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# Stable Kneser Graphs are almost all not weakly Hom-Idempotent\*

Pablo Torres<sup>†</sup>

Mario Valencia-Pabon<sup>‡</sup>

## Abstract

A graph  $G$  is said to be *hom-idempotent* if there is an homomorphism from  $G^2$  to  $G$ , and *weakly hom-idempotent* if for some  $n \geq 1$  there is a homomorphism from  $G^{n+1}$  to  $G^n$ . Larose et al. [*Eur. J. Comb.* 19:867-881, 1998] proved that Kneser graphs  $KG(n, k)$  are not weakly hom-idempotent for  $n \geq 2k + 1$ ,  $k \geq 2$ . We show that 2-stable Kneser graphs  $KG(n, k)_{2\text{-stab}}$  are not weakly hom-idempotent, for  $n \geq 2k + 2$ ,  $k \geq 2$ . Moreover, for  $s, k \geq 2$ , we prove that  $s$ -stable Kneser graphs  $KG(k s + 1, k)_{s\text{-stab}}$  are circulant graphs and so hom-idempotent graphs.

**Keywords:** Cartesian product of graphs, Stable Kneser graphs, Cayley graphs, Hom-idempotent graphs.

## 1 Introduction

Let  $[n]$  denote the set  $\{1, \dots, n\}$ . For positive integers  $n \geq 2k$ , the Kneser graph  $KG(n, k)$  has as vertices the  $k$ -subsets of  $[n]$  and two vertices are connected by an edge if they have empty intersection. In a famous paper, Lovász [8] showed that its chromatic number  $\chi(K(n, k))$  is equal to  $n - 2k + 2$ . After this result, Schrijver [10] proved that the chromatic number remains the same when we consider the subgraph  $KG(n, k)_{2\text{-stab}}$  of  $KG(n, k)$  obtained by restricting the vertex set to the  $k$ -subsets that are 2-stable, that is, that do not contain two consecutive elements of  $[n]$  (where 1 and  $n$  are considered also to be consecutive). Schrijver [10] also proved that the 2-stable Kneser graphs are *vertex critical* (or  $\chi$ -critical), i.e. the chromatic number of any proper subgraph of  $KG(n, k)_{2\text{-stab}}$  is strictly less than  $n - 2k + 2$ ; for this reason, the 2-stable Kneser graphs are also known as the Schrijver graphs. After these general advances, a lot of work has been done concerning properties of Kneser graphs and stable Kneser graphs (see [2, 3, 7, 1, 9] and references therein). For example, it is well known that for  $n \geq 2k + 1$  the automorphism group of the Kneser graph  $KG(n, k)$  is the symmetric group induced by the permutation action on  $[n]$ ; see [4] for a textbook account. More recently, Braun [1] showed that the automorphism group of the 2-stable Kneser graphs  $KG(n, k)_{2\text{-stab}}$  is the dihedral group of order  $2n$ .

The *cartesian product*  $G \square H$  of two graphs  $G$  and  $H$  has vertex set  $V(G) \times V(H)$ , two vertices being joined by an edge whenever they have one coordinate equal and the other adjacent. This product is commutative and associative up to isomorphism.

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An *homomorphism* from a graph  $G$  into a graph  $H$ , denoted by  $G \rightarrow H$ , is an edge-preserving map from  $V(G)$  to  $V(H)$ . If  $H$  is a subgraph of  $G$  and  $\phi : G \rightarrow H$  has the property that  $\phi(u) = u$  for every vertex  $u$  of  $H$ , then  $\phi$  is called a *retraction* and  $H$  is called a *retract* of  $G$ . If  $\phi : G \rightarrow H$  is a bijection and  $\phi^{-1}$  is also a homomorphism from  $H$  to  $G$ , then  $\phi$  is an *isomorphism* and we write  $G \simeq H$ . In particular, if  $G = H$  then  $\phi$  is an *automorphism* if and only if it is bijective. Two graphs  $G$  and  $H$  are *homomorphically equivalent*, denoted by  $G \leftrightarrow H$ , if  $G \rightarrow H$  and  $H \rightarrow G$ . A graph  $G$  is called a *core* if it has no proper retracts, i.e., any homomorphism  $\phi : G \rightarrow G$  is an automorphism of  $G$ . It is well known that any finite graph  $G$  is homomorphically equivalent to at least one core  $G^\bullet$ , as can be seen by selecting  $G^\bullet$  as a retract of  $G$  with a minimum number of vertices. In this way,  $G^\bullet$  is uniquely determined up to isomorphism, and it makes sense to think of it as *the core* of  $G$ . It is widely known that Kneser graphs are core. Moreover, it is not difficult to deduce that any  $\chi$ -critical graph is a core. Therefore, any 2-stable Kneser graph is also a core, because they are  $\chi$ -critical graphs [10].

