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## - To cite this version:

Maurizio Grasselli, Morgan Pierre. ENERGY STABLE AND CONVERGENT FINITE ELEMENT
SCHEMES FOR THE MODIFIED PHASE FIELD CRYSTAL EQUATION. 2015. hal-01118961

HAL Id: hal-01118961

## https://hal.science/hal-01118961

Preprint submitted on 20 Feb 2015

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# ENERGY STABLE AND CONVERGENT FINITE ELEMENT SCHEMES FOR THE MODIFIED PHASE FIELD CRYSTAL EQUATION 

MAURIZIO GRASSELLI AND MORGAN PIERRE


#### Abstract

We propose a space semi-discrete and a fully discrete finite element scheme for the modified phase field crystal equation (MPFC). The space discretization is based on a splitting method and consists in a Galerkin approximation which allows low order (piecewise linear) finite elements. The time discretization is a second-order scheme which has been introduced by Gomez and Hughes for the Cahn-Hilliard equation. The fully discrete scheme is shown to be unconditionally energy stable and uniquely solvable for small time steps, with a smallness condition independent of the space step. Using energy estimates, we prove that in both cases, the discrete solution converges to the unique energy solution of the MPFC equation as the discretization parameters tend to 0 . Using a Lojasiewicz inequality, we also establish that the discrete solution tends to a stationary solution as time goes to infinity. Numerical simulations illustrate the theoretical results.


Keywords: Finite elements, second-order schemes, gradient-like systems, Łojasiewicz inequality.
MSC 2010: 65M60, 65P40, 74N05, 82C26.

## 1. Introduction

In this paper, we analyze finite element discretizations of the modified phase field crystal (MPFC) equation

$$
\begin{equation*}
\beta u_{t t}+u_{t}=\Delta\left[\Delta^{2} u+2 \Delta u+f(u)\right], \quad \text { in } \Omega \times(0,+\infty), \tag{1.1}
\end{equation*}
$$

with periodic boundary conditions on the $d$-parallelepiped $\Omega=\Pi_{k=1}^{d}\left(0, L_{k}\right)$ ( $L_{k}>0$ for $k=1, \ldots, d)$ in space dimension $d=1,2$ or 3 . Equation (1.1) is endowed with initial conditions

$$
\begin{equation*}
u(0, x)=u_{0}(x), \quad u_{t}(0, x)=v_{0}(x), \quad x \in \Omega . \tag{1.2}
\end{equation*}
$$

The unknown function $u$ is the phase function and $\beta>0$ is a relaxation parameter. The nonlinearity $f$ is the derivative of a polynomial potential $F$ (see (2.1)-(2.2) for a precise definition). A relevant example in applications is given by

$$
\begin{equation*}
f(s)=s^{3}+(1-\varepsilon) s, \tag{1.3}
\end{equation*}
$$

where $\varepsilon \in \mathbb{R}$ is constant.
When $\beta=0$, equation (1.1) is known as the phase field crystal (PFC) equation: it has been employed to model and simulate the dynamics of crystalline materials, including crystal growth in a supercooled liquid, dendritic and eutectic solidification, epitaxial growth $[6,7,11,12,31,33]$. In the phase field approach, the number density of atoms is approximated by the phase function $u$. The parameter $\varepsilon$ in (1.3) is
proportional to the undercooling i.e. $\varepsilon \sim T_{e}-T, T_{e}$ being the equilibrium temperature at which the phase transition occurs. The PFC equation is a gradient flow for the free energy

$$
\begin{equation*}
E(u)=\int_{\Omega}\left(\frac{1}{2}|\Delta u|^{2}-|\nabla u|^{2}+F(u)\right) d x . \tag{1.4}
\end{equation*}
$$

It preserves the total mass and can be viewed as an analog of the Swift-Hohenberg equation [36].

The MPFC equation (1.1) (with $\beta>0$ ) was recently proposed in [34] (cf. also [12, $15,16,35]$ ) in order to account for elastic interactions. Equations like (1.1) have also been derived in [14] to take large deviations from thermodynamic equilibrium into account.The MPFC equation is no longer a gradient flow for (1.4), but it is possible to associate to a solution $\left(u, u_{t}\right)$ a "pseudo-energy", obtained by adding to $E$ a kinetic energy term (see (2.7)). This leads in a natural way to the notion of energy solution introduced by Grasselli and Wu [25] for the MPFC equation, following a terminology used for the modified Cahn-Hilliard equation [22, 23, 24]. Existence of a unique energy solution was proved by Grasselli and Wu in [25] as well as the convergence of single trajectories to single stationary states. The analysis of global dynamics (that is, existence of global and exponential attractors) for the MPFC equation was carried out in $[25,26]$.

From the numerical analysis point of view, the MPFC equation has been studied in $[3,4,17,37,38]$, while the literature on the PFC equation is more abundant (see, e.g., $[2,5,10,19,28,39]$ ). In $[3,4,38]$, the authors proposed unconditionally energy stable and unconditionally uniquely solvable finite difference schemes. The time discretization was based on a convex splitting of the pseudo-energy and was either first order or second order. A priori error estimates were proved assuming enough regularity on the solution. A time semi-discrete scheme was used in [37] to establish the existence of a weak solution and of a unique strong solution to the MPFC equation up to any positive final time $T>0$. In [17], an unconditionally energy stable finite element scheme was derived, but no proof of convergence was given.

The main purpose of this paper is to derive and analyze a second order (in time) fully discrete finite element scheme for the MPFC equation. For the space discretization we use a splitting approach which is well known in the context of phase field models (cf., e.g., [9, 21]). This argument allows to consider low order (piecewise linear) finite elements, although the analysis is carried out in a more general setting, namely a Galerkin approximation. For the time discretization, we use a modified Crank-Nicolson scheme which was introduced by Gomez and Hughes for the CahnHilliard equation [18]; their approach represents an interesting alternative to secant schemes (cf. the discussion in [40]).

We prove that our scheme is unconditionally energy stable, solvable for any time step, and uniquely solvable for small enough time steps, with a smallness assumption independent of the space step. Using the energy estimate, and assuming only some natural conditions on the initial conditions, we show that the solution of the fully discrete scheme converges to the energy solution of problem (1.1)-(1.2) as the time step $\tau$ and the space step $h$ tend to 0 . Finally, we prove that the discrete solution tends to a stationary solution as time goes to infinity. This last issue is not trivial
since the set of steady states can be very complicated (see [32] for an analysis of the one dimensional stationary problem, cf. also [8]).

For equations involving a second order time derivative such as (1.1), second order time discretizations are very interesting because they do not regularize in finite time, unlike first order schemes: a fundamental property of the continuous model is therefore reproduced at the discrete level. In contrast with the second order two-step scheme in $[3,4]$, we loose the unconditional unique solvability, but one advantage is that our one-step scheme can be used with variable time steps. Moreover, we do not assume $\varepsilon<1$ in (1.3), and we do not have any restriction on the initial value of $u_{t}$. Since energy solutions have a weaker regularity than the weak solutions, our convergence result as $(h, \tau) \rightarrow(0,0)$ holds with the minimal regularity on the solution.

Our proofs are crucially based on the energy estimates. In order to establish the convergence to equilibrium, we also use the gradient-like structure of the problem and a Łojasiewicz inequality [30], as in the continuous case [25]. Related results have been obtained for first order schemes applied to second-order gradient-like equations in $[1,20]$, but the case of the second-order scheme here is more involved.

The paper is structured as follows. In Section 2 we introduce the functional setting and we recall useful results concerning the continuous problem. In Section 3 we consider the space semi-discrete problem. We show its well-posedness and establish energy estimates which allow us to prove the convergence of the semi-discrete solution to the energy solution of the MPFC problem (1.1)-(1.2). This gives a framework for the fully discrete problem which is treated in Section 4 . Section 5 is concerned with the convergence to equilibrium for the fully discrete problem. In Section 6, numerical simulations in one and two space dimension illustrate the theoretical results.

## 2. The continuous Problem

2.1. Notation and functional spaces. For a real Banach space $V$ with dual $V^{\star}$, we indicate by $\langle\cdot, \cdot\rangle_{V^{\star}, V}$ the duality product between $V$ and $V^{\star}$. We denote by $H_{p e r}^{m}$, $m \in \mathbb{N}$, the space of functions that are in $H_{l o c}^{m}\left(\mathbb{R}^{d}\right)$ and periodic with period $\Omega$. For any $m \in \mathbb{N}, H_{\text {per }}^{m}$ is a Hilbert space for the scalar product

$$
((u, v))_{m}=\sum_{|\kappa| \leq m} \int_{\Omega} D^{\kappa} u(x) D^{\kappa} v(x) d x
$$

( $\kappa$ being a multi-index) and its associated norm $\|u\|_{m}=((u, u))_{m}^{1 / 2}$.
For $m=0, H_{p e r}^{0}=L^{2}(\Omega)$, the inner product as well as the norm on $L^{2}(\Omega)$ are simply indicated by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. For sake of simplicity, we assume that $\int_{\Omega} 1 d x=|\Omega|=1$. The mean value of any function $u \in L^{2}(\Omega)$ is denoted by

$$
\langle u\rangle=\int_{\Omega} u d x
$$

and we set $\dot{u}=u-\langle u\rangle$.
The dual space of $H_{p e r}^{m}$ is denoted by $H_{p e r}^{-m}$, and it is equipped with the operator norm given by

$$
\|\mathcal{T}\|_{-m}=\sup _{\|u\|_{m}=1, u \in H_{p e r}^{m}}|\mathcal{T}(u)| .
$$

For an operator $u \in H_{p e r}^{-m}$, we denote $\langle u\rangle=\langle u, 1\rangle_{H_{p e r}^{-m}, H_{p e r}^{m}}$ and we set $\dot{u}=u-\langle u\rangle$. We denote by $\dot{H}_{p e r}^{m}=\left\{u \in H_{p e r}^{m}:\langle u\rangle=0\right\}(m \in \mathbb{Z})$ the Sobolev spaces for functions with zero mean. We will frequently use the fact that $H_{p e r}^{m}$ is isomorphic to $\mathbb{R} \times \dot{H}_{p e r}^{m}$ $(m \in \mathbb{Z})$ through the decomposition $u=\langle u\rangle+\dot{u}$.

Using the dense and continuous inclusions $H_{p e r}^{1} \subset L^{2}(\Omega) \subset H_{p e r}^{-1}$, the semi-scalar product on $H_{p e r}^{1}$,

$$
(u, v) \mapsto(\nabla u, \nabla v),
$$

defines a linear operator $\mathcal{A}=-\Delta: D(\mathcal{A}) \rightarrow L^{2}(\Omega)$ with domain $D(\mathcal{A})=H_{\text {per }}^{2}$. We denote $\dot{\mathcal{A}}=-\Delta: D(\dot{\mathcal{A}}) \rightarrow L^{2}(\Omega)$ the restriction of $\mathcal{A}$ to $\dot{L}^{2}(\Omega)$, with domain $D(\dot{\mathcal{A}})=\dot{H}_{\text {per }}^{2}$. We observe that $\dot{\mathcal{A}}$ is a positive self-adjoint operator with compact resolvent so that its powers $\dot{\mathcal{A}}^{s}(s \in \mathbb{R})$ are well defined and it is possible to prove that $\dot{H}_{p e r}^{m}=D\left(\dot{\mathcal{A}}^{m / 2}\right)(m \in \mathbb{Z})$.
For $m=-1$, we introduce an equivalent and more convenient norm $|\cdot|_{-1}$ on $\dot{H}_{\text {per }}^{-1}$ associated with the inner product

$$
(\dot{u}, \dot{v})_{-1}=\left(\dot{\mathcal{A}}^{-1 / 2} \dot{u}, \dot{\mathcal{A}}^{-1 / 2} \dot{v}\right),
$$

so that for any $\dot{u} \in \dot{H}_{\text {per }}^{-1}$, we have

$$
|\dot{u}|_{-1}=\left\|\dot{\mathcal{A}}^{-1 / 2} \dot{u}\right\|=\left\|\nabla \dot{\mathcal{A}}^{-1} \dot{u}\right\| .
$$

Similarly, for $m=1$, we will sometimes use the equivalent norm $|\cdot|_{1}$ in $H_{p e r}^{1}$ associated with the inner product

$$
(u, v)_{1}=\langle u\rangle\langle v\rangle+(\nabla u, \nabla v) .
$$

Moreover, $\dot{\mathcal{A}}$ defines a continuous bijection from $\dot{H}_{p e r}^{m}$ onto $\dot{H}_{p e r}^{m-2}$. In particular, for $s=-1$,

$$
(\dot{u}, \dot{v})_{-1}=\left\langle\mathcal{A}^{-1} \dot{u}, \dot{v}\right\rangle_{H_{p e r}^{1}, H_{p e r}^{-1}}=\left\langle\dot{u}, \mathcal{A}^{-1} \dot{v}\right\rangle_{H_{p e r}^{-1}, H_{p e r}^{1}}=\left(\mathcal{A}^{-1 / 2} \dot{u}, \mathcal{A}^{-1 / 2} \dot{v}\right) .
$$

2.2. Energy solutions. The nonlinearity $f$ is a polynomial of odd degree whose leading coefficient is positive and which vanishes at 0 :

$$
\begin{equation*}
f(s)=\sum_{i=1}^{2 p+1} a_{i} s^{i} \quad \forall s \in \mathbb{R} \quad\left(a_{2 p+1}>0\right) \tag{2.1}
\end{equation*}
$$

with $p \in \mathbb{N}^{\star}$ if $d=1$ or $d=2$ and with $p \in\{1,2\}$ if $d=3$. We denote $F$ the antiderivative of $f$ which vanishes at 0 , i.e.,

$$
\begin{equation*}
F(s)=\sum_{i=2}^{2 p+2} \frac{a_{i-1}}{i} s^{i} \quad \forall s \in \mathbb{R} \tag{2.2}
\end{equation*}
$$

We will make use of the Sobolev injection $H_{p e r}^{1} \subset L^{2 p+2}(\Omega)$. In particular, there is a constant $C_{S}=C_{S}(\Omega)$ such that

$$
\begin{equation*}
\|u\|_{L^{2 p+2}(\Omega)} \leq C_{S}|u|_{1}, \quad \forall u \in H_{p e r}^{1}, \tag{2.3}
\end{equation*}
$$

and the map $v \mapsto f(v)$ is Lipschitz continuous on bounded sets of $H_{p e r}^{1}$ with values into $L^{(2 p+2) /(2 p+1)}(\Omega) \subset H_{\text {per }}^{-1}$. We also have $H_{p e r}^{2} \subset C^{0}(\bar{\Omega})$ with continuous injection. Finally, we note that there exist constants $c_{1} \geq 0, c_{2}>0$ and $c_{3} \geq 0$ such that

$$
\begin{align*}
F(s) \geq 2 s^{2}-c_{1} & \forall s \in \mathbb{R},  \tag{2.4}\\
|f(s)| \leq c_{2} F(s)+c_{3} & \forall s \in \mathbb{R} . \tag{2.5}
\end{align*}
$$

We point out that the expression (2.2) includes the standard quartic potential (obtained with $p=1$ )

$$
\begin{equation*}
F(s)=\frac{s^{4}}{4}+\frac{(1-\varepsilon)}{2} s^{2} \tag{2.6}
\end{equation*}
$$

In contrast to some authors, we do not assume that $\varepsilon<1$ (we simply have $\varepsilon \in \mathbb{R}$ ).
In [25], a notion of energy solution was introduced. This is based on the following pseudo-energy:

$$
\begin{equation*}
\mathcal{E}(u, v)=E(u)+\frac{\beta}{2}|\dot{v}|_{-1}^{2}, \tag{2.7}
\end{equation*}
$$

which is well defined for any $(u, v) \in H_{p e r}^{2} \times H_{p e r}^{-1}$.
Definition 2.1. A pair $\left(u, u_{t}\right)$ is called an energy solution to problem (1.1)-(1.2) if

$$
\begin{align*}
&\left(u, u_{t}\right) \in L^{\infty}\left(\mathbb{R}_{+}\right.\left.; H_{p e r}^{2} \times H_{p e r}^{-1}\right), \quad u_{t t} \in L^{\infty}\left(\mathbb{R}_{+} ; H_{p e r}^{-4}\right),  \tag{2.8}\\
& \mathcal{E}\left(u, u_{t}\right) \in L^{\infty}\left(\mathbb{R}_{+}\right), \tag{2.9}
\end{align*}
$$

and the following relations hold:

$$
\begin{gather*}
\left\langle\beta u_{t t}+u_{t}\right\rangle=0  \tag{2.10}\\
\dot{\mathcal{A}}^{-1}\left(\beta u_{t t}+u_{t}\right)+\mathcal{A}^{2} u-2 \mathcal{A} u+f(u)-\langle f(u)\rangle=0, \\
\text { in } \dot{H}_{\text {per }}^{-2}, \quad \text { a.e. in } \mathbb{R}_{+},  \tag{2.11}\\
u(0)=u_{0} \text { in } H_{\text {per }}^{2}, \quad u_{t}(0)=v_{0} \text { in } H_{\text {per }}^{-1} . \tag{2.12}
\end{gather*}
$$

Equation (2.10) can be interpreted as a conservation law for the mass. With this definition, we have

Theorem 2.2 ([25]). For any intial data $\left(u_{0}, v_{0}\right) \in H_{\text {per }}^{2} \times H_{\text {per }}^{-1}$, problem (1.1)-(1.2) has a unique global energy solution $\left(u, u_{t}\right)$. Moreover, any energy solution satifies the strong time continuity property

$$
u \in C^{2}\left(\mathbb{R}_{+} ; H_{\text {per }}^{-4}\right) \cap C^{1}\left(\mathbb{R}_{+} ; H_{\text {per }}^{-1}\right) \cap C\left(\mathbb{R}_{+} ; H_{\text {per }}^{2}\right)
$$

as well as the following energy identity, for all $s, t \in \mathbb{R}_{+}$with $s<t$,

$$
\begin{equation*}
\mathcal{E}\left(u(t), u_{t}(t)\right)=\mathcal{E}\left(u(s), u_{t}(s)\right)-\int_{s}^{t}\left|\dot{u}_{t}(\tau)\right|_{-1}^{2} d \tau+\int_{s}^{t}\left\langle v_{0}\right\rangle e^{-\tau / \beta} \int_{\Omega} f(u(\tau)) d x d \tau \tag{2.13}
\end{equation*}
$$

In particular, whenever $\left\langle v_{0}\right\rangle=0$ then the pseudo-energy is nonincreasing.

## 3. The space semi-discrete problem

3.1. The space semi-discrete scheme. Our space discretization is based on two ideas: first, in view of the time discretization, we write the PDE (1.1) as a first order system; second, in order to use low order finite elements, we split the tri-Laplacian into three terms, in the spirit of a well-known splitting approach in Cahn-Hilliard type equations (see, e.g., [9, 21, 17]). We obtain the following system, which is (formally) equivalent to (1.1):

$$
\left\{\begin{array}{l}
u_{t}=v \\
\beta v_{t}=-v+\Delta w \\
z=-\Delta u \\
w=-\Delta z+2 \Delta u+f(u)
\end{array}\right.
$$

Now, let $V_{h}$ denote a finite-dimensional subspace of $H_{p e r}^{1}$ which contains the constants. In applications, $V_{h}$ will be a space of conforming finite elements (see Section 6). The space $V_{h}$ can also be obtained with a spectral basis.

