\alpha, n+1}), \max(p_{i, \sigma}^{\alpha, n}, p_{i, \sigma}^{\alpha, n+1})]$$

with $p_{i, \sigma}^{\alpha, n}$ defined in (1.10) and

$$D_{i, +}^{\alpha, n} := \sup \left\{ \sup_{p_\alpha \in I_{i, +}^{\alpha, n}} |H'_\alpha(p_\alpha)|, \sup_{p_\alpha \in I_{0, +}^{\alpha, n}} \left\{ -(\nabla \cdot F)(p_1, \dots, p_N) \right\} \right\}. \quad (3.4)$$

The following proposition asserts that if the discrete (space) gradients enjoy suitable estimates, then the discrete time derivative is controlled.

Proposition 3.2 (Discrete time derivative estimate). *Let $n \geq 0$ be fixed and $\Delta x, \Delta t > 0$. Let us consider $(U_{i, \alpha}^{\alpha, n})_{\alpha, i}$ satisfying for some constant $C^n > 0$:*

$$|p_{i, +}^{\alpha, n}| \leq C^n \quad \text{for } i \geq 0, \alpha = 1, \dots, N.$$

We also consider $(U_i^{\alpha, n+1})_{\alpha, i}$ and $(U_i^{\alpha, n+2})_{\alpha, i}$ computed using the scheme (1.9). If

$$D_{i, +}^{\alpha, n} \leq \frac{\Delta x}{\Delta t} \quad \text{for } i \geq 0, \alpha = 1, \dots, N, \quad (3.5)$$

then

$$m^n \leq m^{n+1} \leq M^{n+1} \leq M^n.$$

Proof. We introduce for $n \geq 0, \alpha = 1, \dots, N$,

$$\begin{aligned} \forall i \geq 1, \sigma \in \{+, -\}, \quad C_{i, \sigma}^{\alpha, n} &:= -\sigma \int_0^1 (H_\alpha^{-\sigma})' \left(p_{i, \sigma}^{\alpha, n+1} + \tau(p_{i, \sigma}^{\alpha, n} - p_{i, \sigma}^{\alpha, n+1}) \right) d\tau \geq 0, \\ C_{0, +}^{\beta, n} &= -\int_0^1 \frac{\partial F}{\partial p_\beta} \left(\{p_{0, +}^{\alpha, n+1} + \tau(p_{0, +}^{\alpha, n} - p_{0, +}^{\alpha, n+1})\}_\alpha \right) d\tau \geq 0. \end{aligned} \quad (3.6)$$

Notice that for $i \geq 1, C_{i, \sigma}^{\alpha, n}$ is defined as the integral of $(H_\alpha^{-\sigma})'$ over a convex combination of $p \in I_{i, \sigma}^{\alpha, n}$. Similarly for $C_{0, +}^{\alpha, n}$ which is defined as the integral of F' on a convex combination of

$p \in I_{0,+}^{\alpha,n}$. Hence, in view of (3.5), we have for any $n \geq 0$, $\alpha = 1, \dots, N$ and for any $\sigma \in \{+, -\}$ or for $i = 0$ and $\sigma = +$, we can check that

$$\begin{cases} C_{i,\sigma}^{\alpha,n} \leq \frac{\Delta x}{\Delta t} & \text{if } i \geq 1, \sigma \in \{-, +\} \\ \sum_{\beta=1}^N C_{0,+}^{\beta,n} \leq \frac{\Delta x}{\Delta t}. \end{cases} \quad (3.7)$$

We can also underline that for any $n \geq 0$, $\alpha = 1, \dots, N$ and for any $i \geq 1$, $\sigma \in \{+, -\}$ or for $i = 0$ and $\sigma = +$, we have the following relationship:

$$\frac{p_{i,\sigma}^{\alpha,n} - p_{i,\sigma}^{\alpha,n+1}}{\Delta t} = -\sigma \frac{W_{i+\sigma}^{\alpha,n} - W_i^{\alpha,n}}{\Delta x}. \quad (3.8)$$

Let $n \geq 0$ be fixed and consider $(U_i^{\alpha,n})_{\alpha,i}$ with $\Delta x, \Delta t > 0$ given. We compute $(U_i^{\alpha,n+1})_{\alpha,i}$ and $(U_i^{\alpha,n+2})_{\alpha,i}$ using the scheme (1.9).

Step 1: $(m^n)_n$ is non-decreasing. We want to show that $W_i^{\alpha,n+1} \geq m^n$ for $i \geq 0$ and $\alpha = 1, \dots, N$. Let $i \geq 0$ be fixed and let us distinguish two cases.

Case 1: $i \geq 1$. Let a branch α be fixed and let $\sigma \in \{+, -\}$ be such that

$$\max \left\{ H_{\alpha}^{+}(p_{i,-}^{\alpha,n+1}), H_{\alpha}^{-}(p_{i,+}^{\alpha,n+1}) \right\} = H_{\alpha}^{-\sigma}(p_{i,\sigma}^{\alpha,n+1}). \quad (3.9)$$

We have

$$\begin{aligned} \frac{W_i^{\alpha,n+1} - W_i^{\alpha,n}}{\Delta t} &= \frac{1}{\Delta t} \left(\max \left\{ H_{\alpha}^{+}(p_{i,-}^{\alpha,n}), H_{\alpha}^{-}(p_{i,+}^{\alpha,n}) \right\} - \max \left\{ H_{\alpha}^{+}(p_{i,-}^{\alpha,n+1}), H_{\alpha}^{-}(p_{i,+}^{\alpha,n+1}) \right\} \right) \\ &\geq \frac{1}{\Delta t} \left(H_{\alpha}^{-\sigma}(p_{i,\sigma}^{\alpha,n}) - H_{\alpha}^{-\sigma}(p_{i,\sigma}^{\alpha,n+1}) \right) \\ &= \int_0^1 (H_{\alpha}^{-\sigma})'(p_{i,\sigma}^{\alpha,n+1} + \tau(p_{i,\sigma}^{\alpha,n} - p_{i,\sigma}^{\alpha,n+1})) \left(\frac{p_{i,\sigma}^{\alpha,n} - p_{i,\sigma}^{\alpha,n+1}}{\Delta t} \right) d\tau \\ &= C_{i,\sigma}^{\alpha,n} \left(\frac{W_{i+\sigma}^{\alpha,n} - W_i^{\alpha,n}}{\Delta x} \right) \end{aligned}$$

where we used (3.6) and (3.8) in the last line. Using (3.7), we thus get

$$\begin{aligned} W_i^{\alpha,n+1} &\geq \left(1 - C_{i,\sigma}^{\alpha,n} \frac{\Delta t}{\Delta x} \right) W_i^{\alpha,n} + C_{i,\sigma}^{\alpha,n} \frac{\Delta t}{\Delta x} W_{i+\sigma}^{\alpha,n} \\ &\geq m^n. \end{aligned}$$

