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METASTABILITY FOR THE CONTACT PROCESS ON THE PREFERENTIAL ATTACHMENT GRAPH

VAN HAO CAN

ABSTRACT. We consider the contact process on the preferential attachment graph. The work of Berger, Borgs, Chayes and Saberi [BBCS1] confirmed physicists' predictions that the contact process starting from a typical vertex becomes epidemic for an arbitrarily small infection rate λ with positive probability. More precisely, they showed that with probability $\lambda^{\Theta(1)}$, it survives for a time exponential in the largest degree. Here we obtain sharp bounds for the density of infected sites at a time close to exponential in the number of vertices (up to some logarithmic factor). In addition, a metastable result for the extinction time is also proved.

1. INTRODUCTION

The paper aims at proving metastability results for the contact process on the preferential attachment random graph. We improve the [BBCS1]'s result in two aspects: obtaining a better bound on the extinction time, and estimating more accurately the density of the infected sites. Moreover, we also prove a metastable result for the extinction time.

The contact process is one of the most studied interacting particle systems, see in particular Liggett's book [L], and is also often interpreted as a model to describing how a virus spreads in a network. Mathematically, it can be defined as follows: given a locally finite graph $G = (V, E)$ and $\lambda > 0$, the contact process on G with infection rate λ is a Markov process $(\xi_t)_{t \geq 0}$ on $\{0, 1\}^V$. Vertices of V (also called sites) are regarded as individuals which are either infected (state 1) or healthy (state 0). By considering ξ_t as a subset of V via $\xi_t \equiv \{v : \xi_t(v) = 1\}$, the transition rates are given by

$$\begin{aligned} \xi_t &\rightarrow \xi_t \setminus \{v\} \text{ for } v \in \xi_t \text{ at rate } 1, \text{ and} \\ \xi_t &\rightarrow \xi_t \cup \{v\} \text{ for } v \notin \xi_t \text{ at rate } \lambda \deg_{\xi_t}(v), \end{aligned}$$

where $\deg_{\xi_t}(v)$ denotes the number of edges between v and other infected sites (Note that if G is a simple graph, i.e. contains no multiple edges, then $\deg_{\xi_t}(v)$ is just the number of infected neighbors of v at time t). Given that $A \subset V$, we denote by $(\xi_t^A)_{t \geq 0}$ the contact process with initial configuration A . If $A = \{v\}$ we simply write $(\xi_t^v)_{t \geq 0}$.

Originally the contact process was studied on integer lattices or homogeneous trees. More recently, probabilists started investigating this process on some families of random graphs like the Galton-Watson trees, small world networks, configuration models, random regular graphs, and preferential attachment graphs, see for instance [P, DJ, CD, CS, D, MVY, MV, BBCS1].

The preferential attachment graph (a definition will be given later) is well-known as a pattern of scale-free or social networks. Indeed, it not only shows the power-law degree sequence of a host in real world networks, but also reflects a wisdom that the rich get

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richer - the newbies are more likely to get acquainted with more famous people rather than a relatively unknown person. Therefore there has been great interest in this random graph as well as the processes occurring on it, including the contact process. In [BBCS1], by introducing a new representation of the graph, the authors proved a remarkable result which validated physicists predictions that the phase transition of the contact process occurs at $\lambda = 0$. More precisely, they showed that there are positive constants θ , c and C , such that for all λ small enough

$$(1) \quad \lambda^c \leq \mathbb{P} \left(\xi_{\exp(\theta\lambda^2\sqrt{n})}^u \neq \emptyset \right) \leq \lambda^C,$$

where (ξ_t^u) is the contact process starting from a uniformly chosen vertex.

In this paper we will improve (1) as follows.

Theorem 1.1. *Let (G_n) be the sequential model of the preferential attachment graph with parameters $m \geq 1$ and $\alpha \in [0, 1)$. Consider the contact process $(\xi_t)_{t \geq 0}$ with infection rate $\lambda > 0$ starting from full occupancy on G_n . Then there exist positive constants c and C , such that for λ small enough,*

$$(2) \quad \mathbb{P} \left(c\lambda^{1+\frac{2}{\psi}} |\log \lambda|^{-\frac{1}{\psi}} \leq \frac{|\xi_{t_n}|}{n} \leq C\lambda^{1+\frac{2}{\psi}} |\log \lambda|^{-\frac{1}{\psi}} \right) = 1 - o(1).$$

where $\psi = \frac{1-\alpha}{1+\alpha}$ and (t_n) is any sequence satisfying $t_n \rightarrow \infty$ and $t_n \leq T_n = \exp\left(\frac{c\lambda^2 n}{(\log n)^{1/\psi}}\right)$.

By a well-known property of the contact process called self-duality (see [L], Section I.1) for any $t \geq 0$ we have

$$(3) \quad \sum_{v \in V_n} 1(\{\xi_t^v \neq \emptyset\}) \stackrel{(\mathcal{L})}{=} |\xi_t|.$$

Therefore the survival probability as in (1) is just the expected value of the density of infected sites as in Theorem 1.1, so that our result is a stronger form of the one in [BBCS1]. Additionally we get a more precise estimate of the density and we allow (t_n) to be larger.

Since the empty configuration is the unique absorbing state of the contact process on finite graphs, the contact process on G_n always dies out. Theorem 1.1 shows that before dying, the contact process remains for a long time in a stationary situation in the sense that the density of infected sites is stable for a time stretched exponential in the number of vertices. This metastable behavior for the contact process has been observed in some examples, such as the finite boxes (see [L], Section I.3), the configuration models (see [CD, CS, MVY]), the random regular graphs (see [LS]).

We also prove the following result.

Proposition 1.2. *Let τ_n be the extinction time of the contact process on the sequential preferential attachment graph with infection rate $\lambda > 0$ starting from full occupancy. Then the following convergence in law holds*

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n \rightarrow \infty]{(\mathcal{L})} \mathcal{E}(1),$$

with $\mathcal{E}(1)$ an exponential random variable with mean one.

In the proof of Proposition 1.2, see in particular (69) and (71), we show that after time $\exp(\log^2 n)$, either the contact process dies out, or it equals the contact process starting from full occupancy (and thus the initial configuration is forgotten). Since $\exp(\log^2 n)$ is

much smaller than the extinction time, this shows the presence of the metastability of the contact process.

This kind of metastability result has been studied for different random processes, see for instance [BH, OV] for the history and recent developments of this problem. In particular, it has been proved for the contact process on finite boxes (see [CGOV, M]), and finite homogeneous trees (see [CMMV]). Our method for proving Proposition 1.2 is rather general and only requires some simple hypothesis on the maximal degree and the diameter of the graph, which is satisfied in most scale-free random graphs models, like the configuration model with power law distribution, or the preferential attachment graph. We refer to Proposition 5.2 for more details.

Now let us make some comments on the proof of Theorem 1.1.

First, to obtain the time T_n we will use the maintenance mechanism as in [CD] instead of the one in [BBCS1]. In the latter the authors used that in the preferential attachment graph the maximal degree is of order \sqrt{n} , plus the well-known fact that for any vertex v , the process survives a time exponential in the degree of v , once it is infected, yielding (1). In the former, on the other hand, when considering the contact process on the configuration model, Chatterjee and Durrett employed many vertices with total degree of order $n^{1-\varepsilon}$, for any $\varepsilon > 0$, and derived a much better bound on the extinction time. Here, our strategy is to find vertices with degree larger than $Cd(G_n)$, where $C = C(\lambda) > 0$ is a constant and $d(G_n)$ is the diameter of G_n , which is of order $\log n$. Thanks to Proposition 1 in [CD], we can deduce that the virus propagates along these vertices for a time exponential in their total degree. Moreover, the degree distribution of the graph, denoted by \mathbf{p} , has a power-law with exponent $\nu = 2 + 1/\psi$. Thus the number of these vertices is of order $n(\log n)^{1-\nu}$ and their total degree is of order $n(\log n)^{-1/\psi}$, which explains the bound on t_n in Theorem 1.1.

It is worth noting that for any graph with order n edges (including G_n), the extinction time of the contact process is w.h.p. smaller than $\exp(Cn)$, for some $C > 0$, see for instance Lemma 3.4 in [C]. Hence our bound on t_n is nearly optimal.

Secondly, to gain the precise estimate on the density, we use ideas from [P, BBCS1, CD, MVY]: if the virus starting at a typical vertex wants to survive a long time, it has to infect a big vertex of degree significantly larger than λ^{-2} . Then the virus is likely to survive in the neighborhood of this vertex for a time which is long enough to infect another big vertex, and so on. We can see that the time required for a virus to spread from one big vertex to another is at least $\lambda^{-\Theta(1)}$ (corresponding to the case when the distance between them is constant). Besides, it was shown that if $\deg(v) \geq K/\lambda^2$, then the survival time of the contact process on the star graph formed by v and its neighbors is about $\exp(cK)$. Hence the degree of big vertices should be larger than $C\lambda^{-2}|\log \lambda|$. Then we consider Λ , the set of vertices which have a big neighbor. The probability for a vertex in Λ to infect its big neighbor is of order λ . Moreover, we will show in Section 4 that any big vertex has a positive probability to make the virus survive up to time T_n . This means that the probability for the dual process starting from any vertex in Λ to be active at time T_n is of order λ . Therefore the density of vertices from where the dual process survives up to T_n is about λ times the density of Λ . This is of order $\lambda \times \mathbf{p}([\lambda^{-2}|\log \lambda|, \infty))\lambda^{-2}|\log \lambda| \asymp \lambda^{1+\frac{2}{\psi}}|\log \lambda|^{-\frac{1}{\psi}}$ yielding the desired lower bound.

We notice that the density of big vertices is of order $\mathbf{p}([\lambda^{-2}|\log \lambda|, \infty))$, which is $o(\lambda^{1+\frac{2}{\psi}}|\log \lambda|^{-\frac{1}{\psi}})$. Thus it is not optimal. Hence we need to consider also their neighbors. In fact this idea of using Λ was first introduced in [CD] for the configuration model.

For the upper bound, we adapt the proof in [MVY] for Galton-Watson trees, see the appendix.

It is interesting to note also that if we consider the contact process on the configuration model with the same power-law degree distribution, the density is of order $\lambda^{1+\frac{2}{\psi}}|\log \lambda|^{-\frac{2}{\psi}}$ (see [MVY, Theorem 1.1]), which is slightly smaller than the one in (2). This difference is due to the fact that the distance between big vertices in the configuration model is about $|\log \lambda|$, instead of constant here.

Finally, the above strategy works properly when studying the contact process on the Pólya-point graph. In fact, proving the following result helped us pave the way to potential solutions for Theorem 1.1.

Proposition 1.3. *Let (ξ_t^o) be the contact process on the Pólya-point graph with infection rate $\lambda > 0$ starting from the root o . There exist positive constants c and C , such that for λ small enough,*

$$c\lambda^{1+2/\psi}|\log \lambda|^{-1/\psi} \leq \mathbb{P}(\xi_t^o \neq \emptyset \forall t) \leq C\lambda^{1+2/\psi}|\log \lambda|^{-1/\psi}.$$

Theorem 1.1 (resp. Proposition 1.3) implies that for all $\lambda > 0$, the contact process becomes epidemic (resp. survives forever) with positive probability. We say that the critical values of the contact process on the preferential attachment graph and its weak limit are all zero. This is a new example of a more general phenomena that there is a relationship between the phase transition for the contact process on a sequence of finite graphs and the one on its weak local limit in the sense of Benjamini–Schramm’s convergence. Here are some known results on this topic: the contact process on the integer lattice \mathbb{Z}^d and on finite boxes $[[1, n]^d$ exhibit a phase transition at the same critical value $\lambda_c = \lambda_c(d)$, see [L, Part I] for all $d \geq 1$. The phase transition of the process on the random regular graph of degree d and its limit, the homogeneous tree \mathbb{T}_d , occurs at the same constant $\lambda_1(\mathbb{T}_d)$, see [MV]. The phase transition of the contact process on \mathbb{T}_d^ℓ (the d -homogeneous tree of height ℓ) and its limit, the canopy tree \mathbb{CT}_d , happens at $\lambda_2(\mathbb{T}_d)$, see [CMMV, MV]. The critical value of the contact process on the configuration model with heavy tail degree distributions or on its limit, the Galton-Watson tree, is zero, see [P, CD, MVY, MMVY, CS].

Now the paper is organized as follows. In the next section, based on [BBCS2], we give the definition of the sequential model of the preferential attachment graph as well as its weak local limit, the Pólya-point graph. We also prove preliminary results on the graph structure and fix some notation. In Section 3, we prove Proposition 1.3. The main theorem 1.1 and Proposition 1.2 are proved in Sections 4 and 5 respectively.

2. PRELIMINARIES

2.1. Construction of the random graph and notation. Let us give a definition following [BBCS2] of the sequential model of the preferential attachment graph with parameters $m \geq 1$ and $\alpha \in [0, 1)$. We construct a sequence of graphs (G_n) with vertex set $V_n = \{v_1, \dots, v_n\}$ as follows.

First G_1 contains one vertex v_1 and no edge, and G_2 contains 2 vertices v_1, v_2 and m edges connecting them. Given G_{n-1} , we define G_n the following way. Add the vertex

v_n to the graph, and draw edges between v_n and m vertices $w_{n,1}, \dots, w_{n,m}$ (possibly with repetitions) from G_{n-1} as follows: with probability $\alpha_n^{(i)}$, the vertex $w_{n,i}$ is chosen uniformly at random from V_{n-1} where

$$\alpha_n^{(i)} = \begin{cases} \alpha & \text{if } i = 1, \\ \alpha \frac{2m(n-1)}{2m(n-2) + 2m\alpha + (1-\alpha)(i-1)} = \alpha + \mathcal{O}(n^{-1}) & \text{if } i \geq 2. \end{cases}$$

Otherwise, $w_{n,i} = v_k$ with probability

$$\frac{\deg_{n-1}^{(i)}(v_k)}{Z_{n-1}^{(i)}},$$

where

$$\deg_{n-1}^{(i)}(v_k) = \deg_{n-1}(v_k) + \#\{1 \leq j \leq i-1 : w_{n,j} = v_k\},$$

is the degree of v_k before choosing $w_{n,i}$, and

$$Z_{n-1}^{(i)} = \sum_{k=1}^{n-1} \deg_{n-1}^{(i)}(v_k) = \sum_{k=1}^{n-1} \deg_{n-1}(v_k) + i - 1 = 2m(n-2) + i - 1,$$

with $\deg_{n-1}(v_k)$ the degree of v_k in G_{n-1} .

This construction might seem less natural than in the independent model where with probability α we choose $w_{n,i}$ uniformly from V_{n-1} and with probability $1 - \alpha$ it is chosen according to a simpler rule: $w_{n,i} = v_k$ with probability $\deg_{n-1}(v_k)/2m(n-2)$. However the sequential model constructed above is easier to analyze because it is exchangeable, and as a consequence it admits an alternative representation which contains more independence. In [BBCS2], the authors called it the Pólya urn representation which we now recall in the following theorem. To this end, we denote by $\beta(a, b)$ the Beta distribution, whose density is proportional to $x^{a-1}(1-x)^{b-1}$ on $[0, 1]$, and by $\Gamma(a, b)$ the Gamma distribution, whose density is proportional to $x^{a-1}e^{-bx}$ on $[0, \infty)$. For any $a < b$, $\mathcal{U}([a, b])$ stands for the uniform distribution on $[a, b]$.

Theorem 2.1. [BBCS2, Theorem 2.1] *Fix $m \geq 1$, $\alpha \in [0, 1)$ and $n \geq 1$. Set $r = \alpha/(1 - \alpha)$, $\psi_1 = 1$, and let ψ_2, \dots, ψ_n be independent random variables with law*

$$\psi_j \sim \beta(m + 2mr, (2j - 3)m + 2mr(j - 1)).$$

Define

$$\varphi_j = \psi_j \prod_{t=j+1}^n (1 - \psi_t), \quad S_k = \sum_{j=1}^k \varphi_j, \quad \text{and} \quad I_k = [S_{k-1}, S_k].$$

Conditionally on ψ_1, \dots, ψ_n , let $\{U_{k,i}\}_{k=1, \dots, n, i=1, \dots, m}$ be a sequence of independent random variables, with $U_{k,i} \sim \mathcal{U}([0, S_{k-1}])$. Start with the vertex set $V_n = \{v_1, \dots, v_n\}$. For $j < k$, join v_j and v_k by as many edges as the number of indices $i \in \{1, \dots, m\}$, such that $U_{k,i} \in I_j$. Denote the resulting random graph by G_n .

Then G_n has the same distribution as the sequential model of the preferential attachment graph.

We remark that in [BBCS2], the authors state the theorem for $m \geq 2$. However, their proof also works properly when $m = 1$.

From now on, we always consider the random multi-graph G_n constructed as in this theorem.

We now look at the local structure of G_n . It was shown in [BBCS2] that G_n is locally tree-like, with some subtle degree distribution that we now recall. First we define some constants:

$$\chi = \frac{1+2r}{2+2r} \quad \text{and} \quad \psi = \frac{1-\chi}{\chi} = \frac{1}{1+2r}.$$

Note that $1/2 \leq \chi < 1$ and $0 < \psi \leq 1$. Let $F \sim \Gamma(m+2mr, 1)$ and $F' \sim \Gamma(m+2mr+1, 1)$.

We will construct inductively a random rooted tree (T, o) with vertices identified with elements of $\cup_{\ell \geq 1} \mathbb{N}^\ell$ (where vertices at generation ℓ are elements of \mathbb{N}^ℓ) and a map which associates to each vertex v a position x_v in $[0, 1]$. Additionally each vertex (except the root) will be assigned a type, either R or L.

- The root $o = (0)$ has position $x_o = U_0^\chi$, where $U_0 \sim \mathcal{U}([0, 1])$.
- Given $v \in T$ and its position x_v , define

$$m_v = \begin{cases} m & \text{if } v \text{ is the root or of type L,} \\ m-1 & \text{if } v \text{ is of type R.} \end{cases}$$

and

$$\gamma_v \sim \begin{cases} F & \text{if } v \text{ is the root or of type R,} \\ F' & \text{if } v \text{ is of type L.} \end{cases}$$

The children of v are $(v, 1), \dots, (v, m_v), (v, m_v+1), \dots, (v, m_v+q_v)$, the first m_v 's are of type L and the remaining ones are of type R. Conditionally on $x_v, x_{(v,1)}, \dots, x_{(v,m_v)}$ are i.i.d. uniform random variable in $[0, x_v]$, and $x_{(v,m_v+1)}, \dots, x_{(v,m_v+q_v)}$ are the points of the Poisson point process on $[x_v, 1]$ with intensity

$$\rho_v(x)dx = \gamma_v \frac{\psi x^{\psi-1}}{x_v^\psi} dx.$$

This procedure defines inductively an infinite rooted tree (T, o) , which is called the Pólya-point graph and $(x_v)_{v \in T}$ is called the Pólya-point process.

