HAL open science

COMPLEMENTS ON DISCONNECTED REDUCTIVE GROUPS

François Digne, J Michel

To cite this version:

François Digne, J Michel. COMPLEMENTS ON DISCONNECTED REDUCTIVE GROUPS. 2014. hal-01119016

HAL Id: hal-01119016

https://hal.science/hal-01119016v1

Preprint submitted on 20 Feb 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

COMPLEMENTS ON DISCONNECTED REDUCTIVE GROUPS

F. DIGNE AND J. MICHEL

Dedicated to the memory of Robert Steinberg

Abstract. We present various results on disconnected reductive groups, in particular about the characteristic 0 representation theory of such groups over finite fields.

1. Introduction

Let $\mathbf{G}$ be a (possibly disconnected) linear algebraic group over an algebraically closed field. We assume that the connected component $\mathbf{G}^0$ is reductive, and then call $\mathbf{G}$ a (possibly disconnected) reductive group. This situation was studied by Steinberg in [St] where he introduced the notion of quasi-semi-simple elements.

Assume now that $\mathbf{G}$ is over an algebraic closure $\overline{\mathbb{F}}_q$ of the finite field $\mathbb{F}_q$, defined over $\mathbb{F}_q$ with corresponding Frobenius endomorphism F. Let $\mathbf{G}^1$ be an F-stable connected component of $\mathbf{G}$. We want to study $(\mathbf{G}^0)^F$-class functions on $(\mathbf{G}^1)^F$; if $\mathbf{G}^1$ generates $\mathbf{G}$, they coincide with $\mathbf{G}^F$-class functions on $(\mathbf{G}^1)^F$.

This setting we adopt here is also taken up by Lusztig in his series of papers on disconnected groups [Lu] and is slightly more general than the setting of [DM94], where we assumed that $\mathbf{G}^1$ contains an F-stable quasi-central element. A detailed comparison of both situations is done in the next section.

As the title says, this paper is a series of complements to our original paper [DM94] which are mostly straightforward developments that various people asked us about and that, as far as we know, have not appeared in the literature; we thank in particular Olivier Brunat, Gerhard Hiss, Cheryl Praeger and Karine Sorlin for asking these questions.

In section 2 we show how quite a few results of [DM94] are still valid in our more general setting.

In section 3 we take a "global" viewpoint to give a formula for the scalar product of two Deligne-Lusztig characters on the whole of $\mathbf{G}^F$.

In section 4 we show how to extend to disconnected groups the formula of Steinberg [St, 15.1] counting unipotent elements.

In section 5 we extend the theorem that tensoring Lusztig induction with the Steinberg character gives ordinary induction.

In section 6 we give a formula for the characteristic function of a quasi-semi-simple class, extending the case of a quasi-central class which was treated in [DM94].

In section 7 we show how to classify quasi-semi-simple conjugacy classes, first for a (possibly disconnected) reductive group over an arbitrary algebraically closed field, and then over $\mathbb{F}_q$.

Finally, in section 8 we extend to our setting previous results on Shintani descent.

2010 *Mathematics Subject Classification.* 20G15,20G40,20C33,20G05.

2. Preliminaries

In this paper, we consider a (possibly disconnected) algebraic group $\mathbf{G}$ over $\overline{\mathbb{F}}_q$ (excepted at the beginning of section 7 where we accept an arbitrary algebraically closed field), defined over $\mathbb{F}_q$ with corresponding Frobenius automorphism F. If $\mathbf{G}^1$ is an F-stable component of $\mathbf{G}$, we call class functions on $(\mathbf{G}^1)^F$ the complex-valued functions invariant under $(\mathbf{G}^0)^F$-conjugacy (or equivalently under $(\mathbf{G}^1)^F$-conjugacy). Note that if $\mathbf{G}^1$ does not generate $\mathbf{G}$, there may be less functions invariant by $\mathbf{G}^F$-conjugacy than by $(\mathbf{G}^1)^F$-conjugacy; but the propositions we prove will apply *in particular* to the $\mathbf{G}^F$-invariant functions so we do not lose any generality. The class functions on $(\mathbf{G}^1)^F$ are provided with the scalar product $\langle f, g\rangle_{(\mathbf{G}^1)^F} = |(\mathbf{G}^1)^F|^{-1}\sum_{h\in(\mathbf{G}^1)^F} f(h)\overline{g(h)}$. We call $\mathbf{G}$ reductive when $\mathbf{G}^0$ is reductive.

When $\mathbf{G}$ is reductive, following [St] we call quasi-semi-simple an element which normalizes a pair $\mathbf{T}^0 \subset \mathbf{B}^0$ of a maximal torus of $\mathbf{G}^0$ and a Borel subgroup of $\mathbf{G}^0$. Following [DM94, 1.15], we call quasi-central a quasi-semi-simple element σ which has maximal dimension of centralizer $C_{\mathbf{G}^0}(\sigma)$ (that we will also denote by $\mathbf{G}^{0\sigma}$) amongst all quasi-semi-simple elements of $\mathbf{G}^0 \cdot \sigma$.

In the sequel, we fix a reductive group $\mathbf{G}$ and (excepted in the next section where we take a "global" viewpoint) an F-stable connected component $\mathbf{G}^1$ of $\mathbf{G}$. In most of [DM94] we assumed that $(\mathbf{G}^1)^F$ contained a quasi-central element. Here we do not assume this. Note however that by [DM94, 1.34] $\mathbf{G}^1$ contains an element σ which induces an F-stable quasi-central automorphism of $\mathbf{G}^0$. Such an element will be enough for our purpose, and we fix one from now on.

By [DM94, 1.35] when $H^1(F, Z\mathbf{G}^0) = 1$ then $(\mathbf{G}^1)^F$ contains quasi-central elements. Here is an example where $(\mathbf{G}^1)^F$ does not contain quasi-central elements.

Example 2.1. Take $s = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}$ where $\xi \in \mathbb{F}_q - \mathbb{F}_q^2$, take $\mathbf{G}^0 = \mathrm{SL}_2$ and let $\mathbf{G} = <\mathbf{G}^0, s> \subset \mathrm{GL}_2$ endowed with the standard Frobenius endomorphism on GL_2, so that s is F-stable and $\mathbf{G}^F = \mathrm{GL}_2(\mathbb{F}_q)$. We take $\mathbf{G}^1 = \mathbf{G}^0 \cdot s$. Here quasi-central elements are central and coincide with $\mathbf{G}^0 \cdot s \cap Z\mathbf{G}$ which is nonempty since if $\eta \in \mathbb{F}_{q^2}$ is a square root of ξ then $\begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \in \mathbf{G}^0 \cdot s \cap Z\mathbf{G}$; but $\mathbf{G}^0 \cdot s$ does not meet $(Z\mathbf{G})^F$. □

In the above example $\mathbf{G}^1/\mathbf{G}^0$ is semisimple. No such example exists when $\mathbf{G}^1/\mathbf{G}^0$ is unipotent:

Lemma 2.2. *Let $\mathbf{G}^1$ be an F-stable connected component of $\mathbf{G}$ such that $\mathbf{G}^1/\mathbf{G}^0$ is unipotent. Then $(\mathbf{G}^1)^F$ contains unipotent quasi-central elements.*

Proof. Let $\mathbf{T}^0 \subset \mathbf{B}^0$ be a pair of an F-stable maximal torus of $\mathbf{G}^0$ and an F-stable Borel subgroup of $\mathbf{G}^0$. Then $N_{\mathbf{G}^F}(\mathbf{T}^0 \subset \mathbf{B}^0)$ meets $(\mathbf{G}^1)^F$, since any two F-stable pairs $\mathbf{T}^0 \subset \mathbf{B}^0$ are $(\mathbf{G}^0)^F$-conjugate. Let su be the Jordan decomposition of an element of $N_{(\mathbf{G}^1)^F}(\mathbf{T}^0 \subset \mathbf{B}^0)$. Then $s \in \mathbf{G}^0$ since $\mathbf{G}^1/\mathbf{G}^0$ is unipotent, and u is F-stable, unipotent and still in $N_{(\mathbf{G}^1)^F}(\mathbf{T}^0 \subset \mathbf{B}^0)$ thus quasi-semi-simple, so is quasi-central by [DM94, 1.33]. □

Note, however, that there may exist a unipotent quasi-central element σ which is rational as an automorphism but such that there is no rational element inducing the same automorphism.

Example 2.3. We give an example in $\mathbf{G} = \mathrm{SL}_5 \rtimes <\sigma'>$ where $\mathbf{G}^0 = \mathrm{SL}_5$ has the standard rational structure over a finite field $\mathbb{F}_q$ of characteristic 2 with $q \equiv 1 \bmod 5$ and σ' is the automorphism of $\mathbf{G}^0$ given by $g \mapsto J^t g^{-1} J$ where J is the antidiagonal matrix with all non-zero entries equal to 1, so that σ' stabilizes the pair $\mathbf{T}^0 \subset \mathbf{B}^0$ where $\mathbf{T}^0$ is the maximal torus of diagonal matrices and $\mathbf{B}^0$ the Borel subgroup of upper triangular matrices, hence σ' is quasi-semi-simple. Let t be the diagonal matrix with entries (a, a, a^{-4}, a, a) where a^{q-1} is a non trivial 5-th root of unity $\zeta \in \mathbb{F}_q$. We claim that $\sigma = t\sigma'$ is as announced: it is still quasi-semi-simple; we have $\sigma^2 = t\sigma'(t) = tt^{-1} = 1$ so that σ is unipotent; we have ${}^F\sigma = {}^F t t^{-1}\sigma = \zeta\sigma$, so that σ is rational as an automorphism but not rational. Moreover a rational element inducing the same automorphism must be of the form $z\sigma$ with z central in $\mathbf{G}^0$ and $z \cdot {}^F z^{-1} = \zeta\,\mathrm{Id}$; but the center $Z\mathbf{G}^0$ is generated by $\zeta\,\mathrm{Id}$ and for any $z = \zeta^k\,\mathrm{Id} \in Z\mathbf{G}^0$ we have $z \cdot {}^F z^{-1} = \zeta^{k(q-1)}\,\mathrm{Id} = \mathrm{Id} \neq \zeta\,\mathrm{Id}$. □

As in [DM94] we call "Levi" of $\mathbf{G}$ a subgroup $\mathbf{L}$ of the form $N_{\mathbf{G}}(\mathbf{L}^0 \subset \mathbf{P}^0)$ where $\mathbf{L}^0$ is a Levi subgroup of the parabolic subgroup $\mathbf{P}^0$ of $\mathbf{G}^0$. A particular case is a "torus" $N_{\mathbf{G}}(\mathbf{T}^0, \mathbf{B}^0)$ where $\mathbf{T}^0 \subset \mathbf{B}^0$ is a pair of a maximal torus of $\mathbf{G}^0$ and a Borel subgroup of $\mathbf{G}^0$; note that a "torus" meets all connected components of $\mathbf{G}$, while (contrary to what is stated erroneously after [DM94, 1.4]) this may not be the case for a "Levi".

We call "Levi" of $\mathbf{G}^1$ a set of the form $\mathbf{L}^1 = \mathbf{L} \cap \mathbf{G}^1$ where $\mathbf{L}$ is a "Levi" of $\mathbf{G}$ and the intersection is nonempty; note that if $\mathbf{G}^1$ does not generate $\mathbf{G}$, there may exist several "Levis" of $\mathbf{G}$ which have same intersection with $\mathbf{G}^1$. Nevertheless $\mathbf{L}^1$ determines $\mathbf{L}^0$ as the identity component of $<\mathbf{L}^1>$.

We assume now that $\mathbf{L}^1$ is an F-stable "Levi" of $\mathbf{G}^1$ of the form $N_{\mathbf{G}^1}(\mathbf{L}^0 \subset \mathbf{P}^0)$. If $\mathbf{U}$ is the unipotent radical of $\mathbf{P}^0$, we define $\mathbf{Y}^0_{\mathbf{U}} = \{x \in \mathbf{G}^0 \mid x^{-1} \cdot {}^F x \in \mathbf{U}\}$ on which $(g, l) \in \mathbf{G}^F \times \mathbf{L}^F$ such that $gl \in \mathbf{G}^0$ acts by $x \mapsto gxl$, where $\mathbf{L} = N_{\mathbf{G}}(\mathbf{L}^0, \mathbf{P}^0)$. Along the same lines as [DM94, 2.10] we define

Definition 2.4. *Let $\mathbf{L}^1$ be an F-stable "Levi" of $\mathbf{G}^1$ of the form $N_{\mathbf{G}^1}(\mathbf{L}^0 \subset \mathbf{P}^0)$ and let $\mathbf{U}$ be the unipotent radical of $\mathbf{P}^0$. For λ a class function on $(\mathbf{L}^1)^F$ and $g \in (\mathbf{G}^1)^F$ we set*

$$R_{\mathbf{L}^1}^{\mathbf{G}^1}(\lambda)(g) = |(\mathbf{L}^1)^F|^{-1} \sum_{l \in (\mathbf{L}^1)^F} \lambda(l)\,\mathrm{Trace}((g, l^{-1}) \mid H_c^*(\mathbf{Y}^0_{\mathbf{U}}))$$

and for γ a class function on $(\mathbf{G}^1)^F$ and $l \in (\mathbf{L}^1)^F$ we set

$${}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\gamma)(l) = |(\mathbf{G}^1)^F|^{-1} \sum_{g \in (\mathbf{G}^1)^F} \gamma(g)\,\mathrm{Trace}((g^{-1}, l) \mid H_c^*(\mathbf{Y}^0_{\mathbf{U}})).$$

In the above H_c^* denotes the ℓ-adic cohomology with compact support, where we have chosen once and for all a prime number $\ell \neq p$. In order to consider the virtual character $\mathrm{Trace}(x \mid H_c^*(\mathbf{X})) = \sum_i (-1)^i \,\mathrm{Trace}(x \mid H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_\ell))$ as a complex character we chose once and for all an embedding $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$.

Writing $R_{\mathbf{L}^1}^{\mathbf{G}^1}$ and ${}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}$ is an abuse of notation: the definition needs the choice of a $\mathbf{P}^0$ such that $\mathbf{L}^1 = N_{\mathbf{G}^1}(\mathbf{L}^0 \subset \mathbf{P}^0)$. Our subsequent statements will use an implicit choice. Under certain assumptions we will prove a Mackey formula (Theorem 2.6) which when true implies that $R_{\mathbf{L}^1}^{\mathbf{G}^1}$ and ${}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}$ are independent of the choice of $\mathbf{P}^0$.

By the same arguments as for [DM94, 2.10] (using that $(\mathbf{L}^1)^F$ is nonempty and [DM94, 2.3]) definition 2.4 agrees with the restriction to $(\mathbf{G}^1)^F$ and $(\mathbf{L}^1)^F$ of [DM94, 2.2].