Case 2: $i = 0$. We recall that in this case, we have $U_0^{\beta,n} := U_0^n$ and $W_0^{\beta,n} := W_0^n = \frac{U_0^{n+1} - U_0^n}{\Delta t}$ for any $\beta = 1, \dots, N$. We compute in this case:

$$\begin{aligned} \frac{W_0^{n+1} - W_0^n}{\Delta t} &= \frac{1}{\Delta t} \left(-F(\{p_{0,+}^{\alpha,n+1}\}_{\alpha}) + F(\{p_{0,+}^{\alpha,n}\}_{\alpha}) \right) \\ &= \frac{1}{\Delta t} \int_0^1 \sum_{\beta=1}^N p_{\beta} \frac{\partial F}{\partial p_{\beta}} \left(\{p_{0,+}^{\alpha,n+1} + \tau p_{\alpha}\}_{\alpha} \right) d\tau \quad \text{with } p = (\{p_{0,+}^{\alpha,n} - p_{0,+}^{\alpha,n+1}\}_{\alpha}) \\ &= - \int_0^1 \sum_{\beta=1}^N \frac{\partial F}{\partial p_{\beta}} \left(\{p_{0,+}^{\alpha,n+1} + \tau p_{\alpha}\}_{\alpha} \right) d\tau \left(\frac{W_1^{\beta,n} - W_0^n}{\Delta x} \right) \\ &= \sum_{\beta=1}^N C_{0,+}^{\beta,n} \left(\frac{W_1^{\beta,n} - W_0^n}{\Delta x} \right). \end{aligned}$$

Using (3.7), we argue like in Case 1 and get

$$W_0^{n+1} \geq m^n.$$

Step 2: $(M^n)_n$ is non-increasing. We want to show that $W_i^{\alpha, n+1} \leq M^n$ for $i \geq 0$ and $\alpha = 1, \dots, N$. We argue as in Step 1 by distinguishing two cases.

Case 1: $i \geq 1$. Instead of (3.9) we simply choose σ such that

$$\max\{H_\alpha^+(p_{i,-}^{\alpha, n}), H_\alpha^-(p_{i,+}^{\alpha, n})\} = H_\alpha^{-\sigma}(p_{i,\sigma}^{\alpha, n}) \quad \text{for one } \sigma \in \{+, -\}$$

and argue as in Step 1.

Case 2: $i = 0$. Using (3.5), we can argue exactly as in Step 1. The proof is now complete. \square

3.2 Gradient estimates

The second result needed in the proof of Theorem 3.1 is the following one. It asserts that if the discrete time derivative is controlled from below, then a discrete gradient estimate holds true.

Proposition 3.3 (Discrete gradient estimate). *Let $n \geq 0$ be fixed, consider that $(U_i^{\alpha, n})_{\alpha, i}$ is given and compute $(U_i^{\alpha, n+1})_{\alpha, i}$ using the scheme (1.9)-(1.10). If there exists a constant $K \in \mathbb{R}$ such that for any $i \geq 0$ and $\alpha = 1, \dots, N$,*

$$K \leq W_i^{\alpha, n} := \frac{U_i^{\alpha, n+1} - U_i^{\alpha, n}}{\Delta t}$$

then

$$\begin{cases} \pi_\alpha^-(-K) \leq p_{i,+}^{\alpha, n} \leq \pi_\alpha^+(-K), & \alpha = 1, \dots, N, \quad i \geq 1, \\ \underline{p}_\alpha(-K) \leq p_{0,+}^{\alpha, n} \leq (H_\alpha^+)^{-1}(-K), & \alpha = 1, \dots, N \end{cases}$$

where $p_{i,+}^{\alpha, n}$ is defined in (1.10) and π_α^\pm and \underline{p} are the ‘‘generalized’’ inverse functions of H_α and F , respectively.

Proof. Let $n \geq 0$ be fixed and consider $(U_i^{\alpha, n})_{\alpha, i}$ with $\Delta x, \Delta t > 0$ given. We compute $(U_i^{\alpha, n+1})_{\alpha, i}$ using the scheme (1.9). Let us consider any $i \geq 0$ and $\alpha = 1, \dots, N$.

If $i \geq 1$, the result follows from

$$K \leq W_i^{\alpha, n} = - \max_{\sigma=+,-} H_\alpha^\sigma(p_{i,\sigma}^{\alpha, n}).$$

If $i = 0$, the results follows from

$$K \leq W_0^n = -F\left(\{p_{0,+}^{\alpha, n}\}_\alpha\right).$$

This achieves the proof of Proposition 3.3 \square

3.3 Proof of gradient estimates

Proof of Theorem 3.1. The idea of the proof is to introduce new Hamiltonians \tilde{H}_α and a new junction function \tilde{F} for which it is easier to derive gradient estimates but whose corresponding numerical scheme in fact coincide with the original one.

Step 1: Modification of the Hamiltonians and the junction function. Let the new Hamiltonians \tilde{H}_α for all $\alpha = 1, \dots, N$ be defined as

$$\tilde{H}_\alpha(p) = \begin{cases} H_\alpha(\underline{p}_\alpha) - \frac{C_\alpha}{2}(p - \underline{p}_\alpha) & \text{if } p \leq \underline{p}_\alpha \\ H_\alpha(p) & \text{if } p \in [\underline{p}_\alpha, \bar{p}_\alpha] \\ H_\alpha(\bar{p}_\alpha) + \frac{C_\alpha}{2}(p - \bar{p}_\alpha) & \text{if } p \geq \bar{p}_\alpha \end{cases} \quad (3.10)$$

where \underline{p}_α and \bar{p}_α are defined in (3.2) respectively, and

$$C_\alpha = \sup_{p_\alpha \in [\underline{p}_\alpha, \bar{p}_\alpha]} |H'_\alpha(p_\alpha)|$$

The modified Hamiltonians \tilde{H}_α satisfy (1.4) and

$$\tilde{H}_\alpha \equiv H_\alpha \text{ in } [\underline{p}_\alpha, \bar{p}_\alpha]$$

and

$$\forall p \in \mathbb{R}, \quad |\tilde{H}'_\alpha(p)| \leq \sup_{p_\alpha \in [\underline{p}_\alpha, \bar{p}_\alpha]} |H'_\alpha(p_\alpha)|. \quad (3.11)$$

Let the new \tilde{F} satisfy (1.8), be such that

$$\tilde{F} \equiv F \text{ in } Q_0 := \prod_{\alpha=1}^N [\underline{p}_\alpha, \bar{p}_\alpha]$$

and (See Appendix C)

$$\forall p \in \mathbb{R}^N, \quad (-\nabla \cdot \tilde{F})(p) \leq \sup_{Q_0} (-\nabla \cdot F). \quad (3.12)$$

We then consider the new numerical scheme

$$\begin{cases} \frac{\tilde{U}_i^{\alpha, n+1} - \tilde{U}_i^{\alpha, n}}{\Delta t} + \max\{\tilde{H}_\alpha^+(\tilde{p}_{i,+}^{\alpha, n}), \tilde{H}_\alpha^-(\tilde{p}_{i,+}^{\alpha, n})\} = 0, & i \geq 1, \quad \alpha = 1, \dots, N \\ \tilde{U}_0^{\beta, n} := U_0^n, & i = 0, \quad \beta = 1, \dots, N \\ \frac{\tilde{U}_0^{n+1} - \tilde{U}_0^n}{\Delta t} + \tilde{F}(\tilde{p}_{0,+}^{1, n}, \tilde{p}_{0,+}^{2, n}, \dots, \tilde{p}_{0,+}^{N, n}) = 0 \end{cases}$$

with the same initial condition, namely,

$$\tilde{U}_i^{\alpha, 0} = u_0^\alpha(i\Delta x), \quad i \geq 0, \quad \alpha = 1, \dots, N.$$

In view of (3.11) and (3.12), the CFL condition (1.13) gives that for any $i \geq 0$, $n \geq 0$, and $\alpha = 1, \dots, N$

$$\tilde{D}_{i,+}^{\alpha, n} \leq \sup \left\{ \sup_{\underline{p}_\alpha \leq p \leq \bar{p}_\alpha} |H'_\alpha(p)|; \sup_{Q_0} (-\nabla \cdot F) \right\} \leq \frac{\Delta x}{\Delta t}. \quad (3.13)$$