For any vertex v in a graph G and any integer R , we call $B_G(v, R)$ the ball of radius R around v in G , which contains all vertices at distance smaller than or equal to R from v and all edges connecting them.

Theorem 2.2. [BBCS2, Theorem 2.2] *Assume that the random graph G_n is constructed as in Theorem 2.1. Let u be a vertex chosen uniformly at random in G_n and let R be some fixed constant. Then $B_{G_n}(u, R)$ converges weakly to the ball $B_T(o, R)$ in the Pólya-point graph.*

Note that in [BBCS2] the authors prove this theorem for $m \geq 2$. For $m = 1$, recently, in [BMR] the authors show that the local limit of preferential attachment graph is always the Pólya-point graph regardless the initial (seed) graph (in our case, the initial graph contains two vertices v_1, v_2 and one edge connecting them).

Now we introduce some notation. We call \mathbb{P} a probability measure on a space in which the random graph G_n is defined together with the contact process. Since we will fix λ , we omit it in the notation. We also call \mathbb{P} a probability measure on a space in which the Pólya-point graph as well as the contact process are defined.

We denote the indicator function of a set A by $\mathbf{1}(A)$. For any vertices v and w we write $v \sim w$ if there is an edge between them (in which case we say that they are neighbors or connected), and $v \not\sim w$ otherwise. We denote by $|G|$ the order of G .

A graph in which all vertices have degree one, except one which is connected to all the others is called a **star graph**. The only vertex with degree larger than one is called the center of the star graph, or central vertex.

If f and g are two real functions, we write $f = \mathcal{O}(g)$ if there exists a constant $C > 0$, such that $f(x) \leq Cg(x)$ for all x ; $f \gtrsim g$ (or equivalently $g \lesssim f$) if $g = \mathcal{O}(f)$; $f \asymp g$ if $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$; $f = o(g)$ if $f(x)/g(x) \rightarrow 0$ as $x \rightarrow \infty$. Finally for a sequence of r.v.s (X_n) and a function $f : \mathbb{N} \rightarrow (0, \infty)$, we say that $X_n \asymp f(n)$ holds w.h.p. if there exist positive constants c and C , such that $\mathbb{P}(cf(n) \leq X_n \leq Cf(n)) \rightarrow 1$.

2.2. Preliminary results on the random graph. We first recall a version of the Azuma-Hoeffding inequality for martingales which we will use throughout this paper (see for instance [CL]).

Lemma 2.3. *Let $(X_i)_{i \geq 0}$ be a martingale satisfying $|X_i - X_{i-1}| \leq 1$ for all $i \geq 1$. Then for any n and $t > 0$, we have*

$$\mathbb{P}(|X_n - X_0| \geq t) \leq 2 \exp(-t^2/2n).$$

From this inequality we can deduce a large deviations result. Let $(X_i)_{i \geq 1}$ be a sequence of independent Bernoulli random variables. Assume that $0 < 2p \leq \mathbb{E}(X_i) \leq Mp$ for all i . Then there exists $c = c(M) > 0$, such that for all n

$$(4) \quad \mathbb{P}\left(np \leq \sum_{i=1}^n X_i \leq 2Mnp\right) \geq 1 - \exp(-cnp).$$

Now we present some estimates on the sequences (φ_i) , (ψ_j) and (S_k) .

Lemma 2.4. *Let $(\varphi_i)_i$, $(\psi_j)_j$ and $(S_k)_k$ be sequences of random variables as in Theorem 2.1. Then there exist positive constants μ and θ_0 , such that for all $\theta \leq \theta_0$, the following assertions hold.*

$$(i) \quad \mathbb{E}(\psi_j) = \frac{\lambda}{j} + \mathcal{O}\left(\frac{1}{j^2}\right), \quad \mathbb{E}(\psi_j^2) \asymp \frac{1}{j^2}.$$

(ii) *For any $\varepsilon > 0$, there exists $K = K(\varepsilon) < \infty$, such that*

$$\mathbb{P}(\mathcal{E}_\varepsilon) \geq 1 - \varepsilon,$$

where

$$\mathcal{E}_\varepsilon = \left\{ \left| \frac{S_j}{S_k} - \left(\frac{j}{k}\right)^x \right| \leq \varepsilon (j/k)^x \quad \forall K(\varepsilon) \leq j \leq k \leq n \right\}.$$

(iii) *As n tends to infinity,*

$$\mathbb{P}(i\psi_i \leq 2 \log n \quad \forall i = 1, \dots, n) = 1 - o(1).$$

(iv) $\mathbb{P}(\mu/j \geq \psi_j \geq \theta/j) \geq 2\theta$.

(v) $\mathbb{E}(\varphi_j 1(\psi_j \geq \theta/j)) \geq \theta \mathbb{E}(\varphi_j)$.

Proof. Let us start with Part (i). Observe that if $\psi \sim \beta(a, b)$, then

$$\mathbb{E}(\psi) = \frac{a}{a+b} \quad \text{and} \quad \mathbb{E}(\psi^2) = \frac{a(a+1)}{(a+b)(a+b+1)}.$$

Hence the result follows from the fact that $\psi_j \sim \beta(m + 2mr, (2j - 3)m + 2mr(j - 1))$.

Part (ii) is a direct consequence of Lemma 3.1 in [BBCS2]. We now prove (iii). Since $\psi_i \sim \beta(a, b_i)$ with $a = m + 2mr$ and $b_i = (2m + 2mr)i - (3m + 2mr)$, we have for $i \geq 2 \log n$

$$\begin{aligned} \mathbb{P}\left(\psi_i > \frac{2 \log n}{i}\right) &= \frac{1}{B(a, b_i)} \int_{2 \log n/i}^1 x^{a-1} (1-x)^{b_i-1} dx \\ &\lesssim b_i^a \left(1 - \frac{2 \log n}{i}\right)^{b_i} \\ &\lesssim n^{-2}. \end{aligned}$$

Here, we have used that $B(a, b) \asymp \mathcal{O}(b^{-a})$ when a is fixed. On the other hand, when $i < 2 \log n$, this probability is zero. Therefore

$$\mathbb{P}(i\psi_i \leq 2 \log n \ \forall i = 1, \dots, n) = 1 - o(1).$$

For Part (iv), Chebyshev's inequality gives that for any $\delta \in (0, 1)$

$$\mathbb{P}(|\psi_j - \mathbb{E}(\psi_j)| > (1 - \delta)\mathbb{E}(\psi_j)) \leq \frac{\text{Var}(\psi_j)}{(1 - \delta)^2 \mathbb{E}(\psi_j)^2}.$$

Moreover, if $\psi \sim \beta(a, b)$, then

$$\frac{\text{Var}(\psi)}{\mathbb{E}(\psi)^2} = \frac{\mathbb{E}(\psi^2)}{\mathbb{E}(\psi)^2} - 1 = \frac{(a+1)(a+b)}{a(a+b+1)} - 1 = \frac{b}{a(a+b+1)} \leq \frac{1}{a}.$$

Therefore for any j

$$\mathbb{P}(\psi_j \in (\delta \mathbb{E}(\psi_j), (2 - \delta)\mathbb{E}(\psi_j))) \geq 1 - \frac{1}{(1 - \delta)^2 (m + 2mr)}.$$

Hence thanks to (i) we can choose positive constants μ and θ , such that for all j

$$(5) \quad \mathbb{P}\left(\psi_j \in \left(\frac{\theta}{j}, \frac{\mu}{j}\right)\right) \geq 2\theta.$$

For (v), we notice that by (i), $\mathbb{E}(\psi_j) \asymp 1/j$. Hence

$$(6) \quad \mathbb{E}(\psi_j 1(\psi_j \geq \theta/j)) \geq \mathbb{E}(\psi_j) - (\theta/j) \geq c\mathbb{E}(\psi_j),$$

for some constants c and θ independent of j . Moreover, using the fact that these random variables $(\psi_j)_{j \geq 0}$ are independent, we obtain

$$\begin{aligned} \mathbb{E}(\varphi_j 1(\psi_j \geq \theta/j)) &= \mathbb{E}\left(\psi_j 1(\psi_j \geq \theta/j) \prod_{t=j+1}^n (1 - \psi_t)\right) \\ &= \mathbb{E}(\psi_j 1(\psi_j \geq \theta/j)) \mathbb{E}\left(\prod_{t=j+1}^n (1 - \psi_t)\right) \\ &\geq c\mathbb{E}(\psi_j) \mathbb{E}\left(\prod_{t=j+1}^n (1 - \psi_t)\right) \\ &= c\mathbb{E}(\varphi_j). \end{aligned}$$

Here for the third line, we have used (6). □

The preferential attachment graph is known as a prototype of small world networks whose diameter and typical distance (the distance between two randomly chosen vertices) are of logarithmic order. In fact, these quantities in the independent model were well-studied, see for instance [DVH] or [H]. In the following two lemmas, we prove similar estimates for the sequential model. These estimates are in fact weaker but sufficient for our purpose.

Lemma 2.5. *Let $d(G_n)$ be the diameter of the random graph G_n , i.e. the maximal distance between any pair of vertices in G_n . Then there exists a positive constant b_1 , such that*

$$\mathbb{P}(d(G_n) \leq b_1 \log n) = 1 - o(1).$$

Proof. Let $\varepsilon \in (0, 1/2)$ be given, and recall the definitions of $K(\varepsilon)$ and \mathcal{E}_ε given in Lemma 2.4 (ii). We first bound $d(v_1, v_n)$. Define a decreasing random sequence $(n_i)_{i \geq 0}$ as follows $n_0 = n$ and for $i \geq 1$, n_{i+1} is arbitrarily chosen from indices such that $v_{n_{i+1}}$ receives an edge emanating from v_{n_i} . Define

$$X_i = 1(\{n_i \leq n_{i-1}/2\}) \text{ and } \mathcal{F}_i = \sigma(n_j : j \leq i) \vee \sigma((\varphi_t)).$$

By the construction of the graph in Theorem 2.1, we have

$$\mathbb{E}(X_{i+1} | \mathcal{F}_i) = \mathbb{P}(n_{i+1} \leq n_i/2 | \mathcal{F}_i) = \frac{S_{\lfloor n_i/2 \rfloor}}{S_{n_i-1}},$$

with (S_i) as in Theorem 2.1. We now define

$$\sigma_n = \inf\{i : n_{i+1} \leq \log n\}.$$

If $i \leq \sigma_n$, then $n_i > \log n > 2K(\varepsilon)$ for n large enough. Therefore by Lemma 2.4 (ii), we have on \mathcal{E}_ε for $i \leq \sigma_n$

$$\frac{S_{\lfloor n_i/2 \rfloor}}{S_{n_i-1}} \geq (1 - \varepsilon) \left(\frac{\lfloor n_i/2 \rfloor}{n_i - 1} \right)^x \geq \frac{1}{2^{x+1}} =: p.$$

In other words, we have

$$(7) \quad \mathbb{E}(X_{i+1} | \mathcal{F}_i, \mathcal{E}_\varepsilon) 1(i \leq \sigma_n) \geq p 1(i \leq \sigma_n).$$

Let

$$Y_k = \sum_{i=1}^k (X_i - \mathbb{E}(X_i | \mathcal{F}_{i-1}, \mathcal{E}_\varepsilon)).$$

Then (Y_k) is a martingale with respect to the filtration (\mathcal{F}_k) and $|Y_k - Y_{k-1}| \leq 1$. By using Lemma 2.3 we get that

$$(8) \quad \mathbb{P}(Y_k \leq -kp/2 | (\varphi_t)) \leq 2 \exp(-kp^2/8).$$

Note that there is a slight abuse of notation: the left-hand side of (8) should be a conditional expectation of an indicator variable, but for simplicity we just write it as a conditional probability. Now, it follows from (7) and (8) that

$$(9) \quad \mathbb{P}\left(\sum_{i=1}^k X_i \leq kp/2 | (\varphi_t), \mathcal{E}_\varepsilon, \sigma_n \geq k\right) \leq 2 \exp(-kp^2/8).$$

Moreover, if $\sum_1^k X_i > (\log_2 n - \log_2 \log n)$, then $n_k \leq \log n$ or equivalently $\sigma_n \leq k$. Hence for all $C = C(p)$ large enough

$$\begin{aligned} & \mathbb{P} \left(\sigma_n \geq C \log n, \sum_{i=1}^{C \log n} X_i > (C \log n)p/2 \mid (\varphi_t), \mathcal{E}_\varepsilon \right) \\ & \leq \mathbb{P} \left(\sigma_n \geq C \log n, \sum_{i=1}^{C \log n} X_i > \log_2 n - \log_2 \log n \mid (\varphi_t), \mathcal{E}_\varepsilon \right) = 0 \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathbb{P}(\sigma_n \geq C \log n \mid (\varphi_t), \mathcal{E}_\varepsilon) &= \mathbb{P}_n \left(\sigma_n \geq C \log n, \sum_{i=1}^{C \log n} X_i \leq (C \log n)p/2 \mid (\varphi_t), \mathcal{E}_\varepsilon \right) \\ (10) \quad &\leq 2 \exp(-(C \log n)p^2/8) = \mathcal{O}(n^{-2}). \end{aligned}$$

Here for the last line, we have used (9) for $k = C \log n$, with some C large enough. On the other hand,

$$\{\sigma_n \leq C \log n\} \subset \{\exists k \leq \log n : d(v_n, v_k) \leq C \log n\},$$

and $d(v_1, v_k) \leq \log n$ for all $k \leq \log n$. Therefore,

$$\{\sigma_n \leq C \log n\} \subset \{d(v_n, v_1) \leq (C+1) \log n\}.$$

Hence by (10), we obtain that

$$(11) \quad \mathbb{P}(d(v_1, v_n) \geq (C+1) \log n \mid (\varphi_t), \mathcal{E}_\varepsilon) = \mathcal{O}(n^{-2}).$$

Let $d_{G_k}(v_i, v_j)$ be the distance between v_i and v_j in G_k for $i, j \leq k \leq n$. Note that

$$d_{G_k}(v_i, v_j) \geq d(v_i, v_j) = d_{G_n}(v_i, v_j).$$

Similarly to (11), we deduce that on \mathcal{E}_ε , for all $i \geq C \log n$,

$$\mathbb{P}(d(v_1, v_i) \geq (C+1) \log i \mid (\varphi_t)) \leq \mathbb{P}(d_{G_i}(v_1, v_i) \geq (C+1) \log i \mid (\varphi_t)) = \mathcal{O}(i^{-2}).$$

Hence on \mathcal{E}_ε

$$\mathbb{P}(d(v_1, v_i) \leq (C+1) \log n \mid \forall i \geq C \log n \mid (\varphi_t)) = 1 - o(1).$$

Therefore by taking expectation with respect to (φ_t) and using Lemma 2.4 (ii), we get

$$\mathbb{P}(d(G_n) \leq 2(C+1) \log n) \geq 1 - 2\varepsilon,$$

which proves the result by letting ε tend to 0. \square

Before proving the lower bound on the typical distance, we make a remark which will be used frequently in this paper. It follows from the definition of G_n that for all $i < j$,

$$\mathbb{P}(v_i \not\sim v_j \mid (\varphi_t)) = \left(1 - \frac{\varphi_i}{S_{j-1}}\right)^m.$$

Hence

$$\frac{\varphi_i}{S_{j-1}} \leq \mathbb{P}(v_i \sim v_j \mid (\varphi_t)) \leq \frac{m\varphi_i}{S_{j-1}}.$$

Then by using the following identities

$$S_{j-1} = \sum_{t=1}^{j-1} \varphi_t = \prod_{t=j}^n (1 - \psi_t) \quad \text{and} \quad \varphi_i = \psi_i \prod_{t=i+1}^n (1 - \psi_t),$$

we obtain that

$$(12) \quad \frac{\psi_i S_i}{S_{j-1}} \leq \mathbb{P}(v_i \sim v_j \mid (\varphi_t)) \leq m \frac{\psi_i S_i}{S_{j-1}}.$$

Lemma 2.6. *Let w_1 and w_2 be two uniformly chosen vertices from V_n . Then there is a positive constant b_2 , such that w.h.p.*

$$d(w_1, w_2) \geq \frac{b_2 \log n}{\log \log n}.$$

Proof. Fix $\varepsilon \in (0, 1/2)$ a positive constant. Then we define

$$\mathcal{I}_\varepsilon = \mathcal{E}_\varepsilon \cap \{i\psi_i \leq 2 \log n \ \forall i = 1, \dots, n\},$$

with \mathcal{E}_ε as in Lemma 2.4. It follows from 2.4 (ii) and (iii) that for all n large enough

$$(13) \quad \mathbb{P}(\mathcal{I}_\varepsilon) \geq 1 - 2\varepsilon.$$

We now use an argument from [H, Lemma 7.16] to bound the typical distance. We call a sequence of distinct vertices $\pi = (\pi_1, \dots, \pi_k)$ a self-avoiding path. We write $\pi \subset G_n$ if π_i and π_{i+1} are neighbors for all $1 \leq i \leq k-1$. Let $\Pi(i, j, k)$ be the set of all self-avoiding paths of length k starting from v_i and ending at v_j . We then claim that for all $i, j, k \geq 1$,

$$(i) \quad \mathbb{P}(d(v_i, v_j) = k \mid (\varphi_t), \mathcal{I}_\varepsilon) \leq g_k(i, j) := \sum_{\pi \in \Pi(i, j, k)} \mathbb{P}(\pi \subset G_n \mid (\varphi_t), \mathcal{I}_\varepsilon),$$

$$(ii) \quad g_{k+1}(i, j) \leq \sum_{s \neq i, j} g_1(i, s) g_k(s, j).$$

The first claim is clear, because if $d(v_i, v_j) = k$ then there exists a self-avoiding path in $\Pi(i, j, k)$ which is in G_n . For the second one, we note that for any self-avoiding path $\pi = (\pi_1, \dots, \pi_k)$,

$$\mathbb{P}(\pi \subset G_n \mid (\varphi_t), \mathcal{I}_\varepsilon) = \mathbb{P}(\pi_1 \sim \pi_2 \mid (\varphi_t), \mathcal{I}_\varepsilon) \times \mathbb{P}(\bar{\pi} \subset G_n \mid (\varphi_t), \mathcal{I}_\varepsilon),$$

where $\bar{\pi} = (\pi_2, \dots, \pi_k)$. Indeed, if $j < k$, then the event that $v_j \sim v_k$ depends only on the $(U_{k,i})_{i \leq m}$. Hence this result follows from the facts that the vertices in π are distinct and that the $\{(U_{k,i})_{i \leq m}\}_k$ are independent. We are now in position to prove (ii):

$$\begin{aligned} g_{k+1}(i, j) &= \sum_{s \neq i, j} \sum_{\substack{v_i \notin \bar{\pi} \\ \bar{\pi} \in \Pi(s, j, k)}} \mathbb{P}(v_i \sim v_s, \bar{\pi} \subset G_n \mid (\varphi_t), \mathcal{I}_\varepsilon) \\ &\leq \sum_{s \neq i, j} \sum_{\bar{\pi} \in \Pi(s, j, k)} \mathbb{P}(v_i \sim v_s \mid (\varphi_t), \mathcal{I}_\varepsilon) \times \mathbb{P}(\bar{\pi} \subset G_n \mid (\varphi_t), \mathcal{I}_\varepsilon) \\ &= \sum_{s \neq i, j} g_1(i, s) g_k(s, j). \end{aligned}$$

We prove by induction on k that there is a positive constant C , such that

$$(14) \quad g_k(i, j) \leq \frac{(C \log n)^{2k-1}}{\sqrt{ij}}.$$

For $k = 1$, it follows from (12) that for all $i < j$,

$$(15) \quad \begin{aligned} g_1(i, j) &= \mathbb{P}(v_i \sim v_j \mid (\varphi_t), \mathcal{I}_\varepsilon) \\ &\leq m \mathbb{E} \left(\frac{\psi_i S_i}{S_{j-1}} \mid (\varphi_t), \mathcal{I}_\varepsilon \right). \end{aligned}$$

We now claim that on \mathcal{I}_ε

$$(16) \quad \frac{\psi_i S_i}{S_{j-1}} \leq \frac{4 \log n}{\sqrt{ij}}.$$

Indeed, we recall that on \mathcal{I}_ε ,

$$\psi_i \leq \frac{2 \log n}{i} \quad \forall i = 1, \dots, n \quad \text{and} \quad \left| \frac{S_i}{S_j} - \left(\frac{i}{j} \right)^x \right| \leq \varepsilon \left(\frac{i}{j} \right)^x \quad \forall j \geq i \geq K(\varepsilon),$$

with $K(\varepsilon)$ as in Lemma 2.4 (ii).