The two maps $R_{\mathbf{L}^1}^{\mathbf{G}^1}$ and ${}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}$ are adjoint with respect to the scalar products on $(\mathbf{G}^1)^F$ and $(\mathbf{L}^1)^F$.

We note the following variation on [DM94, 2.6].

Proposition 2.5. *Let su be the Jordan decomposition of an element of $(\mathbf{G}^1)^F$ and λ a class function on $(\mathbf{L}^1)^F$;*

(i) *if s is central, if we set*

$$Q_{\mathbf{L}^0}^{\mathbf{G}^0}(u,v) = \begin{cases} \mathrm{Trace}((u,v) \mid H_c^*(\mathbf{Y}_{\mathbf{U}}^0)) & \text{if } uv \in \mathbf{G}^0 \\ 0 & \text{otherwise} \end{cases},$$

we have

$$(R_{\mathbf{L}^1}^{\mathbf{G}^1}\lambda)(su) = |(\mathbf{L}^0)^F|^{-1} \sum_{v \in (\mathbf{L}^0 \cdot u)_{unip}^F} Q_{\mathbf{L}^0}^{\mathbf{G}^0}(u, v^{-1})\lambda(sv)$$

(ii) *in general*

$$(R_{\mathbf{L}^1}^{\mathbf{G}^1}\lambda)(su) = \sum_{\{h \in (\mathbf{G}^0)^F \mid {}^h\mathbf{L} \ni s\}} \frac{|{}^h\mathbf{L}^0 \cap C_{\mathbf{G}}(s)^{0F}|}{|(\mathbf{L}^0)^F||C_{\mathbf{G}}(s)^{0F}|} R_{{}^h\mathbf{L}^1 \cap C_{\mathbf{G}}(s)^0 \cdot su}^{C_{\mathbf{G}}(s)^0 \cdot su}({}^h\lambda)(su).$$

(iii) *if tv is the Jordan decomposition of an element of $(\mathbf{L}^1)^F$ and γ a class function on $(\mathbf{G}^1)^F$; we have*

$$({}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}\gamma)(tv) = |(\mathbf{G}^{t0})^F|^{-1} \sum_{u \in (\mathbf{G}^{t0} \cdot v)_{unip}^F} Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}(u, v^{-1})\gamma(tu).$$

In the above we abused notation to write ${}^h\mathbf{L} \ni s$ for $< \mathbf{L}^1 > \ni {}^{h^{-1}}s$.

Proof. (i) results from [DM94, 2.6(i)] using the same arguments as the proof of [DM94, 2.10]; we then get (ii) by plugging back (i) in [DM94, 2.6(i)]. □

In our setting the Mackey formula [DM94, 3.1] is still valid in the cases where we proved it [DM94, Théorème 3.2] and [DM94, Théorème 4.5]. Before stating it notice that [DM94, 1.40] remains true without assuming that $(\mathbf{G}^1)^F$ contains quasi-central elements, replacing in the proof $(\mathbf{G}^0)^F.\sigma$ with $(\mathbf{G}^1)^F$, which shows that any F-stable "Levi" of $\mathbf{G}^1$ is $(\mathbf{G}^0)^F$-conjugate to a "Levi" containing σ. This explains why we only state the Mackey formula in the case of "Levis" containing σ.

Theorem 2.6. *If $\mathbf{L}^1$ and $\mathbf{M}^1$ are two F-stable "Levis" of $\mathbf{G}^1$ containing σ then under one of the following assumptions:*

- *$\mathbf{L}^0$ (resp. $\mathbf{M}^0$) is a Levi subgroup of an F-stable parabolic subgroup normalized by $\mathbf{L}^1$ (resp. $\mathbf{M}^1$).*
- *one of $\mathbf{L}^1$ and $\mathbf{M}^1$ is a "torus"*

we have

$${}^*R_{\mathbf{L}^1}^{\mathbf{G}^1} R_{\mathbf{M}^1}^{\mathbf{G}^1} = \sum_{x \in [\mathbf{L}^{\sigma 0 F} \backslash \mathcal{S}_{\mathbf{G}^{\sigma 0}}(\mathbf{L}^{\sigma 0}, \mathbf{M}^{\sigma 0})^F / \mathbf{M}^{\sigma 0 F}]} R_{(\mathbf{L}^1 \cap {}^x\mathbf{M}^1)}^{\mathbf{L}^1} {}^*R_{(\mathbf{L}^1 \cap {}^x\mathbf{M}^1)}^{{}^x\mathbf{M}^1} \operatorname{ad} x$$

where $\mathcal{S}_{\mathbf{G}^{\sigma 0}}(\mathbf{L}^{\sigma 0}, \mathbf{M}^{\sigma 0})$ is the set of elements $x \in \mathbf{G}^{\sigma 0}$ such that $\mathbf{L}^{\sigma 0} \cap {}^x\mathbf{M}^{\sigma 0}$ contains a maximal torus of $\mathbf{G}^{\sigma 0}$.

Proof. We first prove the theorem in the case of F-stable parabolic subgroups $\mathbf{P}^0 = \mathbf{L}^0 \ltimes \mathbf{U}$ and $\mathbf{Q}^0 = \mathbf{M}^0 \ltimes \mathbf{V}$ following the proof of [DM94, 3.2]. The difference is that the variety we consider here is the intersection with $\mathbf{G}^0$ of the variety considered in *loc. cit.*. Here, the left-hand side of the Mackey formula is given by $\overline{\mathbb{Q}}_\ell[(\mathbf{U}^F\backslash(\mathbf{G}^0)^F/\mathbf{V}^F)^\sigma]$ instead of $\overline{\mathbb{Q}}_\ell[(\mathbf{U}^F\backslash(\mathbf{G}^0)^F.<\sigma>/\mathbf{V}^F)^\sigma]$. Nevertheless we can use [DM94, Lemma 3.3] which remains valid with the same proof. As for [DM94, Lemma 3.5], we have to replace it with

Lemma 2.7. *For any $x \in \mathcal{S}_{\mathbf{G}^{\sigma 0}}(\mathbf{L}^{\sigma 0}, \mathbf{M}^{\sigma 0})^F$ the map $(l(\mathbf{L}^0 \cap {}^x\mathbf{V}^F), ({}^x\mathbf{M}^0 \cap \mathbf{U}^F) \cdot {}^x m) \mapsto \mathbf{U}^F lxm\mathbf{V}^F$ is an isomorphism from $(\mathbf{L}^0)^F/(\mathbf{L}^0 \cap {}^x\mathbf{V}^F) \times_{(\mathbf{L}^0 \cap {}^x\mathbf{M}^0)^F} ({}^x\mathbf{M}^0 \cap \mathbf{U}^F)\backslash {}^x(\mathbf{M}^0)^F$ to $\mathbf{U}^F\backslash(\mathbf{P}^0)^F x(\mathbf{Q}^0)^F/\mathbf{V}^F$ which is compatible with the action of $(\mathbf{L}^1)^F \times ((\mathbf{M}^1)^F)^{-1}$ where the action of $(\lambda, \mu^{-1}) \in (\mathbf{L}^1)^F \times ((\mathbf{M}^1)^F)^{-1}$ maps $(l(\mathbf{L}^0 \cap {}^x\mathbf{V}^F), ({}^x\mathbf{M}^0 \cap \mathbf{U}^F) \cdot {}^x m)$ to the class of $(\lambda l \nu^{-1}(\mathbf{L}^0 \cap {}^x\mathbf{V}^F), ({}^x\mathbf{M}^0 \cap \mathbf{U}^F) \cdot \nu {}^x m \mu^{-1})$ with $\nu \in (\mathbf{L}^1)^F \cap {}^x(\mathbf{M}^1)^F$ (independent of ν).*

Proof. The isomorphism of the lemma involves only connected groups and is a known result (see *e.g.* [DM91, 5.7]). The compatibility with the actions is straightforward. □

This allows to complete the proof in the first case.

We now prove the second case following section 4 of [DM94]. We first notice that the statement and proof of Lemma 4.1 in [DM94] don't use the element σ but only its action. In Lemma 4.2, 4.3 and 4.4 there is no σ involved but only the action of the groups $\mathbf{L}^F$ and $\mathbf{M}^F$ on the pieces of a variety depending only on $\mathbf{L}$, $\mathbf{M}$ and the associated parabolics. This gives the second case. □

We now rephrase [DM94, 4.8] and [DM94, 4.11] in our setting, specializing the Mackey formula to the case of two "tori". Let $\mathcal{T}_1$ be the set of "tori" of $\mathbf{G}^1$; if $\mathbf{T}^1 = N_{\mathbf{G}^1}(\mathbf{T}^0, \mathbf{B}^0) \in \mathcal{T}_1^F$ then $\mathbf{T}^0$ is F-stable. We define $\mathrm{Irr}((\mathbf{T}^1)^F)$ as the set of restrictions to $(\mathbf{T}^1)^F$ of extensions to $<(\mathbf{T}^1)^F>$ of elements of $\mathrm{Irr}((\mathbf{T}^0)^F)$.

Proposition 2.8. *If $\mathbf{T}^1, \mathbf{T}'^1 \in \mathcal{T}_1^F$ and $\theta \in \mathrm{Irr}((\mathbf{T}^1)^F), \theta' \in \mathrm{Irr}((\mathbf{T}'^1)^F)$ then*

$$\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta), R_{\mathbf{T}'^1}^{\mathbf{G}^1}(\theta')\rangle_{(\mathbf{G}^1)^F} = 0 \text{ unless } (\mathbf{T}^1, \theta) \text{ and } (\mathbf{T}'^1, \theta') \text{ are } (\mathbf{G}^0)^F\text{-conjugate.}$$

And

(i) *If for some $n \in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)$ and $\zeta \neq 1$ we have ${}^n\theta = \zeta\theta$ then $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) = 0$.*

(ii) *Otherwise $\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta), R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F} = |\{n \in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1) \mid {}^n\theta = \theta\}|/|(\mathbf{T}^1)^F|$.*

If $\mathbf{T}^1 = \mathbf{T}'^1$ the above can be written

$$\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta), R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta')\rangle_{(\mathbf{G}^1)^F} = \langle \mathrm{Ind}_{(\mathbf{T}^1)^F}^{N_{\mathbf{G}^1}(\mathbf{T}^0)^F} \theta, \mathrm{Ind}_{(\mathbf{T}^1)^F}^{N_{\mathbf{G}^1}(\mathbf{T}^0)^F} \theta'\rangle_{N_{\mathbf{G}^1}(\mathbf{T}^0)^F}$$

where when $A^1 \subset B^1$ are cosets of finite groups $A^0 \subset B^0$ and χ is a A^0-class function on A^1 we set $\mathrm{Ind}_{A^1}^{B^1} \chi(x) = |A^0|^{-1} \sum_{\{y \in B^0 \mid {}^y x \in A^1\}} \chi({}^y x)$.

Proof. As noticed above Theorem 2.6 we may assume that $\mathbf{T}^1$ and $\mathbf{T}'^1$ contain σ. By [DM94, 1.39], if $\mathbf{T}^1$ and $\mathbf{T}'^1$ contain σ, they are $(\mathbf{G}^0)^F$ conjugate if and only if they are conjugate under $\mathbf{G}^{\sigma 0^F}$. The Mackey formula shows then that the scalar product vanishes when $\mathbf{T}^1$ and $\mathbf{T}'^1$ are not $(\mathbf{G}^0)^F$-conjugate.

Otherwise we may assume $\mathbf{T}^1 = \mathbf{T}'^1$ and the Mackey formula gives

$$\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta), R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F} = |(\mathbf{T}^{\sigma 0})^F|^{-1} \sum_{n \in N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F} \langle \theta, {}^n\theta\rangle_{(\mathbf{T}^1)^F}.$$

The term $\langle\theta, {}^n\theta\rangle_{(\mathbf{T}^1)^F}$ is 0 unless ${}^n\theta = \zeta_n\theta$ for some constant ζ_n and in this last case $\langle\theta, {}^n\theta\rangle_{(\mathbf{T}^1)^F} = \overline{\zeta}_n$. If ${}^{n'}\theta = \zeta_{n'}\theta$ then ${}^{nn'}\theta = \zeta_{n'}{}^n\theta = \zeta_{n'}\zeta_n\theta$ thus the ζ_n form a group; if this group is not trivial, that is some ζ_n is not equal to 1, we have $\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta), R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F} = 0$ which implies that in this case $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) = 0$. This gives (i) since by [DM94, 1.39], if $\mathbf{T}^1 \ni \sigma$ then $N_{(\mathbf{G}^0)^F}(\mathbf{T}^1) = N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F \cdot (\mathbf{T}^0)^F$, so that if there exists n as in (i) there exists an $n \in N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F$ with same action on θ since $(\mathbf{T}^0)^F$ has trivial action on θ.

In case (ii), for each non-zero term we have ${}^n\theta = \theta$ and we have to check that the value $|((\mathbf{T}^\sigma)^0)^F|^{-1}\{n \in N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F \mid {}^n\theta = \theta\}|$ given by the Mackey formula is equal to the stated value. This results again from [DM94, 1.39] written $N_{(\mathbf{G}^0)^F}(\mathbf{T}^1) = N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^1)^F \cdot (\mathbf{T}^0)^F$, and from $N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^1)^F \cap (\mathbf{T}^0)^F = ((\mathbf{T}^\sigma)^0)^F$.