The space semi-discrete scheme reads: find $u_{h}, v_{h}, z_{h}, w_{h}: \mathbb{R}_{+} \rightarrow V_{h}$ such that

$$
\left\{\begin{array}{l}
\left(\partial_{t} u_{h}, \varphi_{h}\right)=\left(v_{h}, \varphi_{h}\right)  \tag{3.1}\\
\beta\left(\partial_{t} v_{h}, \psi_{h}\right)=-\left(v_{h}, \psi_{h}\right)-\left(\nabla w_{h}, \nabla \psi_{h}\right) \\
\left(z_{h}, \zeta_{h}\right)=\left(\nabla u_{h}, \nabla \zeta_{h}\right) \\
\left(w_{h}, \xi\right)=\left(\nabla z_{h}, \nabla \xi_{h}\right)-2\left(\nabla u_{h}, \nabla \xi_{h}\right)+\left(f\left(u_{h}\right), \xi_{h}\right)
\end{array}\right.
$$

for all $\varphi_{h}, \psi_{h}, \zeta_{h}, \xi_{h}$ in $V_{h}$. This problem is completed with initial conditions

$$
\begin{equation*}
u_{h}(0)=u_{h}^{0}, \quad v_{h}(0)=v_{h}^{0} \tag{3.2}
\end{equation*}
$$

where $u_{h}^{0}$ and $v_{h}^{0}$ are given in $V_{h}$.
It will be convenient to work with an appropriate basis of $V_{h}$. For this purpose, let $\left(e_{h}^{i}\right)_{1 \leq i \leq N_{h}}$ denote an orthonormal basis of $V_{h}$ for the $L^{2}(\Omega)$ scalar product, such that $e_{h}^{1} \equiv 1$. The integer $N_{h}$ is the dimension of $V_{h}$. To every function $r_{h}=\sum_{i=1}^{N_{h}} r_{i} e_{h}^{i} \in$ $V_{h}$ corresponds a unique (column) vector $R=\left(r_{1}, \ldots, r_{N_{h}}\right)^{t}$, represented by the corresponding capital letter. We seek

$$
u_{h}(t)=\sum_{i=1}^{N_{h}} u_{i}(t) e_{h}^{i} \simeq\left(u_{1}(t), \ldots, u_{N_{h}}(t)\right)^{t}=U(t), \quad v_{h} \simeq V, \quad z_{h} \simeq Z, \quad w_{h} \simeq W
$$

Define $A=\left(A_{i j}\right)_{1 \leq i, j \leq N_{h}}$, where

$$
\begin{equation*}
A_{i j}=\left(\nabla e_{h}^{i}, \nabla e_{h}^{j}\right), \quad 1 \leq i, j \leq N_{h}, \tag{3.3}
\end{equation*}
$$

and let

$$
F_{h}(U)=\left(F\left(u_{h}\right), 1\right), \quad \text { so that } \quad \nabla F_{h}(U)=\left(\left(f\left(u_{h}\right), e_{h}^{1}\right), \ldots,\left(f\left(u_{h}\right), e_{h}^{N_{h}}\right)\right)^{t}
$$

By choosing the test functions $\varphi_{h}, \psi_{h}, \zeta_{h}, \xi_{h}$ in (3.1) as the basis functions $e_{h}^{i}$, we obtain the following equivalent system:

$$
\left\{\begin{array}{l}
U_{t}=V  \tag{3.4}\\
\beta V_{t}=-V-A W \\
Z=A U \\
W=A Z-2 A U+\nabla F_{h}(U)
\end{array}\right.
$$

Eliminating $V, Z$ and $W$, we see that (3.4) is equivalent to

$$
\begin{equation*}
\beta U_{t t}+U_{t}=-A\left[A^{2} U-2 A U+\nabla F_{h}(U)\right], \quad t \geq 0 \tag{3.5}
\end{equation*}
$$

Since $A$ is a discretization of $-\Delta$, this is natural space semi-discrete version of (1.1).
Let $U$ denote a solution of (3.5). We notice that the first line and the first column of $A$ are filled with zeros (recall $e_{h}^{1} \equiv 0$, so that $\nabla e_{h}^{1} \equiv 0$ ). Thus, the first component of $U, u_{1}(t)=\left(u_{h}(t), 1\right)$, satisfies

$$
\begin{equation*}
\beta \partial_{t t} u_{1}+\partial_{t} u_{1}=0, \quad t \geq 0 \tag{3.6}
\end{equation*}
$$

Solving (3.6) with initial conditions $u_{1}(0)=\left(u_{h}^{0}, 1\right)=: u_{1}^{0}$ and $\partial_{t} u_{1}(0)=\left(v_{h}^{0}, 1\right)=: v_{1}^{0}$ yields

$$
\begin{equation*}
\partial_{t} u_{1}(t)=v_{1}^{0} e^{-t / \beta}=: a(t) \quad \text { and } \quad u_{1}(t)=M-\beta a(t), \text { with } M=\beta v_{1}^{0}+u_{1}^{0} \tag{3.7}
\end{equation*}
$$

For every vector $R=\left(r_{1}, \ldots, r_{N_{h}}\right)^{t} \in \mathbb{R}^{N_{h}}$, we denote $\dot{R}=\left(r_{2}, \ldots, r_{N_{h}}\right)^{t} \in \mathbb{R}^{N_{h}-1}$. Then $\dot{U}$ satisfies

$$
\begin{equation*}
\beta \dot{U}_{t t}+\dot{U}_{t}=-\dot{A}\left[\dot{A}^{2} \dot{U}-2 \dot{A} \dot{U}+\dot{\nabla} F_{h}(U)\right], \quad t \geq 0 \tag{3.8}
\end{equation*}
$$

where $\dot{A}$ is the submatrix $\dot{A}=\left(A_{i j}\right)_{2 \leq i, j \leq N_{h}}$, and

$$
\dot{\nabla} F_{h}(U)=\left(\left(f\left(u_{h}\right), e_{h}^{2}\right), \ldots,\left(f\left(u_{h}\right), e_{h}^{N_{h}}\right)\right)^{t}
$$

We can also write $\dot{\nabla} F_{h}(U)=\dot{P}\left(\nabla F_{h}\left(u_{1}(t), \dot{U}\right)\right)$, where $\dot{P}: \mathbb{R}^{N_{h}} \rightarrow \mathbb{R}^{N_{h}-1}$ is the projection on the components $2, \ldots, N_{h}$. This shows that $\dot{\nabla} F_{h}(U)$ is a "non autonomous" function of $\dot{U}$ (recall that $u_{1}(t)$ is determined by (3.7)). For later purpose, we note that by (3.3), $\dot{A}$ is symmetric positive definite: in particular, $\dot{A}$ is invertible.

Conversely, any solution $U$ of (3.7)-(3.8) satisfies (3.4), i.e. that the second equation of (3.4) is satisfied with $V, Z$ and $W$ given by the three other equations of the system (3.4).
3.2. Existence, uniqueness, and discrete energy estimate. The standard Euclidean norm in $\mathbb{R}^{N_{h}}$ or $\mathbb{R}^{N_{h}-1}$ will be denoted $|\cdot|$. We also use the following quadratic norm:

$$
\begin{equation*}
|\dot{R}|_{-1}=\left(\dot{R}^{t} \dot{A}^{-1} \dot{R}\right)^{1 / 2} \tag{3.9}
\end{equation*}
$$

defined for all $\dot{R} \in \mathbb{R}^{N_{h}-1}$. Notice that $\left|A^{s} U\right|=\left|\dot{A}^{s} \dot{U}\right|\left(s>0, U \in \mathbb{R}^{N_{h}}\right)$. We set

$$
\begin{align*}
E_{h}(U) & =\frac{1}{2}|A U|^{2}-\left|A^{1 / 2} U\right|^{2}+F_{h}(U)  \tag{3.10}\\
\mathcal{E}_{h}(U, V) & =E_{h}(U)+\frac{\beta}{2}|\dot{V}|_{-1}^{2} \tag{3.11}
\end{align*}
$$

As a shortcut, for a solution $\left(U, U_{t}\right)$ of $(3.5)$, we will write

$$
\mathcal{E}_{h}(t)=\mathcal{E}_{h}\left(U(t), U_{t}(t)\right) .
$$

Notice that by the Cauchy-Schwarz inequality we have

$$
\begin{equation*}
\left|A^{1 / 2} U\right|^{2}=U^{t} A U \leq \frac{1}{4}|A U|^{2}+|U|^{2} \tag{3.12}
\end{equation*}
$$

Then, using also (2.4), we find that

$$
\begin{equation*}
E_{h}(U) \geq \frac{1}{4}|A U|^{2}+|U|^{2}-c_{1} \tag{3.13}
\end{equation*}
$$

We first prove the following
Lemma 3.1. Any solution $U \in C^{2}\left([0, T) ; \mathbb{R}^{N_{h}}\right)$ of (3.5) satisfies the energy equality

$$
\begin{equation*}
\frac{d}{d t} \mathcal{E}_{h}(t)+\left|\dot{U}_{t}\right|_{-1}^{2}=v_{1}^{0} e^{-t / \beta}\left(f\left(u_{h}\right), 1\right) \tag{3.14}
\end{equation*}
$$

and the energy estimate

$$
\begin{equation*}
\mathcal{E}_{h}(t)+\int_{0}^{t}\left|\dot{U}_{t}(s)\right|_{-1}^{2} d s \leq \mathcal{E}_{h}(0) e^{2 c_{2}\left|v_{1}^{0}\right| \beta}+\left(c_{1} c_{2}+c_{3}\right)\left|v_{1}^{0}\right| \beta e^{2 c_{2}\left|v_{1}^{0}\right| \beta} \tag{3.15}
\end{equation*}
$$

for all $t \in[0, T)$, where $c_{1}, c_{2}$ and $c_{3}$ depend only on $f$ (see (2.4)-(2.5)), and where $v_{1}^{0}=\partial_{t} u_{1}(0)$.

Proof. Recall that $\partial_{t} u_{1}(t)=v_{1}^{0} e^{-t / \beta}$ (see (3.7)). On multiplying (3.8) by $\dot{U}_{t}^{t} \dot{A}^{-1}$, and using

$$
\frac{d}{d t}\left[F_{h}(U(t))\right]=\left(\nabla F_{h}(U), U_{t}(t)\right)=\sum_{i=1}^{N_{h}} \partial_{t} u_{i}(t)\left(f\left(u_{h}\right), e_{h}^{i}\right)
$$

we find the energy equality (3.14). Estimate (2.5) yields

$$
\frac{d}{d t} \mathcal{E}_{h}(t)+\left|\dot{U}_{t}\right|_{-1}^{2} \leq\left|v_{1}^{0}\right| e^{-t / \beta}\left(c_{2} F_{h}(U)+c_{3}\right)
$$

By the Cauchy-Schwarz inequality, we have

$$
2\left|A^{1 / 2} U\right|^{2}=2 U^{t} A U \leq|A U|^{2}+|U|^{2}
$$

Thus we get

$$
2 E_{h}(U)=\left(|A U|^{2}-2\left|A^{1 / 2} U\right|^{2}+|U|^{2}\right)+F_{h}(U)+\left(F_{h}(U)-|U|^{2}\right) \geq F_{h}(U)-c_{1}
$$

so that

$$
\begin{equation*}
F_{h}(U) \leq 2 \mathcal{E}_{h}(U, V)+c_{1} \tag{3.16}
\end{equation*}
$$

Therefore we obtain

$$
\frac{d}{d t} \mathcal{E}_{h}(t)+\left|\dot{U}_{t}\right|_{-1}^{2} \leq\left|v_{1}^{0}\right| e^{-t / \beta}\left(2 c_{2} \mathcal{E}_{h}(t)+c_{1} c_{2}+c_{3}\right)
$$

Letting $\eta(t)=\int_{0}^{t} 2 c_{2}\left|v_{1}^{0}\right| e^{-t / \beta}$ and $c_{3}^{\prime}=c_{1} c_{2}+c_{3}$, Gronwall's lemma yields

$$
\mathcal{E}_{h}(t)+\int_{0}^{t}\left|\dot{U}_{t}(s)\right|_{-1}^{2} e^{\eta(t)-\eta(s)} d s \leq \mathcal{E}_{h}(0) e^{\eta(t)}+\int_{0}^{t} c_{3}^{\prime}\left|v_{1}^{0}\right| e^{-s / \beta} e^{\eta(t)-\eta(s)} d s
$$

Since $\eta(t)=2 c_{2}\left|v_{1}^{0}\right| \beta\left(1-e^{-t / \beta}\right) \leq 2 c_{2}\left|v_{1}^{0}\right| \beta$, we deduce the energy estimate (3.15).
Theorem 3.2. For every $U^{0}, V^{0}$ in $\mathbb{R}^{N_{h}}$, there exists a unique solution $U \in$ $C^{2}\left(\mathbb{R}_{+}, \mathbb{R}^{N_{h}}\right)$ of $(3.5)$ such that $U(0)=U^{0}$ and $U_{t}(0)=V^{0}$.

Proof. By the Cauchy-Lipschitz theorem, there exists a unique maximal solution $U \in C^{2}\left(\left[0, T^{+}\right) ; \mathbb{R}^{N_{h}}\right)$ of (3.5) satisfying the given initial conditions. The energy estimate (3.15) shows that $\mathcal{E}_{h}$ is uniformly bounded for $t \geq 0$. By (3.13), $|U|$ and $\left|\dot{U}_{t}\right|_{-1}$ are uniformly bounded for $t \geq 0$. This, together with the estimate (3.7) on the mass, implies that $T^{+}=+\infty$.
3.3. Some additional notation. We assume now that $\left(V_{h}\right)_{h>0}$ is a family of subspaces of $H_{p e r}^{1}$ such that:
(H1) for all $h>0, V_{h}$ has finite dimension and contains all the constants;
(H2) for any $\varphi \in H_{p e r}^{1}$, there exists $\varphi_{h} \in V_{h}$ such that $\varphi_{h} \rightarrow \varphi$ (strongly) in $H_{p e r}^{1}$, as $h$ tends to 0 .
For the convergence result as $h \rightarrow 0$, it will be useful to have $h$-dependent operators and norms. We denote $\mathcal{A}_{h}: V_{h} \rightarrow V_{h}$ the linear operator such that for any $q_{h} \in V_{h}$, $\mathcal{A}_{h} q_{h}$ solves

$$
\begin{equation*}
\left(\mathcal{A}_{h} q_{h}, \zeta_{h}\right)=\left(\nabla q_{h}, \nabla \zeta_{h}\right), \quad \forall \zeta_{h} \in V_{h} . \tag{3.17}
\end{equation*}
$$

The operator $\mathcal{A}_{h}$ is a discrete Laplacian, $\mathcal{A}_{h} \simeq-\Delta_{h}$. Notice that if $q_{h}$ is constant, then $\mathcal{A}_{h} q_{h}=0$ so that $\mathcal{A}_{h}$ is not invertible. In order to define a discrete version of $\dot{\mathcal{A}}^{-1}$, we introduce the subspace

$$
\dot{V}_{h}=\left\{\varphi_{h} \in V_{h}:\left\langle\varphi_{h}\right\rangle=0\right\}
$$

The bilinear form $(\nabla \cdot, \nabla \cdot)$ is symmetric positive definite on $\dot{V}_{h}$. We can define the operator $\dot{\mathcal{S}}_{h}: \dot{V}_{h} \rightarrow \dot{V}_{h}$ such that for any $\dot{r}_{h} \in \dot{V}_{h}, \dot{\mathcal{S}}_{h} \dot{r}_{h}$ is the unique solution of

$$
\begin{equation*}
\left(\nabla \dot{\mathcal{S}}_{h} \dot{r}_{h}, \nabla \dot{\varphi}_{h}\right)=\left(\dot{r}_{h}, \dot{\varphi}_{h}\right), \forall \dot{\varphi}_{h} \in \dot{V}_{h} \tag{3.18}
\end{equation*}
$$

By choosing $\zeta_{h} \equiv 1$ in (3.17), we see that $\mathcal{A}_{h}\left(V_{h}\right) \subset \dot{V}_{h}$, so that the restriction $\dot{\mathcal{A}}_{h}: \dot{V}_{h} \rightarrow \dot{V}_{h}$ of $\mathcal{A}_{h}$ is well defined. Using (3.17)-(3.18), it is easily seen that $\dot{\mathcal{S}}_{h}=\dot{\mathcal{A}}_{h}^{-1}$.

We also define the $L^{2}$-orthogonal projector $P_{h}: L^{2}(\Omega) \rightarrow V_{h}$, i.e.,

$$
\left(P_{h} q, \varphi_{h}\right)=\left(q, \varphi_{h}\right), \quad \forall q \in L^{2}(\Omega), \forall \varphi_{h} \in V_{h}
$$

By Pythagoras' theorem and assumption $(\mathcal{H} 2)$, for any $q \in L^{2}(\Omega)$, we have

$$
\begin{equation*}
\left\|q-P_{h} q\right\|=\inf _{r_{h} \in V_{h}}\left\|q-r_{h}\right\| \rightarrow 0, \text { as } h \rightarrow 0 \tag{3.19}
\end{equation*}
$$

Since $V_{h} \subset H_{p e r}^{1}$, the operator $P_{h}$ has a natural extension to $H_{\text {per }}^{-1}\left(\right.$ also denoted $\left.P_{h}\right)$, by setting

$$
P_{h} q \in V_{h}, \quad\left(P_{h} q, \varphi_{h}\right)=\left\langle q, \varphi_{h}\right\rangle_{H_{p e r}^{-1}, H_{p e r}^{1}}, \quad \forall q \in H_{p e r}^{-1}, \forall \varphi_{h} \in H_{p e r}^{1}
$$

The $H^{1}$-orthogonal projector $\Pi_{h}: H_{p e r}^{1} \rightarrow V_{h}$ is defined as follows: for any $q \in H_{p e r}^{1}$, $\Pi_{h} q \in V_{h}$ is uniquely defined by

$$
\begin{equation*}
\left\langle\Pi_{h} q\right\rangle=\langle q\rangle \quad \text { and } \quad\left(\nabla \Pi_{h} q, \nabla \varphi_{h}\right)=\left(\nabla q, \nabla \varphi_{h}\right), \quad \forall q \in H_{p e r}^{1}, \forall \varphi_{h} \in V_{h} \tag{3.20}
\end{equation*}
$$

By Pythagora's theorem and assumption $(\mathcal{H} 2)$, for any $q \in H_{p e r}^{1}$, we get

$$
\begin{equation*}
\left|q-\Pi_{h} q\right|_{1}=\inf _{r_{h} \in V_{h}}\left|q-r_{h}\right|_{1} \rightarrow 0, \text { as } h \rightarrow 0 \tag{3.21}
\end{equation*}
$$

We point out that the space $\dot{V}_{h}$ is invariant by $\Pi_{h}$ and by $P_{h}$.
By using a $L^{2}$-orthonormal basis $\left(e_{h}^{i}\right)_{1 \leq i \leq N_{h}}$ of $V_{h}$ as in Section 3.1, with $e_{h}^{1} \equiv 1$, we see that the matrix of $\mathcal{A}_{h}$ is $A$, so that the energy $E_{h}$ from (3.10) can be rewritten

$$
E_{h}\left(u_{h}\right)=\frac{1}{2}\left\|\mathcal{A}_{h} u_{h}\right\|^{2}-\left|\dot{u}_{h}\right|_{1}^{2}+\left(F\left(u_{h}\right), 1\right) ;
$$

the norm $|\cdot|_{-1}$ from (3.9) becomes for any element $\dot{r}_{h} \in \dot{V}_{h}$

$$
\left|\dot{r}_{h}\right|_{-1, h}:=\left(\dot{r}_{h}, \dot{\mathcal{A}}_{h}^{-1} \dot{r}_{h}\right)=\left|\dot{\mathcal{A}}_{h}^{-1} \dot{r}_{h}\right|_{1}
$$

It is an Euclidean norm for the following scalar product on $\dot{V}_{h}$

$$
\left(\dot{r}_{h}, \dot{q}_{h}\right)_{-1, h}:=\left(\dot{r}_{h}, \dot{\mathcal{A}}_{h}^{-1} \dot{q}_{h}\right)=\left(\dot{\mathcal{A}}_{h}^{-1} \dot{r}_{r}, \dot{\mathcal{A}}_{h}^{-1} \dot{q}_{h}\right)_{1}
$$

Thus, the discrete pseudo-energy (3.11) reads

$$
\begin{equation*}
\mathcal{E}_{h}\left(u_{h}, v_{h}\right):=E_{h}\left(u_{h}\right)+\frac{\beta}{2}\left|\dot{v}_{h}\right|_{-1, h}^{2}, \quad \forall u_{h}, v_{h} \in V_{h} . \tag{3.22}
\end{equation*}
$$

The Cauchy-Schwarz (3.12) inequality gives

$$
\begin{equation*}
|\dot{u}|_{1}^{2} \leq \frac{1}{4}\left\|\dot{\mathcal{A}}_{h} u_{h}\right\|^{2}+\left\|u_{h}\right\|^{2}, \quad \forall u_{h} \in V_{h} \tag{3.23}
\end{equation*}
$$

and estimate (3.13) becomes

$$
\begin{equation*}
E_{h}\left(u_{h}\right) \geq \frac{1}{4}\left\|\mathcal{A}_{h} u_{h}\right\|^{2}+\left\|u_{h}\right\|^{2}-c_{1}, \quad \forall u_{h} \in V_{h} \tag{3.24}
\end{equation*}
$$

### 3.4. Convergence as $h \rightarrow 0$.

Theorem 3.3. Let $\left(u_{0}, v_{0}\right) \in H_{\text {per }}^{2} \times H_{\text {per }}^{-1}$. Assume that $\left(u_{h}^{0}, v_{h}^{0}\right)_{h>0}$ is a family of functions in $V_{h} \times V_{h}$ such that

$$
\begin{align*}
u_{h}^{0} \rightarrow u_{0} \text { in } H_{p e r}^{1}, & \mathcal{A}_{h} u_{h}^{0} \rightarrow \mathcal{A} u_{0} \text { in } L^{2}(\Omega)  \tag{3.25}\\
\left\langle v_{h}^{0}\right\rangle \rightarrow\left\langle v_{0}\right\rangle \text { in } \mathbb{R}, & \dot{\mathcal{A}}_{h}^{-1} \dot{v}_{h}^{0} \rightarrow \dot{\mathcal{A}}^{-1} \dot{v}_{0} \text { in } \dot{H}_{p e r}^{1} \tag{3.26}
\end{align*}
$$

as $h \rightarrow 0$. Then the solution $\left(u_{h}, \partial_{t} u_{h}\right)$ of the space semi-discrete scheme (3.1)-(3.2) tends to the energy solution $\left(u, u_{t}\right)$ of (1.1)-(1.2) in the following sense:

$$
\begin{aligned}
u_{h} & \rightarrow u \text { weakly } \star \text { in } L^{\infty}\left(\mathbb{R}_{+} ; H_{p e r}^{1}\right), \\
u_{h} & \rightarrow \text { u strongly in } C\left([0, T] ; L^{2}(\Omega)\right), \text { for all } T>0 \\
\mathcal{A}_{h} u_{h} & \rightarrow \mathcal{A} u \text { weakly } \star \text { in } L^{\infty}\left(\mathbb{R}_{+} ; L^{2}(\Omega)\right) \\
\dot{\mathcal{A}}_{h}^{-1} \partial_{t} \dot{u}_{h} & \rightarrow \dot{\mathcal{A}}^{-1} \partial_{t} \dot{u} \text { weakly } \star \text { in } L^{\infty}\left(\mathbb{R}_{+} ; \dot{H}_{p e r}^{1}\right) \text { and weakly in } L^{2}\left(\mathbb{R}_{+} ; \dot{H}_{p e r}^{1}\right) .
\end{aligned}
$$

