FINITE ROTATIONS IN THE REFINED MACRODYNAMICS OF ELASTIC COMPOSITES

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The refined macrodynamics of periodic composites, describing the microstructure length effect on the macro-behaviour of the body, within the framework of the linear elastodynamics was proposed by Woźniak (1993) and investigated in a series of related papers. The main aim of this contribution is to formulate equations of the refined macrodynamics for elastic composite materials subjected to small strains but finite rotations and displacements. The obtained results can be applied to the analysis of geometrically nonlinear problems for thin flexible structural elements, made of composite materials.

1. Introduction

As it is known asymptotic homogenization methods of macro-modelling for elastic composite materials leading to various effective modulae theories, constitute the foundations of analysis and calculations of different engineering problems (cf Jones (1975), Bensoussan et al. (1980), Bakhvalov and Panasenko

[1]The research was supported by the State Committee of Scientific Research (KBN)

(1984), Nemat-Nasser and Hori (1993) and the extensive list of papers on this subject). However, the effective modulus theories of periodic composites, based on the asymptotic approach in which the microstructure of the composite is scaled down, neglect the microstructure length effect of the body. The thesis of the present contribution is that this effect plays a crucial role in a description of nonstationary problems and hence the effective modulae approach to dynamics of composites often leads to incorrect results. In order to obtain the engineering tool describing dynamic problems the refined macrodynamics of composite materials and structures was formulated by Woźniak (1993) and investigated in a series of related papers. In the refined macrodynamics the microstructure length scale effects are included into macro-description of the composite. So far the refined macrodynamics was investigated within the framework of the linear theory of elastic or visco-elastic composites. However, many thin, flexible structural composite elements, like composite rods, plates and shells, can be subjected to small strains but finite rotations. That is why the refined macrodynamics has to be formulated also within the framework of the geometrically nonlinear theory.

The aim of this contribution is threefold. Firstly, we generalize the approach proposed by Woźniak (1993) and derive equations of the refined macroelastodynamics for small strains but finite rotations and displacements. The obtained nonlinear equations can be used as a starting point for formulations of different geometrically nonlinear composite plate and shell theories; investigations related to these problems are reserved for a separate study. Secondly, by scaling the microstructure down we pass to the geometrically nonlinear effective modulus theory. Thirdly, on the simple example it is shown that the effective modulus theory cannot be used as a tool in analysis of dynamic boundary-value problems.

1.1. Denotations

Sub- and superscripts $i, j, k, \ldots$ run over $1,2,3$ and are related to the orthogonal cartesian coordinate system $0x^1x^2x^3$ in the physical space. Points of this space are denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$ and t is the time coordinate. Sub- and superscripts $\alpha, \beta, \gamma, \ldots$ also run over $1,2,3$ but are related to the material coordinates $\boldsymbol{X} = (X^1, X^2, X^3)$, $\boldsymbol{X} \in \Omega_R$, where Ω_R is the known region occupied by the body in the reference configuration. Positions of the points $\boldsymbol{X} \in \Omega_R$ in the natural (unstressed) configuration of the body are denoted by $x^i = P^i(X)$, and their positions at the time instant t are $x^i = p^i(X,t)$.

The displacements and strain components are $u^i = p^i(X,t) - P^i(X)$ and $e_{\alpha\beta} = \frac{1}{2}(p^i{}_{,\alpha} p_{i,\beta} - P^i{}_{,\alpha} P_{i,\beta})$, respectively. The body under consideration is assumed to have V_R-periodic material structure with respect to material coordinates, where $V_R = (0,l_1) \times (0,l_2) \times (0,l_3)$ is the representative volume element in the space of X^α-coordinates. The microstructure length parameter l is defined by $l \equiv \max\{l_1, l_2, l_3\}$ and is assumed to be sufficiently small compared to the minimum characteristic length dimension of Ω_R. For any integrable V_R-periodic function $f(X)$, we denote by

$$< f >= (l_1 l_2 l_3)^{-1} \int\limits_{V_R} f(X)\, dV_R \qquad\qquad dV_R = dX^1 dX^2 dX^3$$

its averaged value. By $s_R^{i\alpha}$ and ρ_R we denote the first Piola-Kirchhoff stress tensor and mass density related to the region Ω_R and by s_R^i the boundary tractions related to $\partial\Omega_R$. The body forces are denoted by b_i and are assumed constant. We also introduce non-tensorial indices $a, b, c, \ldots$, which run over the sequence $1, \ldots, n$. Summation convention holds for all the aforementioned indices.