We now prove the final remark. By definition we have

$$\langle \mathrm{Ind}_{(\mathbf{T}^1)^F}^{N_{\mathbf{G}^1}(\mathbf{T}^0)^F}\theta, \mathrm{Ind}_{(\mathbf{T}^1)^F}^{N_{\mathbf{G}^1}(\mathbf{T}^0)^F}\theta'\rangle_{N_{\mathbf{G}^1}(\mathbf{T}^0)^F} = \\ |N_{\mathbf{G}^1}(\mathbf{T}^0)^F|^{-1}|(\mathbf{T}^1)^F|^{-2}\sum_{x\in N_{\mathbf{G}^1}(\mathbf{T}^0)^F}\ \sum_{\{n,n'\in N_{\mathbf{G}^1}(\mathbf{T}^0)^F \mid {}^nx, {}^{n'}x\in\mathbf{T}^1\}} \theta({}^nx)\overline{\theta({}^{n'}x)}.$$

Doing the summation over $t = {}^nx$ and $n'' = n'n^{-1} \in N_{\mathbf{G}^0}(\mathbf{T}^0)^F$ we get

$$|N_{\mathbf{G}^1}(\mathbf{T}^0)^F|^{-1}|(\mathbf{T}^1)^F|^{-2}\sum_{t\in(\mathbf{T}^1)^F}\ \sum_{n\in N_{\mathbf{G}^1}(\mathbf{T}^0)^F}\ \sum_{\{n''\in N_{\mathbf{G}^0}(\mathbf{T}^0)^F \mid {}^{n''}t\in\mathbf{T}^1\}} \theta(t)\overline{\theta({}^{n''}t)}.$$

The conditions $n'' \in N_{\mathbf{G}^0}(\mathbf{T}^0)^F$ together with ${}^{n''}t \in \mathbf{T}^1$ are equivalent to $n'' \in N_{\mathbf{G}^0}(\mathbf{T}^1)^F$, so that we get $|(\mathbf{T}^1)^F|^{-1}\sum_{n''\in N_{\mathbf{G}^0}(\mathbf{T}^1)^F}\langle\theta, {}^{n''}\theta\rangle_{(\mathbf{T}^1)^F}$. As explained in the first part of the proof, the scalar product $\langle\theta, {}^{n''}\theta\rangle_{(\mathbf{T}^1)^F}$ is zero unless ${}^{n''}\theta = \zeta_{n''}\theta$ for some root of unity $\zeta_{n''}$ and arguing as in the first part of the proof we find that the above sum is zero if there exists n'' such that $\zeta_{n''} \neq 1$ and is equal to $|(\mathbf{T}^1)^F|^{-1}|\{n \in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1) \mid {}^n\theta = \theta\}|$ otherwise. □

Remark 2.9. In the context of Proposition 2.8, if σ is F-stable then we may apply θ to it and for any $n \in N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F$ we have $\theta({}^n\sigma) = \theta(\sigma)$ so for any $n \in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)$ and ζ such that ${}^n\theta = \zeta\theta$ we have $\zeta = 1$. When $H^1(F, Z\mathbf{G}^0) = 1$ we may choose σ to be F-stable, so that $\zeta \neq 1$ never happens.

Here is an example where $\zeta_n = -1$, thus $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) = 0$: we take again the context of Example 2.1 and take $\mathbf{T}^0 = \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}$ and let $\mathbf{T}^1 = \mathbf{T}^0 \cdot s$; let us define θ on $ts \in (\mathbf{T}^1)^F$ by $\theta(ts) = -\lambda(t)$ where λ is the non-trivial order 2 character of $(\mathbf{T}^0)^F$ (Legendre symbol); then for any $n \in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)\backslash\mathbf{T}^0$ we have ${}^n\theta = -\theta$. □

Proposition [DM94, 4.11] extends as follows to our context:

Corollary 2.10 (of 2.8)**.** *Let $p^{\mathbf{G}^1}$ be the projector to uniform functions on $(\mathbf{G}^1)^F$. We have*

$$p^{\mathbf{G}^1} = |(\mathbf{G}^1)^F|^{-1}\sum_{\mathbf{T}^1\in\mathcal{T}_1^F}|(\mathbf{T}^1)^F|R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ {}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}.$$

Proof. We have only to check that for any $\theta \in \mathrm{Irr}((\mathbf{T}^1)^F)$ such that $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) \neq 0$ and any class function χ on $(\mathbf{G}^1)^F$ we have $\langle p^{\mathbf{G}^1}\chi, R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F} = \langle\chi, R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F}$.

By Proposition 2.8, to evaluate the left-hand side we may restrict the sum to tori conjugate to $\mathbf{T}^1$, so we get

$$\begin{aligned}\langle p^{\mathbf{G}^1}\chi, R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F} &= |N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)|^{-1}|(\mathbf{T}^1)^F|\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ {}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}\chi, R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F}\\ &= |N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)|^{-1}|(\mathbf{T}^1)^F|\langle \chi, R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ {}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F}.\end{aligned}$$

The equality to prove is true if $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)=0$; otherwise by Proposition 2.8 we have ${}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) = |(\mathbf{T}^1)^F|^{-1}\sum_{n\in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)} {}^n\theta$, whence in that case

$$R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ {}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) = |(\mathbf{T}^1)^F|^{-1}|N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)|R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta),$$

since $R_{\mathbf{T}^1}^{\mathbf{G}^1}({}^n\theta) = R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)$, whence the result. □

We now adapt the definition of duality to our setting.

Definition 2.11.
- *For a connected reductive group $\mathbf{G}$, we define the $\mathbb{F}_q$-rank as the maximal dimension of a split torus, and define $\varepsilon_{\mathbf{G}} = (-1)^{\mathbb{F}_q\text{-rank of }\mathbf{G}}$ and $\eta_{\mathbf{G}} = \varepsilon_{\mathbf{G}/\operatorname{rad}\mathbf{G}}$.*
- *For an F-stable connected component $\mathbf{G}^1$ of a (possibly disconnected) reductive group we define $\varepsilon_{\mathbf{G}^1} = \varepsilon_{\mathbf{G}^{\sigma 0}}$ and $\eta_{\mathbf{G}^1} = \eta_{\mathbf{G}^{\sigma 0}}$ where σ is a quasi-central element of $\mathbf{G}^1$ which induces an F-stable automorphism of $\mathbf{G}^0$.*

Let us see that these definitions agree with [DM94]: in [DM94, 3.6(i)], we define $\varepsilon_{\mathbf{G}^1}$ to be $\varepsilon_{\mathbf{G}^{0\tau}}$ where τ is any quasi-semi-simple element of $\mathbf{G}^1$ which induces an F-stable automorphism of $\mathbf{G}^0$ and lies in a "torus" of the form $N_{\mathbf{G}^1}(\mathbf{T}_0\subset\mathbf{B}_0)$ where both $\mathbf{T}^0$ and $\mathbf{B}^0$ are F-stable; by [DM94, 1.36(ii)] a σ as above is such a τ.

We fix an F-stable pair $(\mathbf{T}_0\subset\mathbf{B}_0)$ and define duality on $\operatorname{Irr}((\mathbf{G}^1)^F)$ by

$$D_{\mathbf{G}^1} = \sum_{\mathbf{P}^0\supset\mathbf{B}^0}\eta_{\mathbf{L}^1}R_{\mathbf{L}^1}^{\mathbf{G}^1}\circ {}^*R_{\mathbf{L}^1}^{\mathbf{G}^1} \tag{2.12}$$

where in the sum $\mathbf{P}^0$ runs over F-stable parabolic subgroups containing $\mathbf{B}^0$ such that $N_{\mathbf{G}^1}(\mathbf{P}^0)$ is non empty, and $\mathbf{L}^1$ denotes $N_{\mathbf{G}^1}(\mathbf{L}^0\subset\mathbf{P}^0)$ where $\mathbf{L}^0$ is the Levi subgroup of $\mathbf{P}^0$ containing $\mathbf{T}^0$. The duality thus defined coincides with the duality defined in [DM94, 3.10] when σ is in $(\mathbf{G}^1)^F$.

In our context we can define $\mathrm{St}_{\mathbf{G}^1}$ similarly to [DM94, 3.16], as $D_{\mathbf{G}^1}(\mathrm{Id}_{\mathbf{G}^1})$, and [DM94, 3.18] remains true:

Proposition 2.13. $\mathrm{St}_{\mathbf{G}^1}$ *vanishes outside quasi-semi-simple elements, and if $x\in(\mathbf{G}^1)^F$ is quasi-semi-simple we have*

$$\mathrm{St}_{\mathbf{G}^1}(x) = \varepsilon_{\mathbf{G}^1}\varepsilon_{(\mathbf{G}^x)^0}|(\mathbf{G}^x)^0|_p.$$

3. A global formula for the scalar product of Deligne-Lusztig characters.

In this section we give a result of a different flavor, where we do not restrict our attention to a connected component $\mathbf{G}^1$.

Definition 3.1. *For any character θ of $\mathbf{T}^F$, we define $R_{\mathbf{T}}^{\mathbf{G}}$ as in* [DM94, 2.2]. *If for a "torus" $\mathbf{T}$ and $\alpha = g\mathbf{G}^0\in\mathbf{G}/\mathbf{G}^0$ we denote by $\mathbf{T}^{[\alpha]}$ or $\mathbf{T}^{[g]}$ the unique connected component of $\mathbf{T}$ which meets $g\mathbf{G}^0$, this is equivalent for $g\in\mathbf{G}^F$ to*

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = |(\mathbf{T}^0)^F|/|\mathbf{T}^F|\sum_{\{a\in[\mathbf{G}^F/(\mathbf{G}^0)^F]\,|\,{}^ag\in\mathbf{T}^F(\mathbf{G}^0)^F\}} R_{\mathbf{T}^{[{}^ag]}}^{\mathbf{G}^{[{}^ag]}}(\theta)({}^ag)$$

where the right-hand side is defined by 2.4 (see [DM94, 2.3]*).*

We deduce from Proposition 2.8 the following formula for the whole group $\mathbf{G}$:

Proposition 3.2. *Let $\mathbf{T}$, $\mathbf{T}'$ be two "tori" of $\mathbf{G}$ and let $\theta \in \mathrm{Irr}(\mathbf{T}^F), \theta' \in \mathrm{Irr}(\mathbf{T}'^F)$. Then $\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta')\rangle_{\mathbf{G}^F} = 0$ if $\mathbf{T}^0$ and $\mathbf{T}'^0$ are not $\mathbf{G}^F$-conjugate, and if $\mathbf{T}^0 = \mathbf{T}'^0$ we have*

$$\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta')\rangle_{\mathbf{G}^F} = \langle \mathrm{Ind}_{\mathbf{T}^F}^{N_{\mathbf{G}}(\mathbf{T}^0)^F}(\theta), \mathrm{Ind}_{\mathbf{T}'^F}^{N_{\mathbf{G}}(\mathbf{T}^0)^F}(\theta')\rangle_{N_{\mathbf{G}}(\mathbf{T}^0)^F}.$$

Proof. Definition 3.1 can be written

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = |(\mathbf{T}^0)^F|/|\mathbf{T}^F| \sum_{\{a\in[\mathbf{G}^F/(\mathbf{G}^0)^F]\,|\,{}^a g\in \mathbf{T}^F(\mathbf{G}^0)^F\}} R_{({}^{a^{-1}}\mathbf{T})^{[g]}}^{\mathbf{G}^{[g]}}({}^{a^{-1}}\theta)(g).$$

So the scalar product we want to compute is equal to

$$\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta')\rangle_{\mathbf{G}^F} = \frac{1}{|\mathbf{G}^F|}\frac{|(\mathbf{T}^0)^F|}{|\mathbf{T}^F|}\frac{|\mathbf{T}'^{0F}|}{|\mathbf{T}'^F|}$$
$$\sum_{\substack{\alpha\in\mathbf{G}^F/\mathbf{G}^{0F}\\ g\in(\mathbf{G}^0)^F.\alpha}} \sum_{\substack{\{a\in[\mathbf{G}^F/(\mathbf{G}^0)^F]\,|\,{}^a\alpha\in\mathbf{T}^F(\mathbf{G}^0)^F\}\\ \{a'\in[\mathbf{G}^F/(\mathbf{G}^0)^F]\,|\,{}^{a'}\alpha\in\mathbf{T}'^F(\mathbf{G}^0)^F\}}} R_{({}^{a^{-1}}\mathbf{T})^{[\alpha]}}^{\mathbf{G}.\alpha}({}^{a^{-1}}\theta)(g)\overline{R_{({}^{a'^{-1}}\mathbf{T}')^{[\alpha]}}^{\mathbf{G}.\alpha}({}^{a'^{-1}}\theta')(g)},$$

which can be written

$$\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta')\rangle_{\mathbf{G}^F} = \frac{|(\mathbf{G}^0)^F|}{|\mathbf{G}^F|}\frac{|(\mathbf{T}^0)^F|}{|\mathbf{T}^F|}\frac{|\mathbf{T}'^{0F}|}{|\mathbf{T}'^F|}$$
$$\sum_{\alpha\in\mathbf{G}^F/\mathbf{G}^{0F}} \sum_{\substack{\{a\in[\mathbf{G}^F/(\mathbf{G}^0)^F]\,|\,{}^a\alpha\in\mathbf{T}^F(\mathbf{G}^0)^F\}\\ \{a'\in[\mathbf{G}^F/(\mathbf{G}^0)^F]\,|\,{}^{a'}\alpha\in\mathbf{T}'^F(\mathbf{G}^0)^F\}}} \langle R_{({}^{a^{-1}}\mathbf{T})^{[\alpha]}}^{\mathbf{G}.\alpha}({}^{a^{-1}}\theta), R_{({}^{a'^{-1}}\mathbf{T}')^{[\alpha]}}^{\mathbf{G}.\alpha}({}^{a'^{-1}}\theta')\rangle_{(\mathbf{G}^0)^F.\alpha}.$$

By Proposition 2.8 the scalar product on the right-hand side is zero unless $({}^{a^{-1}}\mathbf{T})^{[\alpha]}$ and $({}^{a'^{-1}}\mathbf{T}')^{[\alpha]}$ are $(\mathbf{G}^0)^F$-conjugate, which implies that $\mathbf{T}^0$ and $\mathbf{T}'^0$ are $(\mathbf{G}^0)^F$-conjugate. So we can assume that $\mathbf{T}^0 = \mathbf{T}'^0$. Moreover for each a' indexing a non-zero summand, there is a representative $y \in a'^{-1}(\mathbf{G}^0)^F$ such that $({}^y\mathbf{T}')^{[\alpha]} = ({}^{a^{-1}}\mathbf{T})^{[\alpha]}$. This last equality and the condition on a imply the condition ${}^{a'}\alpha \in \mathbf{T}'^F(\mathbf{G}^0)^F$ since this condition can be written $({}^y\mathbf{T}')^{[\alpha]} \neq \emptyset$. Thus we can do the summation over all such $y \in \mathbf{G}^F$, provided we divide by $|N_{(\mathbf{G}^0)^F}(({}^{a^{-1}}\mathbf{T})^{[\alpha]})|$. So we get, applying Proposition 2.8 that the above expression is equal to

$$\frac{|(\mathbf{G}^0)^F|}{|\mathbf{G}^F|}\frac{|(\mathbf{T}^0)^F|^2}{|\mathbf{T}^F||\mathbf{T}'^F|} \sum_{\alpha\in\mathbf{G}^F/\mathbf{G}^{0F}} \sum_{\{a\in[\mathbf{G}^F/(\mathbf{G}^0)^F]\,|\,{}^a\alpha\in\mathbf{T}^F(\mathbf{G}^0)^F\}} |N_{(\mathbf{G}^0)^F}(({}^{a^{-1}}\mathbf{T})^{[\alpha]})|^{-1}$$
$$\sum_{\{y\in\mathbf{G}^F\,|\,({}^y\mathbf{T}')^{[\alpha]}=({}^{a^{-1}}\mathbf{T})^{[\alpha]}\}} \langle \mathrm{Ind}_{({}^{a^{-1}}\mathbf{T})^{[\alpha]}}^{N_{\mathbf{G}^0.\alpha}({}^{a^{-1}}\mathbf{T}^0)^F}\ {}^{a^{-1}}\theta, \mathrm{Ind}_{({}^{a^{-1}}\mathbf{T})^{[\alpha]}}^{N_{\mathbf{G}^0.\alpha}({}^{a^{-1}}\mathbf{T}^0)^F}\ {}^y\theta'\rangle_{N_{\mathbf{G}^0.\alpha}(\mathbf{T}^0)^F}.$$

We now conjugate everything by a, take ay as new variable y and set $b = {}^a\alpha$. We get

$$\frac{|(\mathbf{T}^0)^F|^2}{|\mathbf{T}^F||\mathbf{T}'^F|} \sum_{b\in\mathbf{T}^F/(\mathbf{T}^0)^F} |N_{(\mathbf{G}^0)^F}(\mathbf{T}^{[b]})|^{-1} \tag{3.3}$$
$$\sum_{\{y\in\mathbf{G}^F\,|\,({}^y\mathbf{T}')^{[b]}=\mathbf{T}^{[b]}\}} \langle \mathrm{Ind}_{\mathbf{T}^{[b]F}}^{N_{\mathbf{G}^0.b}(\mathbf{T}^0)^F}\ \theta, \mathrm{Ind}_{\mathbf{T}^{[b]F}}^{N_{\mathbf{G}^0.b}(\mathbf{T}^0)^F}\ {}^y\theta'\rangle_{N_{\mathbf{G}^0.b}(\mathbf{T}^0)^F},$$

since for $b \in \mathbf{T}^F/(\mathbf{T}^0)^F$ any choice of $a \in \mathbf{G}^F/(\mathbf{G}^0)^F$ gives an $\alpha = {}^{a^{-1}}b$ which satisfies the condition ${}^a\alpha \in \mathbf{T}^F(\mathbf{G}^0)^F$.