Step 2: First gradient bounds. Let $n \geq 0$ be fixed. If \tilde{m}^n and \tilde{M}^n are finite, we have

$$\tilde{m}^n \leq \tilde{W}_i^{\alpha, n} \quad \text{for any } i \geq 0, \quad \alpha = 1, \dots, N.$$

Proposition 3.3 implies that

$$\begin{cases} \tilde{\pi}_\alpha^-(-\tilde{m}^n) \leq \tilde{p}_{i,+}^{\alpha, n} \leq \tilde{\pi}_\alpha^+(-\tilde{m}^n), & i \geq 1, \quad \alpha = 1, \dots, N, \\ \tilde{p}_{\underline{p}_\alpha}^-(-\tilde{m}^n) \leq \tilde{p}_{0,+}^{\alpha, n} \leq \tilde{\pi}_\alpha^+(-\tilde{m}^n), & i \geq 0, \quad \alpha = 1, \dots, N. \end{cases}$$

In particular, we get that

$$|\tilde{p}_{i,+}^{\alpha, n}| \leq C^n \quad \text{for } i \geq 0, \quad \alpha = 1, \dots, N$$

with

$$C^n = \max_{\alpha} \left(\max \left(|\tilde{\pi}_{\alpha}^{-}(-\tilde{m}^n)|, |\tilde{\pi}_{\alpha}^{+}(-\tilde{m}^n)|, |\tilde{p}_{\alpha}^{-}(-\tilde{m}^n)| \right) \right).$$

In view of (3.13), Proposition 3.2 implies that

$$\tilde{m}^n \leq \tilde{m}^{n+1} \leq \tilde{M}^{n+1} \leq \tilde{M}^n \quad \text{for any } n \geq 0. \quad (3.14)$$

In particular, \tilde{m}^{n+1} is also finite. Since $\tilde{m}^0 = m^0$ and \tilde{M}^0 are finite, we conclude that \tilde{m}^n and \tilde{M}^n are finite for all $n \geq 0$ and for all $n \geq 0$,

$$m^0 \leq \tilde{m}^n \leq \tilde{M}^n \leq M^0. \quad (3.15)$$

Step 3: Time derivative and gradient estimates. Now we can repeat the same reasoning but applying Proposition 3.3 with $K = m^0$ and get

$$\begin{cases} \underline{p}_{\alpha} \leq \tilde{p}_{i,+}^{\alpha,n} \leq \bar{p}_{\alpha}, & i \geq 1, & \alpha = 1, \dots, N, \\ \underline{p}_{\alpha}^0 \leq \tilde{p}_{0,+}^{\alpha,n} \leq \bar{p}_{\alpha}, & i \geq 0, & \alpha = 1, \dots, N. \end{cases} \quad (3.16)$$

This implies that $\tilde{U}_i^{\alpha,n} = U_i^{\alpha,n}$ for all $i \geq 0$, $n \geq 0$, $\alpha = 1, \dots, N$. In view of (3.14), (3.15) and (3.16), the proof is now complete. \square

4 Convergence for general junction conditions

This section is devoted to the convergence of the scheme defined by (1.9)-(1.10). In order to do so, we first make precise how to choose \bar{p}_{α} , \underline{p}_{α} and \underline{p}_{α}^0 in the CFL condition (1.13).

4.1 Monotonicity of the scheme

In order to prove the convergence of the numerical solution as the mesh size tends to zero, we need first to prove a monotonicity result. It is common to write the scheme defined by (1.9)-(1.10) under the compact form

$$u^{\varepsilon}(t + \Delta t, x) = S^{\varepsilon}[u^{\varepsilon}(t)](x)$$

where the operator S^{ε} is defined on the set of functions defined in J^{ε} . The scheme is monotone if

$$u \leq v \quad \Rightarrow \quad S^{\varepsilon}[u] \leq S^{\varepsilon}[v].$$

In our cases, if $t = n\Delta t$ and $x = i\Delta x \in J^{\alpha}$ and $U(t, x) = U_i^{\alpha,n}$ for $x \in J^{\alpha}$, then $S^{\varepsilon}[U]$ is defined as follows,

$$\begin{cases} U_i^{\alpha,n+1} = S_{\alpha}[U_{i-1}^{\alpha,n}, U_i^{\alpha,n}, U_{i+1}^{\alpha,n}] \text{ for } i \geq 1, \alpha = 1, \dots, N, \\ U_0^{n+1} = S_0[U_0^n, (U_1^{\beta,n})_{\beta=1, \dots, N}] \end{cases}$$

where

$$\begin{cases} S_{\alpha}[U_{i-1}^{\alpha,n}, U_i^{\alpha,n}, U_{i+1}^{\alpha,n}] := U_i^{\alpha,n} - \Delta t \max \left\{ H_{\alpha}^{+} \left(\frac{U_i^{\alpha,n} - U_{i-1}^{\alpha,n}}{\Delta x} \right), H_{\alpha}^{-} \left(\frac{U_{i+1}^{\alpha,n} - U_i^{\alpha,n}}{\Delta x} \right) \right\}, \\ S_0[U_0^n, (U_1^{\beta,n})_{\beta=1, \dots, N}] := U_0^n - \Delta t F(p_{0,+}^{1,n}, \dots, p_{0,+}^{N,n}) \end{cases} \quad (4.1)$$

Checking the monotonicity of the scheme reduces to checking that S_{α} and S_0 are non-decreasing in all their variables.

Lemma 4.1 (Monotonicity of the numerical scheme). *Let $(U^n) := (U_i^{\alpha,n})_{\alpha,i}$ the numerical solution of (1.9)-(1.11). Under the CFL condition (1.12) the scheme is monotone.*

Proof. We distinguish two cases.

Case 1: $i \geq 1$. It is straightforward to check that, for any $\alpha = 1, \dots, N$, the function S_α is non-decreasing with respect to $U_{i-1}^{\alpha,n}$ and $U_{i+1}^{\alpha,n}$. Moreover,

$$\frac{\partial S_\alpha}{\partial U_i^{\alpha,n}} = \begin{cases} 1 - \frac{\Delta t}{\Delta x} (H_\alpha^+)'(p_{i,-}^{\alpha,n}) & \text{if } \max\{H_\alpha^+(p_{i,-}^{\alpha,n}), H_\alpha^-(p_{i,+}^{\alpha,n})\} = H_\alpha^+(p_{i,-}^{\alpha,n}) \\ 1 + \frac{\Delta t}{\Delta x} (H_\alpha^-)'(p_{i,+}^{\alpha,n}) & \text{if } \max\{H_\alpha^+(p_{i,-}^{\alpha,n}), H_\alpha^-(p_{i,+}^{\alpha,n})\} = H_\alpha^-(p_{i,+}^{\alpha,n}) \end{cases}$$

which is non-negative if the CFL condition (1.12) is satisfied.

Case 2: $i = 0$. Similarly it is straightforward to check that S_0 is non-decreasing with respect to $U_1^{\beta,n}$ for $\beta = 1, \dots, N$. Moreover,

$$\frac{\partial S_0}{\partial U_0^n} = 1 + \frac{\Delta x}{\Delta t} \sum_{\beta=1}^N \frac{\partial F}{\partial p_\beta} \{(p_{0,+}^{\alpha,n})_{\alpha=1}^N\}$$

which is non-negative due to the CFL condition. The proof is now complete. \square

A direct consequence of the previous lemma is the following elementary but useful discrete comparison principle.