2. Modelling procedure

In order to make considerations self consistent, we recall two auxiliary concepts used in the proposed modelling procedure, namely the concept of a regular V-macro function and that of micro-shape functions (cf Woźniak (1993)).

A function $F(X,t)$, $X \in \Omega_R$, will be referred to as the V_R-macro function (related to a certain small macro-accuracy parameter λ_F), provided that for every $X', X'' \in \Omega_R$, if $X' - X'' \in V_R$ then $|F(X',t) - F(X'',t)| < \lambda_F$. If the function F is sufficiently regular and the conditions of this form hold for all derivatives of F (including time derivatives) with macro-accuracy parameters $\lambda_{\nabla F}$, $\lambda_{\dot F}$ etc., then it will be called a regular V_R-macro function. The choice of a parameter λ_F depends on the assumed accuracy of calculations involving function F.

Let $h_a(X)$, $a = 1, \ldots, n$, denote a system of n linear independent functions defined on the space of X^α-coordinates, which are continuous, V_R-periodic, have piecewise continuous first derivatives and satisfy conditions: $< h_a >= 0$, $<\rho_R h_a >= 0$, $h_a(X) \in O(l)$ for every X and $h_{a,\alpha}(X) \in O(1)$, i.e., the values of derivatives of h_a are independent of the microstructure length parameter l.

Let us also assume that the set of all linear combinations of h_a describes all expected disturbances in the displacements field $u^i(X,t)$ related to an arbitrary but fixed cell $(X^1, X^1+l_1) \times (X^2, X^2+l_2) \times (X^3, X^3+l_3)$ in Ω_R and caused by the periodic inhomogeneity of the body under consideration. Under the aforementioned conditions functions $h_a(X)$ will be called micro-shape functions. Roughly speaking, the micro-shape functions have a role similar to that of the known shape functions of the finite element method but are V_R-periodic and their values averaged over V_R are equal to zero.

The starting point of the modelling procedure is the direct description of a V_R-periodic composite, given by the principle of virtual work related to the region Ω

$$\int\limits_{\Omega_R} s_R^{i\alpha} \delta u_{i,\alpha} \, dV_R = \oint\limits_{\partial\Omega_R} s_R^i \delta u_i \, dA_R + \int\limits_{\Omega_R} \rho_R (b_i - \ddot{u}_i) \delta u_i \, dV_R \tag{2.1}$$

which holds for every virtual displacement δu_i and by the constitutive relation for the first Piola-Kirchhoff stress tensor

$$s_R^{i\alpha} = \frac{\partial \varepsilon_R(X; u_{j,\beta})}{\partial u_{i,\alpha}} \tag{2.2}$$

where $\varepsilon_R(\cdot; u_{j,\beta})$ is a V_R-periodic strain energy function. We restrict ourselves to small strains and assume that this function has the form

$$\varepsilon_R = \frac{1}{2} C_R^{\alpha\beta\gamma\delta}(X) e_{\alpha\beta} e_{\gamma\delta}$$

$$e_{\alpha\beta} = \frac{1}{2}(p^i{}_{,\alpha}\, u_{i,\beta} - p^i{}_{,\beta}\, u_{i,\alpha}) + \frac{1}{2} u^i{}_{,\alpha}\, u_{i,\beta}$$

The elastic modulae $C_R^{\alpha\beta\gamma\delta}(X)$ are V_R-periodic functions, constant for every constituent of the composite. Due to the highly oscillating micro-periodic form of these functions, the direct description of the composite body do not constitute a proper tool for analysis and numerical calculations of engineering problems.