Proof. The idea is to use a priori estimates on the mass and on the discrete energy, and to pass to the limit in the equation by a compactness argument. We first consider the conservation law for the mass. By (3.7), we get

$$
a_{h}(t):=\left\langle\partial_{t} u_{h}(t)\right\rangle=\left\langle v_{h}^{0}\right\rangle e^{-t / \beta}, \quad t \geq 0
$$

and

$$
\begin{equation*}
\left\langle u_{h}(t)\right\rangle=\beta\left\langle v_{h}^{0}\right\rangle+\left\langle u_{h}^{0}\right\rangle-\beta a_{h}(t), \quad t \geq 0 \tag{3.27}
\end{equation*}
$$

By assumption, $\left\langle u_{h}^{0}\right\rangle \rightarrow\left\langle u_{0}\right\rangle$ and $\left\langle v_{h}^{0}\right\rangle \rightarrow\left\langle v_{0}\right\rangle$ in $\mathbb{R}$, so that $a_{h}$ converges uniformly on $\mathbb{R}_{+}$to the function $a(t):=\left\langle v_{0}\right\rangle e^{-t / \beta}$, and $\left\langle u_{h}\right\rangle$ converges uniformly on $\mathbb{R}_{+}$to the function $\beta\left\langle v_{0}\right\rangle+\left\langle u_{0}\right\rangle-\beta a(t)$. The estimates below show that $\left(u_{h}\right)_{h>0}$ is bounded in $L^{\infty}\left(\mathbb{R}_{+} ; L^{2}(\Omega)\right)$, so that, up to a subsequence, $u_{h}$ converges weakly $\star$ in $L^{\infty}\left(\mathbb{R}_{+} ; L^{2}(\Omega)\right)$ to some $u$ and so $\left\langle u_{h}\right\rangle \rightarrow\langle u\rangle$ weakly $\star$ in $L^{\infty}\left(\mathbb{R}_{+}\right)$. By uniqueness of the limit, we find

$$
\langle u\rangle=\beta\left\langle v_{0}\right\rangle+\left\langle u_{0}\right\rangle-\beta a(t)
$$

By differentiating, we recover the conservation law for the mass:

$$
\beta \partial_{t t}\langle u\rangle+\partial_{t}\langle u\rangle=0, \quad t \geq 0
$$

We now turn to the energy estimate. As pointed out in Section 3.1, the (unique) solution $\left(u_{h}, \partial_{t} u_{h}\right)$ of (3.1)-(3.2) is in fact a solution $\left(u_{h}, v_{h}, z_{h}, w_{h}\right)$ of (3.1)-(3.2). In particular, $v_{h}=\partial_{t} u_{h}$ and $z_{h}=\mathcal{A}_{h} u_{h}$. We have (recall (3.22)):

$$
\mathcal{E}\left(u_{h}^{0}, v_{h}^{0}\right)=\frac{1}{2}\left\|\mathcal{A}_{h} u_{h}^{0}\right\|^{2}-\left|\dot{u}_{h}^{0}\right|_{1}^{2}+\left(F\left(u_{h}^{0}\right), 1\right)+\frac{\beta}{2}\left|\dot{\mathcal{A}}_{h}^{-1} \dot{v}_{h}^{0}\right|_{1}^{2} .
$$

By using assumptions (3.25)-(3.26) and the Sobolev injection $H_{p e r}^{1} \hookrightarrow L^{2 p+2}(\Omega)$, we see that $\mathcal{E}\left(u_{h}^{0}, v_{h}^{0}\right)$ is uniformly bounded as $h$ tends to 0 . The energy estimate (3.15) shows that there exists a constant $C$ independent of $h$ such that

$$
\mathcal{E}_{h}\left(u_{h}(t), \partial_{t} u_{h}(t)\right)+\int_{0}^{t}\left|\partial_{t} \dot{u}_{h}(s)\right|_{-1, h}^{2} d s \leq C
$$

for all $t \geq 0$. By (3.24) and (3.23), we obtain that $z_{h}=\mathcal{A}_{h} u_{h}$ and $u_{h}$ are uniformly bounded in $L^{2}(\Omega)$, that $\dot{u}_{h}$ and $\dot{r}_{h}:=\dot{\mathcal{A}}_{h}^{-1} \partial_{t} \dot{u}_{h}$ are uniformly bounded in $\dot{H}_{p e r}^{1}$, and that

$$
\int_{0}^{\infty}\left|\dot{\mathcal{A}}_{h}^{-1} \partial_{t} \dot{u}_{h}(t)\right|_{1}^{2} d t \leq C
$$

This implies that $\left(u_{h}\right)_{h>0}$ is precompact in the space $C\left([0, T] ; L^{2}(\Omega)\right)$, for all $T>0$, as a consequence of the Ascoli-Arzelà Theorem. Indeed, let $T>0$. The family $\left(u_{h}\right)_{h>0}$ is uniformly bounded from $[0, T]$ with values in $H_{p e r}^{1}$, and $H_{p e r}^{1}$ is compactly embedded into $L^{2}(\Omega)$ by Rellich's Theorem. Moreover, for all $0 \leq s \leq t \leq T$, we have

$$
\begin{align*}
\left\|\dot{u}_{h}(t)-\dot{u}_{h}(s)\right\|^{2} & =2 \int_{s}^{t}\left(\partial_{t} \dot{u}_{h}(\sigma), \dot{u}_{h}(\sigma)-\dot{u}_{h}(s)\right) d \sigma \\
& =2 \int_{s}^{t}\left(\nabla \dot{r}_{h}(\sigma), \nabla\left[\dot{u}_{h}(\sigma)-\dot{u}_{h}(s)\right]\right) d \sigma \\
& \leq 4\left\|\dot{r}_{h}\right\|_{L^{\infty}\left(\mathbb{R}_{+} ; H_{p e r}^{1}\right)}\left\|\dot{u}_{h}\right\|_{L^{\infty}\left(\mathbb{R}_{+} ; H_{p e r}^{1}\right)}|t-s| . \tag{3.28}
\end{align*}
$$

Moreover, by (3.27) and the mean value theorem, we find

$$
\left|\left\langle u_{h}(t)\right\rangle-\left\langle u_{h}(s)\right\rangle\right| \leq\left|\left\langle v_{h}^{0}\right\rangle\right||t-s|
$$

Thus, $\left(u_{h}\right)_{h>0}$ is uniformly equicontinuous from $[0, T]$ with values into $L^{2}(\Omega)$, and therefore pre-compact in $C\left([0, T] ; L^{2}(\Omega)\right)$, as claimed. Up to a subsequence, we have the following convergence results

$$
\begin{aligned}
u_{h} & \rightarrow u \text { weakly } \star \text { in } L^{\infty}\left(\mathbb{R}_{+} ; H_{p e r}^{1}\right), \\
u_{h} & \rightarrow u \text { strongly in } C\left([0, T] ; L^{2}(\Omega)\right), \text { for all } T>0, \\
u_{h} & \rightarrow u \text { a.e. in } \Omega \times \mathbb{R}_{+}, \\
f\left(u_{h}\right) & \rightarrow f(u) \text { weakly in } L^{q}\left(0, T ; L^{q}(\Omega)\right), \text { for all } T>0, \\
z_{h} & \rightarrow z \text { weakly } \star \text { in } L^{\infty}\left(\mathbb{R}_{+} ; L^{2}(\Omega)\right), \\
\dot{r}_{h} & \rightarrow \dot{r} \text { weakly } \star \text { in } L^{\infty}\left(\mathbb{R}_{+} ; \dot{H}_{p e r}^{1}\right), \\
\dot{r}_{h} & \rightarrow \dot{r} \text { weakly in } L^{2}\left(\mathbb{R}_{+} ; \dot{H}_{p e r}^{1}\right),
\end{aligned}
$$

where $q=(2 p+2) /(2 p+1)>1$. Let now $\dot{\psi} \in \dot{H}_{p e r}^{1}$ and let $\dot{\psi}_{h}=\Pi_{h}(\dot{\psi})$ so that $\dot{\psi}_{h} \rightarrow \dot{\psi}$ strongly in $\dot{H}_{p e r}^{1}$. We have

$$
\left(\partial_{t} \dot{u}_{h}, \dot{\psi}_{h}\right)=\left(\dot{\mathcal{A}}_{h} \dot{r}_{h}, \dot{\psi}_{h}\right)=\left(\nabla \dot{r}_{h}, \nabla \dot{\psi}_{h}\right) \rightarrow(\dot{r}, \dot{\psi})
$$

weakly $\star$ in $L^{\infty}\left(\mathbb{R}_{+}\right)$. On the other hand, $\partial_{t}\left(\dot{u}_{h}, \dot{\psi}_{h}\right) \rightarrow \partial_{t}(\dot{u}, \dot{\psi})$ in $\mathcal{D}^{\prime}(0, \infty)$ (i.e. in the sense of distributions), since $\left(\dot{u}_{h}, \dot{\psi}_{h}\right) \rightarrow(\dot{u}, \dot{\psi})$ in $L^{\infty}\left(\mathbb{R}_{+}\right)$weakly $\star$. Thus,

$$
\begin{equation*}
\partial_{t}(\dot{u}, \dot{\psi})=(\nabla \dot{r}, \nabla \dot{\psi})=\langle\dot{v}, \dot{\psi}\rangle_{H_{p e r}^{-1}, H_{p e r}^{1}} \tag{3.29}
\end{equation*}
$$

with $\dot{v}=\dot{\mathcal{A}} \dot{r} \in L^{\infty}\left(\mathbb{R}_{+} ; \dot{H}_{\text {per }}^{-1}\right)$. This shows that $\partial_{t} \dot{u}=\dot{v} \in L^{\infty}\left(\mathbb{R}_{+} ; \dot{H}_{\text {per }}^{-1}\right)$.
Next, we set $\varphi \in H_{p e r}^{2}$ and we let $\varphi_{h}=\Pi_{h}(\varphi)$, so that $\varphi_{h} \rightarrow \varphi$ strongly in $H_{p e r}^{1}$. Let $\left(e_{h}^{i}\right)_{1 \leq i \leq N_{h}}$ be an orthonormal basis of $V_{h}$ with $e_{h}^{1} \equiv 1$. We let $\varphi_{h}=$ $\sum_{i=1}^{N_{h}} \varphi_{i} e_{h}^{i}$ and $\Phi=\left(\varphi_{1}, \ldots, \varphi_{N_{h}}\right)^{t}$ be the vector associated to $\varphi_{h}$, as in Section 3.1. On multiplying (3.8) by $\dot{\Phi}^{t} \dot{A}^{-1}$ and using $z_{h}=\mathcal{A}_{h} u_{h}$, we find

$$
\begin{equation*}
\beta\left(\partial_{t t} \dot{u}_{h}, \dot{\varphi}_{h}\right)_{-1, h}+\left(\partial_{t} \dot{u}_{h}, \dot{\varphi}_{h}\right)_{-1, h}+\left(\nabla z_{h}, \nabla \varphi_{h}\right)-2\left(\nabla u_{h}, \nabla \varphi_{h}\right)+\left(f\left(u_{h}\right), \dot{\varphi}_{h}\right)=0 \tag{3.30}
\end{equation*}
$$

for all $t \geq 0$. We have that

$$
\left(\partial_{t} \dot{u}_{h}, \dot{\varphi}_{h}\right)_{-1, h}=\left(\dot{r}_{h}, \dot{\varphi}_{h}\right) \rightarrow(\dot{r}, \dot{\varphi})=\left(\partial_{t} \dot{u}, \dot{\varphi}\right)_{-1}
$$

by (3.29). The convergence above holds in $L^{\infty}\left(\mathbb{R}_{+}\right)$weak $\star$, so that

$$
\left(\partial_{t t} \dot{u}_{h}, \dot{\varphi}_{h}\right)_{-1, h}=\partial_{t}\left(\partial_{t} \dot{u}_{h}, \dot{\varphi}_{h}\right)_{-1, h} \rightarrow \partial_{t}\left(\partial_{t} \dot{u}, \dot{\varphi}\right)_{-1} \text { in } \mathcal{D}^{\prime}(0, \infty) .
$$

Moreover,

$$
\left(\nabla u_{h}, \nabla \varphi_{h}\right) \rightarrow(\nabla u, \nabla \varphi) \text { in } L^{\infty}\left(\mathbb{R}_{+}\right) \text {weak } \star .
$$

Since $\varphi_{h}=\Pi_{h} \varphi$, we have

$$
\left(\nabla z_{h}, \nabla \varphi_{h}\right)=\left(\nabla z_{h}, \nabla \varphi\right)=\left(z_{h}, \mathcal{A} \varphi\right) \rightarrow(z, \mathcal{A} \varphi),
$$

in $L^{\infty}\left(\mathbb{R}_{+}\right)$weak $\star$. Concerning the last term in (3.30), we have

$$
\left(f\left(u_{h}\right), \dot{\varphi}_{h}\right) \rightarrow(f(u), \dot{\varphi}) \quad \text { weakly in } L^{q}(0, T), \quad \forall T>0 .
$$

Summing up, we have proved that

$$
\begin{equation*}
\beta \partial_{t}\left(\partial_{t} \dot{u}, \dot{\varphi}\right)_{-1}+\left(\partial_{t} \dot{u}, \dot{\varphi}\right)_{-1}+(z, \mathcal{A} \varphi)-2(\nabla u, \nabla \varphi)+(f(u), \varphi)=\langle f(u)\rangle\langle\varphi\rangle . \tag{3.31}
\end{equation*}
$$

The equality holds in $\mathcal{D}^{\prime}(0, \infty)$, for all $\varphi \in H_{\text {per }}^{2}$. Moreover, $\partial_{t} \dot{u} \in L^{\infty}\left(\mathbb{R}_{+} ; H_{\text {per }}^{-1}\right)$, $z \in L^{\infty}\left(\mathbb{R}_{+} ; L^{2}(\Omega)\right), u \in L^{\infty}\left(\mathbb{R}_{+} ; H_{p e r}^{1}\right)$ and $f(u) \in L^{\infty}\left(\mathbb{R}_{+} ; H_{p e r}^{-1}\right)$ so that $\partial_{t t} u$ belongs to $L^{\infty}\left(\mathbb{R}_{+} ; H_{\text {per }}^{-4}\right)$ and

$$
\dot{\mathcal{A}}^{-1}\left(\beta \partial_{t t} \dot{u}+\partial_{t} \dot{u}\right)+\mathcal{A} z-2 \mathcal{A} u+f(u)-\langle f(u)\rangle=0
$$

in $\dot{H}_{\text {per }}^{-2}$, a.e. in $\mathbb{R}_{+}$. Now, recall that $z_{h}=\mathcal{A}_{h} u_{h}$. Let $\zeta \in H_{p e r}^{1}$ and $\zeta_{h}=\Pi_{h} \zeta$, so that $\zeta_{h} \rightarrow \zeta$ strongly in $H_{\text {per }}^{1}$. We have

$$
\left(z_{h}, \zeta_{h}\right)=\left(\mathcal{A}_{h} u_{h}, \zeta_{h}\right)=\left(\nabla u_{h}, \nabla \zeta_{h}\right) \rightarrow(\nabla u, \nabla \zeta),
$$

on one hand, and $\left(z_{h}, \zeta_{h}\right) \rightarrow(z, \zeta)$, on the other hand. Thus, we deduce

$$
(z, \zeta)=(\nabla u, \nabla \zeta),
$$

in $L^{\infty}\left(\mathbb{R}_{+}\right)$. The equality holds for every $\zeta \in H_{p e r}^{1}$, so $z=\mathcal{A} u, u \in L^{\infty}\left(\mathbb{R}_{+} ; H_{p e r}^{2}\right)$ and $\left(u, u_{t}\right)$ is an energy solution of (1.1)-(1.2). By uniqueness of the limit $\left(u, u_{t}\right)$, the whole family $\left(u_{h}, \partial_{t} u_{h}\right)$ converges to $\left(u, u_{t}\right)$.
Remark 3.4. Let $\left(u_{0}, v_{0}\right) \in H_{p e r}^{2} \times H_{\text {per }}^{-1}$. If $u_{h}^{0}=\Pi_{h}\left(u_{0}\right)$ and $v_{h}^{0}=P_{h}\left(v_{0}\right)$, then assumptions (3.25)-(3.26) are satisfied. Indeed, for all $\varphi_{h} \in V_{h}$,

$$
\left(\mathcal{A} u_{0}, \varphi_{h}\right)=\left(\nabla u_{0}, \nabla \varphi_{h}\right)=\left(\nabla \Pi_{h} u_{0}, \nabla \varphi_{h}\right)=\left(\mathcal{A}_{h}\left(\Pi_{h} u_{0}\right), \varphi_{h}\right) .
$$

Then $\mathcal{A}_{h}\left(\Pi_{h} u_{0}\right)=P_{h}\left(\mathcal{A} u_{0}\right)$ and by (3.19) we obtain

$$
\mathcal{A}_{h}\left(\Pi_{h} u_{0}\right) \rightarrow \mathcal{A} u_{0} \text { in } L^{2}(\Omega), \text { as } h \rightarrow 0 .
$$

By definition, observe that

$$
\left\langle P_{h}\left(v_{0}\right)\right\rangle=\left(P_{h}\left(v_{0}\right), 1\right)=\left\langle v_{0}, 1\right\rangle_{H_{p e r}^{-1}, H_{p e r}^{1}}=\left\langle v_{0}\right\rangle .
$$

Moreover, for all $\varphi_{h} \in V_{h}$, we have

$$
\begin{aligned}
\left(P_{h}\left(\dot{v}_{0}\right), \dot{\varphi}_{h}\right) & =\left\langle\dot{v}_{0}, \dot{\varphi}_{h}\right\rangle_{H_{p e r}^{-1}, H_{p e r}^{1}}=\left(\nabla \dot{\mathcal{A}}^{-1} \dot{v}_{0}, \nabla \dot{\varphi}_{h}\right) \\
& =\left(\nabla \Pi_{h}\left(\dot{\mathcal{A}}^{-1} \dot{v}_{0}\right), \nabla \dot{\varphi}_{h}\right)=\left(\dot{\mathcal{A}}_{h}\left(\Pi_{h}\left(\dot{\mathcal{A}}^{-1} \dot{v}_{0}\right)\right), \dot{\varphi}_{h}\right),
\end{aligned}
$$

so that

$$
P_{h}\left(\dot{v}_{0}\right)=\dot{\mathcal{A}}_{h}\left(\Pi_{h}\left(\dot{\mathcal{A}}^{-1} \dot{v}_{0}\right)\right) .
$$

Thus, thanks to (3.21), we deduce

$$
\dot{\mathcal{A}}_{h}^{-1}\left(P_{h}\left(\dot{v}_{0}\right)\right)=\Pi_{h}\left(\dot{\mathcal{A}}^{-1} \dot{v}_{0}\right) \rightarrow \dot{\mathcal{A}}^{-1} \dot{v}_{0} \text { in } H_{p e r}^{1} .
$$

## 4. The fully discrete problem

4.1. The fully discrete scheme. For the time discretization, we use the following decomposition:
(H3) $F=F_{+}+F_{-}$, where $F_{+}$and $F_{-}$are polynomials such that $F_{+}^{(i v)} \geq 0, F_{-}^{(i v)} \leq$ 0 , and $\operatorname{deg}\left(F_{-}\right)<\operatorname{deg}(F)$ (here, $\operatorname{deg}$ denotes the degree of the polynomial).
As a consequence, $\operatorname{deg}\left(F_{+}\right)=\operatorname{deg}(F)$ and $F_{+}, F$ have the same leading coefficient. We denote $f=F^{\prime}=f_{+}+f_{-}$, where $f_{+}=F_{+}^{\prime}$ and $f_{-}=F_{-}^{\prime}$. For the energy estimate, we will use the fact that there exist two constants $c_{5}>0$ and $c_{6} \geq 0$ which depend only on $f$ and on the decomposition $f=f_{+}+f_{-}$such that

$$
\begin{equation*}
\frac{1}{2}(|f(r)|+|f(s)|)+\frac{1}{12}(s-r)^{2}\left(\left|f_{+}^{\prime \prime}(r)\right|+\left|f_{-}^{\prime \prime}(s)\right|\right) \leq c_{5}(F(r)+F(s))+c_{6} \tag{4.1}
\end{equation*}
$$

for all $r, s \in \mathbb{R}$.
Remark 4.1. A decomposition (H3) is always possible for a polynomial potential such as (2.2). Indeed, for a quartic polynomial (for instance (2.6)), we can always choose $F_{+}=F$ and $F_{-}=0$. For a polynomial with higher degree, we notice that $F^{(i v)}$, being a polynomial of even degree with strictly positive leading coefficient, is bounded from below, i.e.

$$
F^{(i v)}(s) \geq-c_{4} \quad \forall s \in \mathbb{R}
$$

for some constant $c_{4} \geq 0$. A possible (but not unique !) choice is then $F_{+}^{(i v)}=$ $F^{(i v)}+c_{4}$ and $F_{-}^{(i v)}=-c_{4}$, i.e. $F_{+}(s)=F(s)+c_{4} s^{4} / 24$ and $F_{-}(s)=-c_{4} s^{4} / 24$.

We use the same notation as in Section 3. In particular, $V_{h}$ is a family of finitedimensional subspaces of $H_{p e r}^{1}$ which satisfies assumptions (H1)-(H2).