If $i < j \leq K(\varepsilon)$ then (16) holds for all n large enough (the left hand-side is bounded by 1 and the right-hand side tends to infinity). If $j > i \geq K(\varepsilon)$ then

$$(17) \quad \psi_i \frac{S_i}{S_{j-1}} \leq \frac{2 \log n}{i} \times (1 + \varepsilon) \left(\frac{i}{j-1} \right)^x \leq \frac{4 \log n}{\sqrt{ij}},$$

since $i < j$ and $\chi \geq 1/2 > \varepsilon$. If $i \leq K(\varepsilon) < j$ then using $\psi_i \leq 1$ and the fact that the sequence (S_i) is increasing, we obtain

$$\psi_i \frac{S_i}{S_{j-1}} \leq \frac{S_{K(\varepsilon)}}{S_{j-1}} \leq (1 + \varepsilon) \left(\frac{K(\varepsilon)}{j-1} \right)^x \leq \frac{4 \log n}{\sqrt{ij}},$$

for all n large enough. In any case, (16) holds. It follows from (15) and (16) that (14) holds for $k = 1$.

Assume now that (14) is true for some k , and let us prove it for $k + 1$. By using the induction hypothesis and (ii), we get that

$$\begin{aligned} g_{k+1}(i, j) &\leq \sum_{s \neq i, j} g_1(i, s) g_k(s, j) \\ &\leq \sum_{s \neq i, j} \frac{C \log n}{\sqrt{is}} \frac{(C \log n)^{2k-1}}{\sqrt{sj}} \\ &\leq \frac{(C \log n)^{2k+1}}{\sqrt{ij}}, \end{aligned}$$

which proves the induction step. Now it follows from (i) and (14) that

$$\begin{aligned} \mathbb{P}(d(w_1, w_2) \leq K \mid (\varphi_t), \mathcal{I}_\varepsilon) &\leq \frac{1}{n^2} \sum_{k=1}^K \sum_{1 \leq i, j \leq n} g_k(i, j) \\ &\leq \frac{1}{n^2} \sum_{k=1}^K \sum_{1 \leq i, j \leq n} \frac{(C \log n)^{2k-1}}{\sqrt{ij}} \\ &\leq \frac{(C \log n)^{2K}}{n}. \end{aligned}$$

Therefore if $K = \log n / (3C \log \log n)$, then $\mathbb{P}(d(w_1, w_2) \mid (\varphi_t), \mathcal{I}_\varepsilon) \leq K) = o(1)$. Combining this with (13) gives the desired lower bound with $b_2 = 1/(3C)$. \square

Remark 2.7. The bound in Lemma 2.6 is probably not sharp. Indeed for the independent model, it is proved in [DVH] or [H] that if $\alpha > 0$ (or equivalent $\chi > 1/2$), then w.h.p. $d(w_1, w_2) \geq c \log n$; and otherwise, w.h.p. $d(w_1, w_2) \geq \log n / (C + \log \log n)$, for some positive constants c and C .

2.3. Graphical construction of the contact process. We briefly recall here the graphical construction of the contact process (see more in Liggett's book [L]).

Fix $\lambda > 0$ and an oriented graph G (recall that a non-oriented graph can also be seen as oriented by associating to each edge two oriented edges). Then assign independent Poisson point processes D_v of rate 1 to each vertex $v \in V$ and D_e of rate λ to each oriented edge e . Set also $D_{(v,w)} := \cup_{e:v \rightarrow w} D_e$, for each ordered pair (v,w) of vertices, where the notation $e : v \rightarrow w$ means that the oriented edge e goes from v to w .

We say that there is an infection path from (v, s) to (w, t) , and we denote it by

$$(18) \quad (v, s) \longleftrightarrow (w, t),$$

either if $s = t$ and $v = w$, or if $s < t$ and if there is a sequence of times $s = s_0 < s_1 < \dots < s_l < s_{l+1} = t$, and a sequence of vertices $v = v_0, v_1, \dots, v_l = w$ such that for every $i = 1, \dots, l$

$$\begin{cases} s_i \in D_{(v_{i-1}, v_i)} & \text{and} \\ D_{v_i} \cap [s_i, s_{i+1}] = \emptyset. \end{cases}$$

Furthermore, for any A, B two subsets of V_n and I, J two subsets of $[0, \infty)$, we write

$$A \times I \longleftrightarrow B \times J,$$

if there exists $v \in A, w \in B, s \in I$ and $t \in J$, such that (18) holds. Then for any $A \subset V_n$, the contact process with initial configuration A is defined by

$$\xi_t^A := \{v \in V_n : A \times \{0\} \longleftrightarrow (v, t)\},$$

for all $t \geq 0$. It is well known that $(\xi_t^A)_{t \geq 0}$ has the same distribution as the process defined in the introduction. Moreover, this graphic construction allows us to couple contact processes with different initial states with additivity property: for all $t \geq 0$ and $A, B \subset V$,

$$\xi_t^{A \cup B} = \xi_t^A \cup \xi_t^B.$$

2.4. Contact process on star graphs. We will see that star graphs play a crucial role in the conservation of the virus on the preferential attachment graph. Hence, it is important to understand the behavior of the contact process on a single star graph as well as the transmission between them. These have been studied for a long time by many authors, for instance in [P, BBS1, CD, MVY]. The results we need will be summarized in Lemma 2.8 and 2.9 below. We say that a vertex v is **lit** (the term is taken from [CD]) at some time t if the proportion of its infected neighbors at time t is larger than $\lambda/(16e)$.

Lemma 2.8. *Let (ξ_t) be the contact process on a star graph S with center v . There exists a positive constant c^* , such that the following assertions hold.*

$$(i) \quad \mathbb{P}(v \text{ is lit at time } 1 \mid \xi_0(v) = 1) \geq (1 - \exp(-c^* \lambda |S|))/e.$$

$$(ii) \quad \mathbb{P}(\exists t > 0 : v \text{ is lit at time } t \mid \xi_0(v) = 1) \rightarrow 1 \quad \text{as } |S| \rightarrow \infty.$$

$$(iii) \quad \text{If } \lambda^2 |S| \geq 64e^2, \text{ and } v \text{ is lit at time } 0, \text{ then } v \text{ is lit during the time interval } [\exp(c^* \lambda^2 |S|), 2 \exp(c^* \lambda^2 |S|)] \text{ with probability larger than } 1 - 2 \exp(-c^* \lambda^2 |S|).$$

Proof. Parts (i) and (ii) are exactly Lemma 3.1 (i), (iii) in [MVY]. For (iii) we need an additional definition: a vertex v is said to be **hot** at some time t if the proportion of its infected neighbors at time t is larger than $\lambda/(8e)$. Then in [CD] the authors proved (with different constants in the definition of lit and hot vertices, but this does not effect the proof) that

- if v is lit at some time t , then it becomes hot before $t + \exp(c^* \lambda^2 \deg(v))$,
- if v is hot at some time t , then it remains lit until $t + 2 \exp(c^* \lambda^2 \deg(v))$,

with probability larger than $1 - \exp(-c^* \lambda^2 \deg(v))$. Now (iii) follows from these claims. \square

The following result is Lemma 3.2 in [MVY].

Lemma 2.9. *Let us consider the contact process on a graph $G = (V, E)$. There exist positive constants c^* and λ_0 , such that if $0 < \lambda < \lambda_0$, the following holds. Let v and w be two vertices satisfying $\deg(v) \geq \frac{7}{c^*} \frac{1}{\lambda^2} \log\left(\frac{1}{\lambda}\right) d(v, w)$. Assume that v is lit at time 0. Then w is lit before $\exp(c^* \lambda^2 \deg(v))$ with probability larger than $1 - 2 \exp(-c^* \lambda^2 \deg(v))$.*

3. PROOF OF PROPOSITION 1.3

In this section we study the contact process on the Pólya-point graph (T, o) . To prove Proposition 1.2, we have to show that

$$(19) \quad \mathbb{P}(\xi_t^o \neq \emptyset \forall t \geq 0) \lesssim \lambda^{1+\frac{2}{\psi}} |\log \lambda|^{-\frac{1}{\psi}},$$

and

$$(20) \quad \mathbb{P}(\xi_t^o \neq \emptyset \forall t \geq 0) \gtrsim \lambda^{1+\frac{2}{\psi}} |\log \lambda|^{-\frac{1}{\psi}}.$$

The proof of (19) is based on the proof of the upper bound in Proposition 1.4 in [MVY] for the case of the contact process on Galton-Watson trees, and we put it in Appendix.

3.1. Proof of (20). In this part, we first estimate the probability that there is an infinite sequence of vertices, including a neighbor of the root, with larger and larger degree and a small enough distance between any two consecutive elements of the sequence. We then repeatedly apply Lemma 2.8 and 2.9 to bound from below the probability that the virus propagates along these vertices, and like this survives forever. To this end, we denote by

$$(21) \quad \varphi(\lambda) = \frac{7}{c^*} \frac{1}{\lambda^2} \log\left(\frac{1}{\lambda}\right),$$

with c^* as in Lemma 2.8 and 2.9.

We denote by $w_0 = (0)$, and $x_0 = x_{w_0}$. For any $i \geq 1$, let

$$w_i = (0, 1, \dots, 1) \text{ and } x_i = x_{w_i},$$

where w_i has i 1s. Then w_i 's degree conditioned on x_i is $m + 1$ plus a Poisson random variable with parameter

$$\frac{\gamma}{x_i^\psi} \int_{x_i}^1 \psi x^{\psi-1} dx = \gamma \frac{1 - x_i^\psi}{x_i^\psi},$$

where γ is a Gamma random variable with parameters $a = m + 2mr + 1$ and 1. Therefore letting $\kappa = (1 - x_i^\psi)/x_i^\psi$, we have

$$(22) \quad \begin{aligned} \mathbb{P}(\deg(w_i) = m + 1 + k \mid x_i) &= \mathbb{E}\left(\frac{e^{-\gamma\kappa}}{k!} (\gamma\kappa)^k \mid x_i\right) = \frac{\kappa^k \Gamma(k + a)}{(\kappa + 1)^{k+a} \Gamma(a) k!} \\ &= \frac{\Gamma(k + a)}{\Gamma(a) k!} (1 - x_i^\psi)^k x_i^{a\psi}, \end{aligned}$$

where $\Gamma(b) = \int_0^\infty x^{b-1} e^{-x} dx$.

Lemma 3.1. *There is a positive constant c , such that for λ small enough,*

$$\mathbb{P}(\mathcal{N}) \geq c\varphi(\lambda)^{-1/\psi},$$

where

$$\mathcal{N} = \{\exists(j_\ell)_{\ell \geq 1} : j_1 = 1, \deg(w_{j_\ell}) \geq 2^{\ell+1}\varphi(\lambda)/\psi \geq \varphi(\lambda)d(w_{j_\ell}, w_{j_{\ell+1}}) \forall \ell \geq 1\}.$$

Proof. It follows from Markov's inequality that for any $k \geq 1$,

$$(23) \quad \mathbb{P}\left(\prod_{i=1}^k U_i > 2^{-(k+1)/2}\right) \leq 2^{-(k-1)/2},$$

where (U_i) is a sequence of i.i.d. uniform random variables in $[0, 1]$.

Since $\Gamma(k+a)/k! \asymp k^{a-1}$, there is a positive constant C , such that for all $k \geq 1$

$$\frac{\Gamma(k+a)}{\Gamma(a)k!} \leq Ck^{a-1}.$$

Then it follows from (22) that

$$(24) \quad \mathbb{P}(\deg(w_i) \leq m+1 + (c/x_i^\psi) \mid x_i) \leq \sum_{k=0}^{\lfloor c/x_i^\psi \rfloor} Ck^{a-1}x_i^{a\psi} \leq Cc^a,$$

for any $c > 0$. Set $j_1 = 1$ and $j_\ell = \lfloor 4\ell/\psi \rfloor$ for $\ell \geq 2$. Then define

$$\mathcal{N}_\ell = \{x_{j_\ell} \leq (4^\ell \varphi(\lambda)/(c\psi))^{-1/\psi}, \deg(w_{j_\ell}) \geq 2^{\ell+1}\varphi(\lambda)/\psi\}$$

for all $\ell \geq 1$, where c is a positive constant to be chosen later.

Since $2^{\ell+1}\varphi(\lambda)/\psi \geq 4\varphi(\lambda)/\psi = \varphi(\lambda)d(w_{j_\ell}, w_{j_{\ell+1}})$, we have

$$(25) \quad \mathcal{N} \supset \bigcap_{\ell=1}^{\infty} \mathcal{N}_\ell.$$

Since x_{j_ℓ} is distributed as $x_1 U_1 \dots U_{j_\ell-1}$, applying (23) gives that

$$\mathbb{P}(x_{j_\ell} \leq x_1 4^{-\ell/\psi}) \geq 1 - 2(4^{-\ell/\psi}).$$

Therefore

$$(26) \quad \mathbb{P}(x_{j_\ell} \leq (4^\ell \varphi(\lambda)/(c\psi))^{-1/\psi} \mid \mathcal{N}_1) \geq 1 - 2(4^{-\ell/\psi}).$$

By using (24) with $(2^{1-\ell}c)$ instead of c we obtain that

$$(27) \quad \mathbb{P}(\deg(w_{j_\ell}) \geq 2^{\ell+1}\varphi(\lambda)/\psi \mid x_{j_\ell} \leq (4^\ell \varphi(\lambda)/(c\psi))^{-1/\psi}) \geq 1 - C(2^{1-\ell}c)^a.$$

Then it follows from (26) and (27) that

$$(28) \quad \mathbb{P}\left(\bigcap_{\ell=2}^{\infty} \mathcal{N}_\ell \mid \mathcal{N}_1\right) \geq 1 - 2 \sum_{\ell=2}^{\infty} 4^{-\ell/\psi} - C \sum_{\ell=2}^{\infty} (2^{1-\ell}c)^a \geq 1/4,$$

provided c is small enough. We now estimate $\mathbb{P}(\mathcal{N}_1)$. Let

$$\bar{\varphi}(\lambda) = (4\varphi(\lambda)/(c\psi))^{-1/\psi}.$$

Recall that x_1 is uniformly distributed on $[0, x_0]$, with $x_0 \sim U_0^x$ and $U_0 \sim \mathcal{U}([0, 1])$. Therefore

$$\begin{aligned}
\mathbb{P}(x_1 \leq \bar{\varphi}(\lambda)) &= \mathbb{E} \left(\frac{\min\{\bar{\varphi}(\lambda), x_0\}}{x_0} \right) \\
&\geq \bar{\varphi}(\lambda) \mathbb{P}(x_0 \geq \bar{\varphi}(\lambda)) \\
&= \bar{\varphi}(\lambda) (1 - \bar{\varphi}(\lambda)^{1/\chi}) \\
(29) \quad &\geq \bar{\varphi}(\lambda)/2,
\end{aligned}$$

for λ small enough. On the other hand, (24) gives that for c small enough

$$(30) \quad \mathbb{P}(\mathcal{N}_1 \mid x_1 \leq \bar{\varphi}(\lambda)) \geq 1 - Cc^a \geq 1/2.$$

We thus can choose c such that the two inequalities in (28) and (30) are satisfied. Now it follows from (25), (28), (29) and (30) that

$$\mathbb{P}(\mathcal{N}) \gtrsim \bar{\varphi}(\lambda),$$

which implies the result. \square

Proof of (20). By repeatedly applying Lemma 2.9 to the pair of vertices $(w_{i_\ell}, w_{i_{\ell+1}})$, we obtain that

$$\begin{aligned}
\mathbb{P}(\xi_t \neq \emptyset \forall t \geq T \mid \mathcal{N}, w_1 \text{ is lit at some time } T) &\geq 1 - 2 \sum_{\ell=1}^{\infty} \exp(-c^* \lambda^2 2^{\ell+1} \varphi(\lambda)/\psi) \\
&\geq 1 - 2 \sum_{\ell=1}^{\infty} \exp(-7(2^{\ell+1}) |\log \lambda|/\psi) \\
(31) \quad &\geq 1/2,
\end{aligned}$$

for λ small enough. On the other hand, by using Lemma 2.8 (i), we have

$$\begin{aligned}
(32) \quad &\mathbb{P}(w_1 \text{ is lit at some time } T \mid \mathcal{N}, o \text{ is infected at time } 0) \\
&\geq c\lambda \mathbb{E}(1 - \exp(-c^* \lambda \deg(w_1)) \mid \mathcal{N}) \geq c\lambda/2,
\end{aligned}$$

for some $c > 0$ (note that on \mathcal{N} , we have $c^* \lambda \deg(w_1) \geq 7$). Now it follows from (31), (32) and Lemma 3.1 that

$$\mathbb{P}(\xi_t \neq \emptyset \forall t \geq 0) \geq (1/2) \times (c\lambda/2) \times \mathbb{P}(\mathcal{N}) \gtrsim \lambda^{1+2/\psi} |\log \lambda|^{-1/\psi},$$

which proves (20) \square

4. PROOF OF THEOREM 1.1.

By using the self-duality of the contact process (3), we see that to prove (2), it is sufficient to show that

$$(33) \quad \mathbb{P} \left(\frac{1}{n} \sum_{v \in V_n} 1(\{\xi_{t_n}^v \neq \emptyset\}) \leq C\lambda^{1+2/\psi} |\log \lambda|^{-1/\psi} \right) = 1 - o(1),$$

and

$$(34) \quad \mathbb{P} \left(\frac{1}{n} \sum_{v \in V_n} 1(\{\xi_{t_n}^v \neq \emptyset\}) \geq c\lambda^{1+2/\psi} |\log \lambda|^{-1/\psi} \right) = 1 - o(1),$$

for some positive constants c and C . We will prove these two statements in the next two subsections.