In order to pass from Eqs (2.1) and (2.2) to the averaged refined model of a composite (which takes into account the effect of the microstructure length parameter l on the macro-behaviour of a body) we apply the following three macro-modelling hypotheses.

1. *The Kinematic Hypothesis* states that the displacements $u_i(X,t)$ from the natural configuration $P_i(X)$, $X \in \Omega_R$, of the composite can be assumed in the form

$$u_i(X,t) = U_i(X,t) + h_a(X) V_i^a(X,t) \qquad X \in \Omega_R \quad t \geq 0 \tag{2.3}$$

where U_i, V_i^a are arbitrary regular V_R-macro functions, h_a, $a = 1, ..., n$, are postulated a priori micro-shape functions and $P_{i,\alpha}$ are the known regular V_R-macro functions.

Functions U_i are called macro-displacements and V_i^a will be referred to as correctors. In the modelling procedure they constitute new kinematic variables describing the motion of a composite body.

2. *The Virtual Work Hypothesis* states that the principle of virtual work (2.1) is assumed to hold for every virtual displacement

$$\delta u_i(X) = \delta U_i(X) + h_a(X)\delta V_i^a(X) \qquad X \in \Omega_R \tag{2.4}$$

   where δU_i, δV_i^a are arbitrary linear independent regular V_R-macro functions.

   This hypothesis is strictly related to the previous one.

3. *The Macro-Modelling Approximation* states that in the principle of virtual work (obtained by substituting the right-hand sides of Eqs (2.2) $\div$ (2.4) into Eq (2.1)) terms $O(\lambda)$, where λ runs over $\lambda_{\nabla U}$, $\lambda_{\dot{U}}$, λ_V, $\lambda_{\dot{V}}$, $\lambda_{\nabla V}$ can be neglected.

Combining Eqs (2.1) $\div$ (2.4), using the Macro-Modelling Approximation and following the procedure given by Woźniak (1993) we obtain the system of $3 + 3n$ equations in the $3 + 3n$ new basic unknowns U_i and V_i^a. Denoting by $< \varepsilon_R > = < \varepsilon_R > (U_{i,\alpha}, V_i^a)$ the averaged over V_R strain energy $\varepsilon_R(X; U_{i,\alpha} + h_{a,\alpha}(X)V_i^a)$, where terms $h_a V^a_{i,\alpha} \in O(\lambda_V) + lO(\lambda_{\nabla V})$ have been neglected, these equations can be written down in the form

$$\begin{gathered}
\Big[\frac{\partial < \varepsilon_R >}{\partial U^i{}_{,\alpha}}\Big]_{,\alpha} + < \rho_R > b_i - < \rho_R > \ddot{U}_i = 0 \\
\frac{\partial < \varepsilon_R >}{\partial V_i^a} + < \rho_R h_a h_b > \ddot{V}_i^b = 0
\end{gathered} \tag{2.5}$$

We also obtain the natural boundary conditions assuming $\delta u_i = \delta U_i$ on $\partial\Omega_R$

$$\frac{\partial < \varepsilon_R >}{\partial U_{i,\alpha}} n_{R\alpha} = s_R^i \tag{2.6}$$

where $n_{R\alpha}$ is the unit outward normal to $\partial\Omega_R$. It has to be emphasized that Eqs (2.5) involve exclusively V_R-macro functions and hence Eq (2.6) imposes certain restrictions on the boundary tractions s_i.

Now let us observe that under denotations

$$
\begin{aligned}
E_{\alpha\beta} &\equiv \frac{1}{2}(P_{i,\alpha}\, U^i{}_{,\beta} + P_{i,\beta}\, U^i{}_{,\alpha}) + \frac{1}{2} U^i{}_{,\alpha}\, U_{i,\beta} \\
V^a_\alpha &\equiv V^a_i (P^i{}_{,\alpha} + U^i{}_{,\alpha})
\end{aligned}
\tag{2.7}
$$

where $E_{\alpha\beta}$, V^a_α are V_R-macro functions, and introducing matrix $X^\alpha{}_i$ given by the condition $X^\alpha{}_i(P^j{}_{,\alpha} + U^j{}_{,\alpha}) = \delta^j_i$, we obtain the following formula for the strain components

$$
e_{\alpha\beta} = E_{\alpha\beta} + h_{a,(\alpha} V^a_{\beta)} + \frac{1}{2} h_{a,\alpha}\, h_{b,\beta}\, X^\gamma{}_i X^\delta{}_j V^a_\gamma V^b_\delta \delta^{ij} + O(\lambda_V)
$$