Let  $A$  be a group and  $S$  a subset of  $A$  that is closed under inverses and does not contain the identity. The *Cayley graph*  $\text{Cay}(A, S)$  is the graph whose vertex set is  $A$ , two vertices  $u, v$  being joined by an edge if  $u^{-1}v \in S$ . Cayley graphs of cyclic groups are often called *circulants*.

A graph  $G$  is said *vertex-transitive* if its automorphism group  $\text{AUT}(G)$  acts transitively on its vertex-set. It's well known that Cayley graphs and Kneser graphs are vertex-transitive. However, 2-stable Kneser graphs are not vertex-transitive in general. For example, no automorphism in  $KG(6, 2)_{2\text{-stab}}$  sends  $\{1, 3\}$  to  $\{1, 4\}$ , since  $\text{AUT}(KG(6, 2)_{2\text{-stab}})$  is the dihedral group of order 12 acting on the set  $\{1, 2, \dots, 6\}$ .

We write  $G^n$  for the  $n$ -fold cartesian product of a graph  $G$ . A graph  $G$  is said *hom-idempotent* if there is a homomorphism from  $G^2$  to  $G$ , and *weakly hom-idempotent* if for some  $n \geq 1$  there is a homomorphism from  $G^{n+1}$  to  $G^n$ . Larose et al. [7] showed that the Kneser graphs are not weakly hom-idempotent. However, the technique used by Larose et al. [7] cannot be extended directly to the 2-stable Kneser graphs.

A subset  $S \subseteq [n]$  is *s-stable* if any two of its elements are at least "at distance  $s$  apart" on the  $n$ -cycle, that is, if  $s \leq |i - j| \leq n - s$  for distinct  $i, j \in S$ . For  $s, k \geq 2$  and  $n \geq ks$ , the *s-stable* Kneser graph  $KG(n, k)_{s\text{-stab}}$  is the subgraph of  $KG(n, k)$  obtained by restricting the vertex set of  $KG(n, k)$  to the *s-stable*  $k$ -subsets of  $[n]$ .

In this paper, we show that almost all 2-stable Kneser graphs are not weakly hom-idempotent. Moreover, for  $s, k \geq 2$ , we show that *s-stable* Kneser graphs  $KG(ks + 1, k)_{s\text{-stab}}$  are circulant graphs and so hom-idempotent graphs. In the sequel, we will use the term *modulo*  $[n]$  to denote arithmetic operations on the set  $[n]$  where  $n$  represents the 0.

## 2 2-stable Kneser graphs

As we have mentioned in the previous section, Braun [1] showed that the automorphism group of the 2-stable Kneser graph  $KG(n, k)_{2\text{-stab}}$  is the dihedral group  $D_{2n}$  of order  $2n$ . We denote the elements of  $D_{2n}$  as follows (arithmetic operations are taken modulo  $[n]$ ):

- *Rotations*: Let  $\sigma^0$  be the identity permutation on  $[n]$  and, for  $1 \leq i \leq n - 1$ , let  $\sigma^i = \sigma^{i-1} \circ \sigma^1$ , where  $\sigma^1$  is the circular permutation  $(1, 2, \dots, n - 1, n)$ .
- *Reflections*:

- Case  $n$  odd. For  $1 \leq i \leq n$ , let  $\rho_i$  be the permutation formed by the product of the transpositions  $(i+1, i-1)(i+2, i-2) \dots (i + \frac{n-1}{2}, i - \frac{n-1}{2})$ , where  $i$  is a fix point.
- Case  $n$  even. For  $1 \leq i \leq \frac{n}{2}$ , we have two types of reflexions: let  $\rho_i$  be the permutation formed by the product of the transpositions  $(i+1, i-1)(i+2, i-2) \dots (i + \frac{n}{2} - 1, i - \frac{n}{2} + 1)$ , where  $i$  and  $i + \frac{n}{2}$  are fix points; and let  $\delta_i$  be the permutation formed by the product of transpositions  $(i, i-1)(i+1, i-2) \dots (i + \frac{n}{2} - 1, i - \frac{n}{2})$  without fix point.

An automorphism  $\phi$  of a graph  $G$  is called a *shift* of  $G$  if  $\{u, \phi(u)\} \in E(G)$  for each  $u \in V(G)$ . In other words, a shift of  $G$  maps every vertex to one of its neighbors [7].

**Lemma 1.** *Let  $n \geq 2k + 2$ . Then, the only two shifts of the 2-stable Kneser graph  $KG(n, k)_{2\text{-stab}}$  are the rotations  $\sigma^1$  and  $\sigma^{n-1}$ .*

*Proof.* It is very easy to deduce that the circular permutations  $\sigma^1 = (1, 2, 3, \dots, n-1, n)$  and  $\sigma^{n-1} = (\sigma^1)^{-1} = (n, n-1, n-2, \dots, 2, 1)$  are both shifts of the graph  $KG(n, k)_{2\text{-stab}}$ . In order to prove that they are the only two shifts of  $KG(n, k)_{2\text{-stab}}$ , we will proceed by cases. The arithmetic operations are taken modulo  $[n]$ .

- *Rotations.* Clearly, the identity permutation  $\sigma^0$  is not a shift. Now, we claim that for each  $i \in \{2, 3, \dots, n-2\}$ , there exists a vertex  $v_i$  in  $KG(n, k)_{2\text{-stab}}$  such that  $\{1, i+1\} \subseteq v_i$ . In fact, vertex  $v_i$  can be computed as follows:

- If  $2 \leq i \leq 2k-1$  then, set  $v_i = \{1, 1+2.1, 1+2.2, \dots, 1+2.(k-1)\}$  if  $i$  is even, otherwise set  $v_i = \{1, 2+2.1, 2+2.2, \dots, 2+2.(k-1)\}$ .
- If  $2k \leq i \leq n-2$  then, set  $v_i = \{1, 1+2.1, 1+2.2, \dots, 1+2.(k-2), i+1\}$ .

Now, for each  $2 \leq i \leq n-2$ , we know that  $\sigma^i(1) = 1+i$  and therefore,  $1+i \in \sigma^i(v_i)$  which implies that  $\{v_i, \sigma^i(v_i)\}$  is not an edge of  $KG(n, k)_{2\text{-stab}}$ . Thus, for  $2 \leq i \leq n-2$ ,  $\sigma^i$  is not a shift of  $KG(n, k)_{2\text{-stab}}$ .

- *Reflexions.* We consider two cases:

- Case  $n$  odd. For each  $1 \leq i \leq n$ , let  $v_i$  be a vertex in  $KG(n, k)_{2\text{-stab}}$  such that  $i \in v_i$ . Trivially, such vertex  $v_i$  always exists. Now, we know that  $i$  is a fix point under the permutation  $\rho_i$  and thus,  $i \in \rho_i(v_i)$  which implies that  $\{v_i, \rho_i(v_i)\}$  is not an edge of  $KG(n, k)_{2\text{-stab}}$ . Thus, for  $1 \leq i \leq n$ ,  $\rho_i$  is not a shift of  $KG(n, k)_{2\text{-stab}}$ .
- Case  $n$  even. Analogous to the previous case, we can show that  $\rho_i$  is not a shift of  $KG(n, k)_{2\text{-stab}}$ , for  $1 \leq i \leq \frac{n}{2}$ . Now, for each  $1 \leq i \leq \frac{n}{2}$ , let  $v_i = \{i+1, i+1+2.1, i+1+2.2, \dots, i+1+2.(k-2), i-2\}$ . Clearly,  $v_i$  is a 2-stable set, since  $i+1+2.(k-2)$  and  $i-2$  are not consecutive integers modulo  $[n]$ . So,  $v_i$  is a vertex of  $KG(n, k)_{2\text{-stab}}$  such that  $\{i+1, i-2\} \subseteq v_i$ . However,  $\{i+1, i-2\} \subseteq \delta_i(v_i)$  which implies that  $\{v_i, \delta_i(v_i)\}$  is not an edge of  $KG(n, k)_{2\text{-stab}}$ . Thus, for  $1 \leq i \leq \frac{n}{2}$ ,  $\delta_i$  is not a shift of  $KG(n, k)_{2\text{-stab}}$ .

□

Larose et al. [7] showed the following useful results:

**Proposition 1** (Proposition 2.3 in [7]). *A graph  $G$  is hom-idempotent if and only if  $G \leftrightarrow \text{Cay}(\text{Aut}(G^\bullet), S_{G^\bullet})$ , where  $\text{Aut}(G^\bullet)$  denotes the automorphism group of the core  $G^\bullet$  of  $G$  and  $S_{G^\bullet}$  denotes the set of all shifts of  $G^\bullet$ .*

**Theorem 1** (Theorem 5.1 in [7]). *Let  $G$  be a  $\chi$ -critical graph. Then  $G$  is weakly hom-idempotent if and only if it is hom-idempotent.*

**Proposition 2.** *Let  $n \geq 2k + 2$  and let  $G$  denotes the graph  $KG(n, k)_{2\text{-stab}}$ . Then,  $G \not\cong \text{Cay}(\text{Aut}(G), S_G)$ , where  $S_G$  are the shifts of  $G$ .*

*Proof.* We know that the automorphism group of the graph  $KG(n, k)_{2\text{-stab}}$  is the dihedral group  $D_{2n}$  on  $[n]$ . Moreover, by Lemma 1, we known that the only two shifts of  $KG(n, k)_{2\text{-stab}}$  are the circular permutations  $\sigma^1 = (1, 2, \dots, n - 1, n)$  and its inverse permutation  $\sigma^{n-1} = (\sigma^1)^{-1} = (n, n - 1, \dots, 2, 1)$ . Therefore, the Cayley graph  $\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$  is a 2-regular graph, that is, each vertex (i.e. each automorphism in  $D_{2n}$ ) has exactly two neighbors. In fact, notice that the vertex identity  $\sigma^0$  has as neighbors the vertices  $\sigma^1$  and  $\sigma^{n-1}$ , and as it's well known, any Cayley graph is vertex-transitive, and thus, any vertex in  $\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$  has exactly two neighbors. So,  $\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$  is a disjoint union of cycles, which implies that  $2 \leq \chi(\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})) \leq 3$ . Moreover, we known that  $\chi(KG(n, k)_{2\text{-stab}}) = n - 2k + 2 \geq 4$ , because  $n \geq 2k + 2$ . Therefore  $KG(n, k)_{2\text{-stab}} \not\cong \text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$ .  $\square$

As mentioned in the previous section, we know that any 2-stable Kneser graph is a core. Therefore, by Propositions 1 and 2, and by Theorem 1, we have the following result.