4.1. **Proof of (33).** For $r \geq 1$, we define

$$\mathcal{L}_T(o, r) = \{(o, 0) \leftrightarrow B_T(o, r)^c \times \mathbb{R}_+\},$$

the event that the contact process on the Pólya-point graph starting from the root infects vertices outside $B_T(o, r)$. Then we have

$$\{\xi_t^o \neq \emptyset \forall t\} = \cap_{r=1}^{\infty} \mathcal{L}_T(o, r).$$

Hence, it follows from (19) that there are positive constants C and $R = R(\lambda)$, such that

$$(35) \quad \mathbb{P}(\mathcal{L}_T(o, R)) \leq C\lambda^{1+\frac{2}{\psi}} |\log \lambda|^{-\frac{1}{\psi}}.$$

For any $v \in V_n$ and R as in (35), we define

$$\mathcal{L}_n(v, R) = \{(v, 0) \leftrightarrow B_{G_n}(v, R)^c \times \mathbb{R}_+\}$$

and

$$X_v = 1(\mathcal{L}_n(v, R)).$$

Theorem 2.2 yields that

$$(36) \quad \lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{L}_n(u, R)) = \mathbb{P}(\mathcal{L}_T(o, R)),$$

where u is a uniformly chosen vertex from V_n . By combining this with (35) we obtain that

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_u = 1) \leq C\lambda^{1+2/\psi} |\log \lambda|^{-1/\psi},$$

or equivalently

$$(37) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{v \in V_n} \mathbb{P}(X_v = 1) \leq C\lambda^{1+2/\psi} |\log \lambda|^{-1/\psi}.$$

Now, let us consider

$$W_n = \{(v, v') \in V_n \times V_n : d(v, v') \geq 2R + 3\},$$

with R as in (35). Since $R+1 \leq b_2 \log n / (\log \log n)$ for n large enough, Lemma 2.6 implies that

$$\sum_{v, v' \in V_n} \mathbb{P}((v, v') \notin W_n) = o(n^2).$$

On the other hand, if $(v, v') \in W_n$ then X_v and $X_{v'}$ are independent. Therefore

$$(38) \quad \sum_{v, v' \in V_n} \text{Cov}(X_v, X_{v'}) = o(n^2).$$

Thanks to (37) and (38) by using Chebyshev's inequality we get that

$$(39) \quad \mathbb{P}\left(\frac{1}{n} \sum_{v \in V_n} X_v \leq 2C\lambda^{1+2/\psi} |\log \lambda|^{-1/\psi}\right) = 1 - o(1).$$

Since the contact process on a finite ball in the Pólya-point graph a.s. dies out,

$$\lim_{t \rightarrow \infty} \mathbb{P}(\mathcal{L}_T(o, R)^c \cap \{\xi_t^o \neq \emptyset\}) = 0.$$

Hence for any $\varepsilon > 0$, there exists t_ε , such that

$$(40) \quad \mathbb{P}(\mathcal{L}_T(o, R)^c \cap \{\xi_{t_\varepsilon}^o \neq \emptyset\}) \leq \varepsilon.$$

For any $v \in V_n$, define

$$X_{v, \varepsilon} = 1(\mathcal{L}_n(v, R)^c \cap \{\xi_{t_\varepsilon}^v \neq \emptyset\}).$$

Then for n large enough such that $t_n \geq t_\varepsilon$, we have

$$(41) \quad 1(\{\xi_{t_n}^v \neq \emptyset\}) \leq X_v + X_{v,\varepsilon}.$$

It follows from Theorem 2.2 and (40) that

$$\lim_{n \rightarrow \infty} \mathbb{E}_n(X_{u,\varepsilon}) = \mathbb{P}(\mathcal{L}_T(o, R)^c \cap \{\xi_{t_\varepsilon}^o \neq \emptyset\}) \leq \varepsilon.$$

By using this and Markov's inequality we get that for n large enough, and for any $\eta > 0$,

$$(42) \quad \mathbb{P}\left(\frac{1}{n} \sum_{v \in V_n} X_{v,\varepsilon} > \eta\right) \leq \frac{\sum_{v \in V_n} \mathbb{E}(X_{v,\varepsilon})}{n\eta} = \frac{\mathbb{E}(X_{u,\varepsilon})}{\eta} \leq \frac{2\varepsilon}{\eta}.$$

By combining (39), (41) and (42), then letting ε tend to 0, we obtain that

$$\mathbb{P}\left(\frac{1}{n} \sum_{v \in V_n} 1(\{\xi_{t_n}^v \neq \emptyset\}) \leq 3C\lambda^{1+2/\psi} |\log \lambda|^{-1/\psi}\right) = 1 - o(1),$$

which proves (33). \square

4.2. Proof of (34). This subsection is divided into three parts. In the first one, we will show that w.h.p. there are many vertices with large degree (larger than $\varkappa^* \log n$). By using on the other hand that the diameter of the graph is smaller than $b_1 \log n$, we can deduce that if one of these large degree vertices is infected, then the virus survives w.h.p. for a time $\exp(cn/(\log n)^{1/\psi})$, see Proposition 4.2. In the second part, we measure the density of *potential* vertices which are promising for spreading the virus to some of these large degree vertices. In the last part, we estimate the proportion of potential vertices which really send the virus to large degree vertices, getting this way (34).

4.2.1. Lower bound on the extinction time. Our aim in this part is to find large degree vertices as mentioned above. We then prove that if one of them is infected, the virus is likely to survive a long time.

Lemma 4.1. *Let $\varkappa > 0$ be given. Then there exists a positive constant $\bar{c} = \bar{c}(\varkappa)$, such that \mathcal{A}_n holds w.h.p. with*

$$\mathcal{A}_n = \{G_n \text{ contains } \bar{c}n/(\log n)^{1/(1-\varkappa)} \text{ disjoint star graphs of size larger than } \varkappa \log n\}.$$

Proof. Let $\varepsilon \in (0, 1/3)$ be given, and let us recall the event \mathcal{E}_ε and the constant $K = K(\varepsilon)$ defined in Lemma 2.4:

$$\mathcal{E}_\varepsilon = \left\{ \left| \frac{S_j}{S_k} - \left(\frac{j}{k}\right)^\varkappa \right| \leq \varepsilon (j/k)^\varkappa \quad \forall K(\varepsilon) \leq j \leq k \leq n \right\}.$$

Set $a_n = (M \log n)^{1/(1-\varkappa)}$, with M to be chosen later. Denote by

$$A = \{v_i : i \in [n/a_n, 2n/a_n] \text{ and } \psi_i \in (\theta/i, \mu/i)\}$$

and

$$\mathcal{J}_0 = \mathcal{E}_\varepsilon \cap \{|A| \geq \theta n/a_n\},$$

with θ, μ as in Lemma 2.4. Recall that the events $\{\psi_i \in (\theta/i, \mu/i)\}$ are independent and have probability larger than 2θ . Therefore (4) implies that w.h.p. $|A| \geq \theta n/a_n$. Hence by Lemma 2.4 (ii), we have for n large enough

$$\mathbb{P}(\mathcal{J}_0) \geq 1 - 2\varepsilon.$$

We now suppose that \mathcal{J}_0 occurs. Then, the elements of A can be written as $\{v_{j_1}, \dots, v_{j_\ell}\}$ with $\ell \in [\theta n/a_n, n/a_n]$. Then define

$$A_1 = \{v_j : n/2 \leq j \leq n\}.$$

We will show that all vertices in A have a large number of neighbors in A_1 . Indeed, it follows from (12) and Lemma 2.4 (ii) that for $K(\varepsilon) \leq j < k$,

$$\mathbb{P}(v_j \sim v_k \mid (\varphi_t), \mathcal{E}_\varepsilon) \asymp \frac{\psi_j S_j}{S_{k-1}} \asymp \psi_j \left(\frac{j}{k-1} \right)^\varkappa.$$

Hence, for all $v_j \in A$ and $v_k \in A_1$,

$$(43) \quad \mathbb{P}(v_j \sim v_k \mid \mathcal{J}_0) \asymp \frac{a_n^{1-\chi}}{n}.$$

Conditionally on (ψ_j) , the events $\{v_{j_1} \sim v_k\}_{k \in A_1}$ are independent. Hence thanks to (4) we get that there are positive constants θ_1 , c , and C , such that

$$\mathbb{P} \left(ca_n^{1-\chi} \leq \sum_{v_k \in A_1} 1(v_{j_1} \sim v_k) \leq Ca_n^{1-\chi} \mid \mathcal{E}_\varepsilon \right) \geq 1 - \exp(-\theta_1 a_n^{1-\chi}),$$

or equivalently

$$(44) \quad \mathbb{P}(\mathcal{J}_1 \mid \mathcal{J}_0) \geq 1 - \exp(-\theta_1 a_n^{1-\chi}),$$

where

$$\mathcal{J}_1 = \{ca_n^{1-\chi} \leq |B_1| \leq Ca_n^{1-\chi}\}$$

and

$$B_1 = \{v_k \in A_1 : v_{j_1} \sim v_k\}.$$

Note that in this proof, the value of the constant θ_1 may change from line to line. Now let us consider

$$A_2 = A_1 \setminus B_1 \quad \text{and} \quad B_2 = \{v_k \in A_2 : v_{j_2} \sim v_k\}.$$

We notice that on $\mathcal{J}_1 \cap \mathcal{E}$, the cardinality of A_2 is larger than $n/2 - Ca_n^{1-\chi} \geq n/4$. Thus, similarly to (44), there exist positive constants c_1 and C_1 , such that

$$(45) \quad \mathbb{P}(\mathcal{J}_2 \mid \mathcal{J}_1 \cap \mathcal{J}_0) \geq 1 - \exp(-\theta_1 a_n^{1-\chi}),$$

where

$$\mathcal{J}_2 = \{c_1 a_n^{1-\chi} \leq |B_2| \leq C_1 a_n^{1-\chi}\}.$$

Here we can also assume that $c_1 \leq c$ and $C_1 \geq C$. Likewise for all $2 \leq s \leq \ell$, define recursively

$$A_s = A_{s-1} \setminus B_{s-1}, \quad B_s = \{v_k \in A_s : v_{j_s} \sim v_k\}, \\ \mathcal{J}_s = \{c_1 a_n^{1-\chi} \leq |B_s| \leq C_1 a_n^{1-\chi}\}.$$

On $\bigcap_{i=0}^{s-1} \mathcal{J}_i$, we have $|A_s| \geq n/2 - sC_1 a_n^{1-\chi} \geq n/4$. Therefore, similarly to (45)

$$\mathbb{P} \left(\mathcal{J}_s \mid \bigcap_{i=0}^{s-1} \mathcal{J}_i \right) \geq 1 - \exp(-\theta_1 a_n^{1-\chi}).$$

Hence

$$\mathbb{P} \left(\bigcap_{i=1}^{\ell} \mathcal{J}_i \mid \mathcal{J}_0 \right) \geq 1 - n \exp(-\theta_1 a_n^{1-\chi})/a_n.$$

Taking M large enough such that $c_1 a_n^{1-\chi} \geq \varkappa \log n$ and $n \exp(-\theta_1 a_n^{1-\chi}) \leq 1$ yields that

$$(46) \quad \mathbb{P}(|B_s| \geq \varkappa \log n \forall 1 \leq s \leq \ell \mid \mathcal{J}_0) \geq 1 - a_n^{-1}.$$

Moreover, by definition $B_s \cap B_t = \emptyset$ for all $s \neq t$. Hence, all vertices in A have more than $\varkappa \log n$ distinct neighbors. Finally, take \bar{c} such that $\bar{c}n/(\log n)^{1/(1-\chi)} \leq \theta n/a_n$, for instance $\bar{c} \leq \theta M^{-1/(1-\chi)}$. In conclusion, we have shown that for any given $\varepsilon \in (0, 1/3)$,

$$\mathbb{P}(\mathcal{A}_n) \geq 1 - 2\varepsilon - a_n^{-1} \geq 1 - 3\varepsilon,$$

for n large enough. Since this holds for any $\varepsilon > 0$, the result follows. \square

To determine the constant \varkappa in the definition of \mathcal{A}_n , we first recall that

$$\mathbb{P}(\mathcal{B}_n) = 1 - o(1),$$

where

$$\mathcal{B}_n = \{d(G_n) \leq b_1 \log n\}.$$

Hence to apply Lemma 2.9 to the large degree vertices exhibited in the previous lemma, we need

$$\varkappa \log n \geq \frac{7}{c^*} \frac{1}{\lambda^2} \log \left(\frac{1}{\lambda} \right) b_1 \log n.$$

Moreover, in (48), we will use that $\varkappa \geq 3/(c^* \lambda^2)$. So we let

$$(47) \quad \varkappa^* = \max \left\{ \frac{7}{c^*} \frac{1}{\lambda^2} \log \left(\frac{1}{\lambda} \right) b_1, \frac{3}{c^* \lambda^2} \right\}.$$

Then we let $\bar{c}^* = \bar{c}^*(\varkappa^*)$ and \mathcal{A}_n be defined accordingly as in Lemma 4.1.

A set of vertices $V = \{w_1, \dots, w_k\} \subset V_n$ is called **good** if $|S(w_i) \setminus \cup_{j \neq i} S(w_j)| \geq \varkappa^* \log n$ for all $1 \leq i \leq k$, where $S(v)$ denotes the star graph formed by v and its neighbors.

Let V_n^* be a maximal good set i.e. $|V_n^*| = \max\{|V| : V \subset V_n \text{ is good}\}$.

Proposition 4.2. *There exists a positive constant c , such that*

$$\mathbb{P}(\xi_{T_n} \neq \emptyset \mid \xi_0 \cap V_n^* \neq \emptyset) = 1 - o(1),$$

where $T_n = \exp(c\lambda^2 n/(\log n)^{1/\psi})$.

Proof. Thanks to Lemma 2.5 and 4.1, we can assume that $d(G_n) \leq b_1 \log n$ and $|V_n^*| \geq \bar{c}^* n/(\log n)^{1/(1-\chi)}$. Assume also that at time 0 a vertex in V_n^* , say v , is infected.

Due to the definition of V_n^* , for any $w \in V_n^*$, we can select from the set of w 's neighbors a subset $D(w)$ of size $\varkappa^* \log n$, such that $D(w) \cap D(w') = \emptyset$ for all $w \neq w'$.

We say that a vertex w in V_n^* is *infested* at some time t if the proportion of infected sites in $D(w)$ at time t is larger than $\lambda/(16e)$ (the term is taken from [MMVY]).

It follows from Lemma 2.8 (ii) that v becomes infested with probability tending to 1, as $n \rightarrow \infty$. Using Lemma 2.8 (iii) and 2.9 (note that $|D(w)| \geq (7/(c^* \lambda^2)) \log \lambda |d(w, w')|$), we deduce that for any $t \geq 0$ and $w \in V_n^*$,

$$\begin{aligned} & \mathbb{P}(w \text{ is infested at } t + 2 \exp(c^* \lambda^2 \varkappa^* \log n) \mid v \text{ is infested at } t) \\ & \geq 1 - 4 \exp(-c^* \lambda^2 \varkappa^* \log n). \end{aligned}$$

Therefore

$$(48) \quad \begin{aligned} & \mathbb{P}(\text{All vertices in } V_n^* \text{ are infested at } t + 2 \exp(c^* \lambda^2 \varkappa^* \log n) \mid v \text{ is infested at } t) \\ & \geq 1 - 4n \exp(-c^* \lambda^2 \varkappa^* \log n) \\ & \geq 1 - n^{-1}, \end{aligned}$$

where we have used that $c^* \lambda^2 \varkappa^* \geq 3$. Now if all vertices in V_n^* are infested at the same time, then the proof of Proposition 1 in [CD] shows that the virus survives a time

exponential in $\sum_{v \in V_n^*} \deg(v)$. More precisely, let $I_{n,t}$ be the number of infested vertices in V_n^* at time t . Then there is a positive constant η , such that for all $k \leq |V_n^*|$,

$$\mathbb{P}(I_{n,s_k} \geq k/2 \mid I_{n,0} \geq k) \geq 1 - s_k^{-1},$$

where $s_k = \exp(\eta \lambda^2 k \varkappa^* \log n)$. The result follows by taking $k = \lceil \bar{c}^* n / (\log n)^{1/(1-\chi)} \rceil$. \square

4.2.2. Density of potential vertices. In this part we will estimate the proportion of *potential* sites from where the virus can be sent with positive probability to a vertex at distance quite small (of order $(\log \log n)^2$) and with large degree (larger than $\varkappa^* \log n$).

This proportion approximates the probability that there is an infection path from the uniformly chosen vertex u to a vertex with degree larger than $\varkappa^* \log n$. To bound from below this probability, we use the same ideas as in Lemma 3.1. Indeed, we will find a sequence of vertices starting from a neighbor of u and ending at a large degree vertex, satisfying the hypothesis of Lemma 2.9 for spreading the virus from u to the ending vertex (see Lemma 4.4).

Here are just some comments on the order of magnitude above. First, if a vertex with degree larger than $\varkappa^* \log n$ is infected, then w.h.p. it will infect a site in V_n^* , and then we can conclude with Proposition 4.2. Secondly, $(\log \log n)^2$ is the distance from a potential vertex to a large degree vertex and is much smaller than the typical distance between two different potential vertices. Hence the propagation of the virus from these potential vertices to their closest large degree vertex are approximately independent events.

Set

$$R_n = \lceil (\log \log n)^2 \rceil.$$

For $w \in V_n$, define $k_0(w)$ by $w = v_{k_0(w)}$, and for $i \geq 1$ let $k_i(w)$ be chosen arbitrarily from the indices such that $v_{k_i(w)}$ receives an edge emanating from $v_{k_{i-1}(w)}$. We define also

$$\mathcal{H}_n(w) = \{k_0(w) \geq n/\log n\} \cap \{k_{i+1}(w) \geq k_i(w)/\log k_i(w) \geq n^{1/2} \forall 0 \leq i \leq R_n\}.$$

Lemma 4.3. *There is a positive constant θ_0 , such that for all $\theta \leq \theta_0$, for all $\varepsilon \in (0, 1/2)$, and for any vertex w , we have*

(i) *for all $i \leq R_n$*

$$\mathbb{P}\left(\max_{v \in B_{G_n}(w,i)} \deg(v) \geq \theta e^{\theta i} (n/k_0(w))^{1-\chi} \mid \mathcal{E}_\varepsilon \cap \mathcal{H}_n(w)\right) \geq 1 - e^{-\theta i},$$

(ii) $\mathbb{P}(\mathcal{H}_n(w) \mid k_0(w) \geq n/\log n, \mathcal{E}_\varepsilon) = 1 - o(1/\log \log n)$,

with \mathcal{E}_ε as in Lemma 2.4 (ii).