For small strains and finite rotations, the macro-strain measures (2.7) are small and hence in the strain energy terms of the third and higher order of $E_{\alpha\beta}$, V^a_α can be neglected. Bearing in mind the macro-modelling approximation we can assume that

$$
<\varepsilon_R> = \frac{1}{2} A_R^{\alpha\beta\gamma\delta} E_{\alpha\beta} E_{\gamma\delta} + B_{Ra}^{\gamma\alpha\beta} V^a_\gamma E_{\alpha\beta} + \frac{1}{2} C^{\alpha\beta}_{Rab} V^a_\alpha V^b_\beta
\tag{2.8}
$$

where we have denoted

$$
\begin{aligned}
A_R^{\alpha\beta\gamma\delta} &\equiv < C_R^{\alpha\beta\gamma\delta} > \qquad\qquad B_{Ra}^{\gamma\alpha\beta} \equiv < h_{a,\delta}\, C_R^{\gamma\delta\alpha\beta} > \\
C^{\alpha\beta}_{Rab} &\equiv < h_{a,\gamma}\, h_{b,\delta}\, C_R^{\alpha\gamma\beta\delta} >
\end{aligned}
\tag{2.9}
$$

and where denotations (2.7) have to be remembered. From Eq (2.8) it follows that Eqs (2.5) in the case of small strains and finite rotations can be written down in the form

$$
\begin{aligned}
&\Big[\frac{\partial <\varepsilon_R>}{\partial E_{\alpha\beta}} (P^i{}_{,\beta} + U^i{}_{,\beta})\Big]_{,\alpha} + <\rho_R> b_i - <\rho_R> \ddot{U}_i = 0 \\
&\frac{\partial <\varepsilon_R>}{\partial V^a_\alpha} (P_{i,\alpha} + U_{i,\alpha}) + <\rho_R h_a h_b> \ddot{V}^b_i = 0
\end{aligned}
\tag{2.10}
$$

and will be called macro-equations of motion. At the same time the natural boundary conditions (2.6) yield

$$
\frac{\partial <\varepsilon_R>}{\partial E_{\alpha\beta}} (P^i{}_{,\beta} + U^i{}_{,\beta}) n_{R\alpha} = s^i_R
\tag{2.11}
$$

Eqs (2.10) and (2.11) have to be considered jointly with Eqs (2.7) $\div$ (2.9) and constitute the final result of the modelling procedure.

3. Refined macro-elastodynamics

The obtained macro-equations of motion (2.10) will be now transformed to the alternative form of local macro-balance equations and macro-constitutive equations involving explicitly the microstructure length parameter. To this end define $J_{Rab} \equiv <\rho_R h_a h_b> l^{-2}$; since $<\rho_R h_a h_b> \in O(l^2)$ then the inertial modulae J_{Rab} behave as constant under scaling the microstructure down $l \searrow 0$ provided that $l_i/l = \text{const}$. Let us also introduce the fields $S_R^{\alpha\beta}$, H_{Ra}^{α} by means of

$$
\begin{aligned}
S_R^{\alpha\beta} &= A_R^{\alpha\beta\gamma\delta} E_{\gamma\delta} + B_{Ra}^{\gamma\alpha\beta} V_\gamma^a \\
H_{Ra}^{\alpha} &= B_{Ra}^{\alpha\beta\gamma} E_{\beta\gamma} + C_{Rab}^{\alpha\beta} V_\beta^b
\end{aligned}
\tag{3.1}
$$

which are called macro-stresses and micro-dynamic forces, respectively. Hence Eqs (2.10) are