Lemma 4.2 (Discrete Comparison Principle). *Let $(U^n) := (U_i^{\alpha,n})_{\alpha,i}$ and $(V^n) := (V_i^{\alpha,n})_{\alpha,i}$ be such that*

$$\forall n \geq 1, \quad U^{n+1} \leq S^\varepsilon[U^n] \quad \text{and} \quad V^{n+1} \geq S^\varepsilon[V^n].$$

If the CFL condition (1.12) is satisfied and if $U^0 \leq V^0$, then $U^n \leq V^n$ for all $n \in \mathbb{N}$.

Remark 3. The discrete function (U^n) (resp. (V^n)) can be seen as a *super-scheme* (resp. *sub-scheme*).

We finally recall how to derive discrete viscosity inequalities for monotone schemes.

Lemma 4.3 (Discrete viscosity inequalities). *Let u^ε be a solution of (1.9)-(1.11) with $F = F_A$ defined in (1.5). If $u^\varepsilon - \varphi$ has a global maximum (resp. global minimum) on \mathcal{G}_ε at $(\bar{t} + \Delta t, \bar{x})$, then*

$$\delta_t \varphi(\bar{t}, \bar{x}) + \mathcal{H}(\bar{x}, D_+ \varphi(\bar{t}, \bar{x}), D_- \varphi(\bar{t}, \bar{x})) \leq 0. \quad (\text{resp. } \geq 0)$$

where

$$\mathcal{H}(x, p_+, p_-) = \begin{cases} \max\{H_\alpha^+(p_-), H_\alpha^-(p_+)\} & \text{if } \bar{x} \neq 0 \\ \max\{A, \max_\alpha H_\alpha^-(p_+)\} & \text{if } \bar{x} = 0 \end{cases}$$

and

$$\begin{aligned} D_+ \varphi(\bar{t}, \bar{x}) &= \begin{cases} \frac{1}{\Delta x} \{\varphi(\bar{t}, \bar{x} + \Delta x) - \varphi(\bar{t}, \bar{x})\} & \text{if } \bar{x} \neq 0 \\ \left(\frac{1}{\Delta x} \{\varphi^\alpha(\bar{t}, \Delta x) - \varphi^\alpha(\bar{t}, 0)\} \right)_\alpha & \text{if } \bar{x} = 0 \end{cases} \\ D_- \varphi(\bar{t}, \bar{x}) &= \frac{1}{\Delta x} \{\varphi(\bar{t}, \bar{x}) - \varphi(\bar{t}, \bar{x} - \Delta x)\} \\ \delta_t \varphi(\bar{t}, \bar{x}) &= \frac{1}{\Delta t} \{\varphi(\bar{t} + \Delta t, \bar{x}) - \varphi(\bar{t}, \bar{x})\}. \end{aligned}$$

4.2 Stability and Consistency of the scheme

We first derive a local L^∞ bound for the solution of the scheme.

Lemma 4.4 (Stability of the numerical scheme). *Assume that the CFL condition (1.13) is satisfied and let u^ε be the solution of the numerical scheme (1.9)-(1.11). There exists a constant $C_0 > 0$, such that for all $(t, x) \in \mathcal{G}_\varepsilon$,*

$$|u^\varepsilon(t, x) - u_0(x)| \leq C_0 t. \quad (4.2)$$

In particular, the scheme is (locally) stable.

Proof. If C_0 large enough so that

$$\begin{cases} C_0 + \max\{H_\alpha^+(p_{i,-}^{\alpha,0}), H_\alpha^-(p_{i,+}^{\alpha,0})\} \geq 0, & i \geq 1, \quad \alpha = 1, \dots, N \\ C_0 + F(p_{0,+}^{1,0}, p_{0,+}^{2,0}, \dots, p_{0,+}^{N,0}) \geq 0, \end{cases}$$

and

$$\begin{cases} -C_0 + \max\{H_\alpha^+(p_{i,-}^{\alpha,0}), H_\alpha^-(p_{i,+}^{\alpha,0})\} \leq 0, & i \geq 1, \quad \alpha = 1, \dots, N \\ -C_0 + F(p_{0,+}^{1,0}, p_{0,+}^{2,0}, \dots, p_{0,+}^{N,0}) \leq 0, \end{cases}$$

then $\bar{U}_i^{\alpha,n} = U_i^{\alpha,0} + C_0 n \Delta t$ is a super-scheme and $\bar{U}_i^{\alpha,n} = U_i^{\alpha,0} - C_0 n \Delta t$ is a sub-scheme (see Remark 3). The discrete comparison principle, Proposition 4.2, then implies

$$|U_i^{\alpha,n} - U_i^{\alpha,0}| \leq C_0 n \Delta t$$

which is the desired inequality. This achieves the proof. \square

Another condition to satisfy convergence of the numerical scheme (1.9) towards the continuous solution of (1.6) is the consistency of the scheme (which is obvious in our case). In the statement below, we use the short hand notation (2.2) introduced in the preliminary section.

Lemma 4.5 (Consistency of the numerical scheme). *Under the assumptions on the Hamiltonians (1.4), the finite difference scheme is consistent with the continuous problem (1.6), that is to say for any smooth function $\varphi(t, x)$, we have*

$$\frac{S^\varepsilon[\varphi](s, y) - \varphi(s, y)}{\Delta t} \rightarrow H(x, \varphi_x(t, x)) \quad \text{as } \mathcal{G}_\varepsilon \ni (s, y) \rightarrow (t, x).$$

4.3 Convergence of the numerical scheme

In this subsection, we present a sketch of the proof of Theorem 1.1.

Sketch of the proof of Theorem 1.1. Let $T > 0$ and $\varepsilon := (\Delta t, \Delta x)$ satisfying the CFL condition (1.13). We recall that

$$u^\varepsilon(0, x) = u(0, x) \quad \text{for } x \in \mathcal{G}_\varepsilon.$$

We consider \bar{u} and \underline{u} respectively defined as

$$\bar{u}(t, y) = \limsup_{\substack{\varepsilon \rightarrow 0 \\ \mathcal{G}_\varepsilon \ni (t', y') \rightarrow (t, y)}} u_\varepsilon(t', y'), \quad \underline{u}(t, y) = \liminf_{\substack{\varepsilon \rightarrow 0 \\ \mathcal{G}_\varepsilon \ni (t', y') \rightarrow (t, y)}} u_\varepsilon(t', y').$$

By construction, we have $\underline{u} \leq \bar{u}$. Since the scheme is monotone (Lemma 4.1), stable (Lemma 4.4) and consistent (Lemma 4.5), we can follow [11, 4, 10] we can show that \underline{u} (resp. \bar{u}) is a relaxed viscosity super-solution (resp. viscosity sub-solution) of equation (1.2)-(1.3). Using Theorem 2.2, we know that \underline{u} (resp. \bar{u}) is a viscosity super-solution (resp. sub-solution) of (1.6)-(1.3). Moreover, (4.2) implies that

$$\bar{u}(0, x) \leq u_0(x) \leq \underline{u}(0, x).$$

The comparison principle (see Theorem 2.1) then implies that

$$\bar{u} \leq u \leq \underline{u}$$

which achieves the proof. \square

5 $C^{1,1}$ estimates for the vertex test function

In this section, we study the Lipschitz regularity of the gradient of the vertex test function constructed in [17]. It turns out that its gradient is indeed Lipschitz if the flux limiter A is not equal to A_0 , the minimal flux limiter. Such a technical result will be used when deriving error estimates. It is also of independent interest.