Let us now transform the right-hand side of 3.2. Using the definition we have

$$\langle \mathrm{Ind}_{\mathbf{T}^F}^{N_{\mathbf{G}}(\mathbf{T}^0)^F}(\theta), \mathrm{Ind}_{\mathbf{T}'^F}^{N_{\mathbf{G}}(\mathbf{T}^0)^F}(\theta)\rangle_{N_{\mathbf{G}}(\mathbf{T}^0)^F} =$$
$$|\mathbf{T}^F|^{-1}|\mathbf{T}'^F|^{-1}|N_{\mathbf{G}}(\mathbf{T}^0)^F|^{-1} \sum_{\{n,x,x' \in N_{\mathbf{G}}(\mathbf{T}^0)^F \mid {}^xn \in \mathbf{T}, {}^{x'}n \in \mathbf{T}'\}} \theta({}^xn)\overline{\theta'({}^{x'}n)} =$$
$$|\mathbf{T}^F|^{-1}|\mathbf{T}'^F|^{-1}|N_{\mathbf{G}}(\mathbf{T}^0)^F|^{-1}$$
$$\sum_{b,a,a' \in [N_{\mathbf{G}}(\mathbf{T}^0)^F/N_{\mathbf{G}^0}(\mathbf{T}^0)^F]} \sum_{\left\{ \begin{smallmatrix} n \in N_{\mathbf{G}^0}(\mathbf{T}^0)^F b \\ x_0, x_0' \in N_{\mathbf{G}^0}(\mathbf{T}^0)^F \end{smallmatrix} \middle| \begin{smallmatrix} {}^{x_0}n \in ({}^{a^{-1}}\mathbf{T})^{[b]} \\ {}^{x_0'}n \in ({}^{a'^{-1}}\mathbf{T}')^{[b]} \end{smallmatrix} \right\}} {}^{a^{-1}}\theta({}^{x_0}n)\overline{{}^{a'^{-1}}\theta'({}^{x_0'}n)} =$$
$$\frac{|(\mathbf{T}^0)^F|}{|\mathbf{T}^F|}\frac{|\mathbf{T}'^{0F}|}{|\mathbf{T}'^F|}\frac{|N_{\mathbf{G}^0}(\mathbf{T}^0)^F|}{|N_{\mathbf{G}}(\mathbf{T}^0)^F|}$$
$$\sum_{b,a,a' \in [N_{\mathbf{G}}(\mathbf{T}^0)^F/N_{\mathbf{G}^0}(\mathbf{T}^0)^F]} \langle \mathrm{Ind}_{({}^{a^{-1}}\mathbf{T})^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b} {}^{a^{-1}}\theta, \mathrm{Ind}_{({}^{a'^{-1}}\mathbf{T}')^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b} {}^{a'^{-1}}\theta'\rangle_{N_{\mathbf{G}^0}(\mathbf{T}^0)^F b}.$$

We may simplify the sum by conjugating by a the terms in the scalar product to get

$$\langle \mathrm{Ind}_{\mathbf{T}^{[{}^ab]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot {}^ab}\theta, \mathrm{Ind}_{({}^{aa'^{-1}}\mathbf{T}')^{[{}^ab]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot {}^ab} {}^{aa'^{-1}}\theta'\rangle_{N_{\mathbf{G}^0}(\mathbf{T}^0)^F {}^ab}$$

then we may take, given a, the conjugate ${}^ab$ as new variable b, and aa'^{-1} as the new variable a' to get

$$\frac{|(\mathbf{T}^0)^F|}{|\mathbf{T}^F|}\frac{|\mathbf{T}'^{0F}|}{|\mathbf{T}'^F|} \sum_{b,a' \in [\frac{N_{\mathbf{G}}(\mathbf{T}^0)^F}{N_{\mathbf{G}^0}(\mathbf{T}^0)^F}]} \langle \mathrm{Ind}_{\mathbf{T}^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b}\theta, \mathrm{Ind}_{({}^{a'}\mathbf{T}')^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b} {}^{a'}\theta'\rangle_{N_{\mathbf{G}^0}(\mathbf{T}^0)^F b}.$$

Now, by Frobenius reciprocity, for the inner scalar product not to vanish, there must be some element $x \in N_{\mathbf{G}^0}(\mathbf{T}^0)^F$ such that ${}^x({}^{a'}\mathbf{T}')^{[b]F}$ meets $\mathbf{T}^{[b]F}$ which, considering the definitions, implies that $({}^{xa'}\mathbf{T}')^{[b]} = \mathbf{T}^{[b]}$. We may then conjugate the term $\mathrm{Ind}_{({}^{a'}\mathbf{T}')^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b} {}^{a'}\theta'$ by such an x to get $\mathrm{Ind}_{\mathbf{T}^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b} {}^{xa'}\theta'$ and take $y = xa'$ as a new variable, provided we count the number of x for a given a', which is $|N_{\mathbf{G}^0}(\mathbf{T}^{[b]})^F|$. We get

$$\frac{|(\mathbf{T}^0)^F|}{|\mathbf{T}^F|}\frac{|\mathbf{T}'^{0F}|}{|\mathbf{T}'^F|} \sum_{b \in [N_{\mathbf{G}}(\mathbf{T}^0)^F/N_{\mathbf{G}^0}(\mathbf{T}^0)^F]} |N_{\mathbf{G}^0}(\mathbf{T}^{[b]})^F|^{-1}$$
$$\sum_{\{y \in N_{\mathbf{G}}(\mathbf{T}^0)^F \mid ({}^y\mathbf{T}')^{[b]} = \mathbf{T}^{[b]}\}} \langle \mathrm{Ind}_{\mathbf{T}^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b}\theta, \mathrm{Ind}_{\mathbf{T}^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b} {}^y\theta'\rangle_{N_{\mathbf{G}^0}(\mathbf{T}^0)^F b}. \tag{3.4}$$

Since any $b \in [N_{\mathbf{G}}(\mathbf{T}^0)^F/N_{\mathbf{G}^0}(\mathbf{T}^0)^F]$ such that $\mathbf{T}^{[b]F}$ is not empty has a representative in $\mathbf{T}^F$ we can do the first summation over $b \in [\mathbf{T}^F/(\mathbf{T}^0)^F]$ so that 3.3 is equal to 3.4. □

4. Counting unipotent elements in disconnected groups

Proposition 4.1. *Assume $\mathbf{G}^1/\mathbf{G}^0$ unipotent and take $\sigma \in \mathbf{G}^1$ unipotent F-stable and quasi-central (see 2.2). Then the number of unipotent elements of $(\mathbf{G}^1)^F$ is given by $|(\mathbf{G}^{\sigma 0})^F|_p^2 |\mathbf{G}^{0^F}|/|(\mathbf{G}^{\sigma 0})^F|$.*

Proof. Let $\chi_{\mathcal{U}}$ be the characteristic function of the set of unipotent elements of $(\mathbf{G}^1)^F$. Then $|(\mathbf{G}^1)^F_{\mathrm{unip}}| = |(\mathbf{G}^1)^F| \langle \chi_{\mathcal{U}}, \mathrm{Id}\rangle_{(\mathbf{G}^1)^F}$ and

$$\langle \chi_{\mathcal{U}}, \mathrm{Id}\rangle_{(\mathbf{G}^1)^F} = \langle \mathrm{D}_{\mathbf{G}^1}(\chi_{\mathcal{U}}), \mathrm{D}_{\mathbf{G}^1}(\mathrm{Id})\rangle_{(\mathbf{G}^1)^F} = \langle \mathrm{D}_{\mathbf{G}^1}(\chi_{\mathcal{U}}), \mathrm{St}_{\mathbf{G}^1}\rangle_{(\mathbf{G}^1)^F},$$

the first equality since $\mathrm{D}_{\mathbf{G}^1}$ is an isometry by [DM94, 3.12]. According to [DM94, 2.11], for any σ-stable and F-stable Levi subgroup $\mathbf{L}^0$ of a σ-stable parabolic subgroup of $\mathbf{G}^0$, setting $\mathbf{L}^1 = \mathbf{L}^0.\sigma$, we have $R_{\mathbf{L}^1}^{\mathbf{G}^1}(\pi.\chi_{\mathcal{U}}|_{(\mathbf{L}^1)^F}) = R_{\mathbf{L}^1}^{\mathbf{G}^1}(\pi).\chi_{\mathcal{U}}$ and ${}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\varphi).\chi_{\mathcal{U}}|_{(\mathbf{L}^1)^F} = {}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\varphi.\chi_{\mathcal{U}})$, thus, by 2.12, $\mathrm{D}_{\mathbf{G}^1}(\pi.\chi_{\mathcal{U}}) = \mathrm{D}_{\mathbf{G}^1}(\pi).\chi_{\mathcal{U}}$; in particular $\mathrm{D}_{\mathbf{G}^1}(\chi_{\mathcal{U}}) = \mathrm{D}_{\mathbf{G}^1}(\mathrm{Id}).\chi_{\mathcal{U}} = \mathrm{St}_{\mathbf{G}^1}.\chi_{\mathcal{U}}$. Now, by Proposition 2.13, the only unipotent elements on which $\mathrm{St}_{\mathbf{G}^1}$ does not vanish are the quasi-semi-simple (thus quasi-central) ones; by [DM94, 1.37] all such are in the $\mathbf{G}^{0^F}$-class of σ and, again by 2.13 we have $\mathrm{St}_{\mathbf{G}^1}(\sigma) = |(\mathbf{G}^{\sigma 0})^F|_p$. We get

$$\begin{aligned} |(\mathbf{G}^1)^F| \langle \mathrm{D}_{\mathbf{G}^1}\chi_{\mathcal{U}}, \mathrm{St}_{\mathbf{G}^1}\rangle_{(\mathbf{G}^1)^F} &= |(\mathbf{G}^1)^F| \langle \mathrm{St}_{\mathbf{G}^1}.\chi_{\mathcal{U}}, \mathrm{St}_{\mathbf{G}^1}\rangle_{(\mathbf{G}^1)^F} \\ &= |\{\mathbf{G}^{0^F}\text{-class of } \sigma\}| |(\mathbf{G}^{\sigma 0})^F|_p^2 \end{aligned}$$

whence the proposition. □

Example 4.2. The formula of Proposition 4.1 applies in the following cases where σ induces a diagram automorphism of order 2 and q is a power of 2:

- $\mathbf{G}^0 = \mathrm{SO}_{2n}$, $(\mathbf{G}^{\sigma 0})^F = \mathrm{SO}_{2n-1}(\mathbb{F}_q)$;
- $\mathbf{G}^0 = \mathrm{GL}_{2n}$, $(\mathbf{G}^{\sigma 0})^F = \mathrm{Sp}_{2n}(\mathbb{F}_q)$;
- $\mathbf{G}^0 = \mathrm{GL}_{2n+1}$, $(\mathbf{G}^{\sigma 0})^F = \mathrm{SO}_{2n+1}(\mathbb{F}_q) \simeq \mathrm{Sp}_{2n}(\mathbb{F}_q)$;
- $\mathbf{G}^0 = E_6$, $(\mathbf{G}^{\sigma 0})^F = F_4(\mathbb{F}_q)$;

And it applies to the case where $\mathbf{G}^0 = \mathrm{SO}_8$ where σ induces a diagram automorphism of order 3 and q is a power of 3, in which case $(\mathbf{G}^{\sigma 0})^F = G_2(\mathbb{F}_q)$.

5. Tensoring by the Steinberg character

Proposition 5.1. *Let $\mathbf{L}^1$ be an F-stable "Levi" of $\mathbf{G}^1$. Then, for any class function γ on $(\mathbf{G}^1)^F$ we have:*

$${}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\gamma \cdot \varepsilon_{\mathbf{G}^1} \mathrm{St}_{\mathbf{G}^1}) = \varepsilon_{\mathbf{L}^1} \mathrm{St}_{\mathbf{L}^1} \mathrm{Res}_{(\mathbf{L}^1)^F}^{(\mathbf{G}^1)^F} \gamma.$$

Proof. Let su be the Jordan decomposition of a quasi-semi-simple element of $\mathbf{G}^1$ with s semisimple. We claim that u is quasi-central in $\mathbf{G}^s$. Indeed su, being quasi-semi-simple, is in a "torus" $\mathbf{T}$, thus s and u also are in $\mathbf{T}$. By [DM94, 1.8(iii)] the intersection of $\mathbf{T} \cap \mathbf{G}^s$ is a "torus" of $\mathbf{G}^s$, thus u is quasi-semi-simple in $\mathbf{G}^s$, hence quasi-central since unipotent.