Let $\tau>0$ denote the time step, and $\left(u_{h}^{0}, v_{h}^{0}\right)$ in $V_{h} \times V_{h}$ the initial datum. The fully discrete scheme reads: for $n \geq 0$, find $\left(u^{n+1}, v^{n+1}, z^{n+1}, w^{n+1}\right) \in\left(V_{h}\right)^{4}$ such that

$$
\left\{\begin{array}{l}
\left(\left(u_{h}^{n+1}-u_{h}^{n}\right) / \tau, \varphi_{h}\right)=\left(v_{h}^{n+1 / 2}, \varphi_{h}\right)  \tag{4.2}\\
\beta\left(\left(v_{h}^{n+1}-v_{h}^{n}\right) / \tau, \psi_{h}\right)=-\left(v_{h}^{n+1 / 2}, \psi_{h}\right)-\left(\nabla w_{h}^{n+1}, \nabla \psi_{h}\right) \\
\left(z_{h}^{n+1}, \zeta_{h}\right)=\left(\nabla u_{h}^{n+1 / 2}, \nabla \zeta_{h}\right) \\
\left(w_{h}^{n+1}, \xi_{h}\right)=\left(\nabla z_{h}^{n+1}, \nabla \xi_{h}\right)-2\left(\nabla u_{h}^{n+1 / 2}, \nabla \xi_{h}\right)+\left(\left(f\left(u_{h}^{n}\right)+f\left(u_{h}^{n+1}\right)\right) / 2, \xi_{h}\right) \\
\quad-\frac{1}{12}\left(\left(u_{h}^{n+1}-u_{h}^{n}\right)^{2}\left(f_{+}^{\prime \prime}\left(u_{h}^{n}\right)+f_{-}^{\prime \prime}\left(u_{h}^{n+1}\right)\right), \xi_{h}\right)
\end{array}\right.
$$

for all $\varphi_{h}, \psi_{h}, \zeta_{h}, \xi_{h}$ in $V_{h}$. Here, we have denoted

$$
u_{h}^{n+1 / 2}=\left(u_{h}^{n+1}+u_{h}^{n}\right) / 2 \quad \text { and } \quad v_{h}^{n+1 / 2}=\left(v_{h}^{n+1}+v_{h}^{n}\right) / 2
$$

Notice that $z_{h}^{0}$ and $w_{h}^{0}$ do not need to be defined. In fact, $z_{h}^{n+1}$ (resp. $\left(w_{h}^{n+1}\right)$ ) is a second-order (in time) approximation of $z_{h}\left(t_{n+1 / 2}\right)$ (resp. $w_{h}\left(t_{n+1 / 2}\right)$ ), where $t_{n+1 / 2}=(n+1 / 2) \tau$.

Let $\left(e_{h}^{i}\right)_{1 \leq i \leq N_{h}}$ be an $L^{2}$-orthonormal basis of $V_{h}$, with $e_{h}^{1} \equiv 1$, so that we have the identification $V_{h} \ni u_{h} \simeq U \in \mathbb{R}^{N_{h}}$. In $\mathbb{R}^{N_{h}}$, the scheme reads: let $U^{0}, V^{0}$ in $\mathbb{R}^{N_{h}}$
and for $n \geq 0$ find $\left(U^{n+1}, V^{n+1}, Z^{n+1}, W^{n+1}\right) \in\left(\mathbb{R}^{N_{h}}\right)^{4}$ which solves

$$
\left\{\begin{array}{l}
\left(U^{n+1}-U^{n}\right) / \tau=V^{n+1 / 2}  \tag{4.3}\\
\beta\left(V^{n+1}-V^{n}\right) / \tau=-V^{n+1 / 2}-A W^{n+1} \\
Z^{n+1}=A U^{n+1 / 2} \\
W^{n+1}=A Z^{n+1}-2 A U^{n+1 / 2}+\left(\nabla F_{h}\left(U^{n}\right)+\nabla F_{h}\left(U^{n+1}\right)\right) / 2 \\
\quad-G\left(U^{n}, U^{n+1}\right)
\end{array}\right.
$$

where

$$
\begin{equation*}
G\left(U^{n}, U^{n+1}\right)=\frac{1}{12}\left(\left(\left(u_{h}^{n+1}-u_{h}^{n}\right)^{2}\left(f_{+}^{\prime \prime}\left(u_{h}^{n}\right)+f_{-}^{\prime \prime}\left(u_{h}^{n+1}\right)\right), e_{h}^{i}\right)\right)_{1 \leq i \leq N_{h}} \tag{4.4}
\end{equation*}
$$

On eliminating $Z^{n+1}$ and $W^{n+1}$, the scheme becomes

$$
\left\{\begin{array}{l}
\left(U^{n+1}-U^{n}\right) / \tau-V^{n+1 / 2}=0  \tag{4.5}\\
\beta\left(V^{n+1}-V^{n}\right) / \tau+V^{n+1 / 2}+A\left(A^{2} U^{n+1 / 2}-2 A U^{n+1 / 2}\right) \\
\quad+A\left(\left(\nabla F_{h}\left(U^{n}\right)+\nabla F_{h}\left(U^{n+1}\right)\right) / 2-G\left(U^{n}, U^{n+1}\right)\right)=0
\end{array}\right.
$$

In Section 6.1, a numerical example indicates that our fully discrete scheme (4.2) has a second order convergence error in time (and also in space if $V_{h}$ is the space of $P^{1}$ finite elements). By arguing as in Gomez and Hughes [18], we check here that:
Proposition 4.2. The scheme has a second order consistency error in time, i.e. that any solution of the space semi-discrete problem (3.4) satisfies the fully discrete scheme (4.5) with order $O\left(\tau^{2}\right)$.
Proof. Let $(U, V)$ be a solution of (3.4) on a finite time interval $[0, T]$. Since $f$ is a polynomial, by a bootstrap argument, we know that $(U, V) \in C^{\infty}\left([0, T] ; V_{h} \times V_{h}\right)$. In the time discrete scheme (4.5), we replace $U^{n}, U^{n+1}, V^{n}, V^{n+1}$ by $U\left(t_{n}\right), U\left(t_{n+1}\right)$, $V\left(t_{n}\right), V\left(t_{n+1}\right)$ respectively. The purpose of this replacement is to find a local truncation error (or consistency error) in $O\left(\tau^{2}\right)$ in the right-hand side of (4.5), instead of $(0,0)^{t}$. Now consider the midpoint scheme, which is the same as (4.5) without the term $G\left(U^{n}, U^{n+1}\right)$. Standard results show that the midpoint scheme has a second order consistency error, so that it is sufficient to show that $G\left(U\left(t_{n}\right), U\left(t_{n+1}\right)\right)=$ $O\left(\tau^{2}\right)$. This is obvious by definition (4.4). Indeed, using the assumption on $p$ and appropriate Hölder inequalities and Sobolev injections, we have

$$
\left|G\left(U\left(t_{n}\right), U\left(t_{n+1}\right)\right)\right| \leq C\left\|u\left(t_{n+1}\right)-u\left(t_{n}\right)\right\|_{1}^{2}
$$

Moreover, by Taylor expansion, $\left\|\left(u_{h}\left(t_{n+1}\right)-u_{h}\left(t_{n}\right)\right)^{2}\right\|_{1}=O\left(\tau^{2}\right)$, so the local truncation error is $O\left(\tau^{2}\right)$ as well. Notice that the constant in the consistency error depends on $h$, on $T$, and on maximum norms of derivatives of $U, V$ up to order 3 .
4.2. Existence, discrete energy estimate and uniqueness. Let us prove the following
Theorem 4.3 (Existence for any $\tau$ ). For any $\left(u_{h}^{0}, v_{h}^{0}\right) \in V_{h} \times V_{h}$, there exists at least one sequence $\left(u_{h}^{n}, v_{h}^{n}, z_{h}^{n}, w_{h}^{n}\right)_{n \geq 1}$ in $\left(V_{h}\right)^{4}$ which complies with (4.2).
Proof. We work with the $\mathbb{R}^{N_{h}}$ version (4.5). We will show that this problem is variational, and that we can find $U^{n+1}$ by a minimization procedure. Let $\left(U^{n}, V^{n}\right)$ be fixed in $\mathbb{R}^{N_{h}}$. Consider the polynomial of two variables

$$
g(r, s)=\frac{1}{12}(s-r)^{2}\left(f_{+}^{\prime \prime}(r)+f_{-}^{\prime \prime}(s)\right) \quad(r, s \in \mathbb{R})
$$

By assumption (H3), we have $\operatorname{deg}\left(f_{-}\right)<\operatorname{deg}(f)$ and $\operatorname{deg}\left(f_{+}\right)=\operatorname{deg}(f)$ so $\operatorname{deg}\left(f_{-}^{\prime \prime}\right)<$ $\operatorname{deg}(f)-2$, and $\operatorname{deg}\left(f_{+}^{\prime \prime}\right)=\operatorname{deg}(f)-2$. Thus, $g$ is a polynomial of total degree less than or equal to $2 p+1$, and its partial degree with respect to the variable $s$ is strictly less than $2 p+1$. We can write

$$
\begin{equation*}
g(r, s)=\sum_{k, l} b_{k, l} r^{k} s^{l} \tag{4.6}
\end{equation*}
$$

for coefficients $b_{k, l} \in \mathbb{R}$, where $0 \leq k \leq 2 p+1$ and $0 \leq l<2 p+1$. Let us set now

$$
\begin{equation*}
h(r, s)=\sum_{k, l} b_{k, l} r^{k} \frac{s^{l+1}}{l+1} \tag{4.7}
\end{equation*}
$$

so that $\partial_{s} h(r, s)=g(r, s)$. We define $H_{h}^{n}(U)=\left(h\left(u_{h}^{n}, u_{h}\right), 1\right)$ with $u_{h} \simeq U$, so that

$$
\nabla H_{h}^{n}(U)=\left(g\left(u_{h}^{n}, u_{h}\right), e_{i}\right)_{1 \leq i \leq N_{h}}=G\left(U^{n}, U\right)
$$

By (4.7) and Hölder's inequality, we get

$$
\begin{equation*}
\left|H_{h}^{n}(U)\right| \leq C_{n}\left(\left\|u_{h}\right\|_{L^{2 p+2}(\Omega)}^{2 p+1}+1\right) \quad \forall U \in \mathbb{R}^{N_{h}} \tag{4.8}
\end{equation*}
$$

where the constant $C_{n}$ depends on $\left\|u_{h}^{n}\right\|_{L^{2 p+2}(\Omega)}$. Now, by eliminating $V^{n+1}$, we find that (4.5) is equivalent to

$$
\begin{aligned}
& \frac{\beta}{\tau}\left(\frac{2\left(U^{n+1}-U^{n}\right)}{\tau}-V^{n}\right)+\frac{U^{n+1}-U^{n}}{\tau}+A\left[A^{2} \frac{\left(U^{n+1}+U^{n}\right)}{2}\right. \\
& \left.-2 A \frac{\left(U^{n+1}+U^{n}\right)}{2}+\frac{\nabla F_{h}\left(U^{n}\right)+\nabla F_{h}\left(U^{n+1}\right)}{2}-\nabla H_{h}^{n}\left(U^{n+1}\right)\right]=0
\end{aligned}
$$

Writing $U=\left(u_{1}, \dot{U}\right)$, we see that this is equivalent to

$$
\begin{align*}
& \frac{\beta}{\tau}\left(\frac{2\left(u_{1}^{n+1}-u_{1}^{n}\right)}{\tau}-v_{1}^{n}\right)+\frac{u_{1}^{n+1}-u_{1}^{n}}{\tau}=0  \tag{4.9}\\
& \frac{\beta \dot{A}^{-1}}{\tau}\left(\frac{2\left(\dot{U}^{n+1}-\dot{U}^{n}\right)}{\tau}-\dot{V}^{n}\right)+\dot{A}^{-1} \frac{\left(\dot{U}^{n+1}-\dot{U}^{n}\right)}{\tau}+\dot{A}^{2} \frac{\left(\dot{U}^{n+1}+\dot{U}^{n}\right)}{2} \\
& -2 \dot{A} \frac{\left(\dot{U}^{n+1}+\dot{U}^{n}\right)}{2}+\frac{\dot{\nabla} F_{h}\left(U^{n}\right)+\dot{\nabla} F_{h}\left(U^{n+1}\right)}{2}-\dot{\nabla} H_{h}^{n}\left(U^{n+1}\right)=0 . \tag{4.10}
\end{align*}
$$

The first equation determines $u_{1}^{n+1}$ uniquely. The second equation can be solved by letting $\dot{U}^{n+1}$ be a minimizer on $\mathbb{R}^{N_{h}-1}$ of the function:

$$
\begin{aligned}
\mathcal{G}: \dot{U} \mapsto & \frac{\beta}{\tau^{2}}\left|\dot{U}-\dot{U}^{n}\right|_{-1}^{2}-\frac{\beta}{\tau}\left(\dot{V}^{n}\right)^{t} \dot{A}^{-1} \dot{U}+\frac{1}{2 \tau}\left|\dot{U}-\dot{U}^{n}\right|_{-1}^{2}+\frac{1}{4}\left|\dot{A}\left(\dot{U}+\dot{U}^{n}\right)\right|^{2} \\
& -\frac{1}{2}\left|\dot{A}^{1 / 2}\left(\dot{U}+\dot{U}^{n}\right)\right|^{2}+\frac{\left(\dot{\nabla} F_{h}\left(U^{n}\right)\right)^{t}}{2} \dot{U}+\frac{\tilde{F}_{h}^{n}(\dot{U})}{2}-\tilde{H}_{h}^{n}(\dot{U}),
\end{aligned}
$$

where

$$
\tilde{F}_{h}^{n}(\dot{U})=F_{h}\left(u_{1}^{n+1}, \dot{U}\right), \quad \tilde{H}_{h}^{n}(\dot{U})=H_{h}^{n}\left(u_{1}^{n+1}, \dot{U}\right)
$$

By (2.2), we deduce

$$
\tilde{F}_{h}^{n}(\dot{U}) \geq \frac{a_{2 p+1}}{2 p+2}\left\|u_{h}\right\|_{L^{2 p+2}(\Omega)}^{2 p+2}-C_{p}^{\prime}\left(\left\|u_{h}\right\|_{L^{2 p+2}(\Omega)}^{2 p+1}+1\right), \quad \forall \dot{U} \in \mathbb{R}^{N_{h}-1}
$$

where the constant $C_{p}^{\prime}$ depends only on the coefficients of $F$. Thus, by (4.8), we find

$$
\mathcal{G}(\dot{U}) \geq \frac{a_{2 p+1}}{2(2 p+2)}\left\|u_{h}\right\|_{L^{2 p+2}(\Omega)}^{2 p+2}-\left(C_{n}+\frac{C_{p}^{\prime}}{2}\right)\left\|u_{h}\right\|_{L^{2 p+2}(\Omega)}^{2 p+1}+c\left\|\dot{u}_{h}\right\|^{2}-c^{\prime}\left\|\dot{u}_{h}\right\|-c^{\prime \prime}
$$

where $c>0$ and $c^{\prime}, c^{\prime \prime} \geq 0$ depend on $h, F$ and $u_{h}^{n}$. For the quadratic term, we used that all norms are equivalent in $\mathbb{R}^{N_{h}-1}$. As a consequence, $\mathcal{G}(\dot{U}) \rightarrow+\infty$ as $|\dot{U}| \rightarrow+\infty$ : the continuous function $\mathcal{G}$ has a minimizer in $\mathbb{R}^{N_{h}-1}$, and the proof is complete.

The behavior of $\left(u_{1}^{n}, v_{1}^{n}\right)$ is straightforward, thanks to a discrete conservation law for the mass. Indeed, choosing $\psi=1$ in the second equation of (4.2), we find

$$
\begin{equation*}
v_{1}^{n+1}=q v_{1}^{n}, \quad \text { with } \quad q=q(\beta, \tau)=\frac{2 \beta-\tau}{2 \beta+\tau} \tag{4.11}
\end{equation*}
$$

Thus, we obtain

$$
\begin{equation*}
v_{1}^{n}=q^{n} v_{1}^{0} \tag{4.12}
\end{equation*}
$$

We also have $v^{n+1 / 2}=q v^{n-1 / 2}$, so that $v^{n+1 / 2}=q^{n} v^{1 / 2}$. Notice that $|q|<1$, since $\beta>0$ and $\tau>0$, so that $v_{1}^{n} \rightarrow 0$. If $\tau>2 \beta$, then $q<0$.

On choosing $\varphi=1$ in the first equation of (4.2), we find

$$
u_{1}^{n+1}=u_{1}^{n}+\tau v_{1}^{n+1 / 2}
$$

By induction, we deduce

$$
\begin{equation*}
u_{1}^{n}=u_{1}^{0}+\tau\left(\sum_{k=0}^{n-1} q^{k}\right) v_{1}^{1 / 2}=u_{1}^{0}+\tau \frac{1-q^{n}}{1-q} v_{1}^{1 / 2} \tag{4.13}
\end{equation*}
$$

For the energy estimate, we will need a technical lemma, adapted from [18]:
Lemma 4.4. Let $g \in C^{3}([0,1] ; \mathbb{R})$. Then the following identities hold

$$
\begin{align*}
& \int_{0}^{1} g(s) d s=\frac{1}{2}(g(0)+g(1))-\frac{1}{12} g^{\prime \prime}(0)-\frac{1}{2} \int_{0}^{1} k_{2}^{+}(\sigma) g^{(3)}(\sigma) d \sigma  \tag{4.14}\\
& \int_{0}^{1} g(s) d s=\frac{1}{2}(g(0)+g(1))-\frac{1}{12} g^{\prime \prime}(1)+\frac{1}{2} \int_{0}^{1} k_{2}^{-}(\sigma) g^{(3)}(\sigma) d \sigma \tag{4.15}
\end{align*}
$$

where $k_{2}^{+}(\sigma)=(1-\sigma)^{2}(2 \sigma+1) / 6$ and $k_{2}^{-}(\sigma)=\sigma^{2}(3-2 \sigma) / 6$. In particular, $k_{2}^{+}(\sigma) \geq 0$ and $k_{2}^{-}(\sigma) \geq 0$ for all $\sigma \in[0,1]$.
Proof. We prove (4.15) (the proof of (4.14) is similar). For a function $\varphi \in C^{2}([0,1])$, let

$$
\operatorname{Err}(\varphi)=\int_{0}^{1} \varphi(s) d s-\left(\frac{1}{2}(\varphi(0)+\varphi(1))-\frac{1}{12} \varphi^{\prime \prime}(1)\right)
$$

denote the error of the quadrature formula. If $\varphi$ is a polynomial of degree $\leq 2$, a direct computation shows that $\operatorname{Err}(\varphi)=0$. Now, let $g \in C^{3}([0,1])$. The Taylor formula of order 2 at $s=0$ reads

$$
g(s)=p_{2}(s)+\frac{1}{2} \int_{0}^{1}(s-\sigma)_{+}^{2} g^{(3)}(\sigma) d \sigma
$$

with $p_{2}(s)=g(0)+s g^{\prime}(0)+s^{2} g^{\prime \prime}(0) / 2$ and

$$
(s-\sigma)_{+}= \begin{cases}s-\sigma & \text { if } s \geq \sigma \\ 0 & \text { if } s \leq \sigma\end{cases}
$$

In particular, $\operatorname{Err}\left(p_{2}\right)=0$. By linearity of $\operatorname{Err}$,

$$
\operatorname{Err}(g)=\operatorname{Err}\left(p_{2}\right)+\operatorname{Err}\left(s \mapsto \frac{1}{2} \int_{0}^{1}(s-\sigma)_{+}^{2} g^{(3)}(\sigma) d \sigma\right)
$$

By inverting the integration signs, we obtain

$$
\operatorname{Err}(g)=\frac{1}{2} \int_{0}^{1} k_{2}^{-}(\sigma) g^{(3)}(\sigma) d \sigma
$$

where $k_{2}^{-}(s)=\operatorname{Err}\left(s \mapsto(s-\sigma)_{+}^{2}\right)$. Using the definition of Err, we find that for $\sigma \in[0,1)$,

$$
\begin{aligned}
k_{2}^{-}(\sigma) & =\int_{0}^{1}(s-\sigma)_{+}^{2} d s-\left(\frac{1}{2}\left[(0-\sigma)_{+}^{2}+(1-\sigma)_{+}^{2}\right]-\frac{2}{12}\right) \\
& =\sigma^{2} / 2-\sigma^{3} / 3
\end{aligned}
$$

The claim is proved.
We have (compare with Lemma 3.1):
Lemma 4.5 (Energy estimate for any $\tau$ ). If $\left(U^{n}, V^{n}, Z^{n}, W^{n}\right)_{n \geq 1}$ is a sequence in $\left(\mathbb{R}^{N_{h}}\right)^{4}$ which complies with (4.3), then for all $n \geq 0$,

$$
\begin{equation*}
\mathcal{E}_{h}\left(U^{n+1}, V^{n+1}\right)-\mathcal{E}_{h}\left(U^{n}, V^{n}\right)+\tau\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2} \leq \tau v_{1}^{n+1 / 2} w_{1}^{n+1} \tag{4.16}
\end{equation*}
$$

As a consequence, for all $k \geq 0$, we have

$$
\begin{align*}
& \mathcal{E}_{h}\left(U^{N_{0}+k}, V^{N_{0}+k}\right)+\sum_{j=0}^{k-1} \tau\left|\dot{V}^{N_{0}+j+1 / 2}\right|_{-1}^{2} \\
& \leq \exp \left(16 c_{5} \frac{\tau|q|^{N_{0}}}{1-|q|}\left|v_{1}^{1 / 2}\right|\right)\left(\mathcal{E}_{h}\left(U^{N_{0}}, V^{N_{0}}\right)+c_{7} \frac{\tau|q|^{N_{0}}}{1-|q|}\left|v_{1}^{1 / 2}\right|\right) \tag{4.17}
\end{align*}
$$

where $N_{0}=N_{0}\left(\beta, c_{5}, \tau,\left|v_{1}^{0}\right|\right) \in \mathbb{N}$ is such that

$$
\begin{equation*}
2 c_{5} \tau|q|^{N_{0}}\left|v_{1}^{1 / 2}\right| \leq 1 / 2 \tag{4.18}
\end{equation*}
$$