Proposition 5.1 ($C^{1,1}$ estimates for the vertex test function). *Let $\gamma > 0$ and assume that the Hamiltonians satisfy (1.4) and (2.4). The vertex test function G associated with the small parameter γ and with the flux limiter $A_0 + \gamma$ obtained from Theorem 2.4 can be chosen $C^{1,1}(J_K^2)$ for any $K > 0$ where $J_K^2 = \{(x, y) \in J^2 : d(x, y) \leq K\}$. Moreover, there exists C_K such that*

$$\|D^2 G^{A_0 + \gamma}\|_{L^\infty(J_K^2)} \leq \frac{C_K}{\gamma};$$

the constant C_K depends only on K and (1.4).

Proof. In the following A denotes $A_0 + \gamma$. We first get the desired estimate in the smooth convex case and then derive it in the general case.

Step 1: the smooth convex case. We first assume that Hamiltonians satisfy

$$\begin{cases} H_\alpha \in C^2(\mathbb{R}) \\ \min H_\alpha = H_\alpha(0) \\ \min H''_\alpha =: m_\alpha > 0. \end{cases} \quad (5.1)$$

In this case, the vertex test function $G(x, y)$ is constructed in [17] in two different ways if x, y are in the same branch or not. If they are, then G is a regularization of

$$G^0(x, y) = (\max A, H_\alpha)^*(x - y)$$

when x and y are on the branch J_α . This regularization implies an error γ in the viscosity inequalities and the second derivatives of G in J_α^2 can be bounded by $\mathcal{O}(\gamma^{-1})$.

For $x \in J_\alpha$ and $y \in J_\beta$ with $\alpha \neq \beta$, G is defined in [17] by the following formula,

$$G(x, y) = \sup_{\lambda \geq A} \{\pi_\alpha^+(\lambda)x - \pi_\beta^-(\lambda)y - \lambda\}.$$

The supremum is reached for some $\lambda \geq A$ which depends on x and y . In the region where $\lambda = A$, the function G is linear and there is nothing to prove. In $\{\lambda > A\}$, the function $\lambda(x, y)$ is implicitly defined by the following equation

$$(\pi_\alpha^+(\lambda))'x - (\pi_\beta^-(\lambda))'y = 1 \quad (5.2)$$

and the gradient of G is given by

$$\partial_x G(x, y) = \pi_\alpha^+(\lambda), \quad \partial_y G(x, y) = -\pi_\beta^-(\lambda)$$

with $\lambda = \lambda(x, y)$. We thus can easily compute the second order derivatives of G ,

$$\begin{cases} \partial_x^2 G(x, y) = \frac{1}{\frac{x}{H'_\alpha} H''_\alpha + \frac{y}{-H'_\beta} H''_\beta \frac{(H'_\alpha)^2}{(H'_\beta)^2}} \\ \partial_y^2 G(x, y) = \frac{1}{\frac{x}{H'_\alpha} H''_\alpha \frac{(H'_\beta)^2}{(H'_\alpha)^2} + \frac{y}{-H'_\beta} H''_\beta} \\ \partial_{xy}^2 G(x, y) = \frac{1}{-x H'_\beta \frac{H''_\alpha}{(H'_\alpha)^2} + H'_\alpha \frac{H''_\beta}{(H'_\beta)^2} y} \end{cases} \quad (5.3)$$

where H''_α and H''_β respectively denote $H''_\alpha(\pi_\alpha^+(\lambda))$ and $H''_\beta(\pi_\beta^-(\lambda))$. We distinguish cases.

Case 1: $\min H_\alpha > A_0$ **and** $\min H_\beta > A_0$. Then

$$H'_\alpha(\pi_\alpha^+(A_0)) \neq 0 \quad \text{and} \quad H'_\beta(\pi_\beta^-(A_0)) \neq 0$$

and

$$\frac{x}{H'_\alpha} H''_\alpha + \frac{y}{-H'_\beta} H''_\beta \frac{(H'_\alpha)^2}{(H'_\beta)^2} \geq \min \left(H''_\alpha, H''_\beta \frac{(H'_\alpha)^2}{(H'_\beta)^2} \right) \underbrace{\left(\frac{x}{H'_\alpha} - \frac{y}{H'_\beta} \right)}_{=1}.$$

using the fact that H''_α are bounded from below,

$$|\partial_x^2 G| \leq \frac{1}{\min \left(H''_\alpha, H''_\beta \frac{(H'_\alpha)^2}{(H'_\beta)^2} \right)} = \mathcal{O}(1).$$

Similarly,

$$\frac{x}{H'_\alpha} H''_\alpha \frac{(H'_\beta)^2}{(H'_\alpha)^2} + \frac{y}{-H'_\beta} H''_\beta \geq \min \left(H''_\alpha \frac{(H'_\beta)^2}{(H'_\alpha)^2}, H''_\beta \right) \underbrace{\left(\frac{x}{H'_\alpha} - \frac{y}{H'_\beta} \right)}_{=1}$$

implies

$$|\partial_y^2 G| \leq \frac{1}{\min \left(H''_\alpha \frac{(H'_\beta)^2}{(H'_\alpha)^2}, H''_\beta \right)} = \mathcal{O}(1)$$

and

$$\begin{aligned} -x H'_\beta \frac{H''_\alpha}{(H'_\alpha)^2} + H'_\alpha \frac{H''_\beta}{(H'_\beta)^2} y &= \frac{x}{H'_\alpha} \frac{-H'_\beta}{H'_\alpha} H''_\alpha + \frac{-y}{H'_\beta} \frac{H'_\alpha}{-H'_\beta} H''_\beta \\ &\geq \min \left(\frac{-H'_\beta}{H'_\alpha} H''_\alpha, \frac{H'_\alpha}{-H'_\beta} H''_\beta \right) \underbrace{\left(\frac{x}{H'_\alpha} - \frac{y}{H'_\beta} \right)}_{=1} \end{aligned}$$

implies

$$|\partial_{xy}^2 G| \leq \frac{1}{\min \left(\frac{-H'_\beta}{H'_\alpha} H''_\alpha, \frac{H'_\alpha}{-H'_\beta} H''_\beta \right)} = \mathcal{O}(1).$$

Case 2: $\min H_\alpha = A_0$ **and** $\min H_\beta > A_0$. In this case

$$H'_\alpha(\pi_\alpha^+(A)) = 0 \quad \text{and} \quad H'_\beta(\pi_\beta^-(A)) \neq 0.$$

Using a second order Taylor expansion for H_α we assume that

$$H'_\alpha(\pi_\alpha^+(A_0 + \gamma)) \simeq \sqrt{\gamma}.$$

Now reasoning as in the previous case, one can deduce that:

$$\begin{cases} |\partial_x^2 G| = \mathcal{O}\left(\frac{1}{\gamma}\right) \\ |\partial_y^2 G| = \mathcal{O}(\gamma) \\ |\partial_{xy}^2 G| \lesssim \frac{1}{\min(\sqrt{\gamma}, \frac{1}{\sqrt{\gamma}})} = \mathcal{O}\left(\frac{1}{\sqrt{\gamma}}\right). \end{cases}$$

Case 3: $\min H_\alpha > A_0$ and $\min H_\beta = A_0$. In this case

$$H'_\alpha(\pi_\alpha^+(A)) \neq 0 \quad \text{and} \quad H'_\beta(\pi_\beta^-(A)) = 0.$$