Let tv be the Jordan decomposition of an element $l \in (\mathbf{L}^1)^F$ where t is semisimple. Since $\mathrm{St}_{\mathbf{L}^1}$ vanishes outside quasi-semi-simple elements the right-hand side of the proposition vanishes on l unless it is quasi-semi-simple which by our claim

means that v is quasi-central in $\mathbf{L}^t$. By the character formula 2.5 the left-hand side of the proposition evaluates at l to

$$ {}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\gamma\cdot\varepsilon_{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1})(l)=|(\mathbf{G}^{t0})^F|^{-1}\sum_{u\in(\mathbf{G}^{t0}\cdot v)^F_{\mathrm{unip}}}Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}(u,v^{-1})\gamma(tu)\varepsilon_{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1}(tu). $$

By the same argument as above, applied to $\operatorname{St}_{\mathbf{G}^1}$, the only non zero terms in the above sum are for u quasi-central in $\mathbf{G}^t$. For such u, by [DM94, 4.16], $Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}(u,v^{-1})$ vanishes unless u and v are $(\mathbf{G}^{t0})^F$-conjugate. Hence both sides of the equality to prove vanish unless u and v are quasi-central and $(\mathbf{G}^{t0})^F$-conjugate. In that case by [DM94, 4.16] and [DM91, (**) page 98] we have $Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}(u,v^{-1})=Q_{\mathbf{L}^{l0}}^{\mathbf{G}^{l0}}(1,1)=\varepsilon_{\mathbf{G}^{l0}}\varepsilon_{\mathbf{L}^{l0}}|(\mathbf{G}^{l0})^F|_{p'}|(\mathbf{L}^{l0})^F|_p$. Taking into account that the $(\mathbf{G}^{t0})^F$-class of v has cardinality $|(\mathbf{G}^{t0})^F|/|(\mathbf{G}^{l0})^F|$ and that by 2.13 we have $\operatorname{St}_{\mathbf{G}^1}(l)=\varepsilon_{\mathbf{G}^{\sigma 0}}\varepsilon_{\mathbf{G}^{l0}}|(\mathbf{G}^{l0})^F|_p$, the left-hand side of the proposition reduces to $\gamma(l)\varepsilon_{\mathbf{L}^{l0}}|(\mathbf{L}^{l0})^F|_p$, which is also the value of the right-hand side by applying 2.13 in $\mathbf{L}^1$. □

By adjunction, we get

Corollary 5.2. *For any class function λ on $(\mathbf{L}^1)^F$ we have:*

$$ R_{\mathbf{L}^1}^{\mathbf{G}^1}(\lambda)\varepsilon_{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1}=\operatorname{Ind}_{(\mathbf{L}^1)^F}^{(\mathbf{G}^1)^F}(\varepsilon_{\mathbf{L}^1}\operatorname{St}_{\mathbf{L}^1}\lambda) $$

6. Characteristic functions of quasi-semi-simple classes

One of the goals of this section is Proposition 6.4 where we give a formula for the characteristic function of a quasi-semi-simple class which shows in particular that it is uniform; this generalizes the case of quasi-central elements given in [DM94, 4.14].

If $x\in(\mathbf{G}^1)^F$ has Jordan decomposition $x=su$ we will denote by d_x the map from class functions on $(\mathbf{G}^1)^F$ to class functions on $(C_{\mathbf{G}}(s)^0\cdot u)^F$ given by

$$ (d_xf)(v)=\begin{cases} f(sv) & \text{if } v\in(C_{\mathbf{G}}(s)^0\cdot u)^F \text{ is unipotent}\\ 0 & \text{otherwise}\end{cases} $$

Lemma 6.1. *Let $\mathbf{L}^1$ be an F-stable "Levi" of $\mathbf{G}^1$. If $x=su$ is the Jordan decomposition of an element of $(\mathbf{L}^1)^F$ we have $d_x\circ{}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}={}^*R_{C_{\mathbf{L}}(s)^0\cdot u}^{C_{\mathbf{G}}(s)^0\cdot u}\circ d_x$.*

Proof. For v unipotent in $(C_{\mathbf{G}}(s)^0\cdot u)^F$ we have

$$ (d_x{}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}f)(v)=({}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}f)(sv)=({}^*R_{C_{\mathbf{L}}(s)^0\cdot su}^{C_{\mathbf{G}}(s)^0\cdot su}f)(sv)=({}^*R_{C_{\mathbf{L}}(s)^0\cdot u}^{C_{\mathbf{G}}(s)^0\cdot u}d_xf)(v) $$

where the second equality is [DM94, 2.9] and the last is by the character formula 2.5(iii). □

Proposition 6.2. *If $x=su$ is the Jordan decomposition of an element of $(\mathbf{G}^1)^F$, we have $d_x\circ p^{\mathbf{G}^1}=p^{C_{\mathbf{G}}(s)^0\cdot u}\circ d_x$.*

Proof. Let f be a class function on $(\mathbf{G}^1)^F$. For $v\in(C_{\mathbf{G}}(s)^0\cdot u)^F$ unipotent, we have, where the last equality is by 2.10:

$$ (d_xp^{\mathbf{G}^1}f)(v)=p^{\mathbf{G}^1}f(sv)=|(\mathbf{G}^1)^F|^{-1}\sum_{\mathbf{T}^1\in\mathcal{T}_1^F}|(\mathbf{T}^1)^F|(R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ{}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}f)(sv) $$

which by Proposition 2.5(ii) is:

$$\sum_{\mathbf{T}^1\in\mathcal{T}_1^F}\ \sum_{\{h\in(\mathbf{G}^0)^F\,|\,{}^h\mathbf{T}\ni s\}}\frac{|{}^h\mathbf{T}^0\cap C_{\mathbf{G}}(s)^{0F}|}{|(\mathbf{G}^0)^F||C_{\mathbf{G}}(s)^{0F}|}(R_{{}^h\mathbf{T}\cap C_{\mathbf{G}}(s)^0\cdot su}^{C_{\mathbf{G}}(s)^0\cdot su}\circ {}^{h*}R_{\mathbf{T}^1}^{\mathbf{G}^1}f)(sv).$$

Using that ${}^{h*}R_{\mathbf{T}^1}^{\mathbf{G}^1}f={}^*R_{{}^h\mathbf{T}^1}^{\mathbf{G}^1}f$ and summing over the ${}^h\mathbf{T}^1$, this becomes

$$\sum_{\{\mathbf{T}^1\in\mathcal{T}_1^F|\mathbf{T}\ni s\}}\frac{|\mathbf{T}^0\cap C_{\mathbf{G}}(s)^{0F}|}{|C_{\mathbf{G}}(s)^{0F}|}(R_{\mathbf{T}^1\cap C_{\mathbf{G}}(s)^0\cdot su}^{C_{\mathbf{G}}(s)^0\cdot su}\circ {}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}f)(sv).$$

Using that by Proposition 2.5(i) for any class function χ on $\mathbf{T}^1\cap C_{\mathbf{G}}(s)^0\cdot su^F$

$$\begin{aligned}(R_{\mathbf{T}^1\cap C_{\mathbf{G}}(s)^0\cdot su}^{C_{\mathbf{G}}(s)^0\cdot su}\chi)(sv)&=|\mathbf{T}^0\cap C_{\mathbf{G}}(s)^{0F}|^{-1}\sum_{v'\in(\mathbf{T}\cap C_{\mathbf{G}}(s)^0\cdot u)^F_{\mathrm{unip}}}Q_{(\mathbf{T}^s)^0}^{(\mathbf{G}^s)^0}(v,v'^{-1})\chi(sv')\\&=R_{\mathbf{T}\cap C_{\mathbf{G}}(s)^0\cdot u}^{C_{\mathbf{G}}(s)^0\cdot u}(d_x\chi)(v),\end{aligned}$$

and using Lemma 6.1, we get

$$|C_{\mathbf{G}}(s)^0\cdot su^F|^{-1}\sum_{\{\mathbf{T}^1\in\mathcal{T}_1^F|\mathbf{T}\ni s\}}|(\mathbf{T}^s)^{0F}|(R_{\mathbf{T}\cap C_{\mathbf{G}}(s)^0\cdot u}^{C_{\mathbf{G}}(s)^0\cdot u}\circ {}^*R_{\mathbf{T}\cap C_{\mathbf{G}}(s)^0\cdot u}^{C_{\mathbf{G}}(s)^0\cdot u}d_xf)(v)$$

which is the desired result if we apply Corollary 2.10 in $C_{\mathbf{G}}(s)^0\cdot u$ and remark that by [DM94, 1.8 (iv)] the map $\mathbf{T}^1\mapsto\mathbf{T}\cap C_{\mathbf{G}}(s)^0\cdot u$ induces a bijection between $\{\mathbf{T}^1\in\mathcal{T}_1^F\mid\mathbf{T}\ni s\}$ and F-stable "tori" of $C_{\mathbf{G}}(s)^0\cdot u$. □

Corollary 6.3. *A class function f on $(\mathbf{G}^1)^F$ is uniform if and only if for every $x\in(\mathbf{G}^1)^F$ the function d_xf is uniform.*

Proof. Indeed, $f=p^{\mathbf{G}^1}f$ if and only if for any $x\in(\mathbf{G}^1)^F$ we have $d_xf=d_xp^{\mathbf{G}^1}f=p^{C_{\mathbf{G}}(s)^0\cdot u}d_xf$, the last equality by Proposition 6.2. □

For $x\in(\mathbf{G}^1)^F$ we consider the class function $\pi_x^{\mathbf{G}^1}$ on $(\mathbf{G}^1)^F$ defined by

$$\pi_x^{\mathbf{G}^1}(y)=\begin{cases}0 & \text{if } y \text{ is not conjugate to } x\\ |C_{\mathbf{G}^0}(x)^F| & \text{if } y=x\end{cases}$$

Proposition 6.4. *For $x\in(\mathbf{G}^1)^F$ quasi-semi-simple the function $\pi_x^{\mathbf{G}^1}$ is uniform, given by*

$$\begin{aligned}\pi_x^{\mathbf{G}^1}&=\varepsilon_{\mathbf{G}^{x0}}|C_{\mathbf{G}}(x)^0|_p^{-1}\sum_{\{\mathbf{T}^1\in\mathcal{T}_1^F|\mathbf{T}^1\ni x\}}\varepsilon_{\mathbf{T}^1}R_{\mathbf{T}^1}^{\mathbf{G}^1}(\pi_x^{\mathbf{T}^1})\\&=|W^0(x)|^{-1}\sum_{w\in W^0(x)}\dim R_{\mathbf{T}_w}^{C_{\mathbf{G}}(x)^0}(\mathrm{Id})R_{C_{\mathbf{G}^1}(\mathbf{T}_w)}^{\mathbf{G}^1}(\pi_x^{C_{\mathbf{G}^1}(\mathbf{T}_w)})\end{aligned}$$

where in the second equality $W^0(x)$ denotes the Weyl group of $C_{\mathbf{G}}(x)^0$ and $\mathbf{T}_w$ denotes an F-stable torus of type w of this last group.

Proof. First, using Corollary 6.3 we prove that $\pi_x^{\mathbf{G}^1}$ is uniform. Let su be the Jordan decomposition of x. For $y\in(\mathbf{G}^1)^F$ the function $d_y\pi_x^{\mathbf{G}^1}$ is zero unless the semi-simple part of y is conjugate to s. Hence it is sufficient to evaluate $d_y\pi_x^{\mathbf{G}^1}(v)$ for elements y whose semisimple part is equal to s. For such elements $d_y\pi_x^{\mathbf{G}^1}(v)$ is up to a coefficient equal to $\pi_u^{C_{\mathbf{G}}(s)^0\cdot u}$. This function is uniform by [DM94, 4.14],

since u being the unipotent part of a quasi-semi-simple element is quasi-central in $C_{\mathbf{G}}(s)$ (see beginning of the proof of Proposition 5.1).

We have thus $\pi_x^{\mathbf{G}^1} = p^{\mathbf{G}^1}\pi_x^{\mathbf{G}^1}$. We use this to get the formula of the proposition. We start by using Proposition 2.13 to write $\pi_x^{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1} = \varepsilon_{\mathbf{G}^1}\varepsilon_{\mathbf{G}^{x0}}|(\mathbf{G}^{x0})^F|_p\pi_x^{\mathbf{G}^1}$, or equivalently $\pi_x^{\mathbf{G}^1} = \varepsilon_{\mathbf{G}^1}\varepsilon_{\mathbf{G}^{x0}}|(\mathbf{G}^{x0})^F|_p^{-1}p^{\mathbf{G}^1}(\pi_x^{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1})$. Using Corollary 2.10 and that by Proposition 5.1 we have ${}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}(\pi_x^{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1}) = \varepsilon_{\mathbf{G}^1}\varepsilon_{\mathbf{T}^1}\operatorname{St}_{\mathbf{T}^1}\operatorname{Res}_{(\mathbf{T}^1)^F}^{(\mathbf{G}^1)^F}(\pi_x^{\mathbf{G}^1})$, we get

$$p^{\mathbf{G}^1}(\pi_x^{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1}) = \varepsilon_{\mathbf{G}^1}|(\mathbf{G}^1)^F|^{-1}\sum_{\mathbf{T}^1\in\mathcal{T}_1^F}|(\mathbf{T}^1)^F|\varepsilon_{\mathbf{T}^1}R_{\mathbf{T}^1}^{\mathbf{G}^1}(\operatorname{St}_{\mathbf{T}^1}\operatorname{Res}_{(\mathbf{T}^1)^F}^{(\mathbf{G}^1)^F}(\pi_x^{\mathbf{G}^1})).$$

The function $\operatorname{St}_{\mathbf{T}^1}$ is constant equal to 1. Now we have

$$\operatorname{Res}_{(\mathbf{T}^1)^F}^{(\mathbf{G}^1)^F}\pi_x^{\mathbf{G}^1} = |(\mathbf{T}^0)^F|^{-1}\sum_{\{g\in(\mathbf{G}^0)^F|\,{}^gx\in\mathbf{T}^1\}}\pi_{{}^gx}^{\mathbf{T}^1}.$$

To see this, do the scalar product with a class function f on $(\mathbf{T}^1)^F$:

$$\langle\operatorname{Res}_{(\mathbf{T}^1)^F}^{(\mathbf{G}^1)^F}\pi_x^{\mathbf{G}^1}, f\rangle_{(\mathbf{T}^1)^F} = \langle\pi_x^{\mathbf{G}^1}, \operatorname{Ind}_{\mathbf{T}^1}^{\mathbf{G}^1}f\rangle_{(\mathbf{G}^1)^F} = |(\mathbf{T}^0)^F|^{-1}\sum_{\{g\in(\mathbf{G}^0)^F|\,{}^gx\in\mathbf{T}^1\}}f({}^gx).$$

We then get using that $|(\mathbf{T}^0)^F| = |(\mathbf{T}^1)^F|$

$$p^{\mathbf{G}^1}(\pi_x^{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1}) = \varepsilon_{\mathbf{G}^1}|(\mathbf{G}^1)^F|^{-1}\sum_{\mathbf{T}^1\in\mathcal{T}_1^F}\sum_{\{g\in(\mathbf{G}^0)^F|\,{}^gx\in\mathbf{T}^1\}}\varepsilon_{\mathbf{T}^1}R_{\mathbf{T}^1}^{\mathbf{G}^1}(\pi_{{}^gx}^{\mathbf{T}^1}).$$

Taking ${}^{g^{-1}}\mathbf{T}^1$ as summation index we get

$$p^{\mathbf{G}^1}(\pi_x^{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1}) = \varepsilon_{\mathbf{G}^1}\sum_{\{\mathbf{T}^1\in\mathcal{T}_1^F|\mathbf{T}^1\ni x\}}\varepsilon_{\mathbf{T}^1}R_{\mathbf{T}^1}^{\mathbf{G}^1}(\pi_x^{\mathbf{T}^1}),$$

hence

$$\pi_x^{\mathbf{G}^1} = \varepsilon_{\mathbf{G}^{x0}}|(\mathbf{G}^{x0})^F|_p^{-1}\sum_{\{\mathbf{T}^1\in\mathcal{T}_1^F|\mathbf{T}^1\ni x\}}\varepsilon_{\mathbf{T}^1}R_{\mathbf{T}^1}^{\mathbf{G}^1}(\pi_x^{\mathbf{T}^1}),$$

which is the first equality of the proposition.