Proof. We first prove (ii). It follows from the construction of G_n and Lemma 2.4 (ii) that if $k_i(w)/\log k_i(w) \geq K(\varepsilon)$, then

$$\mathbb{P}(k_{i+1}(w) \leq k_i(w)/\log k_i(w) \mid \mathcal{E}_\varepsilon, k_i(w), (\varphi_t)) = \frac{S_{\lfloor k_i(w)/\log k_i(w) \rfloor}}{S_{k_i(w)-1}} \leq (1 + \varepsilon) \left(\frac{1}{\log k_i(w)}\right)^\chi.$$

Hence for all $i \leq R_n$,

$$\begin{aligned} & \mathbb{P}(k_{i+1}(w) \geq n/(\log n)^{i+2} \mid \mathcal{E}_\varepsilon, k_i(w) \geq n/(\log n)^{i+1}) \\ & \geq \mathbb{P}(k_{i+1}(w) \geq k_i(w)/\log k_i(w) \mid \mathcal{E}_\varepsilon, k_i(w) \geq n/(\log n)^{i+1}) \\ & = 1 - o((\log n)^{-\chi/2}). \end{aligned}$$

Therefore

$$\begin{aligned} & \mathbb{P} \left(k_{i+1}(w) \geq k_i(w) / \log k_i(w) \geq n / (\log n)^{i+2} \ \forall i \leq R_n \mid \mathcal{E}_\varepsilon, k_0(w) \geq n / \log n \right) \\ &= 1 - o(R_n (\log n)^{-\chi/2}) = 1 - o(1 / \log \log n). \end{aligned}$$

This implies (ii), since for all $i \leq R_n$

$$n / (\log n)^{i+2} \geq \sqrt{n}.$$

We now prove (i). First, we claim that there is a positive constant c_0 , such that for any $c < c_0$, there exists $c' = c'(c) > 0$, such that for all $i \leq R_n$

$$(a) \ \mathbb{P} \left(k_{\lfloor i/2 \rfloor}(w) \leq e^{-ci} k_0(w) \mid \mathcal{E}_\varepsilon \cap \mathcal{H}_n(w) \right) \geq 1 - e^{-c'i},$$

$$(b) \ \mathbb{P} \left(\exists j \in (i/2, i) : \psi_{k_j(w)} \geq c/k_j(w) \mid \mathcal{E}_\varepsilon \cap \mathcal{H}_n(w) \right) \geq 1 - e^{-c'i},$$

$$(c) \ \mathbb{P}(\deg(v_k) \geq c'(n/k)^{1-\chi} \mid \psi_k \geq c/k, \mathcal{E}_\varepsilon) \geq 1 - \exp(-c'(n/k)^{1-\chi}), \text{ for any } v_k \in V_n.$$

From these claims we can deduce the result. Indeed, (a) and (b) imply that with probability larger than $1 - 2 \exp(-c'i)$ there is an integer $j \in (i/2, i)$ such that $k_j(w) \leq e^{-ci} k_0(w)$ and $\psi_{k_j(w)} \geq c/k_j(w)$. Then (c) gives that $\deg(v_{k_j(w)}) \geq c' e^{c(1-\chi)i} (n/k_0(w))^{1-\chi}$ with probability larger than $1 - \exp(-c' e^{c(1-\chi)i})$. Hence (ii) follows by taking θ small enough.

To prove (a), similarly to Lemma 2.6, we consider

$$X_j(w) = 1(\{k_j(w) \leq k_{j-1}(w)/2\}) \text{ and } \mathcal{F}_j(w) = \sigma(k_t(w) : t \leq j) \vee \sigma((\varphi_t)).$$

On $\mathcal{H}_n(w)$, we have $K(\varepsilon) \leq \sqrt{n} \leq k_j(w)$ for all $j \leq R_n$. Then by using the same argument as in Lemma 2.6 we obtain that on $\mathcal{H}_n(w) \cap \mathcal{E}_\varepsilon$,

$$(49) \quad \mathbb{E}(X_j(w) \mid \mathcal{F}_{j-1}(w)) \geq \frac{S_{\lfloor k_j(w)/2 \rfloor} - S_{\lfloor k_j(w)/\log k_j(w) \rfloor}}{S_{k_j(w)-1}} \geq p$$

and

$$(50) \quad \mathbb{P} \left(\sum_{j=1}^{\lfloor i/2 \rfloor} X_j(w) \geq ip/4 \right) \geq 1 - 2 \exp(-ip^2/16),$$

for some constant $p > 0$. Since $k_{\lfloor i/2 \rfloor}(w) \leq 2^{-ip/4} k_0(w)$ as soon as $\sum_{j=1}^{\lfloor i/2 \rfloor} X_j(w) \geq ip/4$, the claim (a) follows from (50).

We now prove (b). Let θ be the constant as in Lemma 2.4 (v). Fix some $j \in (i/2, i)$ and set $k = k_j(w) - 1$ and $\ell = \lfloor k_j(w) / \log k_j(w) \rfloor$. Then we have

$$\begin{aligned} (51) \quad \mathbb{P} \left(\psi_{k_{j+1}(w)} \geq \theta/k_{j+1}(w) \mid k, \ell \right) &= \mathbb{E} \left(\frac{1}{S_k - S_\ell} \sum_{t=\ell+1}^k \varphi_t 1(\psi_t \geq \theta/t) \mid k, \ell \right) \\ &\geq \mathbb{E} \left(\frac{1}{S_k} \sum_{t=\ell+1}^k \varphi_t 1(\psi_t \geq \theta/t) \mid k, \ell \right) \end{aligned}$$

On $\mathcal{H}_n(w) \cap \mathcal{E}_\varepsilon$, we have $k \geq k_{j+1}(w) \geq \ell \geq \sqrt{n} \geq K(\varepsilon)$. Hence, using Lemma 2.4 (ii) and the fact that $S_n = 1$, we get on $\mathcal{H}_n(w) \cap \mathcal{E}_\varepsilon$,

$$(52) \quad |S_k - (k/n)^\chi| \leq \varepsilon(k/n)^\chi \quad \text{and} \quad |S_\ell - (\ell/n)^\chi| \leq \varepsilon(\ell/n)^\chi.$$

Moreover, by Lemma 2.4 (v) for all t , we have

$$(53) \quad \mathbb{E}[\varphi_t 1(\psi_t \geq \theta/t)] \geq \theta.$$

Now using (52) and (53), we obtain that for all n large enough

$$(54) \quad \begin{aligned} \mathbb{E} \left(\frac{1}{S_k} \sum_{t=\ell+1}^k \varphi_t 1(\psi_t \geq \theta/t) \mid k, \ell, \mathcal{H}_n(w) \cap \mathcal{E}_\varepsilon \right) &\geq \frac{n^\chi}{(1+\varepsilon)k^\chi} \mathbb{E} \left(\theta \sum_{t=\ell+1}^k \varphi_t \mid k, \ell, \mathcal{H}_n(w) \cap \mathcal{E}_\varepsilon \right) \\ &\geq \frac{\theta n^\chi}{(1+\varepsilon)k^\chi} \mathbb{E}(S_k - S_\ell \mid k, \ell, \mathcal{H}_n(w) \cap \mathcal{E}_\varepsilon) \\ &= \frac{\theta n^\chi}{(1+\varepsilon)k^\chi} [(1-\varepsilon)(k/n)^\chi - (1+\varepsilon)(\ell/n)^\chi] \\ &\geq \theta/4, \end{aligned}$$

since $\varepsilon \in (0, 1/2)$ and $\ell = \lceil (k+1)/\log(k+1) \rceil$. It follows from (51) and (54) that

$$(55) \quad \mathbb{P}(\psi_{k_{j+1}(w)} \geq \theta/k_{j+1}(w) \mid k_j(w), \mathcal{H}_n(w) \cap \mathcal{E}_\varepsilon) \geq \theta/4.$$

Now it follows from (55) that

$$\mathbb{P}(\nexists j \in (i/2, i) : \psi_{k_{j+1}(w)} \geq \theta/k_{j+1}(w) \mid \mathcal{H}_n(w) \cap \mathcal{E}_\varepsilon) \leq (1 - \theta/4)^{\lfloor i/2 \rfloor},$$

which implies (b). Finally, (c) can be proved as (44). \square

Lemma 4.4. *Let u be a uniformly chosen vertex from V_n . Then*

$$\mathbb{P}(\mathcal{M}) \gtrsim \lambda \varphi(\lambda)^{-1/\psi},$$

where

$$\mathcal{M} = \{\exists w \in B_{G_n}(u, R_n) : \deg(w) \geq \varkappa^* \log n\} \cap \{(\xi_t^u) \text{ makes } w \text{ lit inside } B_{G_n}(u, R_n)\}.$$

Proof. Define k_0 by $v_{k_0} = u$ and for $i \geq 1$ let k_i be chosen arbitrarily from the indices that v_{k_i} receives an edge emanating from $v_{k_{i-1}}$. Let us denote $u_1 = v_{k_1}$ and define also

$$\mathcal{H}_n := \mathcal{H}_n(u_1) = \{k_1 \geq n/\log n\} \cap \{k_{i+1} \geq k_i/\log k_i \geq \sqrt{n} \forall 1 \leq i \leq R_n + 1\}.$$

In this proof, we assume that $\varepsilon = o(\lambda \varphi(\lambda)^{-1/\psi})$ as $\lambda \rightarrow 0$. Similarly to Lemma 4.3 by using that k_0 is chosen uniformly from $\{1, \dots, n\}$, we have $\mathbb{P}(\mathcal{H}_n \mid \mathcal{E}_\varepsilon) = 1 - o(1/\log \log n)$ and hence $\mathbb{P}(\mathcal{E}_\varepsilon \cap \mathcal{H}_n) = 1 - o(\lambda \varphi(\lambda)^{-1/\psi})$. We assume now that these two events happen.

We recall the claim (c) in the proof of Lemma 4.3: there is a positive constant c_0 , such that for any $c < c_0$, there exists $c' = c'(c) > 0$, such that

$$(56) \quad \mathbb{P}(\deg(v_k) \geq c'(n/k)^{1-\chi} \mid \psi_k \geq c/k) \geq 1 - \exp(-c'(n/k)^{1-\chi}),$$

for any $v_k \in V_n$. Let us consider

$$\mathcal{M}_1 = \{k_1 \leq n/\tilde{\varphi}(\lambda)\},$$

where

$$\tilde{\varphi}(\lambda) = (4\varphi(\lambda)/c'\theta^2)^{1/1-\chi},$$

with θ a small enough constant (smaller than θ_0 as in Lemma 2.4 and 4.3 and than c_0), and $c' = c'(\theta)$. Define

$$\mathcal{M}_2 = \mathcal{M}_1 \cap \{\forall 1 \leq \ell \leq R'_n \exists w_\ell : d(u_1, w_\ell) \leq r_\ell, \deg(w_\ell) \geq \varphi(\lambda) \exp(\theta r_\ell)\},$$

where

$$r_\ell = 4\ell/\theta^2 \quad \text{for } 1 \leq \ell \leq R'_n := \theta^2 R_n/8.$$

By applying Lemma 4.3 for u_1 we get that for any $\ell \leq R'_n$

$$\mathbb{P} \left(\max_{v \in B_{G_n}(u_1, r_\ell)} \deg(v) \geq \theta e^{\theta r_\ell} (n/k_1)^{1-x} \right) \geq 1 - e^{-\theta r_\ell}.$$

If $k_1 \leq n/\tilde{\varphi}(\lambda)$, then

$$\theta \exp(\theta r_\ell) (n/k_1)^{1-x} \geq \theta \exp(\theta r_\ell) \tilde{\varphi}(\lambda)^{1-x} \geq \varphi(\lambda) \exp(\theta r_\ell).$$

Thus

$$\mathbb{P}(\exists v : d(v, u_1) \leq r_\ell, \deg(v) \geq \varphi(\lambda) \exp(\theta r_\ell) \mid \mathcal{M}_1) \geq 1 - e^{-4\ell/\theta}.$$

Hence

$$(57) \quad \mathbb{P}(\mathcal{M}_2 \mid \mathcal{M}_1) \geq 1 - \sum_{\ell=1}^{R'_n} \exp(-4\ell/\theta) \geq 1 - 2 \exp(-4/\theta).$$

Define

$$\mathcal{M}_3 = \mathcal{M}_1 \cap \{\deg(u_1) \geq 4\varphi(\lambda)/\theta^2\}.$$

Similarly to (54), we can show that

$$\mathbb{P}(\psi_{k_1} \geq \theta/k_1 \mid k_1 \leq n/\tilde{\varphi}(\lambda)) \geq \theta/4.$$

Using (56) and the fact that $c'\tilde{\varphi}(\lambda)^{1-x} = 4\varphi(\lambda)/\theta^2$, we get

$$\mathbb{P}(\deg(u_1) \geq c'(n/k_1)^{1-x} \mid k_1 \leq n/\tilde{\varphi}(\lambda), \psi_{k_1} \geq \theta/k_1) \geq 1 - \exp(-4\varphi(\lambda)/\theta^2) \geq 1/2.$$

From the last two inequalities we deduce that

$$\mathbb{P}(\mathcal{M}_3 \mid \mathcal{M}_1) \geq \theta/8.$$

Combining this with (57) we obtain that

$$(58) \quad \mathbb{P}(\mathcal{M}_2 \cap \mathcal{M}_3 \mid \mathcal{M}_1) \geq \theta/8 - 2 \exp(-4/\theta) \geq \theta/16.$$

We now bound from below $\mathbb{P}(\mathcal{M}_1)$. Observe that

$$\begin{aligned} \mathbb{P} \left(k_1 \leq n/\tilde{\varphi}(\lambda) \mid k_0, (\varphi_j) \right) &\geq \frac{S_{[n/\tilde{\varphi}(\lambda)]} 1(k_0 > n/\tilde{\varphi}(\lambda))}{S_{k_0-1}} \\ &\gtrsim \tilde{\varphi}(\lambda)^{-x} \left(\frac{k_0}{n} \right)^x 1(k_0 > n/\tilde{\varphi}(\lambda)). \end{aligned}$$

Since k_0 is distributed uniformly on $\{1, \dots, n\}$, we get

$$\mathbb{E}[(k_0/n)^x 1(k_0 > n/\tilde{\varphi}(\lambda))] \asymp 1.$$

Therefore

$$\mathbb{P}(\mathcal{M}_1) \gtrsim \tilde{\varphi}(\lambda)^{-x} \gtrsim \varphi(\lambda)^{-1/\psi}.$$

This and (58) give that

$$(59) \quad \mathbb{P}(\mathcal{M}_2 \cap \mathcal{M}_3) \gtrsim \varphi(\lambda)^{-1/\psi}.$$

Observe that on $\mathcal{M}_2 \cap \mathcal{M}_3$, we have $\deg(u_1) \geq \varphi(\lambda)r_1 \geq \varphi(\lambda)d(u_1, w_1)$ and

$$\deg(w_\ell) \geq \varphi(\lambda) \exp(\theta r_\ell) \geq 2\varphi(\lambda)r_{\ell+1} \geq \varphi(\lambda)d(w_\ell, w_{\ell+1})$$

for all $1 \leq \ell \leq R'_n$. In other words, u_1 and the vertices (w_ℓ) satisfy the condition in Lemma 2.9, and thus applying this lemma inductively yields that

$$\begin{aligned} & \mathbb{P}(w_{R'_n} \text{ is lit inside } B_{G_n}(u_1, R_n) \mid \mathcal{M}_2 \cap \mathcal{M}_3, u_1 \text{ is lit}) \\ & \geq 1 - \sum_{\ell=1}^{R'_n} \exp(-c^* \lambda^2 \varphi(\lambda) e^{\theta r_\ell}) \\ (60) \quad & \gtrsim 1. \end{aligned}$$

Similarly to (32), the probability that (ξ_t^u) makes u_1 lit is of order λ . It follows from this and (60) that

$$(61) \quad \mathbb{P}((\xi_t^u) \text{ makes } w_{R'_n} \text{ lit inside } B_{G_n}(u, R_n) \mid \mathcal{M}_2 \cap \mathcal{M}_3) \gtrsim \lambda.$$

In addition, $\deg(w_{R'_n}) \geq \varkappa^* \log n$. Therefore

$$\mathcal{M} \supset \mathcal{M}_2 \cap \mathcal{M}_3 \cap \{(\xi_t^u) \text{ makes } w_{R'_n} \text{ lit inside } B_{G_n}(u, R_n)\}.$$

Combining this with (59) and (61) gives the result. \square

4.2.3. *Proof of (34).* For any $v \in V_n$, we define

$$Y_v = 1(\{\exists w \in B_{G_n}(v, R_n) : \deg(w) \geq \varkappa^* \log n\} \cap \{(\xi^v) \text{ makes } w \text{ lit inside } B_{G_n}(v, R_n)\})$$

and

$$Z_v = Y_v 1(\{\xi_{T_n}^v \neq \emptyset\}),$$

where T_n is as in Proposition 4.2. Then

$$(62) \quad \sum_{v \in V_n} Z_v \leq \sum_{v \in V_n} 1(\{\xi_{T_n}^v \neq \emptyset\}).$$

Lemma 4.5. *The following assertions hold:*

- (i) $\mathbb{P}\left(\frac{1}{n} \sum_{v \in V_n} Y_v \geq c \lambda \varphi(\lambda)^{-1/\psi}\right) = 1 - o(1)$, for some $c > 0$, independent of λ .
- (ii) $\mathbb{P}(Z_v = 1 \mid Y_v = 1) \rightarrow 1$, as $n \rightarrow \infty$ uniformly in $v \in V_n$.

Proof. For (i), let $\varepsilon \in (0, 1/2)$ be given. We have to show that the probability in the left-hand side is larger than $1 - 2\varepsilon$ for n large enough. First, Lemma 4.4 implies that

$$\mathbb{P}(Y_u = 1) \gtrsim \lambda \varphi(\lambda)^{-1/\psi},$$

or equivalently

$$\frac{1}{n} \sum_{v \in V_n} \mathbb{P}(Y_v = 1) \gtrsim \lambda \varphi(\lambda)^{-1/\psi}.$$

Using Chebyshev's inequality, the result follows from this and the following claim: on \mathcal{E}_ε

$$(63) \quad \sum_{v, v' \in V_n} \text{Cov}(Y_v, Y_{v'}) = o(n^2).$$

To prove it, we consider

$$\mathcal{V}_n = \{(v_i, v_j) : i, j \geq n/\log n, d(v_i, v_j) \geq 2R_n + 3\}.$$

Since $R_n + 1 \leq b_2 \log n / (\log \log n)$ for n large enough, it follows from Lemma 2.6 that

$$(64) \quad \sum_{v, v' \in V_n} \mathbb{P}((v, v') \notin \mathcal{V}_n) = o(n^2).$$

On the other hand, Lemma 4.3 gives that if $i \geq n/\log n$, then

$$\mathbb{P}(\exists w \in B_{G_n}(v_i, R_n) : \deg(w) \geq \varkappa^* \log n \mid \mathcal{E}_\varepsilon) = 1 - o(1/\log \log n).$$

Moreover, given the graph G_n , Y_v and $Y_{v'}$ only depend on the Poisson processes defined on the vertices and edges on the balls $B_{G_n}(v, R_n)$ and $B_{G_n}(v', R_n)$ respectively. Hence on \mathcal{E}_ε for all $(v, v') \in \mathcal{V}_n$,

$$(65) \quad \text{Cov}(Y_v, Y_{v'}) = o(1/\log \log n).$$

Now (63) follows from (64) and (65).