Arguing as in the previous case, we have

$$\begin{aligned} H'_\beta(\pi_\beta^-(A_0 + \gamma)) &\simeq \sqrt{\gamma} \\ H'_\alpha(\pi_\alpha^+(A_0 + \gamma)) &\simeq \mathcal{O}(1) \end{aligned}$$

from which we deduce

$$\begin{cases} |\partial_x^2 G| = \mathcal{O}(\gamma) \\ |\partial_y^2 G| = \mathcal{O}\left(\frac{1}{\gamma}\right) \\ |\partial_{xy}^2 G| = \mathcal{O}\left(\frac{1}{\min(\sqrt{\gamma}, \frac{1}{\sqrt{\gamma}})}\right) = \mathcal{O}\left(\frac{1}{\sqrt{\gamma}}\right). \end{cases}$$

Case 4: $\min H_\alpha = A_0$ and $\min H_\beta = A_0$. In this case

$$H'_\alpha(\pi_\alpha^+(A)) = 0 \quad \text{and} \quad H'_\beta(\pi_\beta^-(A)) = 0.$$

We have

$$\begin{aligned} H'_\beta(\pi_\beta^-(A_0 + \gamma)) &\simeq \sqrt{\gamma} \\ H'_\alpha(\pi_\alpha^+(A_0 + \gamma)) &\simeq \sqrt{\gamma} \end{aligned}$$

from which we deduce

$$\begin{cases} |\partial_x^2 G| = \mathcal{O}(1), \\ |\partial_y^2 G| = \mathcal{O}(1), \\ |\partial_{xy}^2 G| = \mathcal{O}(1), \end{cases}$$

Step 2: the smooth case. We now weaken (5.1) as

$$\begin{cases} H_\alpha \in C^2(\mathbb{R}), & H''_\alpha(0) := m_\alpha > 0 \\ \pm H'_\alpha(p) > 0 & \text{for } \pm p > 0 \\ H_\alpha(p) \rightarrow +\infty & \text{as } |p| \rightarrow +\infty. \end{cases} \quad (5.4)$$

In this case, it is explained in [17] that the smooth convex case can be used by considering $\hat{H}_\alpha = \beta \circ H_\alpha$ for some C^2 convex function β such that $\beta(0) = 0$ and $\beta' \geq \delta$ for some $\delta > 0$. Indeed,

$$\begin{aligned} (\hat{H}_\alpha)''(p) &= \beta''(\lambda)(H'_\alpha)^2(p) + \beta'(\lambda)H''_\alpha(p) \\ &\geq \frac{m_\alpha}{2}\beta'(\lambda) \\ &\geq m_\alpha \end{aligned}$$

with $\lambda = H_\alpha(p)$ and β such that

$$\forall \lambda > H_\alpha(0), \quad \begin{cases} \frac{\beta''}{\beta'}(\lambda) \geq -\frac{H''_\alpha}{(H'_\alpha)^2} \circ \pi_\alpha^\pm(\lambda) + \frac{m_\alpha}{2(H'_\alpha)^2} \circ \pi_\alpha^\pm(\lambda), \\ \beta' \geq 2 > 0. \end{cases}$$

In this case, the vertex test function studied in Step 1 and associated with the Hamiltonians \hat{H}_α satisfies

$$\hat{H}(y, -G_y(x, y)) \leq \hat{H}(x, G_x(x, y)) + \gamma$$

which implies that

$$H(y, -G_y(x, y)) \leq \beta^{-1}(\beta H(x, G_x(x, y)) + \gamma) \leq H(x, G_x(x, y)) + \frac{\gamma}{2}$$

(where the short hand notation $H(x, p)$ is associated with $A_0 + \gamma$) since $0 \leq (\beta^{-1})' \leq 1/2$. We proved in Step 1 that

$$\|D^2 G^{A_0 + \gamma}\|_{L^\infty(K)} \leq \frac{C_K}{m_\alpha \gamma}.$$

The general case. If the Hamiltonians H_α merely satisfy (1.4), we can construct \tilde{H}_α such that

$$\begin{cases} |H_\alpha - \tilde{H}_\alpha| \leq \gamma/3 \\ \tilde{H}_\alpha''(0) \geq 1 \\ \pm \tilde{H}_\alpha'(p) > 0 \text{ for } \pm p > 0. \end{cases}$$

If we consider now the vertex test function G constructed in the smooth case associated with the small parameter $\gamma/3$ and $A = A_0 + \gamma$, we get

$$\tilde{H}(y, -G_y) \leq \tilde{H}(x, G_x) + \gamma/3.$$

This implies

$$H(y, -G_y) \leq H(x, G_x) + \gamma.$$

Hence G is a vertex test function for the Hamiltonians H_α and it satisfies the desired gradient estimate. The proof is now complete. \square

6 Error estimates

To prove Theorem 1.2, we will need the following result whose classical proof is given in Appendix for the reader's convenience.

Lemma 6.1 (A priori control). *Let $T > 0$ and let u^ε be a solution of the numerical scheme (1.9)-(1.11) and u a super-solution of (1.2)-(1.3) satisfying for some $C_T > 0$,*

$$u(t, x) \geq -C_T(1 + d(0, x)) \quad \text{for } t \in (0, T).$$

Then there exists a constant $C = C(T) > 0$ such that for all $(t, x) \in \mathcal{G}_\varepsilon$, $t \leq T$, and $(s, y) \in [0, T) \times J$, we have

$$u^\varepsilon(t, x) \leq u(s, y) + C(1 + d(x, y)). \quad (6.1)$$

We now turn to the proof of the error estimates in the case of flux-limited junction conditions.

Proof of Theorem 1.2. Before deriving the error estimate, we remark as in [17] that we can assume without loss of generality that the Hamiltonians satisfy the additional condition (2.4). Indeed, if u solves (1.2) then

$$\tilde{u}(t, x) = u(t, x) - p_0^\alpha x \quad \text{for } x \in J^\alpha$$

solves the same equation in which H_α replaced with

$$\tilde{H}_\alpha(p) = H_\alpha(p_0^\alpha + p).$$

We next remark that the solution \tilde{u}_ε of the associated scheme satisfies

$$\tilde{u}^\varepsilon(t, x) = u^\varepsilon(t, x) - p_0^\alpha x \quad \text{for } (t, x) \in \mathcal{G}_\varepsilon.$$

Hence, if

$$\sup_{[0,T) \times J \cap \mathcal{G}_\varepsilon} |\tilde{u}^\varepsilon - \tilde{u}| \leq C(\Delta x)^{1/3}$$

then the same estimate between u^ε and u holds true. We thus assume from now on that (2.4) holds true.

In order to get (1.15), we only prove that

$$u^\varepsilon(t, x) - u(t, x) \leq C_T(\Delta x)^{1/3} \quad \text{in } [0, T) \times \mathbb{R} \cap \mathcal{G}_\varepsilon$$

since the proof of the reverse inequality is very similar. Let

$$M = \sup_{[0,T) \times J \cap \mathcal{G}_\varepsilon} \{u^\varepsilon(t, x) - u(t, x)\}.$$

The remaining of the proof proceeds in several steps.