For the second equality of the proposition, we first use [DM94, 1.8 (iii) and (iv)] to sum over tori of $C_{\mathbf{G}}(x)^0$: the $\mathbf{T}^1\in\mathcal{T}_1^F$ containing x are in bijection with the maximal tori of $C_{\mathbf{G}}(x)^0$ by $\mathbf{T}^1\mapsto(\mathbf{T}^{1x})^0$ and conversely $\mathbf{S}\mapsto C_{\mathbf{G}^1}(\mathbf{S})$. This bijection satisfies $\varepsilon_{\mathbf{T}^1} = \varepsilon_{\mathbf{S}}$ by definition of ε.

We then sum over $(C_{\mathbf{G}}(x)^0)^F$-conjugacy classes of maximal tori, which are parameterized by F-conjugacy classes of $W^0(x)$. We then have to multiply by $|(C_{\mathbf{G}}(x)^0)^F|/|N_{(C_{\mathbf{G}}(x)^0)}(\mathbf{S})^F|$ the term indexed by the class of $\mathbf{S}$. Then we sum over the elements of $W^0(x)$. We then have to multiply the term indexed by w by $|C_{W^0(x)}(wF)|/|W^0(x)|$. Using $|N_{(C_{\mathbf{G}}(x)^0)}(\mathbf{S})^F| = |\mathbf{S}^F||C_{W^0(x)}(wF)|$, and the formula for $\dim R_{\mathbf{T}_w}^{C_{\mathbf{G}}(x)^0}(\mathrm{Id})$ we get the result. □

7. Classification of quasi-semi-simple classes

The first items of this section, before 7.7, apply for algebraic groups over an arbitrary algebraic closed field k.

We denote by $\mathcal{C}(\mathbf{G}^1)$ the set of conjugacy classes of $\mathbf{G}^1$, that is the orbits under $\mathbf{G}^0$-conjugacy, and denote by $\mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}}$ the set of quasi-semi-simple classes.

Proposition 7.1. *For $\mathbf{T}^1 \in \mathcal{T}_1$ write $\mathbf{T}^1 = \mathbf{T}^0 \cdot \sigma$ where σ is quasi-central. Then $\mathcal{C}(\mathbf{G}^1)_{qss}$ is in bijection with the set of $N_{\mathbf{G}^0}(\mathbf{T}^1)$-orbits in $\mathbf{T}^1$, which itself is in bijection with the set of W^σ-orbits in $\mathcal{C}(\mathbf{T}^1)$, where $W = N_{\mathbf{G}^0}(\mathbf{T}^0)/\mathbf{T}^0$. We have $\mathcal{C}(\mathbf{T}^1) \simeq \mathbf{T}^1/\mathcal{L}_\sigma(\mathbf{T}^0)$ where $\mathcal{L}_\sigma$ is the map $t \mapsto t^{-1}.{}^\sigma t$.*

Proof. By definition every quasi-semi-simple element of $\mathbf{G}^1$ is in some $\mathbf{T}^1 \in \mathcal{T}_1$ and $\mathcal{T}_1$ is a single orbit under $\mathbf{G}^0$-conjugacy. It is thus sufficient to find how classes of $\mathbf{G}^1$ intersect $\mathbf{T}^1$. By [DM94, 1.13] two elements of $\mathbf{T}^1$ are $\mathbf{G}^0$-conjugate if and only if they are conjugate under $N_{\mathbf{G}^0}(\mathbf{T}^0)$. We can replace $N_{\mathbf{G}^0}(\mathbf{T}^0)$ by $N_{\mathbf{G}^0}(\mathbf{T}^1)$ since if ${}^g(\sigma t) = \sigma t'$ where $g \in N_{\mathbf{G}^0}(\mathbf{T}^0)$ then the image of g in W lies in W^σ. By [DM94, 1.15(iii)] elements of W^σ have representatives in $\mathbf{G}^{\sigma 0}$. Write $g = s\dot{w}$ where $\dot{w}$ is such a representative and $s \in \mathbf{T}^0$. Then ${}^{s\dot{w}}(t\sigma) = \mathcal{L}_\sigma(s^{-1})^w t\sigma$ whence the proposition. □

Lemma 7.2. $\mathbf{T}^0 = \mathbf{T}^{\sigma 0}.\mathcal{L}_\sigma(\mathbf{T}^0)$.

Proof. This is proved in [DM94, 1.33] when σ is unipotent (and then the product is direct). We proceed similarly to that proof: $\mathbf{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\mathbf{T}^0)$ is finite, since its exponent divides the order of σ (if $\sigma(t^{-1}{}^\sigma t) = t^{-1}{}^\sigma t$ then $(t^{-1}{}^\sigma t)^n = t^{-1}{}^{\sigma^n} t$), and $\dim(\mathbf{T}^{\sigma 0}) + \dim(\mathcal{L}_\sigma(\mathbf{T}^0)) = \dim(\mathbf{T}^0)$ as the exact sequence $1 \to \mathbf{T}^{0\sigma} \to \mathbf{T}^0 \to \mathcal{L}_\sigma(\mathbf{T}^0) \to 1$ shows. □

It follows that $\mathbf{T}^0/\mathcal{L}_\sigma(\mathbf{T}^0) \simeq \mathbf{T}^{\sigma 0}/(\mathbf{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\mathbf{T}^0))$; since the set $\mathcal{C}(\mathbf{G}^{\sigma 0})_{\mathrm{ss}}$ of semi-simple classes of $\mathbf{G}^{\sigma 0}$ identifies with the set of W^σ-orbits on $\mathbf{T}^{\sigma 0}$ this induces a surjective map $\mathcal{C}(\mathbf{G}^{\sigma 0})_{\mathrm{ss}} \to \mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}}$.

Example 7.3. We will describe the quasi-semi-simple classes of $\mathbf{G}^0 \cdot \sigma$, where $\mathbf{G}^0 = \mathrm{GL}_n(k)$ and σ is the quasi-central automorphism given by $\sigma(g) = J\,{}^t g^{-1} J^{-1}$, where, if n is even J is the matrix $\begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}$ with $J_0 = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$ and if n is odd J is the antidiagonal matrix $\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$ (any outer algebraic automorphism of GL_n is equal to σ up to an inner automorphism).

The automorphism σ normalizes the pair $\mathbf{T}^0 \subset \mathbf{B}$ where $\mathbf{T}^0$ is the diagonal torus and $\mathbf{B}$ the group of upper triangular matrices. Then $\mathbf{T}^1 = \mathbf{T}^0 \cdot \sigma \in \mathcal{T}_1$. For $\mathrm{diag}(x_1, \dots, x_n) \in \mathbf{T}^0$, where $x_i \in k^\times$, we have $\sigma(\mathrm{diag}(x_1, \dots, x_n)) = \mathrm{diag}(x_n^{-1}, \dots, x_1^{-1})$. It follows that $\mathcal{L}_\sigma(\mathbf{T}^0) = \{\mathrm{diag}(x_1, x_2, \dots, x_2, x_1)\}$ — here x_{m+1} is a square when $n = 2m+1$ but this is not a condition since k is algebraically closed. As suggested above, we could take as representatives of $\mathbf{T}^0/\mathcal{L}_\sigma(\mathbf{T}^0)$ the set $\mathbf{T}^{\sigma 0}/(\mathbf{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\mathbf{T}^0))$, but since $\mathbf{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\mathbf{T}^0)$ is not trivial (it consists of the diagonal matrices with entries ± 1 placed symmetrically), it is more convenient to take for representatives of the quasi-semi-simple classes the set $\{\mathrm{diag}(x_1, x_2, \dots, x_{\lfloor \frac{n}{2} \rfloor}, 1, \dots, 1)\}\sigma$. In this model the action of W^σ is generated by the permutations of the $\lfloor \frac{n}{2} \rfloor$ first entries, and by the maps $x_i \mapsto x_i^{-1}$, so the quasi-semi-simple classes of $\mathbf{G}^0 \cdot \sigma$ are parameterized by the quasi-semi-simple classes of $\mathbf{G}^{\sigma 0}$.

Proposition 7.4. *Let $s\sigma = \mathrm{diag}(x_1, x_2, \dots, x_{\lfloor \frac{n}{2} \rfloor}, 1, \dots, 1)\sigma$ be a quasi-semi-simple element as above. If char $k = 2$ then $C_{\mathbf{G}^0}(s\sigma)$ is connected. Otherwise, if n is odd, $A(s\sigma) := C_{\mathbf{G}^0}(s\sigma)/C_{\mathbf{G}^0}(s\sigma)^0$ is of order two, generated by $-1 \in Z\mathbf{G}^0 = Z\,\mathrm{GL}_n(k)$.*

If n is even, $A(s\sigma) \neq 1$ if and only if for some i we have $x_i = -1$; then $x_i \mapsto x_i^{-1}$ is an element of W^σ which has a representative in $C_{\mathbf{G}^0}(s\sigma)$ generating $A(s\sigma)$, which is of order 2.

Proof. We will use that for a group G and an automorphism σ of G we have an exact sequence (see for example [St, 4.5])

$$1 \to (ZG)^\sigma \to G^\sigma \to (G/ZG)^\sigma \to (\mathcal{L}_\sigma G \cap ZG)/\mathcal{L}_\sigma ZG \to 1 \tag{7.5}$$

If we take $G = \mathbf{G}^0 = \mathrm{GL}_n(k)$ in 7.5 and $s\sigma$ for σ, since on $Z\mathbf{G}^0$ the map $\mathcal{L}_\sigma = \mathcal{L}_{s\sigma}$ is $z \mapsto z^2$, hence surjective, we get that ${\mathbf{G}^0}^{s\sigma} \to \mathrm{PGL}_n^{s\sigma}$ is surjective and has kernel $(Z\mathbf{G}^0)^\sigma = \{\pm 1\}$.

Assume n odd and take $G = \mathrm{SL}_n(k)$ in 7.5. We have $Z\,\mathrm{SL}_n^\sigma = \{1\}$ so that we get the following diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & \{\pm 1\} \arrow[r] & \mathrm{GL}_n^{s\sigma} \arrow[r] & \mathrm{PGL}_n^{s\sigma} \arrow[r] & 1 \\
 & 1 \arrow[r] & \mathrm{SL}_n^{s\sigma} \arrow[r] \arrow[u, hook] & \mathrm{PGL}_n^{s\sigma} \arrow[r] \arrow[u, equal] & 1
\end{tikzcd}$$

This shows that $\mathrm{GL}_n^{s\sigma} / \mathrm{SL}_n^{s\sigma} \simeq \{\pm 1\}$; by [St, 8.1] $\mathrm{SL}_n^{s\sigma}$ is connected, hence $\mathrm{PGL}_n^{s\sigma}$ is connected thus $\mathrm{GL}_n^{s\sigma} = (\mathrm{GL}_n^{s\sigma})^0 \times \{\pm 1\}$ is connected if and only if char $k = 2$.

Assume now n even; then $(\mathbf{T}^0)^\sigma$ is connected hence $-1 \in (\mathrm{GL}_n^{s\sigma})^0$ for all $s \in \mathbf{T}_0$. Using this, the exact sequence $1 \to \{\pm 1\} \to \mathrm{GL}_n^{s\sigma} \to \mathrm{PGL}_n^{s\sigma} \to 1$ implies $A(s\sigma) = \mathbf{G}^{s\sigma}/{\mathbf{G}^0}^{s\sigma} = \mathrm{GL}_n^{s\sigma}/(\mathrm{GL}_n^{s\sigma})^0 \simeq \mathrm{PGL}_n^{s\sigma}/(\mathrm{PGL}_n^{s\sigma})^0$. To compute this group we use 7.5 with $\mathrm{SL}_n(k)$ for G and $s\sigma$ for σ:

$$1 \to \{\pm 1\} \to \mathrm{SL}_n^{s\sigma} \to \mathrm{PGL}_n^{s\sigma} \to (\mathcal{L}_{s\sigma}\,\mathrm{SL}_n \cap Z\,\mathrm{SL}_n)/\mathcal{L}_\sigma Z\,\mathrm{SL}_n \to 1$$

which, since $\mathrm{SL}_n^{s\sigma}$ is connected, implies that $A(s\sigma) = (\mathcal{L}_{s\sigma}\,\mathrm{SL}_n \cap Z\,\mathrm{SL}_n)/\mathcal{L}_\sigma Z\,\mathrm{SL}_n$ thus is non trivial (of order 2) if and only if $\mathcal{L}_{s\sigma}\,\mathrm{SL}_n \cap Z\,\mathrm{SL}_n$ contains an element which is not a square in $Z\,\mathrm{SL}_n$; thus $A(s\sigma)$ is trivial if char $k = 2$. We assume now char $k \neq 2$. Then a non-square is of the form $\mathrm{diag}(z, \dots, z)$ with $z^m = -1$ if we set $m = n/2$.

The following lemma is a transcription of [St, 9.5].