$c_{7}=2 c_{1} c_{5}+c_{6}$ depends only on $f, f_{+}, f_{-}($see (2.4), (4.1)), and $q$ is defined by (4.11). Proof. Let $\delta u_{h}^{n}=u_{h}^{n+1}-u_{h}^{n}$. Since $f_{+}=F_{+}^{\prime}$ and $f_{-}=F_{-}^{\prime}$, we have

$$
\begin{align*}
& F_{+}\left(u_{h}^{n+1}\right)-F_{+}\left(u_{h}^{n}\right)=\delta u_{h}^{n} \int_{0}^{1} f_{+}\left(u_{h}^{n}+s \delta u_{h}^{n}\right) d s  \tag{4.19}\\
& F_{-}\left(u_{h}^{n+1}\right)-F_{-}\left(u_{h}^{n}\right)=\delta u_{h}^{n} \int_{0}^{1} f_{-}\left(u_{h}^{n}+s \delta u_{h}^{n}\right) d s \tag{4.20}
\end{align*}
$$

Choosing $g(s)=f_{+}\left(u_{h}^{n}+s \delta u_{h}^{n}\right)$ in (4.14), we find

$$
\begin{align*}
\int_{0}^{1} f_{+}\left(u_{h}^{n}+s \delta u_{h}^{n}\right) d s= & \frac{1}{2}\left(f_{+}\left(u_{h}^{n}\right)+f_{+}\left(u_{h}^{n+1}\right)\right)-\frac{\left(\delta u_{h}^{n}\right)^{2}}{12} f_{+}^{\prime \prime}\left(u_{h}^{n}\right) \\
& -\frac{\left(\delta u_{h}^{n}\right)^{3}}{2} \int_{0}^{1} k_{2}^{+}(\sigma) f_{+}^{\prime \prime \prime}\left(u_{h}^{n}+\sigma \delta u_{h}^{n}\right) d \sigma \tag{4.21}
\end{align*}
$$

Setting $g(s)=f_{-}\left(u_{h}^{n}+s \delta u_{h}^{n}\right)$ in (4.15), we find

$$
\begin{align*}
\int_{0}^{1} f_{-}\left(u_{h}^{n}+s \delta u_{h}^{n}\right) d s= & \frac{1}{2}\left(f_{-}\left(u_{h}^{n}\right)+f_{-}\left(u_{h}^{n+1}\right)\right)-\frac{\left(\delta u_{h}^{n}\right)^{2}}{12} f_{-}^{\prime \prime}\left(u_{h}^{n+1}\right) \\
& +\frac{\left(\delta u_{h}^{n}\right)^{3}}{2} \int_{0}^{1} k_{2}^{-}(\sigma) f_{-}^{\prime \prime \prime}\left(u_{h}^{n}+\sigma \delta u_{h}^{n}\right) d \sigma . \tag{4.22}
\end{align*}
$$

Adding (4.19) and (4.20) leads to

$$
\begin{aligned}
F\left(u_{h}^{n+1}\right)-F\left(u_{h}^{n}\right)= & \delta u_{h}^{n}\left[\frac{1}{2}\left(f\left(u_{h}^{n}\right)+f\left(u_{h}^{n+1}\right)\right)\right. \\
& \left.-\frac{\left(\delta u_{h}^{n}\right)^{2}}{12}\left(f_{+}^{\prime \prime}\left(u_{h}^{n}\right)+f_{-}^{\prime \prime}\left(u_{h}^{n+1}\right)\right)\right]-\alpha^{n}
\end{aligned}
$$

where

$$
\alpha^{n}=\frac{\left(\delta u_{h}^{n}\right)^{4}}{2}\left(\int_{0}^{1} k_{2}^{+}(\sigma) f_{+}^{\prime \prime \prime}\left(u_{h}^{n}+\sigma \delta u_{h}^{n}\right) d \sigma-\int_{0}^{1} k_{2}^{-}(\sigma) f_{-}^{\prime \prime \prime}\left(u_{h}^{n}+\sigma \delta u_{h}^{n}\right) d \sigma\right) \geq 0 .
$$

by assumption (H3) on the decomposition. Next, we choose $\xi_{h}=\delta u_{h}^{n}$ in the last equation of (4.2). This gives

$$
\begin{aligned}
\left(w_{h}^{n+1}, \delta u_{h}^{n}\right)-\left(\nabla z_{h}^{n+1},\right. & \left.\nabla \delta u_{h}^{n}\right)+2\left(\nabla u_{h}^{n+1 / 2}, \nabla \delta u_{h}^{n}\right) \\
& =F\left(u_{h}^{n+1}\right)-F\left(u_{h}^{n}\right)+\left(\alpha^{n}, 1\right) .
\end{aligned}
$$

Using the vector form with $\delta U^{n}=U^{n+1}-U^{n}$, and eliminating $z_{h}^{n+1}$, we obtain

$$
\begin{array}{r}
F_{h}\left(U^{n+1}\right)-F_{h}\left(U^{n}\right)+\left(\alpha^{n}, 1\right)=\left(W^{n+1}\right)^{t} \delta U^{n}-\frac{1}{2}\left(\left|A U^{n+1}\right|^{2}-\left|A U^{n}\right|^{2}\right) \\
+\left|A^{1 / 2} U^{n+1}\right|^{2}-\left|A^{1 / 2} U^{n}\right|^{2} . \tag{4.23}
\end{array}
$$

The second equation in (4.2) implies

$$
-\dot{W}=\dot{A}^{-1}\left(\beta \frac{\left(\dot{V}^{n+1}-\dot{V}^{n}\right)}{\tau}+\dot{V}^{n+1 / 2}\right) .
$$

Plugging this in (4.23), together with $\delta U^{n}=\tau V^{n+1 / 2}$, we get

$$
\begin{aligned}
E_{h}\left(U^{n+1}\right)-E_{h}\left(U^{n}\right)+\left(\alpha^{n}, 1\right)+\frac{\beta}{2}\left(\left|\dot{V}^{n+1}\right|_{-1}^{2}-\left|\dot{V}^{n}\right|_{-1}^{2}\right) & +\tau\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2} \\
& =\tau v_{1}^{n+1 / 2} w_{1}^{n+1} .
\end{aligned}
$$

This yields the energy estimate (4.16).
Choosing $r=1$ in the last equation of (4.2), we find

$$
w_{1}^{n+1}=\frac{1}{2}\left(f\left(u_{h}^{n}\right)+f\left(u_{h}^{n+1}\right), 1\right)-\frac{1}{12}\left(\left(u_{h}^{n+1}-u_{h}^{n}\right)^{2}\left(f_{+}^{\prime \prime}\left(u_{h}^{n}\right)+f_{-}^{\prime \prime}\left(u_{h}^{n+1}\right)\right), 1\right) .
$$

Thus, by (4.1), we deduce

$$
\begin{aligned}
\left|w_{1}^{n+1}\right| & \leq\left(\frac{1}{2}\left(\left|f\left(u_{h}^{n}\right)\right|+\left|f\left(u_{h}^{n+1}\right)\right|\right)+\frac{1}{12}\left(\left(u_{h}^{n+1}-u_{h}^{n}\right)^{2}\left(\left|f_{+}^{\prime \prime}\left(u_{h}^{n}\right)\right|+\left|f_{-}^{\prime \prime}\left(u_{h}^{n+1}\right)\right|\right), 1\right)\right) \\
& \leq\left(c_{5}\left(F\left(u_{h}^{n}\right)+F\left(u_{h}^{n+1}\right)\right)+c_{6}, 1\right),
\end{aligned}
$$

As a consequence, by (3.16), we get

$$
\begin{aligned}
& \mathcal{E}_{h}\left(U^{n+1}, V^{n+1}\right)-\mathcal{E}_{h}\left(U^{n}, V^{n}\right)+\tau\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2} \\
& \quad \leq \tau\left|v_{1}^{n+1 / 2}\right|\left(c_{5}\left(F\left(u_{h}^{n}\right), 1\right)+c_{5}\left(F\left(u_{h}^{n+1}\right), 1\right)+c_{6}\right) \\
& \quad \leq \tau\left|v_{1}^{n+1 / 2}\right|\left(2 c_{5} E_{h}\left(U^{n+1}\right)+2 c_{5} E_{h}\left(U^{n}\right)+2 c_{5} c_{1}+c_{6}\right)
\end{aligned}
$$

Let us set

$$
\mathcal{E}_{h}^{n}=E_{h}\left(U^{n}\right)+\frac{\beta}{2}\left|\dot{V}^{n}\right|_{-1}^{2} .
$$

So far, we have proved that

$$
\mathcal{E}_{h}^{n+1}+\tau\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2} \leq \mathcal{E}_{h}^{n}+\tau|q|^{n}\left|v_{1}^{1 / 2}\right|\left(2 c_{5} \mathcal{E}_{h}^{n}+2 c_{5} \mathcal{E}_{h}^{n+1}+c_{7}\right)
$$

where $c_{7}=2 c_{5} c_{1}+c_{6}$. Let $N_{0}=N_{0}\left(\beta, c_{5}, \tau,\left|v_{1}^{0}\right|\right) \in \mathbb{N}$ satisfy (4.18). Then for $n \geq N_{0}$, we have

$$
\left(1-2 c_{5} \tau|q|^{n}\left|v_{1}^{1 / 2}\right|\right) \mathcal{E}_{h}^{n+1}+\tau\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2} \leq\left(1+2 c_{5} \tau|q|^{n}\left|v_{1}^{1 / 2}\right|\right) \mathcal{E}_{h}^{n}+c_{7} \tau|q|^{n}\left|v_{1}^{1 / 2}\right|
$$

We divide by this inequality $\left(1-2 c_{5} \tau|q|^{n}\left|v_{1}^{1 / 2}\right|\right)$ and we use that (by the mean value inequality) for all $x \in[0,1 / 2]$,

$$
1 \leq \frac{1}{1-x} \quad \text { and } \quad \frac{1+x}{1-x} \leq 1+8 x \leq \exp (8 x)
$$

We obtain

$$
\mathcal{E}_{h}^{n+1}+\tau\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2} \leq \exp \left(16 c_{5} \tau|q|^{n}\left|v_{1}^{1 / 2}\right|\right)\left(\mathcal{E}_{h}^{n}+c_{7} \tau|q|^{n}\left|v_{1}^{1 / 2}\right|\right)
$$

for all $n \geq N_{0}$. By induction, for all $k \in \mathbb{N}$, we deduce

$$
\begin{aligned}
& \mathcal{E}_{h}^{N_{0}+k}+\sum_{j=0}^{k-1} \tau\left|\dot{V}^{N_{0}+j+1 / 2}\right|_{-1}^{2} \leq \exp \left(16 c_{5} \tau\left|v_{1}^{1 / 2}\right| \sum_{j=0}^{k-1}|q|^{N_{0}+j}\right) \mathcal{E}_{h}^{N_{0}} \\
& \quad+\sum_{j=0}^{k-1} \exp \left(16 c_{5} \tau\left|v_{1}^{1 / 2}\right|\left(|q|^{N_{0}}+\cdots+|q|^{N_{0}+k-1-j}\right)\right) c_{7} \tau|q|^{N_{0}+j}\left|v_{1}^{1 / 2}\right|
\end{aligned}
$$

Estimate (4.17) follows by using the inequality $\sum_{j=0}^{k-1}|q|^{N_{0}+j} \leq|q|^{N_{0}} /(1-|q|)$.
Theorem 4.6 (Uniqueness for small $\tau$ ). For any $\left(u_{h}^{0}, v_{h}^{0}\right) \in V_{h} \times V_{h}$, there exists $\tau^{\star}=$ $\tau^{\star}(h)>0$ such that for any $\tau \in\left(0, \tau^{\star}\right)$, there is a unique sequence $\left(u_{h}^{n}, v_{h}^{n}, z_{h}^{n}, w_{h}^{n}\right)_{n \geq 1}$ which complies with (4.2). Moreover, $\tau^{\star}$ can be made independent of $h$ if $\left(u_{h}^{0}, v_{h}^{0}\right)_{h>0}$ is a family such that

$$
\begin{equation*}
\left|\left\langle v_{h}^{0}\right\rangle\right|+\mathcal{E}_{h}\left(u_{h}^{0}, v_{h}^{0}\right) \leq C_{1} \tag{4.24}
\end{equation*}
$$

for some constant $C_{1}$ independent of $h$.
Proof. Assume that $\left(u_{h}^{n}, v_{h}^{n}\right)$ is uniquely determined for some $n \geq 0$. We have seen that $u_{1}^{n+1}=\left\langle u_{h}^{n+1}\right\rangle$ is uniquely determined (see (4.9)). It is sufficient to show that $\dot{u}_{h}^{n+1} \simeq \dot{U}^{n+1}$ is uniquely determined by (4.10), for $\tau$ sufficiently small and independent of $n$. Then $v_{h}^{n+1}$ can be recovered by the first equation in (4.5).

Assume that (4.10) has two solutions $\dot{u}_{h}^{n+1} \simeq \dot{U}^{n+1}$ and $\underline{\dot{u}}_{h}^{n+1} \simeq \underline{U}^{n+1}$. We subtract the two resulting systems (4.10), and we multiply by $\delta \dot{U}=\dot{U}^{n+1}-\dot{\underline{U}}^{n+1} \simeq \delta \dot{u}_{h}=\delta u_{h}$. We obtain

$$
\begin{align*}
& \frac{2 \beta}{\tau^{2}}\left|\delta \dot{u}_{h}\right|_{-1, h}^{2}+\frac{1}{\tau}\left|\delta \dot{u}_{h}\right|_{-1, h}^{2}+\frac{1}{2}\left\|\mathcal{A}_{h} \delta \dot{u}_{h}\right\|^{2}-\left|\delta \dot{u}_{h}\right|_{1}^{2} \\
& +\frac{1}{2}\left(f\left(u_{h}^{n+1}\right)-f\left(\underline{u}_{h}^{n+1}\right), \delta u_{h}\right)-\left(g\left(u_{h}^{n}, u_{h}^{n+1}\right)-g\left(u_{h}^{n}, \underline{u}_{h}^{n+1}\right), \delta u_{h}\right)=0 . \tag{4.25}
\end{align*}
$$

By (2.1), $f^{\prime}$ is a polynomial of even degree with strictly positive leading coefficient, so that $f^{\prime}$ is bounded from below. There exists an (optimal) constant $c_{f} \geq 0$ such that

$$
\begin{equation*}
f^{\prime}(s) \geq-c_{f} \quad \forall s \in \mathbb{R} \tag{4.26}
\end{equation*}
$$

By the mean value theorem,

$$
\left(f\left(u_{h}^{n+1}\right)-f\left(\underline{u}_{h}^{n+1}\right), \delta u_{h}\right) \geq-c_{f}\left\|\delta \dot{u}_{h}\right\|^{2} .
$$

On the other hand, by (4.6), we have

$$
g\left(u_{h}^{n}, u_{h}^{n+1}\right)-g\left(u_{h}^{n}, \underline{u}_{h}^{n+1}\right)=\sum_{0 \leq k+l \leq 2 p+1} b_{k, l}\left(u_{h}^{n}\right)^{k}\left[\left(u_{h}^{n+1}\right)^{l}-\left(\underline{u}_{h}^{n+1}\right)^{l}\right]
$$

so that by Hölder's inequality,

$$
\left|\left(g\left(u_{h}^{n}, u_{h}^{n+1}\right)-g\left(u_{h}^{n}, \underline{u}_{h}^{n+1}\right), \delta u_{h}\right)\right| \leq C_{n}^{\prime}\left\|\delta \dot{u}_{h}\right\|_{L^{2 p+2}(\Omega)}^{2},
$$

where

$$
C_{n}^{\prime}=C^{\prime}\left(\left\|u_{h}^{n}\right\|_{L^{2 p+2}(\Omega)},\left\|u_{h}^{n+1}\right\|_{L^{2 p+2}(\Omega)},\left\|\underline{u}_{h}^{n+1}\right\|_{L^{2 p+2}(\Omega)}\right),
$$

and $C^{\prime}$ is a nondecreasing function of its arguments. Thus, by (2.3), equation (4.25) implies

$$
\begin{equation*}
\frac{2 \beta}{\tau^{2}}\left|\delta \dot{u}_{h}\right|_{-1, h}^{2}+\frac{1}{\tau}\left|\delta \dot{u}_{h}\right|_{-1, h}^{2}+\frac{1}{2}\left\|\mathcal{A}_{h} \delta \dot{u}_{h}\right\|^{2} \leq \frac{c_{f}}{2}\left\|\delta \dot{u}_{h}\right\|^{2}+\left(C_{n}^{\prime} C_{S}+1\right)\left|\delta \dot{u}_{h}\right|_{1}^{2} \tag{4.27}
\end{equation*}
$$

Let $\left(u_{h}^{0}, v_{h}^{0}\right)$ be a given initial data and let $\bar{\tau}=\min \left\{2 \beta,\left(4 c_{5}\left|v_{1}^{0}\right|\right)^{-1}\right\}$. Then for $\tau \in(0, \bar{\tau}), q=(2 \beta-\tau) /(2 \beta+\tau) \in(0,1)$ and $(4.18)$ is satisfied for $N_{0}=0$ since $\left|v_{1}^{1 / 2}\right|=\left|(1+q) v_{1}^{0} / 2\right| \leq\left|v_{1}^{0}\right|$. Moreover,

$$
\frac{\tau}{1-q}=\beta+\frac{\tau}{2} \leq 2 \beta
$$

By the energy estimate (4.17), $C_{n}^{\prime}$ is bounded by a constant independent of $n$ and $\tau$. Since all norms are equivalent in $V_{h}$, estimate (4.27) implies that for $\tau>0$ small enough (but dependent on $h!$ ), $\delta u_{h}=0$.

Now, assume that the bound (4.24) is satisfied, and replace $\bar{\tau}$ by

$$
\bar{\tau}=\min \left\{2 \beta,\left(4 c_{5} C_{1}\right)^{-1}\right\}
$$

By the energy estimate (4.17), $\mathcal{E}_{h}\left(U^{n}, V^{n}\right)$ is bounded by a constant independent of $h, n$ and $\tau$. Thus, $C_{n}^{\prime}$ is bounded by a constant $\overline{C^{\prime}}$ independent of $h$ and $n$. We apply Lemma 4.7 below with $\varepsilon_{1}=1 /\left(4\left(\overline{C^{\prime}} C_{S}+1\right)\right)$ and $\varepsilon_{2}=1 /\left(2 c_{f}\right)$, and we obtain

$$
\frac{2 \beta}{\tau^{2}}\left|\delta \dot{u}_{h}\right|_{-1, h}^{2}+\frac{1}{\tau}\left|\delta \dot{u}_{h}\right|_{-1, h}^{2} \leq\left(\frac{c_{f}}{2}\left(\frac{1}{4 \varepsilon_{2}^{2}}+\frac{1}{4}\right)+\frac{\overline{C^{\prime}} C_{S}+1}{4 \varepsilon_{1}^{2}}\right)\left|\delta \dot{u}_{h}\right|_{-1, h}^{2} .
$$

We see that for $\tau>0$ small enough (independent of $h$ now), $\delta \dot{u}_{h}=0$ and the proof is complete.