Step 1: Penalization procedure. For $\sigma > 0$, $\eta > 0$, $\delta > 0$ let us define

$$M_{\varepsilon, \delta} = \sup_{\substack{(t,x) \in \mathcal{G}_\varepsilon, \\ (s,y) \in [0,T) \times J}} \left\{ u^\varepsilon(t, x) - u(s, y) - \varepsilon G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - \frac{(t-s)^2}{2\nu} - \frac{\delta}{2} d^2(y, 0) - \frac{\eta}{T-s} - \sigma s \right\} \quad (6.2)$$

where the vertex test function $G \geq 0$ is given by Theorem 2.4 associated with the small parameter γ and with the flux limiter $A_\gamma = \max(A, A_0 + \gamma)$. Thanks to Lemma 6.1 and the superlinearity of G (see Theorem 2.4), we deduce that

$$0 < \frac{M}{2} \leq C(1 + d(x, y)) - \varepsilon g\left(\frac{d(x, y)}{\varepsilon}\right) - \frac{(t-s)^2}{2\nu} - \frac{\delta}{2} d^2(y, 0) - \frac{\eta}{T-s} - \sigma s$$

which implies in particular that the supremum is reached at some point (t, x, s, y) and

$$\delta d^2(y, 0) \leq 2 \sup_{d>0} (C(1+d) - \varepsilon g(d/\varepsilon)) \leq \tilde{C}$$

where \tilde{C} only depends on g (in particular, it does not depend on ε). This estimate together with the fact that $-G_y(x/\varepsilon, y/\varepsilon) - \delta d(y, 0)$ lies in the viscosity subdifferential of $u(t, \cdot)$ at x implies that there exists $K > 0$ only depending on $\|\nabla u\|_\infty$ (see Theorem 2.3) and g such that the point (t, x, s, y) realizing the maximum

$$d\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) + \left|G_x\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\right| + \left|G_y\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\right| \leq K. \quad (6.3)$$

We want to prove that for $\sigma > \sigma^*$ (to be determined) that the supremum in (6.2) is attained for $t = 0$ or $s = 0$. We assume that $t > 0$ and $s > 0$ and we prove that $\sigma \leq \sigma^*$.

Step 2: Viscosity inequalities. Since $t > 0$ and $s > 0$, we can use Lemma 4.3 and get the following viscosity inequalities. If $x \neq 0$, then

$$\frac{t-s}{\nu} + \frac{\Delta t}{2\nu} + \max \left\{ H_\alpha^- \left(\frac{\varepsilon}{\Delta x} \left\{ G\left(\frac{x+\Delta x}{\varepsilon}, \frac{y}{\varepsilon}\right) - G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \right\} \right), \right. \\ \left. H_\alpha^+ \left(\frac{\varepsilon}{\Delta x} \left\{ G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \right\} \right) - G\left(\frac{x-\Delta x}{\varepsilon}, \frac{y}{\varepsilon}\right) \right\} \leq 0.$$

If $x = 0$, then

$$\frac{t-s}{\nu} + \frac{\Delta t}{2\nu} + \max \left\{ A, \max_\beta \left\{ H_\beta^- \left(\frac{\varepsilon}{\Delta x} \left\{ G^\beta\left(\frac{\Delta x}{\varepsilon}, \frac{y}{\varepsilon}\right) - G^\beta\left(0, \frac{y}{\varepsilon}\right) \right\} \right) \right\} \right\} \leq 0.$$

If $y \neq 0$, then

$$-\frac{\eta}{(T-s)^2} + \frac{t-s}{\nu} - \sigma + H\left(y, -G_y\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - \delta d(y, 0)\right) \geq 0.$$

If $y = 0$, then

$$-\frac{\eta}{(T-s)^2} + \frac{t-s}{\nu} - \sigma + F_A(-G_y(\frac{x}{\varepsilon}, 0)) \geq 0.$$

By Proposition 5.1, Estimate 6.3 and the fact that $|F_A - F_{A_\gamma}| \leq \gamma$, we obtain,

$$\text{if } x \neq 0, \quad \frac{t-s}{\nu} + \frac{\Delta t}{2\nu} + H_\alpha\left(G_x\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\right) + \mathcal{O}\left(\frac{\Delta x}{\varepsilon\gamma}\right) \leq 0 \quad (6.4)$$

$$\text{if } x = 0, \quad \frac{t-s}{\nu} + \frac{\Delta t}{2\nu} + F_{A_\gamma}\left(G_x\left(0, \frac{y}{\varepsilon}\right)\right) + \gamma + \mathcal{O}\left(\frac{\Delta x}{\varepsilon\gamma}\right) \leq 0 \quad (6.5)$$

$$\text{if } y \neq 0, \quad \frac{t-s}{\nu} + H_\beta\left(-G_y\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\right) + \mathcal{O}(\sqrt{\delta}) \geq \sigma \quad (6.6)$$

$$\text{if } y = 0, \quad \frac{t-s}{\nu} + F_{A_\gamma}\left(-G_y\left(\frac{x}{\varepsilon}, 0\right)\right) + \gamma \geq \sigma. \quad (6.7)$$

Combining these viscosity inequalities, we get in all cases:

$$\sigma \leq \frac{\Delta t}{2\nu} + \mathcal{O}\left(\frac{\Delta x}{\varepsilon\gamma}\right) + \mathcal{O}(\sqrt{\delta}) + \gamma =: \sigma^*.$$

Step 3: Estimate of the supremum. We proved in the previous step that, if $\sigma > \sigma^*$, then $M_{\varepsilon, \delta}$ is reached either for $t = 0$ or $s = 0$.

If $t = 0$, then

$$M_{\varepsilon, \delta} \leq u_0(x) - u_0(y) + Cs - \frac{s^2}{2\nu}.$$

Using the fact that u_0 is L_0 -Lipschitz and $d(x, y) = \mathcal{O}(\varepsilon)$ (see (6.3)) one can deduce

$$\begin{aligned} M_{\varepsilon, \delta} &\leq L_0 d(x, y) + \sup_{r>0} \left(Cr - \frac{r^2}{2\nu} \right) \\ &\leq \mathcal{O}(\varepsilon) + \mathcal{O}(\nu). \end{aligned}$$

If $s = 0$, then we can argue similarly (by using (4.2)) and get

$$M_{\varepsilon, \delta} \leq \mathcal{O}(\varepsilon) + \mathcal{O}(\nu).$$

Step 4: Conclusion. We proved that for $\sigma > \sigma^*$, $M_{\varepsilon, \delta} \leq \mathcal{O}(\varepsilon) + \mathcal{O}(\nu)$. This implies that for all $(t, x) \in \mathcal{G}_\varepsilon$, $t \leq T/2$, we have

$$u^\varepsilon(t, x) - u(t, x) \leq \varepsilon G\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon}\right) + \frac{\delta}{2} d^2(x, 0) + \frac{2\eta}{T} + \sigma t + \mathcal{O}(\varepsilon) + \mathcal{O}(\nu)$$

Replacing σ by $2\sigma^*$, say, and recalling that $G(x, x) \leq \gamma$ for all $x \in J$, we deduce that for $(t, x) \in \mathcal{G}_\varepsilon$ and $t \leq T/2$ (after letting $\delta \rightarrow 0$ and $\eta \rightarrow 0$),

$$u^\varepsilon(t, x) - u(t, x) \leq \mathcal{O}\left(\frac{\Delta t}{\nu}\right) + \mathcal{O}\left(\frac{\Delta x}{\varepsilon\gamma}\right) + \mathcal{O}(\gamma) + \mathcal{O}(\varepsilon) + \mathcal{O}(\nu).$$

Using the CFL condition (1.13) and optimizing with respect to ε and ν yields the desired result. \square

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A Relation between the junction and BLN conditions

Consider the following scalar conservation law posed on $(0, +\infty)$,

$$\begin{cases} \partial_t v + \partial_x(H(v)) = 0, & t > 0, x > 0, \\ v(t, 0) = v_b(t), & t > 0, \\ v(0, x) = v_0(x), & x > 0. \end{cases}$$

The usual BLN condition asserts that the trace v_τ of the entropy solution at $x = 0$ (if it exists) of the previous scalar conservation law should satisfy

$$\forall \kappa \in [\min(v_b, v_\tau), \max(v_b, v_\tau)], \quad \text{sgn}(v_\tau - v_b)(H(v_\tau) - H(\kappa)) \leq 0.$$

If H is quasi-convex, this reduces to

$$H(v_\tau) = \max(H^-(v_\tau), H^+(v_b)).$$

This corresponds to a flux limiter $A = H^+(v_b)$.