Lemma 7.6. *Let σ be a quasi-central automorphism of the connected reductive group $\mathbf{G}$ which stabilizes the pair $\mathbf{T} \subset \mathbf{B}$ of a maximal torus and a Borel subgroup; let W be the Weyl group of $\mathbf{T}$ and let $s \in \mathbf{T}$. Then $\mathbf{T} \cap \mathcal{L}_{s\sigma}(\mathbf{G}) = \{\mathcal{L}_w(s^{-1}) \mid w \in W^\sigma\} \cdot \mathcal{L}_\sigma(\mathbf{T})$.*

Proof. Assume $t = \mathcal{L}_{s\sigma}(x)$ for $t \in \mathbf{T}$, or equivalently $xt = {}^{s\sigma}x$. Then if x is in the Bruhat cell $\mathbf{B}w\mathbf{B}$, we must have $w \in W^\sigma$. Taking for w a σ-stable representative $\dot{w}$ and writing the unique Bruhat decomposition $x = u_1 \dot{w} t_1 u_2$ where $u_2 \in \mathbf{U}, t_1 \in \mathbf{T}$ and $u_1 \in \mathbf{U} \cap {}^w\mathbf{U}^-$ where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$ and $\mathbf{U}^-$ the unipotent radical of the opposite Borel, the equality $xt = {}^{s\sigma}x$ implies that $\dot{w}t_1 t = {}^{s\sigma}(\dot{w}t_1)$ or equivalently $t = \mathcal{L}_{w^{-1}}(s^{-1})\mathcal{L}_\sigma(t_1)$, whence the lemma. □

From this lemma we get $\mathcal{L}_{s\sigma}(\mathrm{SL}_n) \cap Z\,\mathrm{SL}_n = \{\mathcal{L}_w(s^{-1}) \mid w \in W^\sigma\} \cdot \mathcal{L}_\sigma(\mathbf{T}) \cap Z\,\mathrm{SL}_n$, where $\mathbf{T}$ is the maximal torus of SL_n. The element $\mathrm{diag}(x_1, x_2, \dots, x_m, 1, \dots, 1)\sigma$ is conjugate to $s\sigma = \mathrm{diag}(y_1, y_2, \dots, y_m, y_m^{-1}, \dots, y_1^{-1})\sigma \in \mathbf{T}^\sigma \cdot \sigma$ where $y_i^2 = x_i$. It will have a non connected centralizer if and only if for some $w \in W^\sigma$ and some $t \in \mathbf{T}$ we have $\mathcal{L}_w(s^{-1}) \cdot \mathcal{L}_\sigma(t) = \mathrm{diag}(z, \dots, z)$ with $z^m = -1$ and then an appropriate representative of w (multiplying if needed by an element of $Z\,\mathrm{GL}_n$) will be

in $C_{\mathbf{G}^0}(s\sigma)$ and have a non-trivial image in $A(s\sigma)$. Since s and w are σ-fixed, we have $\mathcal{L}_w(s) \in \mathbf{T}^\sigma$, thus is of the form $\mathrm{diag}(a_1, \dots, a_m, a_m^{-1}, \dots a_1^{-1})$. Since $\mathcal{L}_\sigma(\mathbf{T}) = \{\mathrm{diag}(t_1, \dots, t_m, t_m \dots, t_1) \mid t_1 t_2 \dots t_m = 1\}$, we get $z = a_1 t_1 = a_2 t_2 = \dots = a_m t_m = a_m^{-1} t_m = \dots = a_1^{-1} t_1$; in particular $a_i = \pm 1$ for all i and $a_1 a_2 \dots a_m = -1$. We can take w up to conjugacy in W^σ since $\mathcal{L}_{vwv^{-1}}(s^{-1}) = {}^v\mathcal{L}_w({}^{v^{-1}}s^{-1})$ and $\mathcal{L}_\sigma(\mathbf{T})$ is invariant under W^σ-conjugacy. We see W^σ as the group of permutations of $\{1, 2, \dots, m, -m, \dots, -1\}$ which preserves the pairs $\{i, -i\}$. A non-trivial cycle of w has, up to conjugacy, the form either $(1, -1)$ or $(1, -2, 3, \dots, (-1)^{i-1} i, i+1, i+2, \dots, k)$ with $0 \le i \le k \le n$ and i even, or $(1, -2, 3, \dots, (-1)^{i-1} i, -(i+1), -(i+2), \dots, -k, -1, 2, -3, \dots, k)$ and i odd. The contribution to $a_1 \dots a_m$ of the orbit $(1, -1)$ is $a_1 = y_1^2$ hence is 1 except if $y_1^2 = x_1 = -1$. Let us consider an orbit of the second form. The k first coordinates of $\mathcal{L}_w(s^{-1})$ are $(y_1 y_2, \dots, y_i y_{i+1}, y_{i+1}/y_{i+2}, \dots, y_k/y_1)$. Hence there must exist signs ε_j such that $y_2 = \varepsilon_1/y_1$, $y_3 = \varepsilon_2/y_2, \dots$, $y_{i+1} = \varepsilon_i/y_i$ and $y_{i+2} = \varepsilon_{i+1} y_{i+1}, \dots$, $y_k = \varepsilon_{k-1} y_{k-1}$, $y_1 = \varepsilon_k y_k$. This gives $y_1 = \begin{cases} \varepsilon_1 \dots \varepsilon_k y_1 & \text{if } i \text{ is even} \\ \varepsilon_1 \dots \varepsilon_k / y_1 & \text{if } i \text{ is odd} \end{cases}$. The contribution of the orbit to $a_1 \dots a_m$ is $\varepsilon_1 \dots \varepsilon_k$ thus is 1 if i is even and $x_1 = y_1^2$ if i is odd. Again, we see that one of the x_i must equal -1 to get $a_1 \dots a_m = -1$. Conversely if $x_1 = -1$, for any z such that $z^m = -1$, choosing t such that $\mathcal{L}_\sigma(t) = \mathrm{diag}(-z, z, z, \dots, z, -z)$ and taking $w = (1, -1)$ we get $\mathcal{L}_w(s^{-1}) \mathcal{L}_\sigma(t) = \mathrm{diag}(z, \dots, z)$ as desired. □

We now go back to the case where $k = \overline{\mathbb{F}}_q$, and in the context of Proposition 7.1, we now assume that $\mathbf{T}^1$ is F-stable and that σ induces an F-stable automorphism of $\mathbf{G}^0$.

Proposition 7.7. *Let $\mathbf{T}^{1rat} = \{s \in \mathbf{T}^1 \mid \exists n \in N_{\mathbf{G}^0}(\mathbf{T}^1), {}^{nF}s = s\}$; then $\mathbf{T}^{1rat}$ is stable by $\mathbf{T}^0$-conjugacy, which gives a meaning to $\mathcal{C}(\mathbf{T}^{1rat})$. Then $c \mapsto c \cap \mathbf{T}^1$ induces a bijection between $(\mathcal{C}(\mathbf{G}^1)_{qss})^F$ and the W^σ-orbits on $\mathcal{C}(\mathbf{T}^{1rat})$.*

Proof. A class $c \in \mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}}$ is F-stable if and only if given $s \in c$ we have ${}^F s \in c$. If we take $s \in c \cap \mathbf{T}^1$ then ${}^F s \in c \cap \mathbf{T}^1$ which as observed in the proof of 7.1 implies that ${}^F s$ is conjugate to s under $N_{\mathbf{G}^0}(\mathbf{T}^1)$, that is $s \in \mathbf{T}^{1\mathrm{rat}}$. Thus c is F-stable if and only if $c \cap \mathbf{T}^1 = c \cap \mathbf{T}^{1\mathrm{rat}}$. The proposition then results from Proposition 7.1 observing that $\mathbf{T}^{1\mathrm{rat}}$ is stable under $N_{\mathbf{G}^0}(\mathbf{T}^1)$-conjugacy and that the corresponding orbits are the W^σ-orbits on $\mathcal{C}(\mathbf{T}^{1\mathrm{rat}})$. □

Example 7.8. When $\mathbf{G}^1 = \mathrm{GL}_n(\overline{\mathbb{F}}_q) \cdot \sigma$ with σ as in Example 7.3, the map

$$\mathrm{diag}(x_1, x_2, \dots, x_{\lfloor \frac{n}{2} \rfloor}, 1, \dots, 1) \mapsto \mathrm{diag}(x_1, x_2, \dots, x_{\lfloor \frac{n}{2} \rfloor}, \dagger, x_{\lfloor \frac{n}{2} \rfloor}^{-1}, \dots, x_2^{-1}, x_1^{-1})$$

where $\dagger$ represents 1 if n is odd and an omitted entry otherwise, is compatible with the action of W^σ as described in 7.3 on the left-hand side and the natural action on the right-hand side. This map induces a bijection from $\mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}}$ to the semi-simple classes of $(\mathrm{GL}_n^\sigma)^0$ which restricts to a bijection from $(\mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}})^F$ to the F-stable semi-simple classes of $(\mathrm{GL}_n^\sigma)^0$.

We will now compute the cardinality of $(\mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}})^F$.

Proposition 7.9. *Let f be a function on $(\mathcal{C}(\mathbf{G}^1)_{qss})^F$. Then*

$$\sum_{c \in (\mathcal{C}(\mathbf{G}^1)_{qss})^F} f(c) = |W^\sigma|^{-1} \sum_{w \in W^\sigma} \tilde{f}(w)$$

where $\tilde{f}(w) := \sum_s f(s)$, *where* s *runs over representatives in* ${\mathbf{T}^1}^{wF}$ *of* ${\mathbf{T}^1}^{wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}$.

Proof. We have $\mathcal{C}(\mathbf{T}^{1\mathrm{rat}}) = \bigcup_{w\in W^\sigma}\{s\mathcal{L}_\sigma(\mathbf{T}^0) \in \mathbf{T}^1/\mathcal{L}_\sigma(\mathbf{T}^0) \mid s\mathcal{L}_\sigma(\mathbf{T}^0) \text{ is } wF\text{-stable}\}$. The conjugation by $v \in W^\sigma$ sends a wF-stable coset $s\mathcal{L}_\sigma(\mathbf{T}^0)$ to a $vwFv^{-1}$-stable coset; and the number of w such that $s\mathcal{L}_\sigma(\mathbf{T}^0)$ is wF-stable is equal to $N_{W^\sigma}(s\mathcal{L}_\sigma(\mathbf{T}^0))$. It follows that

$$\sum_{c\in(\mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}})^F} f(c) = |W^\sigma|^{-1}\sum_{w\in W^\sigma}\ \sum_{s\mathcal{L}_\sigma(\mathbf{T}^0)\in(\mathbf{T}^1/\mathcal{L}_\sigma(\mathbf{T}^0))^{wF}} f(s\mathcal{L}_\sigma(\mathbf{T}^0)).$$

The proposition follows since, $\mathcal{L}_\sigma(\mathbf{T}^0)$ being connected, we have $(\mathbf{T}^1/\mathcal{L}_\sigma(\mathbf{T}^0))^{wF} = {\mathbf{T}^1}^{wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}$. □

Corollary 7.10. *We have* $|(\mathcal{C}(\mathbf{G}^1)_{qss})^F| = |(\mathcal{C}(\mathbf{G}^{\sigma 0})_{ss})^F|$.

Proof. Let us take $f = 1$ in 7.9. We need to sum over $w \in W^\sigma$ the value $|{\mathbf{T}^1}^{wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}|$. First note that $|{\mathbf{T}^1}^{wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}| = |{\mathbf{T}^0}^{wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}|$. By Lemma 7.2 we have the exact sequence

$$1 \to \mathbf{T}^{\sigma 0}\cap\mathcal{L}_\sigma(\mathbf{T}^0) \to \mathbf{T}^{\sigma 0}\times\mathcal{L}_\sigma(\mathbf{T}^0) \to \mathbf{T}^0 \to 1$$

whence the Galois cohomology exact sequence:

$$1 \to (\mathbf{T}^{\sigma 0}\cap\mathcal{L}_\sigma(\mathbf{T}^0))^{wF} \to {\mathbf{T}^{\sigma 0}}^{wF}\times(\mathcal{L}_\sigma(\mathbf{T}^0))^{wF} \to {\mathbf{T}^0}^{wF} \to H^1(wF, (\mathbf{T}^{\sigma 0}\cap\mathcal{L}_\sigma(\mathbf{T}^0))) \to 1.$$

Using that for any automorphism τ of a finite group G we have $|G^\tau| = |H^1(\tau, G)|$, we have $|(\mathbf{T}^{\sigma 0}\cap\mathcal{L}_\sigma(\mathbf{T}^0))^{wF}| = |H^1(wF, (\mathbf{T}^{\sigma 0}\cap\mathcal{L}_\sigma(\mathbf{T}^0)))|$. Together with the above exact sequence it implies that $|{\mathbf{T}^0}^{wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}| = |{\mathbf{T}^{\sigma 0}}^{wF}|$ whence

$$|(\mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}})^F| = |W^\sigma|^{-1}\sum_{w\in W^\sigma}|{\mathbf{T}^{\sigma 0}}^{wF}|.$$

The corollary follows by either applying the same formula for the connected group $\mathbf{G}^{\sigma 0}$, or referring to [Le, Proposition 2.1]. □

8. Shintani descent.

We now look at Shintani descent in our context; we are able to obtain a result when $\mathbf{G}^1/\mathbf{G}^0$ is semi-simple. We should mention previous work on this subject: Eftekhari ([E96, II. 3.4]) has the same result in the case of a torus; he does not need to assume p good but needs q to be large enough to apply results of Lusztig identifying Deligne-Lusztig induction with induction of character sheaves; Digne ([D99, 1.1]) has the result in the same generality excepted that he needs the assumption that $\mathbf{G}^1$ contains an F-stable quasi-central element; however a defect of his proof is the use without proof of the property given in Lemma 8.4 below.

As above $\mathbf{G}^1$ denotes an F-stable connected component of $\mathbf{G}$ of the form $\mathbf{G}^0\cdot\sigma$ where σ induces a quasi-central automorphism of $\mathbf{G}^0$ commuting with F.

Applying Lang's theorem, one can write any element of $\mathbf{G}^1$ as $x\cdot{}^{\sigma F}x^{-1}\sigma$ for some $x \in \mathbf{G}^0$, or as ${}^{\sigma F}x^{-1}\cdot x$ for some $x \in \mathbf{G}^0$. Using that σ, as automorphism, commutes with F, it is easy to check that the correspondence $x\cdot{}^{\sigma F}x^{-1}\sigma \mapsto {}^{\sigma F}x^{-1}\cdot x$ induces a bijection $n_{F/\sigma F}$ from the $(\mathbf{G}^0)^F$-conjugacy classes of $(\mathbf{G}^1)^F$ to the ${\mathbf{G}^0}^{\sigma F}$-conjugacy classes of $(\mathbf{G}^1)^{\sigma F}$ and that $|{\mathbf{G}^0}^{\sigma F}||c| = |(\mathbf{G}^0)^F||n_{F/\sigma F}(c)|$ for any $(\mathbf{G}^0)^F$-class c in $(\mathbf{G}^1)^F$. It follows that the operator $\mathrm{sh}_{F/\sigma F}$ from $(\mathbf{G}^0)^F$-class functions on $(\mathbf{G}^1)^F$

to $\mathbf{G}^{0\sigma F}$-class functions on $(\mathbf{G}^1)^{\sigma F}$ defined by $\mathrm{sh}_{F/\sigma F}(\chi)(n_{F/\sigma F}x) = \chi(x)$ is an isometry.