Lemma 4.7. Let $\varepsilon_{1}, \varepsilon_{2}>0$. Then, for all $\dot{u}_{h} \in \dot{V}_{h}$, there hold

$$
\begin{align*}
\left|\dot{u}_{h}\right|_{1}^{2} & \leq \varepsilon_{1}\left\|\dot{\mathcal{A}}_{h} \dot{u}_{h}\right\|^{2}+\frac{1}{4 \varepsilon_{1}^{2}}\left|\dot{u}_{h}\right|_{-1, h}^{2}  \tag{4.28}\\
\left\|\dot{u}_{h}\right\|^{2} & \leq \varepsilon_{2}\left\|\dot{\mathcal{A}}_{h} \dot{u}_{h}\right\|^{2}+\left(\frac{1}{4 \varepsilon_{2}^{2}}+\frac{1}{4}\right)\left|\dot{u}_{h}\right|_{-1, h}^{2} \tag{4.29}
\end{align*}
$$

Proof. By arguing as in (3.12), we see that

$$
\left|\dot{A}^{1 / 2} \dot{U}\right|=(\dot{A} \dot{U})^{t} \dot{U} \leq \frac{\varepsilon_{1}}{2}|\dot{A} \dot{U}|^{2}+\frac{1}{2 \varepsilon_{1}}|\dot{U}|^{2}
$$

Let $\varepsilon>0$. Similarly, we have

$$
\begin{equation*}
|\dot{U}|^{2}=\left|\left(\dot{A}^{1 / 2} \dot{U}\right)^{t} \dot{A}^{-1 / 2} \dot{U}\right| \leq \varepsilon\left|\dot{A}^{1 / 2} \dot{U}\right|^{2}+\frac{1}{4 \varepsilon}\left|\dot{A}^{-1 / 2} \dot{U}\right|^{2} \tag{4.30}
\end{equation*}
$$

Thus, we get

$$
\left|\dot{A}^{1 / 2} \dot{U}\right| \leq \frac{\varepsilon_{1}}{2}|\dot{A} \dot{U}|^{2}+\frac{1}{2 \varepsilon_{1}}\left(\varepsilon\left|\dot{A}^{1 / 2} \dot{U}\right|^{2}+\frac{1}{4 \varepsilon}\left|\dot{A}^{-1 / 2} \dot{U}\right|^{2}\right)
$$

By choosing $\varepsilon=\varepsilon_{1}$, we obtain (4.28). Next, we plug (4.28) into (4.30), with $\varepsilon=1$ and $\varepsilon_{1}=\varepsilon_{2}$, and we deduce (4.29).
4.3. Convergence as $(h, \tau) \rightarrow(0,0)$. For a time step $\tau>0$, let $\left(u_{h}^{n}, v_{h}^{n}, z_{h}^{n}, w_{h}^{n}\right)_{n \geq 1}$ be a solution of the fully discrete scheme (4.2). We define the following functions from $\mathbb{R}_{+}$into $V_{h}$ :

$$
\begin{aligned}
u_{h}^{\tau}(t) & =((n+1)-t / \tau) u_{h}^{n}+(t / \tau-n) u_{h}^{n+1}, \quad t \in[n \tau,(n+1) \tau) \quad(n \in \mathbb{N}) \\
\bar{u}_{h}^{\tau}(t) & =u_{h}^{n+1}, \quad t \in[n \tau,(n+1) \tau) \quad(n \in \mathbb{N}) \\
\underline{u}_{h}^{\tau}(t) & =u_{h}^{n}, \quad t \in[n \tau,(n+1) \tau) \quad(n \in \mathbb{N}) \\
\hat{u}_{h}^{\tau}(t) & =\left(u_{h}^{n}+u_{h}^{n+1}\right) / 2, \quad t \in[n \tau,(n+1) \tau) \quad(n \in \mathbb{N})
\end{aligned}
$$

We define similarly the functions $v_{h}^{\tau}, \bar{v}_{h}^{\tau}, \underline{v}_{h}^{\tau}, \hat{v}_{h}^{\tau}$ associated to the sequence $\left(v_{h}^{n}\right)_{n \geq 0}$ and the functions $\bar{z}_{h}^{\tau}, \bar{w}_{h}^{\tau}$. Notice that $\hat{u}_{h}^{\tau}=\left(\bar{u}_{h}^{\tau}+\underline{u}_{h}^{\tau}\right) / 2$ for all $t \in \mathbb{R}_{+}$and that

$$
\partial_{t} u_{h}^{\tau}=\left(u_{h}^{n+1}-u_{h}^{n}\right) / \tau \quad \text { in } \quad \mathcal{D}^{\prime}\left((0, \infty) ; V_{h}\right)
$$

The convergence results is as follows:
Theorem 4.8. Let $\left(u_{0}, v_{0}\right) \in H_{\text {per }}^{2} \times H_{\text {per }}^{-1}$. Assume that $\left(u_{h}^{0}, v_{h}^{0}\right)_{h>0}$ is a family of functions in $V_{h} \times V_{h}$ which satisfies assumptions (3.25)-(3.26) as $h \rightarrow 0$. Then the solution $\left(u_{h}^{\tau}, v_{h}^{\tau}\right)$ associated to the fully discrete scheme (4.2) tends to the energy solution of problem (1.1)-(1.2) in the following sense, as $(h, \tau) \rightarrow(0,0)$ :

$$
\begin{aligned}
u_{h}^{\tau} & \rightarrow \text { u weakly } \star \text { in } L^{\infty}\left(\mathbb{R}_{+} ; H_{p e r}^{1}\right), \\
u_{h}^{\tau} & \rightarrow \text { u strongly in } C\left([0, T], L^{2}(\Omega)\right), \text { for all } T>0 \\
\mathcal{A}_{h} u_{h}^{\tau} & \rightarrow \mathcal{A} \text { weakly } \star \text { in } L^{\infty}\left(\mathbb{R}_{+} ; L^{2}(\Omega)\right) \\
\dot{\mathcal{A}}_{h}^{-1} \partial_{t} \dot{u}_{h}^{\tau} & \rightarrow \dot{\mathcal{A}}^{-1} \partial_{t} \text { weakly } \star \text { in } L^{\infty}\left(\mathbb{R}_{+} ; \dot{H}_{p e r}^{1}\right) \text { and weakly in } L^{2}\left(\mathbb{R}_{+} ; \dot{H}_{\text {per }}^{1}\right) .
\end{aligned}
$$

Proof. We proceed as in the proof of Theorem 3.3. We first consider the conservation law for the mass. By choosing $\varphi_{h}=1$ and $\psi_{h}=1$ in (4.2), we find

$$
\left\{\begin{array}{l}
\left(\partial_{t} u_{h}^{\tau}, 1\right)=\left\langle\hat{v}_{h}^{\tau}\right\rangle  \tag{4.31}\\
\beta \partial_{t}\left\langle v_{h}^{\tau}\right\rangle+\left\langle\hat{v}_{h}^{\tau}\right\rangle=0
\end{array}\right.
$$

in $\mathcal{D}^{\prime}((0, \infty))$. The estimates below show that $\left(u_{h}^{\tau}\right)_{h>0, \tau>0}$ is bounded in $L^{\infty}\left(\mathbb{R}_{+} ; H_{p e r}^{1}\right)$, so that, up to a subsequence, $u_{h}^{\tau} \rightarrow u$ in $L^{\infty}\left(\mathbb{R}_{+} ; H_{p e r}^{1}\right)$ weakly $\star$, and so

$$
\left(\partial_{t} u_{h}^{\tau}, 1\right) \rightarrow\left(\partial_{t} u, 1\right) \quad \text { in } \mathcal{D}^{\prime}((0, \infty)), \text { as }(h, \tau) \rightarrow(0,0)
$$

By (4.12), $\left|v_{1}^{n}\right| \leq\left|v_{1}^{0}\right|$ for all $n$. Thus, $\left\langle\hat{v}_{h}^{\tau}\right\rangle$ is bounded in $L^{\infty}\left(\mathbb{R}_{+}\right)$, and so, up to a subsequence, $\left\langle\hat{v}_{h}^{\tau}\right\rangle$ converges weakly $\star$ in $L^{\infty}\left(\mathbb{R}_{+}\right)$to some function $a \in L^{\infty}\left(\mathbb{R}_{+}\right)$. Moreover, by (4.12), we have

$$
\left|v_{1}^{n+1}-v_{1}^{n}\right|=|q|^{n}|1-q|\left|v_{1}^{0}\right|=|q|^{n} \frac{2 \tau}{2 \beta+\tau}\left|v_{1}^{0}\right| \leq \frac{\tau}{\beta}\left|v_{1}^{0}\right|,
$$

Observe now that

$$
\begin{equation*}
v_{h}^{\tau}-\hat{v}_{h}^{\tau}=(t / \tau-(n+1 / 2))\left(v_{h}^{n+1}-v_{h}^{n}\right), \quad t \in[n \tau,(n+1) \tau), \quad(n \in \mathbb{N}) \tag{4.32}
\end{equation*}
$$

Therefore we get $\left|\left\langle v_{h}^{\tau}\right\rangle-\left\langle\hat{v}_{h}^{\tau}\right\rangle\right| \leq \tau\left|v_{1}^{0}\right| /(2 \beta)$ and so $\left|\left\langle v_{h}^{\tau}\right\rangle-\left\langle\hat{v}_{h}^{\tau}\right\rangle\right|$ converges uniformly to 0 in $\mathbb{R}_{+}$, as $(h, \tau) \rightarrow(0,0)$. Hence $\left\langle v_{h}^{\tau}\right\rangle$ converges to $a$ weakly $\star$ in $L^{\infty}\left(\mathbb{R}_{+}\right)$. We can pass to the limit in (4.31) in the sense of distributions on $(0, \infty)$ and we find

$$
\left\{\begin{array}{l}
\left(\partial_{t} u, 1\right)=a(t) \\
\beta \partial_{t} a(t)+a(t)=0,
\end{array}\right.
$$

which is the conservation law for the mass.
We now turn to the energy estimate. Let

$$
\tau^{\star}=\min \left\{2 \beta,\left(4 c_{5} \sup _{h>0}\left|\left\langle v_{h}^{0}\right\rangle\right|\right)^{-1}\right\}
$$

If $\tau \leq \tau^{\star}$, then (4.18) is satisfied for $N_{0}=0$ (and for all $h>0$ ). Since

$$
\frac{\tau}{1-q}=\beta+\tau / 2 \leq 2 \beta
$$

the energy estimate (4.17) implies

$$
\begin{equation*}
\mathcal{E}_{h}\left(u_{h}^{n}, v_{h}^{n}\right)+\sum_{k=0}^{n-1} \tau\left|v_{h}^{k+1 / 2}\right|_{-1, h}^{2} \leq \exp \left(32 \beta c_{5}\left|v_{1}^{0}\right|\right)\left(\mathcal{E}_{h}\left(u_{h}^{0}, v_{h}^{0}\right)+2 \beta c_{7}\left|v_{1}^{0}\right|\right) \tag{4.33}
\end{equation*}
$$

for all $n \geq 0$. Assumptions (3.25)-(3.26) imply that $\mathcal{E}_{h}\left(u_{h}^{0}, v_{h}^{0}\right)$ and $\left|v_{1}^{0}\right|$ are bounded by a constant independent of $h$. The right-hand side of (4.33) is bounded by a constant independent of $h$ and $\tau$. Thus, $u_{h}^{\tau}$ is uniformly bounded in $H_{p e r}^{1}, \bar{z}_{h}^{\tau}=\mathcal{A}_{h} \hat{u}_{h}^{\tau}$ is uniformly bounded in $L^{2}(\Omega)$, and

$$
\dot{r}_{h}^{\tau}:=\dot{\mathcal{A}}_{h}^{-1} \partial_{t} \dot{u}_{h}^{\tau}=\dot{\mathcal{A}}_{h}^{-1} \dot{\hat{v}}_{h}^{\tau}
$$

is uniformly bounded in $H_{p e r}^{1}$. By arguing as in (3.28), we see that for all $0 \leq s \leq t$,

$$
\left\|\dot{u}_{h}^{\tau}(t)-\dot{u}_{h}^{\tau}(s)\right\|^{2} \leq 4\left\|\dot{r}_{h}^{\tau}\right\|_{L^{\infty}\left(\mathbb{R}_{+} ; H_{p e r}^{1}\right)}\left\|\dot{u}_{h}^{\tau}\right\|_{L^{\infty}\left(\mathbb{R}_{+} ; H_{p e r}^{1}\right)}|t-s| .
$$

Moreover, for all $0 \leq s \leq t$, observe that

$$
\left|\left\langle u_{h}^{\tau}(t)\right\rangle-\left\langle u_{h}^{\tau}(s)\right\rangle\right|=\left|\int_{s}^{t}\left\langle\hat{v}_{h}^{\tau}(\sigma)\right\rangle d \sigma\right| \leq\left|\left\langle v_{h}^{0}\right\rangle\right||t-s|
$$

Thus, for all $T>0$, there is a constant $C_{T}$ independent of $(h, \tau)$ such that

$$
\begin{equation*}
\left\|u_{h}^{\tau}(t)-u_{h}^{\tau}(s)\right\| \leq C_{T}|t-s|^{1 / 2} \tag{4.34}
\end{equation*}
$$

for all $0 \leq s \leq t \leq T$. By the Ascoli-Arzelà Theorem, $\left(u_{h}^{\tau}\right)$ is precompact in the space $C\left([0, T] ; L^{2}(\Omega)\right)$, for all $T>0$. Applying (4.34) with $s=n \tau$ and $t=(n+1) \tau$ yields $\left\|u_{h}^{n+1}-u_{h}^{n}\right\| \leq C_{T} \tau^{1 / 2}$, so that

$$
\left\|u_{h}^{\tau}-\bar{u}_{h}^{\tau}\right\|_{L^{\infty}\left(0, T ; L^{2}(\Omega)\right)} \rightarrow 0 \quad \text { and } \quad\left\|u_{h}^{\tau}-\underline{u}_{h}^{\tau}\right\|_{L^{\infty}\left(0, T ; L^{2}(\Omega)\right)} \rightarrow 0
$$

as $\tau \rightarrow 0$. Up to a subsequence, we have

$$
\begin{aligned}
u_{h}^{\tau}, \hat{u}_{h}^{\tau} & \rightarrow u \text { weakly } \star \text { in } L^{\infty}\left(\mathbb{R}_{+} ; H_{p e r}^{1}\right), \\
\bar{u}_{h}^{\tau}, \underline{u}_{h}^{\tau} & \rightarrow u \text { strongly in } C\left([0, T] ; L^{2}(\Omega)\right), \text { for all } T>0, \\
\bar{u}_{h}^{\tau}, \underline{u}_{h}^{\tau} & \rightarrow u \text { a.e. in } \Omega \times \mathbb{R}_{+}, \\
\bar{z}_{h}^{\tau} & \rightarrow \text { weakly } \star \text { in } L^{\infty}\left(\mathbb{R}_{+} ; L^{2}(\Omega)\right), \\
\dot{r}_{h}^{\tau} & \rightarrow \dot{r} \text { weakly } \star \text { in } L^{\infty}\left(\mathbb{R}_{+} ; \dot{H}_{p e r}^{1}\right) \text { and weakly in } L^{2}\left(\mathbb{R}_{+} ; \dot{H}_{p e r}^{1}\right),
\end{aligned}
$$

as $(h, \tau) \rightarrow(0,0)$. Let $\varphi \in H_{p e r}^{1}$ and set $\varphi_{h}=\Pi_{h}(\varphi)$, so that $\varphi_{h} \rightarrow \varphi$ strongly in $H_{p e r}^{1}$. The first equation in (4.2) reads

$$
\left(\partial_{t} u_{h}^{\tau}, \varphi_{h}\right)=\left(\hat{v}_{h}^{\tau}, \varphi_{h}\right)
$$

By arguing as in (3.29) and letting $(h, \tau) \rightarrow(0,0)$, we obtain that

$$
\partial_{t}(\dot{u}, \dot{\varphi})=(\nabla \dot{r}, \nabla \dot{\varphi})=\langle\dot{v}, \dot{\varphi}\rangle_{H_{p e r}^{-1}, H_{p e r}^{1}}
$$

in $\mathcal{D}^{\prime}((0, \infty))$, with $\dot{v}=\dot{\mathcal{A}} \dot{r}$. This shows that $\partial_{t} \dot{u}=\dot{v} \in L^{\infty}\left(\mathbb{R}_{+} ; H_{\text {per }}^{1}\right)$.
Next, we set $\psi \in H_{\text {per }}^{2}$ and we let $\psi_{h}=\Pi_{h}(\psi)$ so that $\psi_{h} \rightarrow \psi$ strongly in $H_{p e r}^{1}$. We have $\psi_{h}=\sum_{i=1}^{N_{h}} \Psi_{i} e_{h}^{i}$ and $\Psi=\left(\psi_{1}, \ldots, \psi_{N_{h}}\right)^{t}$ is the vector associated to $\psi_{h}$. On multiplying (4.5) by $\dot{\Psi}^{t} \dot{A}^{-1}$, we find

$$
\begin{align*}
& \beta\left(\partial_{t} \dot{v}_{h}^{\tau}, \dot{\psi}_{h}\right)_{-1, h}+\left(\dot{\hat{v}}_{h}^{\tau}, \dot{\psi}_{h}\right)_{-1, h}+\left(\nabla \bar{z}_{h}^{\tau}, \nabla \psi_{h}\right)-2\left(\nabla \hat{u}_{h}^{\tau}, \nabla \psi_{h}\right) \\
& +\frac{1}{2}\left(f\left(\underline{u}_{h}^{\tau}\right)+f\left(\bar{u}_{h}^{\tau}\right), \dot{\psi}_{h}\right)-\frac{1}{12}\left(\left(\bar{u}_{h}^{\tau}-\underline{u}_{h}^{\tau}\right)^{2}\left(f_{+}^{\prime \prime}\left(\underline{u}_{h}^{\tau}\right)+f_{-}^{\prime \prime}\left(\bar{u}_{h}^{\tau}\right)\right), \dot{\psi}_{h}\right)=0 \tag{4.35}
\end{align*}
$$

By arguing as in the proof of Theorem 3.3, we get

$$
\left(\dot{\hat{v}}_{h}^{\tau}, \dot{\psi}_{h}\right)_{-1, h} \rightarrow(v, \psi)_{-1},\left(\nabla \bar{z}_{h}^{\tau}, \nabla \psi_{h}\right) \rightarrow(z, \mathcal{A} \psi),\left(\nabla \hat{u}_{h}^{\tau}, \nabla \psi_{h}\right) \rightarrow(\nabla u, \nabla \psi)
$$

in $\mathcal{D}^{\prime}((0, \infty))$. Thanks to the Sobolev injection $H_{p e r}^{1} \hookrightarrow L^{2 p+2}(\Omega)$, for all $T>0$, the terms

$$
\left\|f\left(\underline{u}_{h}^{\tau}\right)\right\|_{L^{q}\left(0, T ; L^{q}(\Omega)\right)},\left\|f\left(\bar{u}_{h}^{\tau}\right)\right\|_{L^{q}\left(0, T ; L^{q}(\Omega)\right)}
$$

and

$$
\left\|\left(\bar{u}_{h}^{\tau}-\underline{u}_{h}^{\tau}\right)^{2}\left(f_{+}^{\prime \prime}\left(\underline{u}_{h}^{\tau}\right)+f_{-}^{\prime \prime}\left(\bar{u}_{h}^{\tau}\right)\right)\right\|_{L^{q}\left(0, T ; L^{q}(\Omega)\right)}
$$

are bounded by a constant independent of $h$ and $\tau$, for $q=(2 p+2) /(2 p+1) \in(1,2)$. We can therefore pass to the limit in the nonlinear terms, and we find that

$$
\frac{1}{2}\left(f\left(\underline{u}_{h}^{\tau}\right)+f\left(\bar{u}_{h}^{\tau}\right), \dot{\psi}_{h}\right) \rightarrow(f(u), \dot{\psi})
$$

and

$$
\frac{1}{12}\left(\left(\bar{u}_{h}^{\tau}-\underline{u}_{h}^{\tau}\right)^{2}\left(f_{+}^{\prime \prime}\left(\underline{u}_{h}^{\tau}\right)+f_{-}^{\prime \prime}\left(\bar{u}_{h}^{\tau}\right)\right), \dot{\psi}_{h}\right) \rightarrow 0
$$

in $\mathcal{D}^{\prime}((0, \infty))$. Thus, the first term in equation (4.35) has a limit in $\mathcal{D}^{\prime}((0, \infty))$,

$$
\beta\left(\partial_{t} \dot{v}_{h}^{\tau}, \dot{\psi}_{h}\right)_{-1, h} \rightarrow \eta_{\psi}
$$

As a consequence,

$$
\left(\dot{\bar{v}}_{h}^{\tau}-\underline{\dot{\underline{v}}}_{h}^{\tau}, \dot{\psi}_{h}\right)_{-1, h}=\tau\left(\partial_{t} \dot{v}_{h}^{\tau}, \dot{\psi}_{h}\right)_{-1, h} \rightarrow 0
$$

in $\mathcal{D}^{\prime}((0, \infty))$. Thus, by (4.32), we have

$$
\left(\dot{v}_{h}^{\tau}, \dot{\psi}_{h}\right)_{-1, h} \rightarrow(\dot{v}, \dot{\psi})_{-1} .
$$

Summing up, we have proved that

$$
\beta \partial_{t}(\dot{v}, \dot{\psi})_{-1}+(\dot{v}, \dot{\psi})_{-1}+(z, \mathcal{A} \psi)-2(\nabla u, \nabla \psi)+(f(u), \dot{\psi})=0,
$$

with $v=\partial_{t} u$ and $z=\mathcal{A} u$. We conclude as in Theorem 3.3 that $\left(u, u_{t}\right)$ is an energy solution of (1.1)-(1.2). Note that the whole family converges to $\left(u, u_{t}\right)$ due to the uniqueness of the limit.

## 5. Convergence to equilibrium

In this section, we prove that any solution of the fully discrete scheme converges to a single equilibrium, for any time step $\tau>0$. The parameter $h$ is fixed (so that assumption (H2) is not relevant). We adapt the proof from [25] in a discrete setting. The main idea is to use the gradient-like flow structure of the problem and a suitable Łojasiewicz inequality. In three space dimensions, in addition to (H1) and (H3), we need the following assumption:
(H4) If $d=3$, then either $V_{h} \subset L^{\infty}(\Omega)$ or $p=1$.
Theorem 5.1. Let $\tau>0$ denote the time step and let $\left(U^{n}, V^{n}\right)_{n>0}$ denote any sequence in $\mathbb{R}^{N_{h}} \times \mathbb{R}^{N_{h}}$ which complies with (4.5). Then $\left(U^{n}, V^{n}\right)$ converges to $\left(U^{\infty}, 0\right)$, where $U^{\infty}=\left(u_{1}^{\infty}, \dot{U}^{\infty}\right)$ is a stationary solution with average constraint, i.e.,

$$
\left\{\begin{array}{l}
u_{1}^{\infty}=M=u_{1}^{0}+\beta v_{1}^{0},  \tag{5.1}\\
\dot{A}^{2} \dot{U}^{\infty}-2 \dot{A} \dot{U}^{\infty}+\dot{\nabla} F_{h}\left(U^{\infty}\right)=0 .
\end{array}\right.
$$

We first prove the following
Lemma 5.2. Let the assumptions of Theorem 5.1 hold. Then $V^{n+1 / 2} \rightarrow 0$.
Proof. Since $|q|<1$, estimate (4.18) is satisfied for $N_{0}$ large enough. By the energy estimate (4.17), $\sum_{n=N_{0}}^{\infty}\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2}<\infty$. In particular, $\dot{V}^{n+1 / 2} \rightarrow 0$ in $\mathbb{R}^{N_{h}-1}$. Moreover, $v_{1}^{n+1 / 2}=q^{n} v_{1}^{1 / 2}$ by (4.12), so $V^{n+1 / 2} \rightarrow 0$, as claimed.