We now prove (ii). If $Y_v = 1$, then there exists a vertex w such that $\deg(w) \geq \varkappa^* \log n$ and w is lit at some time. Besides, on \mathcal{B}_n the diameter of the graph is bounded by $b_1 \log n$ w.h.p. Hence similarly to Lemma 2.9, we can show that on \mathcal{B}_n

$$\mathbb{P}(w \text{ infects a vertex in } V_n^*) \geq 1 - \exp(-c^* \varkappa^* \lambda^2 \log n).$$

If one of the vertices in V_n^* is infected, it follows from Proposition 4.2 that w.h.p. the virus survives up to time T_n . Hence we obtain (ii) by using that \mathcal{B}_n holds w.h.p. \square

It follows from (62), Lemma 4.5 and Markov's inequality that w.h.p.

$$\frac{1}{n} \sum_{v \in V_n} 1(\{\xi_{T_n}^v \neq \emptyset\}) \lesssim \lambda \varphi(\lambda)^{-1/\psi},$$

which proves (34). \square

5. PROOF OF PROPOSITION 1.2

Let us first recall a result which we will use below.

Proposition 5.1. [MMVY, Lemma A.1] *Let $(G_n) = (V_n, E_n)$ be a sequence of graphs, such that $|G_n| = n$, for all n . Let $(\xi_t)_{t \geq 0}$ be the contact process on G_n starting from full occupancy and let τ_n be the extinction time of this process. Assume that there exist sequences of positive numbers $(a_n), (b_n)$ satisfying*

- (i) $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \infty, \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0;$
- (ii) $\lim_{n \rightarrow \infty} \sup_{A \subset V_n} \mathbb{P}(\xi_{a_n}^A \neq \xi_{a_n}, \xi_{a_n} \neq \emptyset) = 0;$
- (iii) $\lim_{n \rightarrow \infty} \mathbb{P}(\tau_n < b_n) = 0.$

Then, $\tau_n/\mathbb{E}(\tau_n)$ converges in distribution, as $n \rightarrow \infty$, to the exponential distribution of parameter 1.

This result for the case of finite boxes in \mathbb{Z}^d is proved by Mountford in [M]. Then it is stated for general case as above. We will apply this result to prove the following proposition 5.2 which implies Proposition 1.2.

Proposition 5.2. *Let (G_n) be a sequence of connected graphs, such that $|G_n| = n$, for all n . Let τ_n denote the extinction time of the contact process on G_n starting from full occupancy. Assume that*

$$(66) \quad \frac{D_{n,\max}}{d_n \vee \log n} \rightarrow \infty,$$

with $D_{n,\max}$ the maximum degree and d_n the diameter of G_n . Then

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n \rightarrow \infty]{(\mathcal{L})} \mathcal{E}(1).$$

Proof. First we set $\bar{\lambda} = \lambda \wedge 1$. Using Lemma 2.8, we get w.h.p.

$$(67) \quad \tau_n \geq \exp(c\bar{\lambda}^2 D_{n,\max}),$$

with $c = c^*/2$ and c^* as Lemma 2.8.

Hence, according to Proposition 5.1, it suffices to show that there exists a sequence (a_n) , such that

$$(68) \quad a_n = o(\exp(c\bar{\lambda}^2 D_{n,\max}))$$

and

$$(69) \quad \sup_{v \in V_n} \mathbb{P}(\xi_{a_n}^v \neq \xi_{a_n}, \xi_{a_n}^v \neq \emptyset) = o(1),$$

where $(\xi_t)_{t \geq 0}$ denotes the process starting from full occupancy.

Using (66), we can find a sequence (k_n) tending to infinity, such that

$$(70) \quad \frac{D_{n,\max}}{(\log n \vee d_n)k_n} \rightarrow \infty.$$

Now define

$$(71) \quad b_n = \exp(c\bar{\lambda}^2 (\log n \vee d_n)k_n) \quad \text{and} \quad a_n = 4b_n + 1.$$

Then (67) and (70) show that a_n satisfies (68), so it amounts now to prove (69) for this choice of (a_n) . To this end it is convenient to introduce the dual contact process. Given some positive real t and A a subset of the vertex set V_n of G_n , the dual process $(\hat{\xi}_s^{A,t})_{s \leq t}$ is defined by

$$\hat{\xi}_s^{A,t} = \{v \in V_n : (v, t-s) \longleftrightarrow A \times \{t\}\},$$

for all $s \leq t$. It follows from the graphical construction that for any v ,

$$(72) \quad \begin{aligned} & \mathbb{P}(\xi_{a_n}^v \neq \xi_{a_n}, \xi_{a_n}^v \neq \emptyset) \\ &= \mathbb{P}(\exists w \in V_n : \xi_{a_n}^v(w) = 0, \xi_{a_n}^v \neq \emptyset, \hat{\xi}_{a_n}^{w, a_n} \neq \emptyset) \\ &\leq \sum_{w \in V_n} \mathbb{P}(\xi_{a_n}^v \neq \emptyset, \hat{\xi}_{a_n}^{w, a_n} \neq \emptyset, \text{ and } \hat{\xi}_{a_n-t}^{w, a_n} \cap \xi_t^v = \emptyset \text{ for all } t \leq a_n), \end{aligned}$$

So let us prove now that the last sum above tends to 0 when $n \rightarrow \infty$. Set

$$\beta_n = [k_n(d_n \vee \log n)],$$

and let u be a vertex with degree larger than β_n . Let then $S(u)$ be a star graph of size β_n centered at u . As in Proposition 4.2, we say that u is infested if the number of its infested neighbors in $S(u)$ is larger than $\bar{\lambda}\beta_n/(16e)$. We first claim that

$$(73) \quad \mathbb{P}(\xi_{b_n}^v \neq \emptyset, u \text{ is not infested before } b_n) = o(1/n).$$

To see this, define $K_n = [b_n/d_n]$ and for any $0 \leq k \leq K_n - 1$

$$A_k := \{\xi_{kd_n}^v \neq \emptyset\},$$

and

$$B_k := \{\xi_{kd_n}^v \times \{kd_n\} \longleftrightarrow (u, (k+1)d_n - 1)\} \cap \{u \text{ is infested at time } (k+1)d_n\}.$$

Note that

$$(74) \quad \{\xi_{b_n}^v \neq \emptyset, u \text{ is not infested before } b_n\} \subset \bigcap_{k=0}^{K_n-1} A_k \cap B_k^c.$$

It is not difficult to see that

$$\mathbb{P}((z, t) \longleftrightarrow (z', t + d_n - 1)) \geq \exp(-Cd_n) \quad \text{for any } z, z' \in V_n \text{ and } t \geq 0,$$

for some constant $C > 0$. On the other hand, Lemma 2.8 (i) implies that if u is infected at time t then it is infested at time $t + 1$ with probability larger than $1/3$, if n is large enough. Therefore for any $k \leq K_n - 1$,

$$\mathbb{P}(B_k^c \mid \mathcal{G}_k) \mathbf{1}(A_k) \leq 1 - \exp(-Cd_n)/3,$$

with \mathcal{G}_k the sigma-field generated by the contact process up to time kd_n . Iterating this, we get

$$\mathbb{P}\left(\bigcap_{k=0}^{K_n-1} A_k \cap B_k^c\right) \leq (1 - \exp(-Cd_n)/3)^{K_n-1} = o(1/n),$$

where the last equality follows from the definition of b_n . Together with (74) this proves our claim (73). Then by using Lemma 2.8 (iii) we get

$$(75) \quad \mathbb{P}(\xi_{b_n}^v \neq \emptyset, u \text{ is not infested at time } 2b_n) = o(1/n).$$

Therefore, if we define

$$\mathcal{A}(v) = \{\xi_{b_n}^v \neq \emptyset, u \text{ is infested at time } 2b_n\},$$

we get

$$\mathbb{P}(\mathcal{A}(v)^c, \xi_{b_n}^v \neq \emptyset) = o(1/n).$$

Likewise if

$$\hat{\mathcal{A}}(w) = \{\hat{\xi}_{b_n}^{w, 4b_n+1} \neq \emptyset, \exists U \subset S(u) : |U| \geq \frac{\bar{\lambda}}{16e} \beta_n \text{ and } (x, 2b_n + 1) \leftrightarrow (w, 4b_n + 1) \forall x \in U\}.$$

then

$$\mathbb{P}(\hat{\mathcal{A}}(w)^c, \hat{\xi}_{b_n}^{w, 4b_n+1} \neq \emptyset) = o(1/n).$$

Moreover, $\mathcal{A}(v)$ and $\hat{\mathcal{A}}(w)$ are independent for all v, w . Now the result will follow if we can show that for any $A, B \subset S(u)$ with $|A|, |B|$ larger than $\bar{\lambda}\beta_n/(16e)$

$$(76) \quad \mathbb{P}(A \times \{2b_n\} \xleftrightarrow{S(u)} B \times \{2b_n + 1\}) = 1 - o(1/n),$$

where the notation

$$A \times \{2b_n\} \xleftrightarrow{S(u)} B \times \{2b_n + 1\}$$

means that there is an infection path inside $S(u)$ from a vertex in A at time $2b_n$ to a vertex in B at time $2b_n + 1$. To prove (76), define

$$\bar{A} = \{x \in A \setminus \{u\} : \bar{A}_x \text{ holds}\},$$

$$\bar{B} = \{y \in B \setminus \{u\} : \bar{A}_y \text{ holds}\},$$

where

$$\bar{A}_x = \{\text{there is no recovery at } x \text{ between } 2b_n \text{ and } 2b_n + 1\}.$$

Observe that

$$\mathbb{P}(\bar{A}_x \text{ holds}) = 1 - e^{-1}.$$

Therefore, the standard large deviations results show that $|\bar{A}|$ and $|\bar{B}|$ are larger than $(1 - e^{-1})\bar{\lambda}\beta_n/(32e)$, with probability at least $1 - o(1/n)$. Now let

$$\mathcal{E} = \{|\bar{A}| \geq (1 - e^{-1})\bar{\lambda}\beta_n/(32e)\} \cap \{|\bar{B}| \geq (1 - e^{-1})\bar{\lambda}\beta_n/(32e)\}.$$

We set

$$\varepsilon_n = \frac{1}{(\log n)\sqrt{k_n}} \quad \text{and} \quad J_n = \left\lceil \frac{(\log n)\sqrt{k_n}}{2} \right\rceil,$$

with (k_n) as in (70), and define for $0 \leq j \leq J_n - 1$

$$\begin{aligned} C_j &= \{ \text{there is no recovery at } u \text{ between } 2b_n + 2j\varepsilon_n \text{ and } 2b_n + (2j+2)\varepsilon_n \} \\ &\cap \{ \exists x \in \bar{A} : \text{there is an infection from } x \text{ to } u \text{ between } 2b_n + 2j\varepsilon_n \text{ and } 2b_n + (2j+1)\varepsilon_n \} \\ &\cap \{ \exists y \in \bar{B} : \text{there is an infection from } u \text{ to } y \text{ between } 2b_n + (2j+1)\varepsilon_n \text{ and } 2b_n + (2j+2)\varepsilon_n \}. \end{aligned}$$

Observe that

$$(77) \quad \bigcup_{j=0}^{J_n-1} C_j \subset \left\{ A \times \{2b_n\} \overset{S(u)}{\longleftrightarrow} B \times \{2b_n + 1\} \right\}.$$

Moreover, conditionally on \bar{A} and \bar{B} , the events (C_j) are independent, and

$$\begin{aligned} \mathbb{P}(C_j \mid \bar{A}, \bar{B}) &= e^{-2\varepsilon_n} \mathbb{P}(\text{Bin}(|\bar{A}|, 1 - e^{-\varepsilon_n}) \geq 1) \times \mathbb{P}(\text{Bin}(|\bar{B}|, 1 - e^{-\varepsilon_n}) \geq 1) \\ &\geq 1/2, \end{aligned}$$

on the event \mathcal{E} , if n is large enough. Therefore

$$\mathbb{P} \left(\mathcal{E}, \bigcap_{j=0}^{J_n-1} C_j^c \right) \leq 2^{-J_n} = o(1/n).$$

This together with (77) imply (76), and concludes the proof of the proposition. \square

Proof of Proposition 1.2. Using the same arguments for (44), we have for any sequence (a_n) tending to infinity

$$\mathbb{P} \left(\exists \ell \in [cn/a_n, n/a_n] : \deg(v_\ell) \geq ca_n^{1-\chi} \right) = 1 - o(1),$$

for some $c > 0$. By taking $a_n = \sqrt{n}$, we obtain that

$$(78) \quad \mathbb{P}(D_{n,\max} \geq cn^{(1-\chi)/2}) = 1 - o(1).$$

Now Proposition 1.2 follows from Proposition 5.2, Lemma 2.5 and (78). \square

6. APPENDIX: PROOF OF (19)

We closely follow the proof given in [MVY] for Galton-Watson trees. However, we have to take care that in our situation the degrees of the vertices are not independent as in Galton-Watson trees. This leads to some complications.

To simplify the computation, we consider a modified version of the Pólya-point graph defined as follows: we use the same construction except that now $m_v = m$ and $\gamma_v \sim F'$ for all vertices. Then, the new tree stochastically dominates the original tree (note that $F \preceq F'$) and thus, it is sufficient to prove the upper bound for the contact process on this new tree. In this appendix, we always consider this modified graph, and for simplicity, we use the same notation as for the Pólya-point graph. Now our goal is to prove that

$$(79) \quad \mathbb{P}(\xi_t^o \neq \emptyset \forall t \geq 0) \lesssim \lambda^{1+\frac{2}{\psi}} |\log \lambda|^{-\frac{1}{\psi}},$$

where $(\xi_t^o)_{t \geq 0}$ is the contact process on the modified Pólya-point graph starting from the root o .

To describe more precisely the distribution of the Pólya-point graph, we recall a basic fact of Poisson processes (see for example [DV, Chapter 2]).

Claim. For any $a < b$, let $\Lambda(a, b)$ be the set of arrivals of the Poisson process on $[a, b]$ with intensity $f(x)$. Then conditional on $|\Lambda(a, b)| = k$, these k arrivals are independently distributed on $[a, b]$ with the same density $f(x)/\int_a^b f(t)dt$.

We observe that for any vertex v in the Pólya-point graph, conditioned on its position x_v and its number of descendants $m + k$, $x_{(v,1)}, \dots, x_{(v,m)}$ are i.i.d. uniform random variables on $[0, x_v]$, and $x_{(v,m+1)}, \dots, x_{(v,m+k)}$ are arrivals of a Poisson process on $[x_v, 1]$ with intensity $\gamma_v \frac{\psi x^{\psi-1}}{1-x_v^\psi} dx$ (conditional on having k arrivals). Therefore $x_{(v,m+1)}, \dots, x_{(v,m+k)}$ are independently distributed on $[x_v, 1]$ with the same density $\frac{\psi x^{\psi-1}}{1-x_v^\psi} dx$.

On the other hand, similarly to (22), for all v with $a = m + 2mr + 1$,

$$\begin{aligned} p(k, x) &= \mathbb{P}(\text{number of descendants of } v = m + k \mid x_v = x) \\ &= \frac{\Gamma(k + a)}{\Gamma(a)k!} (1 - x^\psi)^k x^{a\psi} \\ (80) \quad &\asymp k^{a-1} (1 - x^\psi)^k x^{a\psi}, \end{aligned}$$

since $\Gamma(k + a)/k! \asymp k^{a-1}$. Moreover,

$$\begin{aligned} \sum_{k \leq M} k^{a-1} (1 - x^\psi)^k &\asymp \sum_{k=0}^{M \wedge [x^{-\psi}]} k^{a-1} + \int_{M \wedge x^{-\psi}}^M \exp(-tx^\psi) t^{a-1} dt \\ &\asymp (M \wedge x^{-\psi})^a + x^{-a\psi} \int_{Mx^\psi \wedge 1}^{Mx^\psi} \exp(-u) u^{a-1} du \\ &\asymp (M \wedge x^{-\psi})^a. \end{aligned}$$

Therefore

$$(81) \quad \sum_{k \leq M} p(k, x) \asymp (M \wedge x^{-\psi})^a x^{a\psi}.$$

Similarly,

$$(82) \quad \sum_{k \leq M} kp(k, x) \asymp (M \wedge x^{-\psi})^{a+1} x^{a\psi},$$

$$(83) \quad \sum_{k \leq M} k^2 p(k, x) \asymp (M \wedge x^{-\psi})^{a+2} x^{a\psi},$$

Using (82) and taking M tend to infinity, we have

$$(84) \quad \sum_{k \geq 0} kp(k, x) = \mathcal{O}(x^{-\psi}).$$

Let $r > 0$ and $M \in \mathbb{N}$ be given. For any vertex v , define the truncated tree starting from v as

$$\begin{aligned} T_{r,M}^v &= \{v\} \cup \{w \text{ descendant of } v : d(v, w) \leq r, \deg(y) \leq M \\ &\quad \text{for all } y \notin \{v, w\} \text{ in the geodesic from } v \text{ to } w\}, \end{aligned}$$

and for $1 \leq i \leq r$, set

$$\begin{aligned} T_{i,r,M}^v &= \{w \in T_{r,M}^v : d(v, w) = i\}. \\ S_{i,r,M}^v &= \{w : d(v, w) = i \text{ and } w \text{ is a leaf of } T_{r,M}^v\}. \end{aligned}$$

If $v = o$, we simply write $T_{r,m}$, $T_{i,r,m}$, and $S_{i,r,m}$. By definition, if $v \neq o$, then

$$(85) \quad |T_{1,r,M}^v| = \deg(v) - 1,$$

$$(86) \quad |S_{1,r,M}^v| = |\{w : w \text{ is a child of } v \text{ with } \deg(w) > M\}|,$$

$$(87) \quad |T_{i+1,r,M}^v| = \sum_{j=1}^{\deg(v)-1} |T_{i,r,M}^{(v,j)}| \mathbf{1}(\deg((v, j)) \leq M) \quad \text{for } 1 \leq i \leq r-1,$$

$$(88) \quad |S_{i+1,r,M}^v| = \sum_{j=1}^{\deg(v)-1} |S_{i,r,M}^{(v,j)}| \mathbf{1}(\deg((v, j)) \leq M) \quad \text{for } 1 \leq i \leq r-2,$$

$$(89) \quad |S_{r,r,M}^v| = |T_{r,r,M}^v|.$$

If $v = o$, we just replace $\deg(o) - 1$ by $\deg(o)$ in these equations.