**Theorem 2.** *For any  $n \geq 2k + 2$ , the 2-stable Kneser graphs  $KG(n, k)_{2\text{-stab}}$  are not weakly hom-idempotent.*

### 3 $s$ -stable Kneser graphs $KG(ks + 1, k)_{s\text{-stab}}$

Let  $\overline{G}$  denote the complement graph of the graph  $G$ , i.e.  $\overline{G}$  has the same vertex set of  $G$  and two vertices are adjacent in  $\overline{G}$  if and only if they are not adjacent in  $G$ . Let  $p$  be a positive integer. The  $p$ th power of a graph  $G$ , that we denoted by  $G^{(p)}$ , is the graph having the same vertex set as  $G$  and where two vertices are adjacent in  $G^{(p)}$  if the distance between them in  $G$  is at most equal to  $p$ , where the distance of two vertices in a graph  $G$  is the number of edges on the shortest path connecting them.

Let  $n \geq 2k$  be positive integers. The Cayley graphs  $\text{Cay}(\mathbb{Z}_n, \{k, k + 1, \dots, n - k\})$ , that we denoted by  $G(n, k)$ , are known as *circular graphs* [11, 6], where  $\mathbb{Z}_n$  denote the cyclic group of order  $n$ . It is well known that the Kneser graph  $KG(n, k)$  contains an induced subgraph isomorphic to  $G(n, k)$ . In fact, let  $C(n, k)$  be the subgraph of  $KG(n, k)$  obtained by restricting the vertex set of  $KG(n, k)$  to the shifts modulo  $[n]$  of the  $k$ -subset  $\{1, 2, \dots, k\}$ , that is,  $\{1, 2, \dots, k\}, \{2, 3, \dots, k + 1\}, \dots, \{n, 1, 2, \dots, k - 1\}$ . Define  $\phi : G(n, k) \rightarrow C(n, k)$  by putting  $\phi(u) = \{u + 1, u + 2, \dots, u + k\}$  where the arithmetic operations are taken modulo  $[n]$ . Clearly,  $\phi$  is a graph isomorphism. Notice also that the graph  $G(n, k)$  is isomorphic to the graph  $\overline{C_n^{(k-1)}}$ , i.e. the complement graph of the  $(k - 1)$ th power of a cycle  $C_n$ . Vince [11] has shown that  $\chi(G(n, k)) = \lceil \frac{n}{k} \rceil$ .

In the remaining of this section, we will always assume w.l.o.g. that any vertex  $v = \{v_1, v_2, \dots, v_k\}$  of the  $s$ -stable Kneser graph  $KG(ks + 1, k)_{s\text{-stab}}$  is such that  $v_1 < v_2 < \dots < v_k$ , where  $s, k \geq 2$ . For  $i \in [k - 1]$ , let  $l_i(v) = v_{i+1} - v_i$  and  $l_k(v) = v_1 + (ks + 1) - v_k$ . If  $C$  is the cycle on  $ks + 1$  points labeled by integers  $1, 2, \dots, ks + 1$  in the clockwise direction and  $v = \{v_1, v_2, \dots, v_k\}$  is a vertex of the  $s$ -stable Kneser graph  $KG(ks + 1, k)_{s\text{-stab}}$ , then  $l_i(v)$  gives the distance in the clockwise direction between  $v_i$  and  $v_{i+1}$  in  $C$ .

**Lemma 2.** Let  $s, k \geq 2$  and let  $v = \{v_1, v_2, \dots, v_k\}$  be a vertex of  $KG(ks+1, k)_{s\text{-stab}}$ . Then,  $l_i(v) \in \{s, s+1\}$  for all  $i \in [k]$ . Moreover, there exists exactly one  $i' \in [k]$  such that  $l_{i'}(v) = s+1$ .

*Proof.* By definition,  $l_i(v) \geq s$  for any  $i \in [k]$ . The result follows from the fact that  $\sum_{i=1}^k l_i(v) = ks+1$ .  $\square$

**Lemma 3.** Let  $s, k \geq 2$ . The number of vertices of the graph  $KG(ks+1, k)_{s\text{-stab}}$  is equal to  $ks+1$ .