$$
\begin{aligned}
&[S_R^{\alpha\beta}(P^i{}_{,\beta} + U^i{}_{,\beta})]_{,\alpha} - <\rho_R> \ddot{U}_i + <\rho_R> b_i = 0 \\
&H_{Ra}^{\alpha}(P^i{}_{,\alpha} + U^i{}_{,\alpha}) + l^2 J_{Rab} \ddot{V}_i^b = 0
\end{aligned}
\tag{3.2}
$$

and the natural boundary condition (2.11) yields

$$
S_R^{\alpha\beta}(P^i{}_{,\beta} + U^i{}_{,\beta}) n_{R\alpha} = s_R^i
\tag{3.3}
$$

Eqs (3.1) and (3.2) are called macro-constitutive equations and local macro-balance equations, respectively. They involve exclusively V_R-macro fields, characterize material properties of the micro-periodic medium by the averaged modulae (2.9) inertial properties by the averaged densities $<\rho_R>$, J_{Rab} and describe the effect of the microstructure length parameter l on the behaviour of the body. That is why Eqs (3.1) $\div$ (3.3), together with Eqs (2.7), will be referred to as the governing equations of the refined macro-elastodynamics. The basic unknown fields in these equations are microdisplacements U_i and correctors V_i^a. It has to be emphasized that formulae for correctors are ordinary differential equations. Hence V_i^a are independent of boundary conditions and can be interpreted as certain internal balance variables. It can be shown that for homogeneous bodies and under trivial initial conditions for correctors: $V_i^a(\boldsymbol{X},0) = 0$, $\dot{V}_i^a(\boldsymbol{X},0) = 0$, $\boldsymbol{X} \in \Omega_R$, we obtain that $V_i^a(\boldsymbol{X},t) = 0$ for every $\boldsymbol{X} \in \Omega_R$ and $t \geq 0$. Hence the correctors describe, from the quantitative viewpoint, the effect of inhomogeneity of the composite of its behaviour. The effect of finite rotations in Eqs (3.2) and (3.3) is described by deformation gradients $P^i{}_{,\alpha} + U^i{}_{,\alpha}$ and leads to the nonlinear terms $S_R^{\alpha\beta} U^i{}_{,\beta}$, $H_{Ra}^{\alpha} U^i{}_{,\alpha}$. Neglecting these terms and assuming that $P^i(X) = \delta^i_\alpha X^\alpha$ (i.e., that Ω_R is the region occupied by the body in its natural configuration) we arrive at the equations obtained by Woźniak (1993).

Eqs (3.2) and (3.3) involve densities related to the unit volume element of Ω_R. Denoting $\mathcal{J} \equiv \det(P^i{}_{,\alpha} + U^i{}_{,\alpha})$ and setting

$$
\begin{aligned}
&S^{\alpha\beta} \equiv \mathcal{J}^{-1} S_R^{\alpha\beta} \qquad && H_a^\alpha \equiv \mathcal{J}^{-1} H_{Ra}^\alpha \qquad && \mu \equiv \mathcal{J}^{-1} <\rho_R> \\
&J_{ab} \equiv \mathcal{J}^{-1} J_{Rab} && b^\alpha \equiv X^{\alpha i} b_i &&
\end{aligned}
$$

where $X^{\alpha i}$ are given by $X^\alpha{}_i(P^j{}_{,\alpha} + U^j{}_{,\alpha}) = \delta^{ji}$, we can transform Eqs (3.2) to the form

$$
\begin{aligned}
S^{\alpha\beta}\Big|_\beta - \mu \ddot{U}_i X^{\alpha i} + \mu b^\alpha &= 0 \\
H_a^\alpha + l^2 J_{ab} \ddot{V}_i^b X^{\alpha i} &= 0
\end{aligned}
\tag{3.4}
$$

where $S^{\alpha\beta}\Big|_\beta$ stands for the covariant derivative of $S^{\alpha\beta}$ in the metric tensor $C_{\alpha\beta} \equiv (P^i{}_{,\alpha} + U^i{}_{,\alpha})(P_{i,\beta} + U_{i,\beta})$ of the actual configuration of the body (at the instant t). All densities in Eqs (3.4) are related to this configuration and X^α can be treated as the convective coordinates. Similarly, Eqs (3.3) can be transformed to the form related to the actual configuration