For any $M \in \mathbb{R}$, we introduce the auxiliary function $F_{M}(y)=F(M+y)$ and the following functionals

$$
\begin{align*}
F_{M, h}(\dot{U}) & =\left(F_{M}\left(\dot{u}_{h}\right), 1\right),  \tag{5.2}\\
E_{M, h}(\dot{U}) & =\frac{1}{2}|\dot{A} \dot{U}|^{2}-\left|\dot{A}^{1 / 2} \dot{U}\right|^{2}+F_{M, h}(\dot{U}),  \tag{5.3}\\
\mathcal{E}_{M, h}(\dot{U}, \dot{V}) & =E_{M, h}(\dot{U})+\frac{\beta}{2}|\dot{V}|_{-1}^{2}, \tag{5.4}
\end{align*}
$$

defined for every $\dot{U} \simeq \dot{u}_{h}$ and every $\dot{V}$ in $\mathbb{R}^{N_{h}-1}$.
For any $M \in \mathbb{R}$, we also consider

$$
\mathfrak{S}_{M}=\left\{U \in \mathbb{R}^{N_{h}}: U \text { satisfies (5.1) }\right\} .
$$

For any sequence $\left(U^{n}, V^{n}\right)_{n \geq 0}$ in $\mathbb{R}^{N_{h}} \times \mathbb{R}^{N_{h}}$, we define its $\omega$-limit set in $\mathbb{R}^{N_{h}} \times \mathbb{R}^{N_{h}}$ :

$$
\omega\left(\left(U^{n}, V^{n}\right)_{n \geq 0}\right)=\left\{\left(U^{\star}, V^{\star}\right): \exists n_{j} \nearrow \infty,\left(U^{n_{j}}, V^{n_{j}}\right) \rightarrow\left(U^{\star}, V^{\star}\right) \mid\right\}
$$

Similarly, we set

$$
\omega\left(\left(U^{n}\right)_{n \geq 0}\right)=\left\{U^{\star}: \exists n_{j} \nearrow \infty, U^{n_{j}} \rightarrow U^{\star}\right\}
$$

We have:
Proposition 5.3. Let the assumptions of Theorem 5.1 hold. Then $\omega\left(\left(U^{n}, V^{n}\right)_{n \geq 0}\right)$ is a nonempty compact and connected set such that

$$
\omega\left(\left(U^{n}, V^{n}\right)_{n \geq 0}\right)=\omega\left(\left(U^{n}\right)_{n \geq 0}\right) \times\{0\} \subset\left\{\left(U^{\star}, 0\right): U^{\star} \in \mathfrak{S}_{M}\right\}
$$

with $M=u_{1}^{0}+\beta v_{1}^{0}$. Moreover, $E_{M, h}$ is constant on $\omega\left(\left(U^{n}\right)_{n}\right)$.
This result implies in particular that $V^{n} \rightarrow 0$, as proved below.
Proof. Since $q=(2 \beta-\tau) /(2 \beta+\tau)$, we can rewrite (4.13) as

$$
\begin{equation*}
u_{1}^{n}=u_{1}^{0}+\left(1-q^{n}\right) \beta v_{1}^{0} \tag{5.5}
\end{equation*}
$$

Let $M=u_{1}^{0}+\beta v_{1}^{0}$. We introduce the auxiliary functions

$$
f_{M}(y)=f(M+y) \quad \text { and } \quad \hat{f}_{M}(r, s)=\hat{f}(M+r, M+s)
$$

where

$$
\hat{f}(r, s)=\frac{1}{2}(f(r)+f(s))-\frac{1}{12}(s-r)^{2}\left(f_{+}^{\prime \prime}(r)+f_{-}^{\prime \prime}(s)\right)
$$

We also set

$$
F_{M, \pm}(y)=F_{ \pm}(M+y) \quad \text { and } f_{M, \pm}(y)=f_{ \pm}(M+y)
$$

so that $F_{M}=F_{M,+}+F_{M,-}$ and $F_{M,+}^{(i v)} \geq 0, F_{M,-}^{(i v)} \leq 0$. In particular, the function $F_{M}$ satisfies the decomposition (H3), and we have

$$
\hat{f}_{M}(r, s)=\frac{1}{2}\left(f_{M}(r)+f_{M}(s)\right)-\frac{1}{12}(s-r)^{2}\left(f_{M,+}^{\prime \prime}(r)+f_{M,-}^{\prime \prime}(s)\right)
$$

Then we rewrite the second equation in (4.5) in the following form:

$$
\begin{array}{r}
\beta\left(\dot{V}^{n+1}-\dot{V}^{n}\right) / \tau+\dot{V}^{n+1 / 2}+\dot{A}\left(\dot{A}^{2} \dot{U}^{n+1 / 2}-2 \dot{A} \dot{U}^{n+1 / 2}+\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right)\right) \\
=\dot{A}\left(\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right)-\dot{J}\left(U^{n}, U^{n+1}\right)\right) \tag{5.6}
\end{array}
$$

where

$$
\begin{aligned}
\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right) & =\left(\left(\hat{f}_{M}\left(\dot{u}_{h}^{n}, \dot{u}_{h}^{n+1}\right), e_{i}\right)\right)_{2 \leq i \leq N_{h}} \\
\dot{J}\left(U^{n}, U^{n+1}\right) & =\left(\left(\hat{f}\left(u_{h}^{n}, u_{h}^{n+1}\right), e_{i}\right)\right)_{2 \leq i \leq N_{h}}
\end{aligned}
$$

Multiplying (5.6) by $\left(\dot{U}^{n+1}-\dot{U}^{n}\right)^{t} \dot{A}^{-1}$, using that (4.5) implies

$$
\begin{equation*}
\dot{U}^{n+1}-\dot{U}^{n}=\tau \dot{V}^{n+1 / 2}, \tag{5.7}
\end{equation*}
$$

we find

$$
\begin{align*}
& \frac{\beta}{2}\left(\left|\dot{V}^{n+1}\right|_{-1}^{2}-\left|\dot{V}^{n}\right|_{-1}^{2}\right)+\tau\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2}+\frac{1}{2}\left(\left|\dot{A} \dot{U}^{n+1}\right|^{2}-\left|\dot{A} \dot{U}^{n}\right|^{2}\right) \\
& -\left(\left|\dot{A}^{1 / 2} \dot{U}^{n+1}\right|^{2}-\left|\dot{A}^{1 / 2} \dot{U}^{n}\right|^{2}\right)+\left\langle\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right), \dot{U}^{n+1}-\dot{U}^{n}\right\rangle \\
& =\left(\hat{f}_{M}\left(\dot{u}_{h}^{n}, \dot{u}_{h}^{n+1}\right)-\hat{f}\left(u_{h}^{n}, u_{h}^{n+1}\right), \dot{u}_{h}^{n+1}-\dot{u}_{h}^{n}\right) \tag{5.8}
\end{align*}
$$

Using now Lemma 4.4 and arguing as in the proof of Lemma 4.5, we obtain

$$
\begin{equation*}
\left(F_{M}\left(\dot{u}_{h}^{n+1}\right), 1\right)-\left(F_{M}\left(\dot{u}_{h}^{n}\right), 1\right) \leq\left\langle\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right), \dot{U}^{n+1}-\dot{U}^{n}\right\rangle \tag{5.9}
\end{equation*}
$$

By (5.5), for any solution $u_{h}^{n} \simeq U^{n}$ of (4.5), we have

$$
\hat{f}\left(u_{h}^{n}, u_{h}^{n+1}\right)=\hat{f}_{M}\left(\dot{u}_{h}^{n}-\beta q^{n} v_{1}^{0}, \dot{u}_{h}^{n+1}-\beta q^{n+1} v_{1}^{0}\right)
$$

Thus we get

$$
\begin{aligned}
& \left(\hat{f}_{M}\left(\dot{u}_{h}^{n}, \dot{u}_{h}^{n+1}\right)-\hat{f}\left(u_{h}^{n}, u_{h}^{n+1}\right), \dot{u}_{h}^{n+1}-\dot{u}_{h}^{n}\right) \\
& =\left(\hat{f}_{M}\left(\dot{u}_{h}^{n}, \dot{u}_{h}^{n+1}\right)-\hat{f}_{M}\left(\dot{u}_{h}^{n}-\beta q^{n} v_{1}^{0}, \dot{u}_{h}^{n+1}-\beta q^{n+1} v_{1}^{0}\right), \dot{u}_{h}^{n+1}-\dot{u}_{h}^{n}\right) \\
& =-\beta q^{n} v_{1}^{0}\left(\int_{0}^{1} \partial_{r} \hat{f}_{M}\left(\tilde{u}_{h}^{n}(s), \tilde{u}_{h}^{n+1}(s)\right)+q \partial_{s} \hat{f}_{M}\left(\tilde{u}_{h}^{n}(s), \tilde{u}_{h}^{n+1}(s)\right) d s, \dot{u}_{h}^{n+1}-\dot{u}_{h}^{n}\right)
\end{aligned}
$$

where $\tilde{u}_{h}^{n}(s)=\dot{u}_{h}^{n}-s \beta q^{n} v_{1}^{0}$ for $s \in[0,1]$. By the energy estimate (4.17), the sequence $\left(u_{h}^{n}\right)_{n \geq 0}$ is bounded in $H_{p e r}^{1}$. Moreover, the function $\hat{f}_{M}$ is a polynomial in $(r, s)$ of total degree equal to $2 p+1$. Using the Sobolev injection $H_{p e r}^{1} \hookrightarrow L^{2 p+1}(\Omega)$, we obtain

$$
\begin{align*}
& \left|\left(\hat{f}_{M}\left(\dot{u}_{h}^{n}, \dot{u}_{h}^{n+1}\right)-\hat{f}\left(u_{h}^{n}, u_{h}^{n+1}\right), \dot{u}_{h}^{n+1}-\dot{u}_{h}^{n}\right)\right| \\
& \quad \leq \beta|q|^{n}\left|v_{1}^{0}\right| C\left(\left\|u_{h}^{n}\right\|_{1},\left\|u_{h}^{n+1}\right\|_{1}\right)\left\|\dot{u}_{h}^{n+1}-\dot{u}_{h}^{n}\right\|_{1}  \tag{5.10}\\
& \quad \leq \frac{1}{4 \tau}\left|\dot{U}^{n+1}-\dot{U}^{n}\right|_{-1}^{2}+C_{0}|q|^{2 n} . \tag{5.11}
\end{align*}
$$

Here and in the following, $C_{k}(k=0,1, \ldots)$ denotes a constant independent of $n$ (but which may depend on $\tau, h$ and other parameters of the problem). In the last inequality we have used that all norms are equivalent in $\dot{V}_{h}$. Adding up (5.8), (5.9) and (5.11), we find

$$
\begin{equation*}
\mathcal{E}_{M, h}\left(\dot{U}^{n+1}, \dot{V}^{n+1}\right)-\mathcal{E}_{M, h}\left(\dot{U}^{n}, \dot{V}^{n}\right)+\frac{3 \tau}{4}\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2} \leq C_{0}|q|^{2 n} \tag{5.12}
\end{equation*}
$$

for all $n \geq 0$.
Set now

$$
\mathcal{G}^{n}=\left\langle\dot{A}^{-1} \dot{V}^{n}, \dot{A}^{-1}\left(\dot{A}^{2} \dot{U}^{n}-2 \dot{A} \dot{U}^{n}+\dot{\nabla} F_{M, h}\left(\dot{U}^{n}\right)\right\rangle_{-1},\right.
$$

where $\langle\dot{U}, \dot{V}\rangle_{-1}=\dot{U}^{t} \dot{A}^{-1} \dot{V}$, for all $\dot{U}, \dot{V} \in \mathbb{R}^{N_{h}-1}$. We have

$$
\begin{aligned}
& \mathcal{G}^{n+1}-\mathcal{G}^{n}=\left\langle\dot{A}^{-1}\left(\dot{V}^{n+1}-\dot{V}^{n}\right), \dot{A}^{-1}\left(\dot{A}^{2} \dot{U}^{n}-2 \dot{A} \dot{U}^{n}+\dot{\nabla} F_{M, h}\left(\dot{U}^{n}\right)\right\rangle_{-1}\right. \\
& +\left\langle\dot{A}^{-1} \dot{V}^{n+1}, \dot{A}^{-1}\left(\left(\dot{A}^{2}-2 \dot{A}\right)\left(\dot{U}^{n+1}-\dot{U}^{n}\right)+\dot{\nabla} F_{M, h}\left(\dot{U}^{n+1}\right)-\dot{\nabla} F_{M, h}\left(\dot{U}^{n}\right)\right\rangle_{-1}\right.
\end{aligned}
$$

Let $\delta \mathcal{G}_{1}^{n}$ denote the first term on the right-hand side of this equality and denote the second by $\delta \mathcal{G}_{2}^{n}$. Using (5.6), $\dot{U}^{n}=\dot{U}^{n+1 / 2}-\left(\dot{U}^{n+1}-\dot{U}^{n}\right) / 2$ and $\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n}\right)=$ $\dot{\nabla} F_{M, h}\left(\dot{U}^{n}\right)$, we obtain

$$
\begin{aligned}
& \delta \mathcal{G}_{1}^{n}=-\frac{\tau}{\beta}\left\langle\dot{A}^{-1} \dot{V}^{n+1 / 2}+\dot{S}^{n}-\left(\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right)-\dot{J}\left(U^{n}, U^{n+1}\right)\right), \dot{A}^{-1} \dot{S}^{n}\right\rangle_{-1} \\
& \frac{\tau}{2 \beta}\left\langle\dot{A}^{-1} \dot{V}^{n+1 / 2}+\dot{S}^{n}-\left(\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right)-\dot{J}\left(U^{n}, U^{n+1}\right)\right), \dot{A}^{-1} \dot{T}_{1}^{n}\right\rangle_{-1}
\end{aligned}
$$

where

$$
\begin{aligned}
\dot{S}^{n} & =\dot{A}^{2} \dot{U}^{n+1 / 2}-2 \dot{A} \dot{U}^{n+1 / 2}+\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right) \\
\dot{T}_{1}^{n} & =\left(\dot{A}^{2}-2 \dot{A}\right)\left(\dot{U}^{n+1}-\dot{U}^{n}\right)+2 \dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right)-2 \dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n}\right)
\end{aligned}
$$

Using that all norms are equivalent in $\dot{V}_{h}$, the Cauchy-Schwarz inequality, Young's inequality and (5.7), we deduce

$$
\begin{aligned}
\delta \mathcal{G}_{1}^{n}+\frac{3 \tau}{4 \beta}\left|\dot{A}^{-1} \dot{S}^{n}\right|^{2} \leq & C_{1} \tau\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2}+C_{2}\left|\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right)-\dot{J}\left(U^{n}, U^{n+1}\right)\right|^{2} \\
& +C_{3}\left|\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right)-\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n}\right)\right|^{2} .
\end{aligned}
$$

By Bessel's inequality, we get

$$
\left|\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right)-\dot{J}\left(U^{n}, U^{n+1}\right)\right|^{2} \leq\left\|\hat{f}_{M}\left(u_{h}^{n}, u_{h}^{n+1}\right)-\hat{f}\left(u_{h}^{n}, u_{h}^{n+1}\right)\right\|^{2} .
$$

Arguing as in (5.10), and using assumption (H4), we find that

$$
\begin{equation*}
\left|\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right)-\dot{J}\left(U^{n}, U^{n+1}\right)\right|^{2} \leq C_{4}|q|^{2 n} . \tag{5.13}
\end{equation*}
$$

Similarly, we have

$$
\begin{equation*}
\left|\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right)-\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n}\right)\right|^{2} \leq C_{5} \tau\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2} \tag{5.14}
\end{equation*}
$$

Summing up, we have proved

$$
\delta \mathcal{G}_{1}^{n}+\frac{3 \tau}{4 \beta}\left|\dot{A}^{-1} \dot{S}^{n}\right|^{2} \leq\left(C_{1}+C_{3} C_{5}\right) \tau\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2}+C_{2} C_{4}|q|^{2 n}
$$

for all $n \geq 0$. We now consider the term $\delta \mathcal{G}_{2}^{n}$. Using $V^{n+1}=V^{n+1 / 2}+\left(V^{n+1}-V^{n}\right) / 2$, equation (5.6), and arguing as for $\delta G_{1}^{n}$, we obtain

$$
\delta \mathcal{G}_{2}^{n} \leq \frac{\tau}{4 \beta}\left|\dot{A}^{-1} \dot{S}^{n}\right|^{2}+C_{6} \tau\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2}+C_{7}|q|^{2 n} .
$$

Thus, we get

$$
\begin{equation*}
\mathcal{G}^{n+1}-\mathcal{G}^{n}+\frac{\tau}{2 \beta}\left|\dot{A}^{-1} \dot{S}^{n}\right|^{2} \leq C_{8} \tau\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2}+C_{9}|q|^{2 n} \tag{5.15}
\end{equation*}
$$

for all $n \geq 0$, with $C_{8}=C_{1}+C_{3} C_{5}+C_{6}$ and $C_{9}=C_{2} C_{4}+C_{7}$.
Let us introduce the sequence

$$
W^{n}=2 \mathcal{E}_{M, h}\left(\dot{U}^{n}, \dot{V}^{n}\right)+\nu \mathcal{G}^{n},
$$

where $\nu>0$ is sufficiently small so that $\nu C_{8} \leq 1 / 2$. From estimates (5.12) and (5.15), it follows that

$$
\begin{equation*}
W^{n+1}-W^{n}+\tau\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2}+\frac{\nu \tau}{2 \beta}\left|\dot{A}^{-1} \dot{S}^{n}\right|^{2} \leq C_{10}|q|^{2 n} \tag{5.16}
\end{equation*}
$$

with $C_{10}=2 C_{0}+\nu C_{9}$. By the energy estimate (4.17), the sequence $\left(U^{n}, V^{n}\right)$ is bounded, so $\left(W^{n}\right)_{n \geq 0}$ is bounded. This implies that $W^{n}$ converges to some real number $W^{\infty}$ as $n$ tends to $\infty$. Indeed, let

$$
\widetilde{W}^{n}=W^{n}+\frac{C_{10}}{1-|q|^{2}}|q|^{2 n} .
$$

Using (5.16), we see that $\widetilde{W}^{n+1}-\widetilde{W}^{n} \leq 0$, i.e. $\widetilde{W}^{n}$ is nonincreasing. and since $\widetilde{W}$ is bounded, $\widetilde{W}^{n}$ has a limit $\widetilde{W}^{\infty}=W^{\infty}$.

Adding up estimate (5.16), we obtain that $\sum_{n=0}^{\infty}\left|\dot{A}^{-1} \dot{S}^{n}\right|^{2}<\infty$. In particular, $\dot{S}^{n} \rightarrow 0$. Moreover, by (5.13), we have

$$
\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right)-\dot{J}\left(U^{n}, U^{n+1}\right) \rightarrow 0
$$

as $n \rightarrow \infty$. From (5.6) and Lemma 5.2, it follows that $\dot{V}^{n+1}-\dot{V}^{n} \rightarrow 0$ and that

$$
\dot{V}^{n}=\dot{V}^{n+1 / 2}-\left(\dot{V}^{n+1}-\dot{V}^{n}\right) / 2 \rightarrow 0 .
$$

If $\left(\dot{U}^{n^{\prime}}\right)_{n^{\prime}}$ is a subsequence which converges to some $\dot{U}^{\star}$, then $\dot{U}^{n^{\prime}+1} \rightarrow \dot{U}^{\star}$ as well, since $\dot{U}^{n^{\prime}+1}-\dot{U}^{n^{\prime}}=\tau \dot{V}^{n^{\prime}+1 / 2}$. Since $\dot{S}^{n^{\prime}} \rightarrow 0$ and $\dot{J}_{M}$ is continuous at $\left(\dot{U}^{\star}, \dot{U}^{\star}\right)$ with $\dot{J}_{M}\left(\dot{U}^{\star}, \dot{U}^{\star}\right)=\nabla F_{M, h}\left(\dot{U}^{\star}\right)$, we obtain that

$$
\dot{A}^{2} \dot{U}^{\star}-2 \dot{A} \dot{U}^{\star}+\dot{\nabla} F_{M, h}\left(\dot{U}^{\star}\right)=0 .
$$

Using the conservation law (5.5), we see that $U^{\star} \in \mathfrak{S}_{M}$. Finally, the sequence $\left(U^{n}, V^{n}\right)$ is bounded, and we have seen that $U^{n+1}-U^{n} \rightarrow 0, V^{n} \rightarrow 0$ so the $\omega$-limit set $\omega\left(\left(U^{n}, V^{n}\right)_{n \geq 0}\right)$ is a nonempty compact and connected subset of $\mathfrak{S}_{M} \times\{0\}$, equal to $\omega\left(\left(U^{n}\right)_{n}\right) \times\{0\}$.

Since $\dot{V}^{n} \rightarrow 0$ and $W^{n} \rightarrow W^{\infty}$, we have $\mathcal{G}^{n} \rightarrow 0$ and

$$
\mathcal{E}_{M, h}\left(\dot{U}^{n}, \dot{V}^{n}\right)=(1 / 2)\left(W^{n}-\nu \mathcal{G}^{n}\right) \rightarrow W^{\infty} / 2 .
$$

By definition, $E_{M, h}\left(\dot{U}^{n}\right)=\mathcal{E}_{M, h}\left(\dot{U}^{n}, \dot{V}^{n}\right)-\frac{\beta}{2}\left|\dot{V}^{n}\right|_{-1}^{2}$, so $E_{M, h}\left(\dot{U}^{n}\right) \rightarrow W^{\infty} / 2$. This implies that $E_{M, h}$ is constant and equal to $E^{\infty}:=W^{\infty} / 2$ on $\omega\left(\left(U^{n}\right)_{n}\right)$. The proof of Proposition 5.3 is complete.