*Proof.* Again, let  $C$  be the cycle on  $ks+1$  points labeled by integers  $1, 2, \dots, ks+1$  in the clockwise direction. From Lemma 2, we have that each vertex of  $KG(ks+1, k)_{s\text{-stab}}$  is uniquely determined by a clockwise circular interval of length  $s+1$  in  $C$ . Trivially there exist  $ks+1$  distinct clockwise circular intervals of length  $s+1$  in  $C$  and the lemma holds.  $\square$

**Proposition 3.** Let  $s, k \geq 2$ . Then,  $G(ks+1, k) \simeq KG(ks+1, k)_{s\text{-stab}}$ .

*Proof.* Let  $C$  be a cycle on  $ks+1$  points. We assume that the vertices of  $G(ks+1, k)$  are disposed over  $C$  in clockwise increasing order from 0 to  $ks$ . By Lemmas 2 and 3, we are able to define the application  $\phi : G(ks+1, k) \rightarrow KG(ks+1, k)_{s\text{-stab}}$  as follows: let  $u$  be a vertex of  $G(ks+1, k)$  such that  $u = jk+i$ , where  $0 \leq j \leq s-1$  and  $0 \leq i \leq k-1$ . Then,  $\phi(u) = \{u_1, \dots, u_k\}$  where,

$$u_r = \begin{cases} j+1+(r-1)s, & \text{if } 1 \leq r \leq k-i \\ j+2+(r-1)s, & \text{if } k-i+1 \leq r \leq k. \end{cases}$$

Finally, define  $\phi(ks) = \{s+1, 2s+1, \dots, ks+1\}$ . It is not difficult to prove that  $\phi$  is a bijective function. It remains to show that  $\phi$  is indeed a graph isomorphism. Let  $u, v$  be two vertices in  $C(ks+1, k)$ . In the sequel, we assume that  $v > u$ . In fact, if  $u > v$  we can always swap  $u$  and  $v$ . Let  $u = jk+i$ , where  $0 \leq j \leq s-1$  and  $0 \leq i \leq k-1$ . Let  $t = v-u$ , where  $1 \leq t \leq k-1$ . By construction,  $\phi(v) \setminus \{v_p : k-i-t+1 \leq p \leq k-i\} \subset \phi(u)$ , where the arithmetic operations are taken modulo  $[k]$ . As  $t \leq k-1$  then, we must have that  $\phi(u) \cap \phi(v) \neq \emptyset$ . Analogously, let  $k(s-1)+2 \leq v-u \leq ks$  and let  $t = ks+1-(v-u)$ . Then,  $\phi(u) \setminus \{u_p : k-i-t+1 \leq p \leq k-i\} \subset \phi(v)$ , where the arithmetic operations are taken modulo  $[k]$ . Again, as  $t \leq k-1$  then, we must have that  $\phi(u) \cap \phi(v) \neq \emptyset$ . Notice that, by the previous argument, if  $v-u = k$  or  $v-u = k(s-1)+1$  then,  $\phi(u) \cap \phi(v) = \emptyset$ . Therefore, let  $t = v-u$ , where  $k+1 \leq t \leq k(s-1)$  and assume that there exists  $r \in [k]$  such that  $u_r \in \phi(u) \cap \phi(v)$ . We consider the following cases:

- Case  $1 \leq r \leq k-i$ . As  $u = jk+i$  then,  $v = jk+i+xk+y$ , where  $0 \leq j \leq s-1$ ,  $0 \leq i \leq k-1$ ,  $1 \leq x \leq s-1$ , and  $0 \leq i+y \leq k-1$ . If  $u_r = v_{r'}$ , with  $1 \leq r' \leq k-(i+y)$  then,  $u_r = j+1+(r-1)s = j+x+1+(r'-1)s = v_{r'}$  and thus  $x = (r-r')s$ . As  $x \geq 1$  then,  $r-r' > 0$  and thus  $x \geq s$  which is a contradiction. If  $k-(i+y)+1 \leq r' \leq k$  then,  $j+1+(r-1)s = j+x+2+(r'-1)s$  which implies that  $x = (r-r')s-1$ . However,  $r \leq k-i$  and  $r' \geq k-i-y+1$  and so,  $x \leq (y-1)s-1$ . Moreover, as  $x \geq 1$  then,  $y \geq 2$  and thus  $x = s-1$ , but it implies that  $t \geq k(s-1)+2$  which is a contradiction.
- Case  $k-i+1 \leq r \leq k$ . As the previous case,  $v = (j+x)k+(i+y)$ , with  $1 \leq x \leq s-1$  and  $0 \leq i+y \leq k-1$ . If  $u_r = v_{r'}$ , with  $1 \leq r' \leq k-(i+y)$  then,  $x = (r-r')s+1$  which implies that  $x \geq (y+1)s+1 \geq s$ , that is a contradiction. If  $k-(i+y)+1 \leq r' \leq k$  then,  $j+2+(r-1)s = j+x+2+(r'-1)s$ . Then,  $x = (r-r')s$  and as  $x \geq 1$  then  $r-r' > 0$ , but it implies that  $x \geq s$  which is again a contradiction.

Therefore, vertices  $u, v$  in  $C(k s + 1, k)$  are adjacent if and only if vertices  $\phi(u), \phi(v)$  in  $KG(k s + 1, k)_{s\text{-stab}}$  are adjacent.  $\square$

Notice that if  $n \geq 2r + 3$  the automorphism group of the  $r^{\text{th}}$  power of a  $n$ -cycle,  $C_n^{(r)}$ , is the dihedral group  $D_{2n}$  (see Remark 1) and thus,  $\text{AUT}(C_n^{(r)})$  is also  $D_{2n}$ .

**Remark 1.** For  $r \geq 1$  and  $n \geq 2r + 3$  the automorphism group of  $C_n^{(r)}$  is the dihedral group  $D_{2n}$ .

In fact, it is not hard to see that  $D_{2n}$  is a subgroup of  $\text{AUT}(C_n^{(r)})$ . Let  $\alpha \in \text{AUT}(C_n^{(r)})$  and let  $N[i]$  denote the closed neighborhood of a vertex  $i$  in  $C_n^{(r)}$ . Assume that  $\alpha$  sends two consecutive vertices  $i, j$  in the cycle to non consecutive vertices  $i', j'$  in the cycle, respectively. Observe that  $|N[i] \cap N[j]| = 2r$ , but  $|N[i'] \cap N[j']| \leq 2r - 1$  since there exists at least two vertices in  $N[i'] \setminus N[j']$ . Therefore, any automorphism of  $C_n^{(r)}$  sends consecutive vertices to consecutive vertices. From this fact it is easy to see that any automorphism of  $C_n^{(r)}$  belongs to  $D_{2n}$ .

Therefore, by Proposition 3, the automorphism group of the graph  $KG(k s + 1, k)_{s\text{-stab}}$  is the dihedral group  $D_{2(k s + 1)}$ . However, the problem to compute the automorphism group of the graph  $KG(n, k)_{s\text{-stab}}$  for  $k \geq 2$ ,  $s > 2$ , and  $n > k s + 1$  is still open.

Another direct consequence of Proposition 3 is that  $\chi(KG(k s + 1, k)_{s\text{-stab}}) = s + 1$ . This fact has been found also by Meunier (see Proposition 1 in [9]).

Let  $\text{Cay}(A, S)$  be a Cayley graph. If  $a^{-1} S a = S$  for all  $a \in A$ , then  $\text{Cay}(A, S)$  is called a *normal Cayley graph*.

**Lemma 4** ([5]). *Any normal Cayley graph is hom-idempotent.*

Note that all Cayley graphs on abelian groups are normal, and thus hom-idempotents. In particular, the *circulant* graphs are Cayley graphs on cyclic groups (i.e., cycles, powers of cycles, complements of powers of cycles, complete graphs, etc). Therefore, by Proposition 3 and Lemma 4, we have the following result.

**Theorem 3.** *Let  $s, k \geq 2$ . Then,  $KG(k s + 1, k)_{s\text{-stab}}$  is hom-idempotent.*

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