$$
S^{\alpha\beta} \widetilde{n}_\beta = s^\alpha \tag{3.5}
$$

where $\widetilde{n}_\beta$ is the unit outward normal to the boundary of the region occupied by the body in the actual configuration and s^α are the pertinent boundary tractions. It has to be emphasized that by the actual configuration of a composite body we understand here its configuration, given by the set of positions $x_i = P_i(X) + U_i(X, t)$, $X \in \Omega$, of all material points at the time t. This configuration will be referred to as the macro-configuration since it is described by means of V_R-macro functions U_i.

Eqs (3.1) and (3.2) or (3.1) and (3.4) have to be considered together with boundary conditions for U_i and initial conditions for U_i, V_i^a. It has to be emphasized that solutions to the pertinent initial-boundary value problems have physical sense only if U_i, V_i^a are sufficiently regular V_R-macro functions.

4. Effective modulus theory

The effective modulus theory of elastic composite materials can be obtained from the refined theory by scaling the microstructure down. Setting $l \searrow 0$ in Eqs (3.2) we obtain $H^{\alpha}_{Ra} = 0$ and from Eqs (3.1) it follows that $V^{b}_{\beta} = -(C_R^{-1})^{ba}_{\beta\alpha} B^{\alpha\gamma\delta}_{Ra} E_{\gamma\delta}$, where $(C_R^{-1})^{ba}_{\beta\alpha}$ describe the linear transformation $R^{3n} \rightarrow R^{3n}$ inverse to that given by $C^{\alpha\beta}_{Rab}$. Hence, the governing equations of the effective modulus theory are

$$[S_R^{\alpha\beta}(P^i{}_{,\beta} + U^i{}_{,\beta})]_{,\alpha} - <\rho_R> \ddot{U}_i + <\rho_R> b_i = 0$$

$$S_R^{\alpha\beta} = A^{\alpha\beta\mu\nu}_{Reff} E_{\mu\nu} \tag{4.1}$$

$$E_{\alpha\beta} = \frac{1}{2}(P_{i,\alpha} U^i{}_{,\beta} + P_{i,\beta} U^i{}_{,\alpha}) + \frac{1}{2} U_{i,\alpha} U^i{}_{,\beta}$$

where $A^{\alpha\beta\mu\nu}_{Reff}$ are called the effective elastic modulae, related to the reference configuration, defined by

$$A^{\alpha\beta\mu\nu}_{Reff} = A_R^{\alpha\beta\mu\nu} - B^{\gamma\alpha\beta}_{Ra}(C_R^{-1})^{ab}_{\gamma\delta} B^{\delta\mu\nu}_{Rb} \tag{4.2}$$

The above equations were derived by the asymptotic approximation of the refined macro-elastodynamics. Using the asymptotic homogenization approach (cf Bensoussan et al. (1980), Bakhvalov and Panasenko (1984)) the effective modulae have to be calculated on the basis of a certain boundary value problem on representative volume element, but the governing equations have the form similar to that of Eqs (4.1). The basic unknowns in Eqs (4.1) and (4.2) are macrodisplacements U_i. The obtained nonlinear effective modulus theory takes into account the finite rotations of composite elements.

5. Refined versus effective modulus theory

Let us observe that for stationary problems the refined macro-elastodynamics and the effective modulus theory coincide. One can suppose that the effective modulus theory can be taken as a certain good approximation of the refined macro-elastodynamics also in non-stationary boundary-value problems. In order to prove that this statement is not true, we shall formulate a simple counterexample. For the sake of simplicity we restrict considerations to the linear theory and assume that $P^i(X) = \delta^i_\alpha X^\alpha$ for every $\boldsymbol{X} \in \Omega_R$. As the example let us consider a thick periodically laminated layer, bounded by coordinate planes $x_1 = 0$ and $x_1 = H$, in which the representative sublayer is made of two homogeneous laminae of thicknesses l', l''