B Proofs of some technical results

In order to prove Lemma 6.1, we need the following one.

Lemma B.1 (A priori control at the same time). *Assume that u_0 is Lipschitz continuous. Let $T > 0$ and let u^ε be a sub-solution of (1.9)-(1.11) and u be a super-solution of (1.2)-(1.3). Then there exists a constant $C = C_T > 0$ such that for all $t \in [0, T]$, $x, y \in J$, we have*

$$u^\varepsilon(t, x) \leq u(t, y) + C_T(1 + d(x, y)). \quad (\text{B.1})$$

We first derive Lemma 6.1 from Lemma B.1.

Proof of Lemma 6.1. Let us fix some ε and let us consider the sub-solution u^- of (1.9) and the super-solution u^+ of (1.2) defined as :

$$\begin{aligned} u^+(t, x) &= u_0(x) + C_0 t \\ u^-(n\Delta t, i\Delta x) &= u_0(i\Delta x) - C_0 n\Delta t \end{aligned}$$

where

$$C_0 = \max \left\{ |A|, \max_{\alpha=1, \dots, N} \max_{|p_\alpha| \leq L_0} |H_\alpha(p_\alpha)|; \max_{|p_\alpha| \leq L_0} F(p_1, \dots, p_N) \right\}$$

and L_0 denotes the Lipschitz constant of u_0 . We have for all $(t, x) \in [0, T] \times J$, $(s, y) \in \mathcal{G}_\varepsilon$

$$u^-(t, x) - u^+(s, y) \leq 2C_0 T + L_0 d(x, y).$$

We first apply Lemma B.1 to control $u^\varepsilon(t, x) - u^-(t, x)$ and then apply Lemma 6.1 to control $u^+(s, y) - u(s, y)$. Finally we get the control on $u^\varepsilon(t, x) - u(s, y)$. \square

We can now prove Lemma B.1.

Proof of Lemma B.1. We define φ in J^2 as

$$\varphi(x, y) = \sqrt{1 + d^2(x, y)}.$$

Since,

$$d^2(x, y) = \begin{cases} (x - y)^2 & \text{if } (x, y) \in J_\alpha \times J_\alpha \\ (x + y)^2 & \text{if } (x, y) \in J_\alpha \times J_\beta \end{cases}$$

we see that d^2 (and consequently φ) is in $C^{1,1}$ in J^2 . Moreover φ satisfies

$$|\varphi_x(x, y)|, |\varphi_y(x, y)| \leq 1. \quad (\text{B.2})$$

For constants $C_1, C_2 > 0$ to be chosen let us consider

$$M = \sup_{t \in [0, T], x \in \mathcal{G}_\varepsilon, y \in J} (u^\varepsilon(t, x) - u(t, y) - C_2 t - C_1 \varphi(x, y)).$$

The result follows if we show that M is non-positive for C_1 and C_2 large enough. Assume by contradiction that $M > 0$ for any C_1 and C_2 . Then for $\eta, \delta > 0$ small enough, we have $M_{\eta, \delta} \geq \frac{M}{2} > 0$ with

$$M_{\eta, \delta} = \sup_{t \in [0, T], x \in \mathcal{G}_\varepsilon, y \in J} \left(u^\varepsilon(t, x) - u(t, y) - C_2 t - C_1 \varphi(x, y) - \frac{\eta}{T-t} - \frac{\delta}{2} d^2(y, 0) \right) \quad (\text{B.3})$$

Recalling that there exists $C > 0$ such that

$$|u^\varepsilon(t, x) - u_0(x)| \leq Ct \quad \text{and} \quad |u(t, y) - u_0(y)| \leq Ct$$

(see Theorem 2.3 and (4.2)) and using that u_0 is Lipschitz continuous, we see that $M_{\eta, \delta}$ is reached for C_1 large enough (larger than the Lipschitz constant of u_0) and

$$\delta d^2(y, 0) \leq C. \quad (\text{B.4})$$

Then the classical time penalization (or doubling variable technique) implies the existence of $a, b \in \mathbb{R}$ such that

$$\begin{aligned} a + H(x, C_1 \varphi_x(x, y)) &\leq C \Delta x \\ b + H(y, -C_1 \varphi_y(x, y) - \delta d(0, y)) &\geq 0 \end{aligned}$$

with

$$a - b = C_2 + \eta(T-t)^{-2} \geq C_2.$$

Subtracting these inequalities yields

$$C_2 \leq H(y, -C_1 \varphi_y(x, y) - \delta d(0, y)) - H(x, C_1 \varphi_x(x, y)) + S \Delta x$$

Using bounds (B.2) and (B.4) yields to a contradiction for C_2 large enough. \square

C Construction of \tilde{F}

Lemma C.1. *There exists \tilde{F} , such that*

1. \tilde{F} satisfies (1.8);
2. $F = \tilde{F}$ in Q_0 ;
3. For a.e. $p \in \mathbb{R}^N$, $(-\nabla \cdot \tilde{F})(p) \leq \sup_{Q_0} (-\nabla \cdot F)$.

Proof. Let I_α denote $[p_\alpha^0; \bar{p}_\alpha]$ so that $Q_0 = \prod_\alpha I_\alpha$. We first define \tilde{F} for p in the following set

$$D_\alpha = \{p \in \mathbb{R}^N : \exists \alpha \in \{1, \dots, N\} \text{ such that } p_\alpha \notin I_\alpha \text{ and } \forall \beta \neq \alpha, p_\beta \in I_\beta\}.$$

For $p \in D_\alpha$, we then define

$$\tilde{F}(p) = F(p_1, \dots, \mathcal{P}_\alpha(p_\alpha), \dots, p_N) - C_\alpha(p_\alpha - \mathcal{P}_\alpha(p_\alpha))$$

where

$$C_\alpha = \min_{p_\alpha \in I_\alpha} \left(-\frac{\partial F}{\partial p_\alpha}(p_1, \dots, \mathcal{P}_\alpha(p_\alpha), \dots, p_N) \right),$$

and

$$\mathcal{P}_\alpha(r) = \begin{cases} \underline{p}_\alpha^0 & \text{if } r < \underline{p}_\alpha^0, \\ r & \text{if } r \in I_\alpha, \\ \bar{p}_\alpha & \text{if } r > \bar{p}_\alpha. \end{cases}$$

Remark that in view of the assumptions made on F , we have $C_\alpha^r > 0$ which will ensure that (1.8) holds true.

For $p \notin \cup_{\beta=1}^N D_\beta$, let \bar{p}_α denote $p_\alpha - \mathcal{P}_\alpha(p_\alpha)$ and $\bar{p} = (\bar{p}_1, \dots, \bar{p}_N)$. We next define

$$\lambda_\alpha = \frac{|\bar{p}_\alpha|}{|\bar{p}_1| + \dots + |\bar{p}_N|}.$$

We first remark that $\lambda_\alpha = 0$ if $p_\alpha \in I_\alpha$. We next remark that for all α , there exist $P_\alpha \in D_\alpha$ such that

$$p = \sum_{\alpha=1}^N \lambda_\alpha P_\alpha.$$

Moreover, P_α is unique if $\lambda_\alpha \neq 0$. We thus define

$$\tilde{F}(p) = \sum_{\alpha=1}^N \lambda_\alpha \tilde{F}(P_\alpha).$$

It is now easy to check that (1.8) and Item 3 are satisfied. This ends the proof of the Lemma. \square

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