As in [MVY] (more precisely, Sections 6.1, 6.2 and 6.3), we can see that the proof of (79) follows from the four following lemmas.

Lemma 6.1. *There is a positive constant C , such that for all $1 \leq i \leq r$ and $M \geq m + 1$,*

$$\mathbb{E}(|T_{i,r,M}|) \leq C^i (\log M)^{i-1}.$$

Furthermore for all $1 \leq i \leq r-1$ and $M \geq m + 1$,

$$\mathbb{E}(|S_{i,r,M}|) \leq C^i (\log M)^{i-1} M^{-1/\psi}.$$

In fact, the bound for $\mathbb{E}(|T_{i,r,M}|)$ plays the same role as the estimate (6.2) in [MVY], and the bound for $\mathbb{E}(|S_{i,r,M}|)$ is similar to (6.1) in [MVY].

Lemma 6.2. *For all $r > 0$ and $M \geq m + 1$, we have*

$$\mathbb{E}(|S_{1,r,M}| \mathbf{1}(|S_{1,r,M}| \geq 2) \mid \deg(o) \leq M) = \mathcal{O}(M^{-1-1/\psi} \log M).$$

This result is used in the estimate in (6.7) in [MVY].

Lemma 6.3. *For all $r > 0$ and $M' \geq M \geq m + 1$, we have*

$$\mathbb{P}(\deg(o) \leq M, |S_{1,r,M}| = 1) = \mathcal{O}(M^{-1/\psi}),$$

$$\mathbb{P}(\deg(o^*) \geq M' \mid \deg(o) \leq M, S_{1,r,M} = \{o^*\}) = \mathcal{O}\left((M/M')^{1/\psi}\right).$$

The first estimate is used for the event B_4^4 in [MVY], and the second one is an analogue of the bound for $q[M', \infty)/q[M, \infty)$ in Proposition 6.3 (at the first line of page 27).

For any $r > 0$ and $M' \geq M \geq m + 1$, we define the conditional probability measures

$$\begin{aligned} \mathbb{Q}_1(\cdot) &= \mathbb{P}(\cdot \mid \deg(o) \leq M, |S_{1,r,M}| = 1), \\ \mathbb{Q}_2(\cdot) &= \mathbb{P}(\cdot \mid \deg(o) \leq M, S_{1,r,M} = \{o^*\}, \deg(o^*) = M'). \end{aligned}$$

We call T^* the tree T rooted at o^* . Similarly as for T , we also define $T_{i,r,M}^*$, $S_{i,r,M}^*$.

Lemma 6.4. *There is a positive constant C , such that for all $1 \leq i \leq r$ and $M' \geq M \geq m + 1$*

$$\mathbb{E}_{\mathbb{Q}_1}(|T_{i,r,M}|) \leq (C \log M)^i.$$

$$\mathbb{E}_{\mathbb{Q}_2}(|T_{i,r,M}^*|) \leq C^i (\log M)^{i-1} M'.$$

Furthermore for all $1 \leq i \leq r - 1$ and $M \geq m + 1$,

$$\mathbb{E}_{\mathbb{Q}_2}(|S_{i,r,M}^*|) \leq C^i (\log M)^{i-1} M' M^{-1}.$$

The bound for $\mathbb{E}_{\mathbb{Q}_1}(|T_{i,r,M}|)$ is used in (6.12) in [MVY], the bounds for $\mathbb{E}_{\mathbb{Q}_2}(|T_{i,r,M}^*|)$ and $\mathbb{E}_{\mathbb{Q}_2}(|S_{i,r,M}^*|)$ are used in their Section 6.3.

Assume that Lemmas 6.1–6.4 hold for a moment, we now prove the upper bound of the survival probability of the contact process on T .

Proof of (79). Using the same notation in [MVY], we set

$$M = \left\lceil \frac{1}{8\lambda^2} \right\rceil \quad \text{and} \quad R = \left\lceil \frac{2/\psi + 5}{2/\psi - 1} \right\rceil + 1.$$

and define

$$\begin{aligned} B_1^4 &= \{\deg(o) > M\}, \\ B_2^4 &= \left\{ \deg(o) \leq M, (o, 0) \leftrightarrow \left(\bigcup_{i=2}^R S_{i,r,M} \right) \times \mathbb{R}_+ \text{ in } T_{R,M} \right\}, \\ B_3^4 &= \{\deg(o) \leq M, |S_{1,R,M}| \geq 2, (o, 0) \leftrightarrow S_{1,R,M} \times \mathbb{R}_+ \text{ in } T_{R,M}\}. \end{aligned}$$

When $S_{1,R,M} = \{o^*\}$, let $0 < t^* < t^{**}$ be the first two arrival times of the process $D_{(o,o^*)}$ (the Poisson process of intensity λ representing the infections from o to o^*). Then, we define

$$B_4^4 = \{\deg(o) \leq M, |S_{1,R,M}| = 1, t^{**} < \inf D_o\}.$$

We say that o^* becomes infected *directly* if $t^* < \inf D_o$. We say that it becomes infected *indirectly* if there are infection paths from o to o^* but all these paths must visit at least one vertex different from o and o^* . Define

$$B_5^4 = \{\deg(o) \leq M, |S_{1,R,M}| = 1, \exists y \in T_{R,M}, 0 < s < t : (o, 0) \leftrightarrow (y, s) \leftrightarrow (o, t) \text{ inside } T_{R,M}\}.$$

Note that if o^* becomes infected indirectly, then B_5^4 must occur. Let us define

$$B_6^4 = \{\deg(o) \leq M, |S_{1,R,M}| = 1, t^* < \inf D_o, (o^*, t^*) \leftrightarrow B(o, R)^c \times [t^*, \infty)\}.$$

Then

$$(90) \quad \{(o, 0) \leftrightarrow B(o, R)^c \times \mathbb{R}_+\} \subset \bigcup_{i=1}^6 B_i^4.$$

Event B_1^4 . We observe that by (80)

$$\begin{aligned}
\mathbb{P}(\deg(o) > M \mid x_o = x) &\asymp x^{\alpha\psi} \sum_{k>M} k^{\alpha-1} (1-x^\psi)^k \\
&\lesssim y^{\alpha\psi} \int_M^\infty t^{\alpha-1} \exp(-tx^\psi) dt \\
&\asymp \int_{Mx^\psi}^\infty t^{\alpha-1} \exp(-t) dt \\
&\lesssim \exp(-Mx^\psi/2).
\end{aligned}$$

By combining this with the fact that x_o is distributed on $[0, 1]$ with density $(\psi + 1)x^\psi dx$, we get

$$\begin{aligned}
\mathbb{P}(B_1^4) &= \int_0^1 \mathbb{P}(\deg(o) > M \mid x_o = x) (\psi + 1) x^\psi dx \\
&\lesssim \int_0^1 \exp(-Mx^\psi/2) x^\psi dx \\
&\lesssim M^{-1-1/\psi} = \mathcal{O}(\lambda^{2+2/\psi}).
\end{aligned}$$

Event B_2^4 . As in [MVY], we have

$$\begin{aligned}
\mathbb{P}(B_2^4) &\leq \sum_{i=2}^R (2\lambda)^i \mathbb{E}(|S_{i,R,M}|) \\
&\leq M^{-1/\psi} \sum_{i=2}^{R-1} (C\lambda \log M)^i + (C\lambda \log M)^R \\
&\lesssim \lambda^{-2/\psi} \sum_{i=2}^{R-1} (\lambda |\log \lambda|)^i + (\lambda |\log \lambda|)^R \\
&\lesssim \lambda^{3/2+2/\psi}.
\end{aligned}$$

Here, for the first inequality we have used Lemma 6.1 and (89).

Event B_3^4 . As in [MVY], we have for λ small enough

$$\begin{aligned}
\mathbb{P}(B_3^4) &\leq (2\lambda) \mathbb{E}(|S_{1,R,M}| \mathbf{1}(|S_{1,R,M}| \geq 2) \mid \deg(o) \leq M) \\
&= \mathcal{O}(\lambda M^{-1-1/\psi} \log M) = \mathcal{O}(\lambda \times \lambda^{2+2/\psi} |\log \lambda|) \\
&= \mathcal{O}(\lambda^{2+2/\psi}),
\end{aligned}$$

where we have used Lemma 6.2 in the second line.

Event B_4^4 . The number of transmissions from o to o^* before time t has Poisson distribution with parameter λt . Thus,

$$\begin{aligned}
\mathbb{P}(B_4^4) &= \mathbb{P}(\deg(o) \leq M, |S_{1,R,M}| = 1) \int_0^\infty \mathbb{P}(\text{Poi}(\lambda t) \geq 2) e^{-t} dt \\
&= \mathcal{O}(\lambda^{2/\psi}) \int_0^\infty \lambda^2 t^2 e^{-t} dt \\
&= \mathcal{O}(\lambda^{2+2/\psi}).
\end{aligned}$$

Note that in the second line, we have used Lemma 6.3 to estimate the first term and the fact that $\mathbb{P}(\text{Poi}(u) \geq 2) \leq u^2$ to bound the second one.

Event B_5^4 . As in [MVY],

$$\mathbb{P}(B_5^4) \leq \mathbb{P}(\text{deg}(o) \leq M, |S_{1,R,M}| = 1) \mathbb{P} \left(\begin{array}{c} \exists y \in T_{R,M}, 0 < s < t : \\ (o, 0) \leftrightarrow (y, s) \leftrightarrow (o, t) \text{ inside } T_{R,M} \end{array} \middle| \begin{array}{c} \text{deg}(o) \leq M, \\ |S_{1,R,M}| = 1 \end{array} \right).$$

By Lemma 6.3, the first term is $\mathcal{O}(\lambda^{2/\psi})$. On the other hand, by the same argument in [MVY] the second one is bounded by

$$\begin{aligned} & \sum_{i=1}^R \lambda^{2i} \times \mathbb{E}(|\{x \in T_{R,M} : d(o, x) = i\}| \mid \text{deg}(o) \leq M, |S_{1,R,M}| = 1) \\ &= \sum_{i=1}^R \lambda^{2i} \times \mathbb{E}_{\mathbb{Q}_1}(|T_{i,R,M}|) \\ &\leq \sum_{i=1}^R (C\lambda^2 |\log \lambda|)^i \\ &= \mathcal{O}(\lambda^{3/2}). \end{aligned}$$

Thus we have

$$\mathbb{P}(B_5^4) = \mathcal{O}(\lambda^{3/2+2/\psi}).$$

Event B_6^4 . We have

$$\begin{aligned} \mathbb{P}(B_6^4) &\leq \mathbb{P}(\text{deg}(o) \leq M, |S_{R,M}^1| = 1) \times \mathbb{P}(t^* < \inf D_o) \\ &\quad \times \mathbb{P}((o^*, t^*) \leftrightarrow B(o, R)^c \times [t^*, \infty) \mid \text{deg}(o) \leq M, |S_{R,M}^1| = 1, t^* < \inf D_o) \\ &\lesssim \lambda^{2/\psi} \times \lambda \times \mathbb{P}((o^*, 0) \leftrightarrow B(o, R)^c \times \mathbb{R}_+ \mid \text{deg}(o) \leq M, |S_{R,M}^1| = 1). \end{aligned}$$

Note that we have used Lemma 6.3 to bound the first probability. Now, it remains to bound the third term. We observe that for any $M' > M$,

$$\begin{aligned} & \mathbb{P}((o^*, 0) \leftrightarrow B(o, R)^c \times \mathbb{R}_+ \mid \text{deg}(o) \leq M, |S_{1,R,M}| = 1) \\ &= \sum_{k=M+1}^{\infty} \mathbb{P}((o^*, 0) \leftrightarrow B(o, R)^c \times \mathbb{R}_+ \mid \text{deg}(o^*) = k, \text{deg}(o) \leq M, S_{1,R,M} = \{o^*\}) \\ &\quad \times \mathbb{P}(\text{deg}(o^*) = k \mid \text{deg}(o) \leq M, S_{1,R,M} = \{o^*\}) \\ &\leq \mathbb{P}((o^*, 0) \leftrightarrow B(o, R)^c \times \mathbb{R}_+ \mid \text{deg}(o^*) = M', \text{deg}(o) \leq M, S_{1,R,M} = \{o^*\}) \\ &\quad + \mathbb{P}(\text{deg}(o^*) \geq M' \mid \text{deg}(o) \leq M, S_{1,R,M} = \{o^*\}) \\ &\lesssim \mathbb{Q}_2((o^*, 0) \leftrightarrow B(o, R)^c \times \mathbb{R}_+) + (M/M')^{1/\psi}. \end{aligned}$$

Here, we used Lemma 6.3 to bound the second term and \mathbb{Q}_2 is the conditional probability depending on M' which was defined in Lemma 6.4.

As in [MVY], we take

$$M' = \lceil \varepsilon_1 \lambda^{-2} |\log \lambda| \rceil,$$

with

$$\varepsilon_1 = \varepsilon'_1/64 \quad \text{and} \quad \varepsilon'_1 = \min\{(2/\psi - 1), 2\}/4.$$

Then the second term is of order

$$(M/M')^{1/\psi} \asymp |\log \lambda|^{-1/\psi}.$$

To bound the first term, we notice that

$$\mathbb{Q}_2((o^*, 0) \leftrightarrow B(o, R)^c \times \mathbb{R}_+) \leq \mathbb{Q}_2((o^*, 0) \leftrightarrow B(o^*, R-1)^c \times \mathbb{R}_+).$$

Hence, it remains to prove the following result.

Lemma 6.5. *There exists $\delta > 0$, such that*

$$\mathbb{Q}_2((o^*, 0) \leftrightarrow B(o^*, R-1)^c \times \mathbb{R}_+) < \lambda^\delta.$$

Proof. We follow the proof and notation in [MVY], let $R' = R - 1$, and $L_1 = \lceil \lambda^{-\epsilon'_1/2} \rceil$. Then we define

$$\begin{aligned} \phi(T^*) &= \sum_{i=1}^{R'} (2\lambda)^i |S_{i,R',M}^*(T)|, \\ \psi(T^*) &= \sum_{i=2}^{R'} (2\lambda)^{2i} |T_{i,R',M}^*|, \end{aligned}$$

where T^* is the tree T rooted at o^* and $T_{i,r,M}^*, S_{i,r,M}^*$ are defined in Lemma 6.4. We now define

$$\begin{aligned} B_1^5 &= \{\phi(T^*) > \lambda^{\epsilon'_1}\}, & B_2^5 &= \{\psi(T^*) > \lambda^{\epsilon'_1}\}, \\ B_3^5 &= (B_1^5 \cup B_2^5)^c \cap \left\{ \{o^*\} \times [0, L_1] \leftrightarrow \left(\bigcup_{i=1}^{R'} S_{i,R',M}^* \right) \times \mathbb{R}_+ \right\}, \\ B_4^5 &= (B_1^5 \cup B_2^5)^c \cap \{ \exists z : d(o^*, z) \geq 2, \{o^*\} \times [0, L_1] \leftrightarrow (z, s) \leftrightarrow \{o^*\} \times [s, \infty) \\ & \hspace{15em} \text{inside } T(z) \cap T_{R',M} \}, \\ B_5^5 &= \{B(o^*, 1) \times \{0\} \leftrightarrow B(o^*, 1) \times \{L_1\} \text{ inside } B(o^*, 1)\}. \end{aligned}$$

It is explained in [MVY] that

$$\{(o^*, 0) \leftrightarrow B(o^*, R')^c \times \mathbb{R}_+\} \subset \bigcup_{i=1}^5 B_i^5.$$

Event B_1^5 . Similarly to B_1^4 , using Lemma 6.4, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_2}(\phi(T^*)) &\lesssim \sum_{i=1}^{R'-1} (2\lambda)^i (\log M)^{i-1} M' M^{-1} + (2\lambda)^{R'} M' M^{-1} (\log M)^{R'} \\ &\lesssim \lambda^{3/4}. \end{aligned}$$

Then using Markov's inequality we get

$$\mathbb{Q}_2(B_1^5) = \mathcal{O}(\lambda^{3/4-\epsilon'_1}) = \mathcal{O}(\lambda^{1/4}),$$

since $\epsilon'_1 \leq 1/2$.

Event B_2^5 . We have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_2}(\psi(T^*)) &\lesssim M' \sum_{i=2}^{R'} (2\lambda)^{2i} (\log M)^{(i-1)} \\ &\lesssim \lambda^{3/2}. \end{aligned}$$

Then it follows from Markov's inequality that

$$\mathbb{Q}_2(B_2^5) = \mathcal{O}(\lambda^{3/2-\epsilon'_1}) = \mathcal{O}(\lambda).$$

The events B_3^5, B_4^5, B_5^5 are exactly the same as in [MVY]. \square

We now conclude the proof of (19). By (90) and the estimates of $(B_i^4)_{i \leq 5}$, we obtain

$$\begin{aligned} \mathbb{P}((o, 0) \leftrightarrow B(o, R)^c \times \mathbb{R}_+) &\lesssim \mathbb{P}(B_6^4) \\ &\lesssim \lambda^{1+2/\psi} (\lambda^\delta + |\log \lambda|^{-1/\psi}) \\ &= \mathcal{O}(\lambda^{1+2/\psi} |\log \lambda|^{-1/\psi}), \end{aligned}$$

which proves the desired result. \square

Proof of Lemma 6.1. Since we fix r and M , we omit it in the notation. Let us define for $1 \leq i \leq r$

$$f_i(x) = \mathbb{E}(|T_i^v| \mathbf{1}(\deg(v) \leq M) \mid x_v = x),$$

where v is any vertex different from the root o . For $1 \leq i \leq r-1$,

$$\begin{aligned} f_{i+1}(x) &= \mathbb{E}(|T_{i+1}^v| \mathbf{1}(\deg(v) \leq M) \mid x_v = x) \\ &= \sum_{k \leq M-m-1} \mathbb{E}(|T_{i+1}^v| \mathbf{1}(\deg(v) = m+1+k) \mid x_v = x) \\ &= \sum_{k \leq M-m-1} \mathbb{E}(|T_{i+1}^v| \mid x_v = x, \deg(v) = m+1+k) p(k, x) \\ &= \sum_{k \leq M-m-1} \sum_{j=1}^{m+k} \mathbb{E}(|T_i^{(v,j)}| \mathbf{1}(\deg((v,j)) \leq M) \mid x_v = x, \deg(v) = m+1+k) p(k, x), \end{aligned}$$

where for the last line, we used (87).