We notice that the functional $E_{M, h}$ is a polynomial of the variables $\left(u_{2}, \ldots, u_{N_{h}}\right)$ of total degree $2 p+2$, so the following Lojasiewciz inequality holds:

Lemma 5.4 (Lojasiewicz' inequality [30]). Let $\dot{U}^{\star} \in \mathbb{R}^{N_{h}-1}$ be a critical point of $E_{M, h}$. Then there exist constants $\theta \in(0,1 / 2)$ and $\delta>0$ such that for any $\dot{U} \in \mathbb{R}^{N_{h}-1}$ satisfying $\left|\dot{U}-\dot{U}^{\star}\right|<\delta$, there holds

$$
\begin{equation*}
\left|E_{M, h}(\dot{U})-E_{M, h}\left(\dot{U}^{\star}\right)\right|^{1-\theta} \leq\left|\dot{A}^{2} \dot{U}-2 \dot{A} \dot{U}+\dot{\nabla} F_{M, h}(\dot{U})\right| . \tag{5.17}
\end{equation*}
$$

Proof of Theorem 5.1. Let $M=u_{1}^{0}+\beta v_{1}^{0}$ as previously. By Lemma 5.4, for every $U^{\infty} \in \omega\left(\left(U^{n}\right)_{n}\right)$, there exist some $\delta>0$ and $\theta \in(0,1 / 2)$ that may depend on $\dot{U}^{\infty}$ such that the inequality (5.17) holds for all $\dot{U}$ in

$$
\mathbf{B}_{\delta}\left(\dot{U}^{\infty}\right)=\left\{\dot{U} \in \mathbb{R}^{N_{h}-1}:\left|\dot{U}-\dot{U}^{\infty}\right|<\delta\right\}
$$

and $\left|E_{M, h}(\dot{U})-E_{M, h}\left(\dot{U}^{\infty}\right)\right| \leq 1$. The union of balls $\left\{\mathbf{B}_{\delta}\left(\dot{U}^{\infty}\right): \dot{U}^{\infty} \in \omega\left(\left(\dot{U}^{n}\right)_{n}\right)\right\}$ forms an open covering of $\omega\left(\left(\dot{U}^{n}\right)_{n}\right)$. Due to the compactness of $\omega\left(\left(\dot{U}^{n}\right)_{n}\right)$ in $\mathbb{R}^{N_{h}-1}$, we can find a finite sub-covering $\left\{\mathbf{B}_{\delta_{i}}\left(\dot{U}_{i}^{\infty}\right)\right\}_{i=1,2, \ldots, m}$, where the constants $\delta_{i}, \theta_{i}$ corresponding to $\dot{U}_{i}^{\infty}$ in Lemma 5.4 are indexed by $i$.

From the definition of $\omega\left(\left(\dot{U}^{n}\right)_{n}\right)$, we know that there exists a sufficiently large $n_{0}$ such that $\dot{U}^{n} \in \mathcal{U}:=\cup_{i=1}^{m} \mathbf{B}_{\delta_{i}}\left(\dot{U}_{i}^{\infty}\right)$ for $n \geq n_{0}$. Taking $\theta=\min _{i=1}^{m}\left\{\theta_{i}\right\} \in(0,1 / 2)$, we infer from Lemma 5.4 that, for all $n \geq n_{0}$,

$$
\begin{equation*}
\left|E_{M, h}\left(\dot{U}^{n}\right)-E^{\infty}\right|^{1-\theta} \leq\left|\dot{A}^{2} \dot{U}^{n}-2 \dot{A} \dot{U}^{n}+\dot{\nabla} F_{M, h}\left(\dot{U}^{n}\right)\right|, \tag{5.18}
\end{equation*}
$$

where $E^{\infty}=W^{\infty} / 2$ is the constant value of $E_{M, h}$ on $\omega\left(\left(\dot{U}^{n}\right)_{n}\right)$.
Let us now set

$$
\begin{equation*}
a_{n}=\left(\frac{\tau}{2}\left|\dot{V}^{n+1 / 2}\right|_{-1}^{2}+\frac{\nu \tau}{4 \beta}\left|\dot{A}^{-1} \dot{S}^{n}\right|^{2}\right)^{1 / 2}+|q|^{n} \tag{5.19}
\end{equation*}
$$

From (5.16), it follows that

$$
\sum_{k=n}^{\infty} a_{k}^{2} \leq W^{n}-W^{\infty}+C_{11}|q|^{2 n}
$$

On the other hand, using the Łojasiewicz inequality (5.18) and the fact $1 /(1-\theta)<2$, we deduce that, for all $n \geq n_{0}$ (changing $n_{0}$ into a larger integer if necessary),

$$
\begin{align*}
\left|W^{n}-W^{\infty}\right| \leq & 2\left|E_{M, h}\left(\dot{U}^{n}\right)-E^{\infty}\right|+\beta\left|\dot{V}^{n}\right|_{-1}^{2}+\nu\left|\mathcal{G}^{n}\right| \\
\leq & 2\left|\dot{A}^{2} \dot{U}^{n}-2 \dot{A} \dot{U}^{n}+\dot{\nabla} F_{M, h}\left(\dot{U}^{n}\right)\right|^{1 /(1-\theta)}+\beta\left|\dot{V}^{n}\right|_{-1}^{2} \\
& +C_{12}\left|\dot{V}^{n}\right|_{-1}\left|\dot{A}^{2} \dot{U}^{n}-2 \dot{A} \dot{U}^{n}+\dot{\nabla} F_{M, h}\left(\dot{U}^{n}\right)\right| \\
\leq & C_{13}\left(\left|\dot{A}^{2} \dot{U}^{n}-2 \dot{A} \dot{U}^{n}+\dot{\nabla} F_{M, h}\left(\dot{U}^{n}\right)\right|^{1 /(1-\theta)}+\left|\dot{V}^{n}\right|_{-1}^{1 /(1-\theta)}\right)(5 \tag{5.20}
\end{align*}
$$

Using $\dot{V}^{n}=\dot{V}^{n+1 / 2}-\left(\dot{V}^{n+1}-\dot{V}^{n}\right) / 2$, we deduce from (5.6) and (5.13) that

$$
\begin{equation*}
\left|\dot{V}^{n}\right|_{-1} \leq C_{13}\left(\left|\dot{V}^{n+1 / 2}\right|_{-1}+\left|\dot{A}^{-1} \dot{S}^{n}\right|+|q|^{n}\right) \tag{5.21}
\end{equation*}
$$

Similarly, from

$$
\begin{aligned}
\dot{A}^{2} \dot{U}^{n}-2 \dot{A} \dot{U}^{n}+\dot{\nabla} F_{M, h}\left(\dot{U}^{n}\right)= & \dot{S}^{n}+\left(\dot{A}^{2}-2 \dot{A}\right)\left(\dot{U}^{n}-\dot{U}^{n+1 / 2}\right) \\
& +\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n}\right)-\dot{J}_{M}\left(\dot{U}^{n}, \dot{U}^{n+1}\right)
\end{aligned}
$$

$\dot{U}^{n+1}-\dot{U}^{n}=\tau \dot{V}^{n+1 / 2}$ and (5.14), we infer that

$$
\begin{equation*}
\left|\dot{A}^{2} \dot{U}^{n}-2 \dot{A} \dot{U}^{n}+\dot{\nabla} F_{M, h}\left(\dot{U}^{n}\right)\right| \leq C_{14}\left(\left|\dot{A}^{-1} \dot{S}^{n}\right|+\left|\dot{V}^{n+1 / 2}\right|_{-1}\right) \tag{5.22}
\end{equation*}
$$

Collecting (5.20), (5.21) and (5.22), we obtain

$$
\left|W^{n}-W^{\infty}\right| \leq C_{15}\left(\left|\dot{V}^{n+1 / 2}\right|_{-1}^{1 /(1-\theta)}+\left|\dot{A}^{-1} \dot{S}^{n}\right|^{1 /(1-\theta)}+\left(|q|^{n}\right)^{1 /(1-\theta)}\right)
$$

for all $n \geq n_{0}$. This gives

$$
\sum_{k=n}^{\infty} a_{k}^{2} \leq C_{16} a_{n}^{1 /(1-\theta)}, \quad \forall n \geq n_{0}
$$

From Lemma 5.5 below, we conclude that $\sum_{n=0}^{\infty} a_{n}<\infty$. In particular, we have

$$
\tau \sum_{n=0}^{\infty}\left|\dot{V}^{n+1 / 2}\right|_{-1}=\sum_{n=0}^{\infty}\left|\dot{U}^{n+1}-\dot{U}^{n}\right|<\infty
$$

This shows that the whole sequence $\left(\dot{U}^{n}\right)_{n}$ has a limit $\dot{U}^{\infty}$ as $n \rightarrow \infty$. From (5.5), we know that $u_{1}^{n} \rightarrow M$. Thus, $\left(U^{n}\right)_{n}$ tends to some $U^{\infty}$ in $\mathbb{R}^{N_{h}}$, and the proof is complete.

For the following lemma and its proof, we adapt Lemma 4.1 in [29] in a discrete setting (see also Lemma 7.1 in [13]).

Lemma 5.5. Let $0<\theta<1 / 2$. Assume that $\left(a_{n}\right)_{n \geq 0}$ is a sequence of nonnegative real numbers such that $\sum_{n=0}^{\infty} a_{n}^{2}<\infty$, and there are a constant $C>0$ and an integer $n_{0}$ such that

$$
\begin{equation*}
\sum_{k=n}^{\infty} a_{k}^{2} \leq C a_{n}^{1 /(1-\theta)} \text { for all } n \geq n_{0} \tag{5.23}
\end{equation*}
$$

Then $\sum_{n=0}^{\infty} a_{n}<\infty$.

Proof. First replacing $a_{n}$ by $\max \left\{a_{n}, 1\right\}$ for $0 \leq n<n_{0}$, and then taking $C$ large enough to ensure $C \geq \sum_{n=0}^{\infty} a_{n}^{2}$, we observe that (5.23) becomes valid for all $n \geq 0$. So we may assume $n_{0}=0$. Set now

$$
\rho_{n}:=\sum_{k=n}^{\infty} a_{k}^{2} \quad \text { and } \quad \sigma_{n}=\sum_{k=0}^{n} a_{k} \quad \text { for } n \geq 0
$$

Given any $n \geq 0$, we first raise inequality (5.23) to the power $1-\theta>0$ :

$$
\rho_{n}^{1-\theta} \leq C^{1-\theta} a_{n}
$$

Next, we sum this relation and we obtain

$$
\sum_{k=0}^{n} \rho_{k}^{1-\theta} \leq C_{1} \sum_{k=0}^{n} a_{k}=C_{1} \sigma_{n}
$$

We now apply a discrete integration-by-parts on the left-hand side

$$
\sum_{k=0}^{n}[(k+2)-(k+1)] \rho_{k}^{1-\theta}=(n+2) \rho_{n}^{1-\theta}-\rho_{0}^{1-\theta}+\sum_{k=1}^{n}(k+1)\left(\rho_{k-1}^{1-\theta}-\rho_{k}^{1-\theta}\right)
$$

Next, we notice that

$$
\rho_{k-1}^{1-\theta}-\rho_{k}^{1-\theta}=\int_{\rho_{k}}^{\rho_{k-1}}(1-\theta) s^{-\theta} d s \geq(1-\theta) a_{k-1}^{2} \rho_{k-1}^{-\theta}
$$

since $\rho_{k-1}=\rho_{k}+a_{k-1}^{2}$. This gives

$$
(n+2) \rho_{n}^{1-\theta}-\rho_{0}^{1-\theta}+(1-\theta) \sum_{k=1}^{n}(k+1) a_{k-1}^{2} \rho_{k-1}^{-\theta} \leq C_{1} \sigma_{n}
$$

It follows that, for every $n \geq 0$,

$$
(n+1) \rho_{n}^{1-\theta} \leq C_{2}\left(1+\sigma_{n}\right)
$$

and

$$
\sum_{k=1}^{n}(k+1) a_{k-1}^{2} \rho_{k-1}^{-\theta} \leq C_{2}\left(1+\sigma_{n}\right)
$$

where $C_{2}>0$ is a constant independent of $n$. Since the sequence $\left(\sigma_{n}\right)$ is nondecreasing, the former estimate yields

$$
\rho_{k-1} \leq C_{3}\left(1+\sigma_{n}\right)^{1 /(1-\theta)} k^{-1 /(1-\theta)}, \quad 1 \leq k \leq n<\infty
$$

which we insert into the latter one, thus arriving at

$$
\sum_{k=1}^{n} k^{1+\theta /(1-\theta)} a_{k-1}^{2} \leq C_{4}\left(1+\sigma_{n}\right)^{1+\theta /(1-\theta)}
$$

The constants $C_{3}, C_{4}$ are independent of $n \geq 0$. As a consequence, using the CauchySchwarz inequality, we obtain for $n \geq 1$,

$$
\begin{aligned}
\sigma_{n-1}=\sum_{k=1}^{n} a_{k-1} & \leq\left(\sum_{k=1}^{n} k^{1 /(1-\theta)} a_{k-1}^{2}\right)^{1 / 2}\left(\sum_{k=1}^{n} k^{-1 /(1-\theta)}\right)^{1 / 2} \\
& \leq C_{5}\left(1+\sigma_{n}\right)^{1 / 2(1-\theta)}
\end{aligned}
$$

We conclude that the sequence $\left(\sigma_{n}\right)_{n}$ must be bounded, since $2(1-\theta)>1$. Indeed, assume by contradiction that $\left(\sigma_{n}\right)_{n}$ is unbounded, and set $r_{n}=1+\sigma_{n} \geq 1$. Then $\left(r_{n}\right)$ is nondecreasing, $r_{n} \rightarrow \infty$ and

$$
r_{n-1} \leq C_{6} r_{n}^{1 / 2(1-\theta)}
$$

so that $r_{n-1} / r_{n} \rightarrow 0$, and we deduce that $r_{n} \rightarrow 0$. Contradiction. Thus $\left(\sigma_{n}\right)$ is bounded, i.e. $\sum_{n=0}^{\infty} a_{n}<\infty$, as claimed.

Remark 5.6. In the proof of convergence to equilibrium, we have used the fact that all norms are equivalent in $V_{h}$. An interesting open question would be to prove a similar result for the time semi-discrete version of our problem.

Remark 5.7. By arguing as in the continuous case (see [25]), using the energy estimate (cf. Lemma 3.1) and the Łojasiewicz inequality 5.17 , it is possible to prove that any solution $\left(u_{h}, v_{h}\right)$ of the space semi-discrete scheme (3.1) converges to a single equilibrium, provided assumptions (H1) and (H4) hold.

## 6. NumERICAL RESULTS

We present some numerical results in one space dimension (obtained with the Scilab software $^{1}$ and in two space dimensions (obtained with the Freefem++ software [27]). In every case, the nonlinearity $f$ is given by (1.3) for some parameter $\varepsilon$, and we set $f_{+}=f, f_{-}=0$ in assumption (H3). The space $V_{h}$ is the space of piecewise linear $\left(P^{1}\right)$ finite elements.
6.1. Simulations in one space dimension. We first choose an interval $\Omega=(0, L)$ with $L=4 \pi$. In Table 1, we compute the error in the $C^{0}\left([0, T] ; L^{2}(\Omega)\right)$-norm (which appears in Theorem 4.8). The parameters are $\varepsilon=0.5, \beta=0.5$ and $T=2$. We use a uniform grid with space stepsize $h=L / M$ and time stepsize $\tau=T / N$. The initial value $\left(u_{h}^{0}, v_{h}^{0}\right)$ is the $P^{1}$-interpolate of $u_{0}(x)=\cos (x)+0.3 \cos (3 x), v_{0}(x)=0.1$.

Table 1. Convergence error for the time (left) and for the space (right) discretization

| $N=T / \tau$ | $\operatorname{err}_{h}(\tau)$ | ratio |
| :---: | :---: | :---: |
| 80 | 0.5018208 | - |
| 160 | 0.1455507 | 3.45 |
| 320 | 0.0368516 | 3.95 |
| 640 | 0.0091325 | 4.04 |
| 1280 | 0.0022163 | 4.12 |


| $M=L / h$ | $\operatorname{err}^{\tau}(h)$ | ratio |
| :---: | :---: | :---: |
| 40 | 0.7770682 | - |
| 80 | 0.2735932 | 2.84 |
| 160 | 0.0706798 | 3.87 |
| 320 | 0.0175677 | 4.02 |
| 640 | 0.0041882 | 4.20 |

For the error of the time discretization, $h=L / 160$ is fixed. Since the exact solution $u_{h}$ of the space semi-discrete scheme (3.1)-(3.2) is unknown, we use instead the solution on a fine grid with stepsize $\tau_{\text {sol }}=T / 5120$. Table 1 (left) shows the error

$$
\operatorname{err}_{h}(\tau)=\max _{0 \leq k \leq 5120}\left\|u_{h}^{\tau}\left(t_{k}\right)-u_{h}^{\tau_{s o l}}\left(t_{k}\right)\right\|_{L^{2}(0, L)}
$$

evaluated on the fine grid $t_{k}=k \tau_{\text {sol }}(k=0,1, \ldots, 5120)$, and the ratio

$$
\operatorname{err}_{h}(\tau) / \operatorname{err}_{h}(\tau / 2)
$$

[^0]of consecutive errors. The computed ratio is close to 4 , which means that the convergence error for the time discretization is $O\left(\tau^{2}\right)$, as expected.

For the error of the space discretization, $\tau=T / 160$ is fixed. We use again the solution $u_{h_{\text {sol }}}^{\tau}$ on a fine grid with stepsize $h=L / 2560$ for the comparison. Table (1) (right) shows the error

$$
\operatorname{err}^{\tau}(h)=\max _{0 \leq k \leq 160}\left\|u_{h}^{\tau}\left(t_{k}\right)-u_{h_{\text {sol }}}^{\tau}\left(t_{k}\right)\right\|_{L^{2}(0, L)}
$$

evaulated on the grid $t_{k}=k \tau(k=0,1, \ldots, 160)$ and the ratio $\operatorname{err}^{\tau}(h) / \operatorname{err}^{\tau}(h / 2)$ of consecutive errors. Again, the computed ratio is close to 4: the convergence error for the space discretization is $O\left(h^{2}\right)$, as expected.


Figure 1. (Pseudo-)Energy vs time
Figure 1 shows the plot of the pseudo-energy $\mathcal{E}_{h}\left(u_{h}^{n}, v_{h}^{n}\right)$ (see (3.22)) versus the time $t$ (in solid line). The domain is $\Omega=(0, L)$ with $L=4 \pi$. The parameters are $\varepsilon=0.5, \beta=5, h=L / 320$ and $d t=0.005$. The initial condition is the $P^{1}$ interpolate of $u_{0}(x)=0.1 /(1+0.7 \cos (x)), v_{0}=0.034$. The black color corresponds to the second-order scheme (4.2); the blue color corresponds to a first-order scheme obtained by applying to the space semi-discrete scheme (3.1) the time discretization proposed by Wang and Wise [37, 38]. Both schemes are unconditionally stable.

The left figure shows the pseudo-energy on the interval [ 0,40 ]. If we had $\left\langle v_{0}\right\rangle=0$, then by (4.16), the pseudo-energy would be nonincreasing in both cases. Here, the pseudo-energy exhibits oscillations due to the fact that $\left\langle v_{0}\right\rangle \neq 0$. In both cases, the evolution is driven to a stationary state, as predicted by the theory (see Theorem 5.1). We notice that the first-order scheme has a smoothing effect which creates more dissipation, especially at the beginning of the evolution. This is seen in the right figure which shows the energy $E_{h}\left(u_{h}^{n}\right)$ on the interval [ $0,2.5$ ] (in dashed and dashdot), in addition to the pseudo-energy $\mathcal{E}_{h}\left(u_{h}, v_{h}\right)$ (in solid line). The difference $\mathcal{E}_{h}\left(u_{h}^{n}, v_{h}^{n}\right)$ $E_{h}\left(u_{h}^{n}\right)=(\beta / 2)\left|\dot{v}_{h}^{n}\right|_{-1, h}^{2}$ can be interpreted as a "kinetic energy".
6.2. Simulations in two space dimensions. The domain $\Omega$ is the square $(0,6 \pi) \times$ $(0,6 \pi)$. It is decomposed in $50 \times 50$ squares, and each square is divided along the lower left/upper right diagonal, resulting in a uniform triangulation of $\omega$. The parameters are $\beta=0.1, \varepsilon=2$ and the time step is $\tau=0.25$. The initial condition is the $P^{1}$ interpolate of $u_{0}(x, y)=0.2+0.2 \cos (x) \cos (y)$ and $v_{0}=0$. Figures 2 and 3 show the
evolution from stripes to a triangular distribution of drops. Numerical tests up to time $t=1250$ indicate that the triangular distribution of drops is the steady state for this simulation.

For the continuous problem (1.1), using the translation invariance, from a triangular distribution of drops we easily build a two dimensional continuum of steady states. For the fully discrete scheme (4.2), the translation invariance is broken by the space discretization, but we expect a large number of steady states. This simulation illustrate the convergence to equilibrium result (Theorem 5.1) in a situation where the steady state is not unique.


Figure 2. $\quad t=5$ (left) and $t=26.25$ (right)


Figure 3. $\quad t=27.75$ (left) and $t=50$ (right)

## Acknowledgments

The first author thanks the Laboratoire de Mathématiques et Applications for the kind hospitality and the Université de Poitiers which supported his stay in May 2014. The first author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The second author thanks the Dipartimento di Matematica of the Politecnico di Milano for the invitation in February 2014.

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