The end of this section is devoted to the proof of the following

Proposition 8.1. *Let $\mathbf{L}^1 = N_{\mathbf{G}^1}(\mathbf{L}^0 \subset \mathbf{P}^0)$ be a "Levi" of $\mathbf{G}^1$ containing σ, where $\mathbf{L}^0$ is F-stable; we have $\mathbf{L}^1 = \mathbf{L}^0 \cdot \sigma$. Assume that σ is semi-simple and that the characteristic is good for $\mathbf{G}^{\sigma 0}$. Then*

$$\mathrm{sh}_{F/\sigma F} \circ {}^*R_{\mathbf{L}^1}^{\mathbf{G}^1} = {}^*R_{\mathbf{L}^1}^{\mathbf{G}^1} \circ \mathrm{sh}_{F/\sigma F} \quad \text{and} \quad \mathrm{sh}_{F/\sigma F} \circ R_{\mathbf{L}^1}^{\mathbf{G}^1} = R_{\mathbf{L}^1}^{\mathbf{G}^1} \circ \mathrm{sh}_{F/\sigma F}\,.$$

Proof. The second equality follows from the first by adjunction, using that the adjoint of $\mathrm{sh}_{F/\sigma F}$ is $\mathrm{sh}_{F/\sigma F}^{-1}$. Let us prove the first equality.

Let χ be a $(\mathbf{G}^0)^F$-class function on $\mathbf{G}^1$ and let $\sigma l u = u \sigma l$ be the Jordan decomposition of an element of $(\mathbf{L}^1)^{\sigma F}$ with u unipotent and σl semi-simple. By the character formula 2.5(iii) and the definition of $Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}$ we have

$$({}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}\, \mathrm{sh}_{F/\sigma F}(\chi))(\sigma l u) = \\ |(\mathbf{G}^{\sigma l})^{0\sigma F}|^{-1} \sum_{v \in (\mathbf{G}^{\sigma l})^{0\sigma F}_{\mathrm{unip}}} \mathrm{sh}_{F/\sigma F}(\chi)(\sigma l v)\, \mathrm{Trace}((v, u^{-1}) | H_c^*(Y_{\mathbf{U},\sigma F})),$$

where v (resp. u) acts by left- (resp. right-) translation on $Y_{\mathbf{U},\sigma F} = \{x \in (\mathbf{G}^{\sigma l})^0 \mid x^{-1} \cdot {}^{\sigma F}x \in \mathbf{U}\}$ where $\mathbf{U}$ denotes the unipotent radical of $\mathbf{P}^0$; in the summation v is in the identity component of $\mathbf{G}^{\sigma l}$ since, σ being semi-simple, u is in $\mathbf{G}^0$ hence in $(\mathbf{G}^{\sigma l})^0$ by [DM94, 1.8 (i)] since σl is semisimple.

Let us write $l = {}^F\lambda^{-1} \cdot \lambda$ with $\lambda \in \mathbf{L}^0$, so that $\sigma l = n_{F/\sigma F}(l'\sigma)$ where $l' = \lambda \cdot {}^{\sigma F}\lambda^{-1}$.

Lemma 8.2. *For $v \in (\mathbf{G}^{\sigma l})^{0\sigma F}_{unip}$ we have $\sigma l v = n_{F/\sigma F}((\sigma l \cdot v')^{\sigma^F \lambda^{-1}})$ where $v' = n_{\sigma F/\sigma F} v \in (\mathbf{G}^{\sigma l})^{0\sigma F}$ is defined by writing $v = {}^{\sigma F}\eta \cdot \eta^{-1}$ where $\eta \in (\mathbf{G}^{\sigma t})^0$ and setting $v' = \eta^{-1} \cdot {}^{\sigma F}\eta$.*

Proof. We have $\sigma l v = \sigma l {}^{\sigma F}\eta \cdot \eta^{-1} = {}^{\sigma F}\eta \sigma l \eta^{-1} = \sigma^F(\eta\lambda^{-1})\lambda\eta^{-1}$, thus $\sigma l v = n_{F/\sigma F}((\lambda\eta^{-1}) \cdot {}^{\sigma F}(\eta\lambda^{-1})\sigma)$. And we have $(\lambda\eta^{-1}) \cdot {}^{\sigma F}(\eta\lambda^{-1})\sigma = \lambda v' {}^{\sigma F}\lambda^{-1}\sigma = {}^F\lambda l v' \sigma^F \lambda^{-1} = (\sigma l v')^{\sigma^F\lambda^{-1}}$, thus $\mathrm{sh}_{F/\sigma F}(\chi)(\sigma l v) = \chi((\sigma l v')^{\sigma^F\lambda^{-1}})$. □

Lemma 8.3. (i) *We have $(\sigma l)^{\sigma^F\lambda^{-1}} = l'\sigma$.*

(ii) *The conjugation $x \mapsto x^{\sigma^F\lambda^{-1}}$ maps $\mathbf{G}^{\sigma l}$ and the action of σF on it, to $\mathbf{G}^{l'\sigma}$ with the action of F on it; in particular it induces bijections $(\mathbf{G}^{\sigma l})^{0\sigma F} \xrightarrow{\sim} (\mathbf{G}^{l'\sigma})^{0^F}$ and $Y_{\mathbf{U},\sigma F} \xrightarrow{\sim} Y_{\mathbf{U},F}$, where $Y_{\mathbf{U},F} = \{x \in (\mathbf{G}^{l'\sigma})^0 \mid x^{-1\,F}x \in \mathbf{U}\}$.*

Proof. (i) is an obvious computation and shows that if $x \in \mathbf{G}^{\sigma l}$ then $x^{\sigma^F\lambda^{-1}} \in \mathbf{G}^{l'\sigma}$. To prove (ii), it remains to show that if $x \in \mathbf{G}^{\sigma l}$ then ${}^F(x^{\sigma^F\lambda^{-1}}) = ({}^{\sigma F}x)^{\sigma^F\lambda^{-1}}$. From $x^\sigma = x^{l^{-1}} = x^{\lambda^{-1}\cdot {}^F\lambda}$, we get $x^{\sigma^F\lambda^{-1}} = x^{\lambda^{-1}}$, whence ${}^F(x^{\sigma^F\lambda^{-1}}) = ({}^Fx)^{{}^F\lambda^{-1}} = (({}^{\sigma F}x)^\sigma)^{{}^F\lambda^{-1}} = ({}^{\sigma F}x)^{\sigma^F\lambda^{-1}}$. □

Applying lemmas 8.2 and 8.3 we get

$$({}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}\, \mathrm{sh}_{F/\sigma F}(\chi))(\sigma l u) = \\ |(\mathbf{G}^{\sigma l})^{0\sigma F}|^{-1} \sum_{v \in (\mathbf{G}^{\sigma l})^{0\sigma F}_{\mathrm{unip}}} \chi((\sigma l v')^{\sigma^F\lambda^{-1}})\, \mathrm{Trace}((v^{\sigma^F\lambda^{-1}}, (u^{\sigma^F\lambda^{-1}})^{-1}) \mid H_c^*(Y_{\mathbf{U},F})).$$

Lemma 8.4. *Assume that the characteristic is good for $\mathbf{G}^{\sigma 0}$, where σ is a quasi-central element of $\mathbf{G}$. Then it is also good for $(\mathbf{G}^s)^0$ where s is any quasi-semi-simple element of $\mathbf{G}^0 \cdot \sigma$.*

Proof. Let Σ_σ (resp. Σ_s) be the root system of $\mathbf{G}^{\sigma 0}$ (resp. $(\mathbf{G}^s)^0$). By definition, a characteristic p is good for a reductive group if for no closed subsystem of its root system the quotient of the generated lattices has p-torsion. The system Σ_s is not a closed subsystem of Σ_σ in general, but the relationship is expounded in [DM02]: let Σ be the root system of $\mathbf{G}^0$ with respect to a σ-stable pair $\mathbf{T} \subset \mathbf{B}$ of a maximal torus and a Borel subgroup of $\mathbf{G}^0$. Up to conjugacy, we may assume that s also stabilizes that pair. Let $\overline{\Sigma}$ the set of sums of the σ-orbits in Σ, and Σ' the set of averages of the same orbits. Then Σ' is a non-necessarily reduced root system, but Σ_σ and Σ_s are subsystems of Σ' and are reduced. The system $\overline{\Sigma}$ is reduced, and the set of sums of orbits whose average is in Σ_σ (resp. Σ_s) is a closed subsystem that we denote by $\overline{\Sigma}_\sigma$ (resp. $\overline{\Sigma}_s$).

We need now the following generalization of [Bou, chap VI, §1.1, lemme1]

Lemma 8.5. *Let $\mathcal{L}$ be a finite set of lines generating a vector space V over a field of characteristic 0; then two reflections of V which stabilize $\mathcal{L}$ and have a common eigenvalue $\zeta \neq 1$ with ζ-eigenspace the same line of $\mathcal{L}$ are equal.*

Proof. Here we mean by reflection an element $s \in \mathrm{GL}(V)$ such that $\ker(s-1)$ is a hyperplane. Let s and s' be reflections as in the statement. The product $s^{-1}s'$ stabilizes $\mathcal{L}$, so has a power which fixes $\mathcal{L}$, thus is semisimple. On the other hand $s^{-1}s'$ by assumption fixes one line $L \in \mathcal{L}$ and induces the identity on V/L, thus is unipotent. Being semi-simple and unipotent it has to be the identity. □

It follows from 8.5 that two root systems with proportional roots have same Weyl group, thus same good primes; thus:

- Σ_s and $\overline{\Sigma}_s$ have same good primes, as well as Σ_σ and $\overline{\Sigma}_\sigma$.
- The bad primes for $\overline{\Sigma}_s$ are a subset of those for $\overline{\Sigma}$, since it is a closed subsystem.

It only remains to show that the good primes for $\overline{\Sigma}$ are the same as for $\overline{\Sigma}_\sigma$, which can be checked case by case: we can reduce to the case where Σ is irreducible, where these systems coincide excepted when Σ is of type A_{2n}; but in this case $\overline{\Sigma}$ is of type B_n and Σ_σ is of type B_n or C_n, which have the same set $\{2\}$ of bad primes. □

Since the characteristic is good for $\mathbf{G}^{\sigma 0}$, hence also for $(\mathbf{G}^{\sigma l})^0$ by lemma 8.4, the elements v' et v are conjugate in $(\mathbf{G}^{\sigma l})^{0^{\sigma F}}$ (see [DM85, IV Corollaire 1.2]). By Lemma 8.3(ii), the element $v^{\sigma^F \lambda^{-1}}$ runs over the unipotent elements of $(\mathbf{G}^{l'\sigma})^{0^F}$ when v runs over $(\mathbf{G}^{\sigma l})^{0^{\sigma F}}_{\mathrm{unip}}$. Using moreover the equality $|(\mathbf{G}^{\sigma l})^{0^{\sigma F}}| = |(\mathbf{G}^{l'\sigma})^{0^F}|$ we get

$$(*) \quad ({}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}\,\mathrm{sh}_{F/\sigma F}(\chi))(\sigma l u) = \frac{1}{|(\mathbf{G}^{l'\sigma})^{0^F}|} \sum_{u_1 \in (\mathbf{G}^{l'\sigma})^{0^F}_{\mathrm{unip}}} \chi(u_1 l' \sigma) \mathrm{Trace}((u_1, (u^{\sigma^F \lambda^{-1}})^{-1}) | H_c^*(Y_{\mathbf{U},F}))$$

On the other hand by Lemma 8.2 applied with $v = u$, we have

$$(\mathrm{sh}_{F/\sigma F}\, {}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\chi))(\sigma l u) = {}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\chi)((\sigma l u)^{\sigma^F \lambda^{-1}}) = {}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\chi)(l'\sigma \cdot u^{\sigma^F \lambda^{-1}}),$$

the second equality by Lemma 8.3(i). By the character formula this is equal to the right-hand side of formula (*). □

References

[Bou] N. Bourbaki, Groupes et algèbres de Lie, Chap. 4,5 et 6, *Masson* (1981).

[D99] F. Digne, Descente de Shintani et restriction des scalaires, *J. LMS* **59** (1999) 867–880.

[DM85] F. Digne et J. Michel, Fonctions L des variétés de Deligne-Lusztig et descente de Shintani, *Mémoires SMF* **20** (1985), p. 1–144

[DM91] F. Digne and J. Michel, Representations of finite groups of Lie type, *LMS Student texts* **21** (1991).

[DM94] F. Digne et J. Michel, Groupes réductifs non connexes, *Annales de l'École Normale Supérieure* **20** (1994), p. 345–406.

[DM02] F. Digne et J. Michel, Points fixes des automorphismes quasi-semi-simples, *C.R.A.S.* **334** (2001), p. 1055–1060.

[E96] M. Eftekhari, Faisceaux caractères sur les groupes non connexes, *J. Algebra* **184** (1996) 516–537.

[Le] G. Lehrer, Rational tori, semisimple orbits and the topology of hyperplane complements, *Comment. Math. Helvetici* **67**, (1992) p. 226–251.

[Lu] G. Lusztig, Character sheaves on disconnected groups I–X, *Representation Theory* **7** (2003) 374–403, **8** (2004) 72–178 and 346–413, **9** (2005) 209–266, **10** (2006) 314–379 and **13** (2009) 82–140.

[St] R. Steinberg, Endomorphisms of linear algebraic groups, *Memoirs of the AMS* **80** (1968).

(J. Michel) Institut de Mathématiques de Jussieu – Paris rive gauche, Université Denis Diderot, Bâtiment Sophie Germain, 75013, Paris France.

E-mail address: `jean.michel@imj-prg.fr`

URL: `webusers.imj-prg.fr/~jean.michel`

(F. Digne) Laboratoire Amiénois de Mathématique Fondamentale et Appliquée, CNRS UMR 7352, Université de Picardie-Jules Verne, 80039 Amiens Cedex France.

E-mail address: `digne@u-picardie.fr`

URL: `www.lamfa.u-picardie.fr/digne`