and bounded by planes $x_1 = 0$, $x_1 = l'$ and $x_1 = l'$, $x_1 = l = l' + l''$, respectively, where l is very small compared to H; $l \ll H$. In this case we introduce only one micro-shape function $h = h(x_1)$, which is l-periodic and in $[0, l]$ takes the values $h(0) = h(l) = -l/2$, $h(l') = l/2$, being linear in the intervals $[0, l']$ and $[l', l]$. Let the layer be subjected to the boundary conditions $U^i(0,t) = 0$, $U^2(H,t) = U^3(H,t) = 0$, $U^1(H,t) = \delta$ for every $t > 0$, where $\delta =$ const, and let the initial conditions be independent of x_2, x_3 coordinates. Then the problem under consideration is independent of x_2, x_3 coordinates. Let us denote $U \equiv U^1$, $V \equiv V^1$, $A \equiv A^{1111}$, $B \equiv B_1^{111}$, $C \equiv C_{11}^{11}$ and $J \equiv J_{R11}$, where now $J_{R11} = <\rho_R> /12$. Eqs (2.7), (3.1) and (3.2) of the refined macro-elastodynamics, after the linearization and neglecting body forces, yield

$$
\begin{aligned}
&AU_{,11} + BV_{,1} - <\rho> \ddot{U} = 0 \\
&BU_{,1} + CV + l^2 J \ddot{V} = 0
\end{aligned}
$$

For the initial conditions: $U(x_1, 0) = \delta H^{-1} x_1$, $\dot{U}(x_1, 0) = 0$, $V(x_1, 0) = \dot{V}(x_1, 0) = 0$, $x_1 \in (0, H)$ and the aforementioned boundary conditions: $U(0,t) = 0$, $U(H,t) = \delta$, $t > 0$, denoting $\varepsilon \equiv \delta H^{-1}$, $\kappa^2 \equiv CJ^{-1}l^{-2}$, we obtain $U = \varepsilon x_1$, $V = \varepsilon BC^{-1}(\cos \kappa\tau - 1)$, and the macro-stresses $S = S_R^{11}$ given by the first one from Eqs (3.1), are equal to

$$
S = A_{eff}\varepsilon + \varepsilon B^2 C^{-1} \cos \kappa\tau \qquad A_{eff} \equiv A - B^2 C^{-1} \tag{5.1}
$$

Eqs (4.1) of the effective modulus theory reduce to the simple form

$$
A_{eff} U_{,11} - <\rho_R> \ddot{U} = 0
$$

and for similar boundary and initial conditions for $U(x_1, t)$ yield $U = \varepsilon x_1$ and

$$
S = A_{eff}\varepsilon \tag{5.2}
$$

Comparing solutions (5.1) and (5.2) it can be seen that they coincide only for a homogeneous body, for which $B = 0$. It follows that for a composite body effective modulus theory leads to the incorrect time independent solution (5.2) while the refined macro-elastodynamics describes the time oscillations of macrostresses (and also the oscillations of boundary tractions on $x_1 = 0$ and $x_1 = H$, cf Eq (3.3)) caused by the micro-inhomogeneity of the medium. Hence the trivial conclusion that in investigations of dynamic boundary-value problems for elastic composite materials the effective modulus theory can lead to incorrect solutions and should be replaced by the refined macro-elastodynamics.

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Skończone obroty w mikro-dynamice sprężystych kompozytów

Streszczenie

Mikrodynamika periodycznych kompozytów, opisująca wplyw wielkości mikrostruktury na makro-własności ośrodka, w ramach liniowej elastodynamiki zostala zaproponowana w pracy Woźniaka (1993), oraz stosowana w serii dalszych opracowań. Głównym celem tej pracy jest sformułowanie równań mikro-makrodynamiki dla sprężystych kompozytów poddanych małym odkształceniom przy skończonych obrotach i przemieszczeniach. Otrzymane wyniki zostaną zastosowane do analizy problemów geometrycznie nieliniowych dla wiotkich elementów konstrukcji z materiałów kompozytowych.