On the event $\{x_v = x, \deg(v) = m+1+k\}$, $x_{(v,1)}, \dots, x_{(v,m)}$ are uniformly distributed on $[0, x]$ and $x_{(v,m+1)}, \dots, x_{(v,m+k)}$ are distributed on $[x, 1]$ with density $\frac{\psi y^{\psi-1}}{1-x^\psi} dy$. Therefore

$$\begin{aligned} f_{i+1}(x) &= \sum_{k \leq M-m-1} \left(\frac{m}{x} \int_0^x f_i(y) dy + \frac{k}{1-x^\psi} \int_x^1 \psi y^{\psi-1} f_i(y) dy \right) p(k, x) \\ (91) \quad &\leq \frac{m}{x} \int_0^x f_i(y) dy + \frac{F(M, x)}{1-x^\psi} \int_x^1 \psi y^{\psi-1} f_i(y) dy, \end{aligned}$$

where

$$F(M, x) = \sum_{k \leq M-m-1} k p(k, x).$$

Moreover, it follows from (85) that

$$(92) \quad f_1(x) = \mathbb{E}((\deg(v) - 1) \mathbf{1}(\deg(v) \leq M) \mid x_v = x) = m + F(M, x).$$

Hence by (82),

$$(93) \quad f_1(x) \lesssim F^*(M, x) := (M \wedge x^{-\psi})^{a+1} x^{a\psi} + 1.$$

After some simple computations, we have

$$(94) \quad \frac{1}{x} \int_0^x F^*(M, y) dy \lesssim F^*(M, x) \log M.$$

and

$$(95) \quad \frac{1}{1-x^\psi} \int_x^1 \psi y^{\psi-1} F^*(M, y) dy \lesssim \log M.$$

From (91), (92), (93), (94) and (95), we can prove by induction that for $1 \leq i \leq r$,

$$(96) \quad f_i(x) \leq C^i (\log M)^{i-1} F^*(M, x).$$

for some constant $C > 0$. Similarly to (91), we also have

$$\mathbb{E}(|T_i| \mid x_o = x) \leq \frac{m}{x} \int_0^x f_{i-1}(y) dy + \sum_{k \geq 0} \frac{kp(k, x)}{1 - x^\psi} \int_x^1 \psi y^{\psi-1} f_{i-1}(y) dy.$$

It follows from this estimate, (84) and (96) that

$$\mathbb{E}(|T_i| \mid x_o = x) \leq C^i (\log M)^{i-1} (F^*(M, x) + x^{-\psi}).$$

Hence using that $x_o \sim \mathcal{U}([0, 1])^x$ with $\chi = 1/(\psi + 1)$, we get that for $i \leq r$

$$\begin{aligned} \mathbb{E}(|T_i|) &= (\psi + 1) \int_0^1 \mathbb{E}(|T_i| \mid x_o = x) x^\psi dx \\ &\leq C^i (\log M)^{i-1}. \end{aligned}$$

Now, to estimate $\mathbb{E}(|S_i|)$, we define for $1 \leq i \leq r - 1$

$$g_i(x) = \mathbb{E}(|S_i^v| 1(\deg(v) \leq M) \mid x_v = x).$$

As for $f_i(x)$, we also have for $1 \leq i \leq r - 2$,

$$(97) \quad g_{i+1}(x) \leq \frac{m}{x} \int_0^x g_i(y) dy + \frac{F(M, x)}{1 - x^\psi} \int_x^1 \psi y^{\psi-1} g_i(y) dy,$$

and by (86),

$$(98) \quad \begin{aligned} g_1(x) &= \mathbb{E}(|\{w : w \text{ is a child of } v \text{ with } \deg(w) > M\}| 1(\deg(v) \leq M) \mid x_v = x) \\ &\leq \frac{m}{x} \int_0^x g_0(y) dy + \frac{F(M, x)}{1 - x^\psi} \int_x^1 \psi y^{\psi-1} g_0(y) dy, \end{aligned}$$

where

$$g_0(y) = \mathbb{P}(\deg(w) > M \mid x_w = y).$$

It follows from (80) that

$$\begin{aligned} g_0(y) &\asymp y^{a\psi} \sum_{k > M} k^{a-1} (1 - y^\psi)^k \\ &\lesssim y^{a\psi} \int_M^\infty t^{a-1} \exp(-ty^\psi) dt \\ &\asymp \int_{My^\psi}^\infty t^{a-1} \exp(-t) dt \\ &\lesssim \exp(-My^\psi/2). \end{aligned}$$

On the other hand, if $1 > My^\psi > 1/2$ then

$$\begin{aligned} g_0(y) &\gtrsim y^{a\psi} \sum_{k=M+1}^{2M} k^{a-1} (1 - y^\psi)^k \\ &\gtrsim 1. \end{aligned}$$

Therefore

$$(99) \quad 1(1/2 < My^\psi < 1) \lesssim g_0(y) \lesssim \exp(-My^\psi/2).$$

Let us define for $M \geq 2$

$$(100) \quad \alpha = p_{L,M}(x) = \mathbb{P}(\deg((v, 1)) > M \mid x_v = x) = \frac{1}{x} \int_0^x g_0(y) dy$$

$$(101) \quad \beta = p_{R,M}(x) = \mathbb{P}(\deg((v, m+1)) > M \mid x_v = x) = \frac{1}{1-x^\psi} \int_x^1 \psi y^{\psi-1} g_0(y) dy.$$

Then using (99) we obtain that

$$(102) \quad (Mx^\psi)^{-1/\psi} \mathbf{1}(Mx^\psi \geq 1) \lesssim \alpha \lesssim \mathbf{1}(Mx^\psi < 1) + (Mx^\psi)^{-1/\psi} \mathbf{1}(Mx^\psi \geq 1)$$

$$(103) \quad \beta \lesssim M^{-1} \exp(-Mx^\psi/2).$$

Define

$$G^*(M, x) = \mathbf{1}(Mx^\psi < 1) + (Mx^\psi)^{-1/\psi} \mathbf{1}(Mx^\psi \geq 1).$$

Then

$$\beta F(M, x) = \mathcal{O}(G^*(M, x)).$$

Therefore using this, (98), (102) and (103) we get

$$(104) \quad g_1(x) = \mathcal{O}(G^*(M, x)).$$

Furthermore,

$$(105) \quad \frac{1}{x} \int_0^x G^*(M, y) dy = \mathcal{O}((\log M)G^*(M, x)),$$

$$(106) \quad \frac{1}{1-x^\psi} \int_x^1 \psi y^{\psi-1} G^*(M, y) dy = \mathcal{O}(M^{-1} \mathbf{1}(Mx^\psi < 1) + M^{-1/\psi} \mathbf{1}(Mx^\psi \geq 1)).$$

Hence

$$\frac{m}{x} \int_0^x G^*(M, y) dy + \frac{F(M, x)}{1-x} \int_x^1 G^*(M, y) dy = \mathcal{O}((\log M)G^*(M, x)).$$

From this estimate, (97) and (104), we can prove by induction that for $1 \leq i \leq r-1$

$$(107) \quad g_i(x) = \mathcal{O}((\log M)^{i-1} G^*(M, x)).$$

We now have

$$\begin{aligned} \mathbb{E}(|S_i| \mid x_o) &\leq \frac{m}{x_o} \int_0^{x_o} g_{i-1}(y) dy + \sum_{k \geq 0} \frac{kp(k, x_o)}{1-x_o^\psi} \int_{x_o}^1 \psi y^{\psi-1} g_{i-1}(y) dy \\ &= \mathcal{O} \left(\frac{1}{x_o} \int_0^{x_o} g_{i-1}(y) dy + \frac{x_o^{-\psi}}{(1-x_o^\psi)} \int_{x_o}^1 \psi y^{\psi-1} g_{i-1}(y) dy \right) \\ &= \mathcal{O} \left((\log M)^{i-1} [G^*(M, x_o) + x_o^{-\psi} (M^{-1} \mathbf{1}(Mx^\psi < 1) + M^{-1/\psi} \mathbf{1}(Mx^\psi \geq 1))] \right). \end{aligned}$$

Finally,

$$\mathbb{E}[G^*(M, x_o) + x_o^{-\psi} (M^{-1} \mathbf{1}(Mx^\psi < 1) + M^{-1/\psi} \mathbf{1}(Mx^\psi \geq 1))] = \mathcal{O}(M^{-1/\psi}).$$

Then the result follows from the last two estimates. \square

Proof of Lemma 6.2. We also omit here r and M in the notation. Then

$$\begin{aligned} &\mathbb{E}(|S_1| \mathbf{1}(|S_1| \geq 2) \mid \deg(o) \leq M, x_o = x) \\ &\leq \sum_{k \leq M} \mathbb{E}(|S_1| \mathbf{1}(|S_1| \geq 2) \mid x_o = x, \deg(o) = m+k) p(k, x). \end{aligned}$$

Conditionally on the event $\{x_o = x, \deg(o) = m + k\}$, $|S_1|$ has the same distribution as

$$X = X_1 + \dots + X_m + Y_1 + \dots + Y_k,$$

where (X_i) and (Y_j) are independent Bernoulli random variables with mean $\alpha = p_{L,M}(x)$ and $\beta = p_{R,M}(x)$ respectively, as defined in (100) and (101). Then

$$\begin{aligned} \mathbb{E}(X1(X \geq 2)) &= \mathbb{E}(X) - \mathbb{P}(X = 1) \\ &= m\alpha + k\beta - m\alpha(1 - \alpha)^{m-1}(1 - \beta)^k - k\beta(1 - \alpha)^m(1 - \beta)^{k-1} \\ &\leq (m\alpha + k\beta)^2 \\ &\leq 2(m^2\alpha^2 + k^2\beta^2). \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E}(|S_1|1(|S_1| \geq 2) \mid \deg(o) \leq M, x_o = x) &\leq \sum_{k \leq M} (2m^2\alpha^2 + 2k^2\beta^2)p(k, x) \\ &\leq 2m^2\alpha^2 + 2\beta^2 \sum_{k \leq M} k^2 p(k, x). \end{aligned}$$

We now take the expectation with respect to x_o . Since it has density $(\psi + 1)x^\psi dx$ on $[0, 1]$, the expectation of the first term is of order

$$\begin{aligned} \int_0^1 \alpha^2 x^\psi dx &\lesssim \int_0^1 [1(Mx^\psi < 1) + (Mx^\psi)^{-2/\psi} 1(Mx^\psi \geq 1)] x^\psi dx \\ &\lesssim \int_0^{M^{-1/\psi}} x^\psi dx + M^{-2/\psi} \int_{M^{-1/\psi}}^1 x^{\psi-2} dx \\ &= \mathcal{O}(M^{-1-1/\psi} \log M). \end{aligned}$$

By (83) and (103), the expectation of the second term is equivalent to

$$\begin{aligned} \int_0^1 \beta^2 (M \wedge x^{-\psi})^{a+2} x^{(a+1)\psi} dx &\lesssim M^{-2} \int_0^1 \exp(-Mx^\psi) (M \wedge x^{-\psi})^{a+2} x^{(a+1)\psi} dx \\ &\lesssim M^{-2} \int_0^{M^{-1/\psi}} M^{a+2} x^{(a+1)\psi} dx + M^{-2} \int_{M^{-1/\psi}}^1 \exp(-Mx^\psi) x^{-\psi} dx \\ &\lesssim M^{-1-1/\psi} + M^{-2} \int_1^M e^{-u} \left(\frac{M}{u}\right) M^{-1/\psi} u^{-1+1/\psi} du \\ &= \mathcal{O}(M^{-1-1/\psi}), \end{aligned}$$

where for the third line we used the change of variables $u = Mx^\psi$. Combining these last two estimates, we get the lemma. \square

Proof of Lemma 6.3. With the same α and β as in the previous lemma, we have

$$\begin{aligned} &\mathbb{P}(\deg(o) \leq M, |S_1| = 1 \mid x_o = x) \\ &= \sum_{k \leq M-m} [m\alpha(1 - \alpha)^{m-1}(1 - \beta)^k + k\beta(1 - \alpha)^m(1 - \beta)^{k-1}] p(k, x) \\ &\leq \sum_{k \leq M} [m\alpha + k\beta] p(k, x) \\ &\lesssim \alpha + \beta(M \wedge x^{-\psi})^{a+1} x^{a\psi}. \end{aligned}$$

Then we take expectation in x_o and get

$$\mathbb{P}(\deg(o) \leq M, |S_1| = 1) = \mathcal{O}(M^{-1/\psi}),$$

which proves the first estimate. For the second one, we note that $(1 - \beta)^k \asymp 1$ for $1 \leq k \leq M$ since $\beta = \mathcal{O}(M^{-1})$. Hence using (81), we get

$$\mathbb{P}(\deg(o) \leq M, |S_1| = 1 \mid x_o = x) \gtrsim \alpha(1 - \alpha)^{m-1}(M \wedge x^{-\psi})^a x^{\alpha\psi}.$$

It follows from (102) and the fact that $\alpha \leq 1$ that there is a positive constant C independent of x , such that

$$\alpha \leq 1(Mx^\psi < 1) + (C(Mx^\psi)^{-1/\psi} \wedge 1)1(Mx^\psi \geq 1)$$

Hence, there is a positive constant $c = c(C)$, such that

$$(108) \quad (1 - \alpha)^{m-1} \geq c1(Mx^\psi \geq 1/c).$$

Therefore

$$(109) \quad \begin{aligned} \mathbb{P}(\deg(o) \leq M, |S_1| = 1) &= (\psi + 1) \int_0^1 \mathbb{P}(\deg(o) \leq M, |S_1| = 1 \mid x_o = x) x^\psi dx \\ &\gtrsim \int_0^1 \alpha(1 - \alpha)^{m-1}(M \wedge x^{-\psi})^a x^{\alpha\psi} x^\psi dx \\ &\gtrsim \int_{(cM)^{-1/\psi}}^1 (Mx^\psi)^{-1/\psi} x^\psi dx \\ &\gtrsim M^{-1/\psi}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathbb{P}(\deg(o) \leq M, S_1 = \{o^*\}, \deg(o^*) \geq M' \mid x_o = x) &\leq \sum_{k \leq M-m} [m\alpha' + k\beta'] p(k, x). \\ &\leq m\alpha' + \beta' F(M, x), \end{aligned}$$

with

$$\alpha' = p_{L, M'}(x) \quad \text{and} \quad \beta' = p_{R, M'}(x).$$

Similarly to the calculus for α and β , we get

$$\int_0^1 \alpha' x^\psi dx = \mathcal{O}((M')^{-1/\psi}),$$

and

$$\begin{aligned} \int_0^1 \beta' F(M, x) x^\psi &\lesssim \frac{1}{M'} \int_0^1 e^{-M'x^\psi/2} (M \wedge x^{-\psi})^{a+1} x^{(a+1)\psi} dx \\ &\lesssim (M')^{-1-1/\psi}. \end{aligned}$$

Hence

$$(110) \quad \begin{aligned} \mathbb{P}(\deg(o) \leq M, S_1 = \{o^*\}, \deg(o^*) \geq M') &= \mathcal{O}\left(\int_0^1 (m\alpha' + \beta' F(M, x)) x^\psi dx\right) \\ &= \mathcal{O}((M')^{-1/\psi}). \end{aligned}$$

The result now follows from (109) and (110). □

Proof of Lemma 6.4. We start with the estimate for $|T_{i,r,M}^*|$. Let us define

$$A = \{\deg(o) \leq M\} \quad \text{and} \quad B = \{S_{1,r,M} = \{o^*\}, \deg(o^*) = M'\}.$$

Similarly to the previous lemma,

$$(111) \quad \begin{aligned} & \mathbb{P}(\deg(o) \leq M, S_{1,r,M} = \{o^*\}, \deg(o^*) = M' \mid x_o = x) \\ &= \sum_{k \leq M-m} [m\alpha''(1-\alpha)^{m-1}(1-\beta)^k + k\beta''(1-\alpha)^m(1-\beta)^{k-1}] p(k, x) \end{aligned}$$

$$(112) \quad \gtrsim \alpha''(1-\alpha)^{m-1}(M \wedge x^{-\psi})^a x^{a\psi},$$

where

$$\alpha'' := \mathbb{P}(\deg((0, 1)) = M' \mid x_o = x) = \frac{1}{x} \int_0^x p(M', y) dy$$

and

$$\beta'' := \mathbb{P}(\deg((0, m+1)) = M' \mid x_o = x) = \frac{1}{1-x^\psi} \int_x^1 p(M', y) \psi y^{\psi-1} dy.$$

Similarly to the calculus for α and β , we have

$$(113) \quad \begin{aligned} (M')^{-1-1/\psi} x^{-1} \mathbf{1}(M'x^\psi \geq 1) &\lesssim \alpha'' \lesssim x^\psi (M'x^\psi)^{a-1} \mathbf{1}(M'x^\psi < 1) \\ &\quad + (M')^{-1-1/\psi} x^{-1} \mathbf{1}(M'x^\psi \geq 1), \end{aligned}$$

$$(114) \quad \beta'' \lesssim (M')^{-2} e^{-M'x^\psi/2}.$$

Hence, it follows from (108), (112) and (113) that

$$\mathbb{P}(A \cap B \mid x_o = x) \gtrsim (M')^{-1-1/\psi} x^{-1} \mathbf{1}(Mx^\psi \geq 1/c).$$

Therefore

$$(115) \quad \mathbb{P}(A \cap B) \gtrsim (M')^{-1-1/\psi}.$$

We now prove that

$$\mathbb{E}(|T_{i,r,M}^*| \mathbf{1}(A) \mathbf{1}(B)) \leq C^i (\log M)^{i-1} M' (M')^{-1-1/\psi}.$$

For $i = 1$, observe that on $A \cap B$, $|T_{1,r,M}^*| = \deg(o^*) = M'$. It follows from (111) that

$$\mathbb{P}(A \cap B \mid x_o = x) \leq m\alpha'' + \beta'' F(M, x).$$

Then using that

$$\begin{aligned} \int_0^1 \alpha'' x^\psi dx &= \mathcal{O}((M')^{-1-1/\psi}) \\ \int_0^1 \beta'' F(M, x) x^\psi dx &= \mathcal{O}((M')^{-2-1/\psi}), \end{aligned}$$

we obtain

$$(116) \quad \mathbb{P}(A \cap B) = \mathcal{O}((M')^{-1-1/\psi}).$$

Therefore

$$\mathbb{E}(|T_{1,r,M}^*| \mathbf{1}(A) \mathbf{1}(B)) = \mathcal{O}(M' (M')^{-1-1/\psi}).$$

For $i \geq 2$, we notice that

$$(117) \quad |T_{i,r,M}^*| = |T_{i,r,M}^{o^*}| + \sum_{(0,j) \neq o^*} |T_{i-2,r-2,M}^{(0,j)}|,$$

with the convention $|T_0| = 1$. We see that for any $1 \leq i \leq r$

$$\begin{aligned}
h_{1,i}(x) &= \mathbb{E} \left(\sum_{(0,j) \neq o^*} |T_{i,r,M}^{(0,j)}| 1(A) 1(B) \mid x_o = x \right) \\
&= \sum_{k=0}^{M-m} \mathbb{E} \left(\sum_{(0,j) \neq o^*} |T_{i,r,M}^{(0,j)}| 1(B) \mid x_o = x, \deg(o) = m+k \right) p(k, x) \\
(118) \quad &= \sum_{k=0}^{M-m} \mathbb{E} \left(\sum_{1 \leq j \neq s \leq m+k} |T_{i,r,M}^{(0,j)}| 1(B_{s,k}) \mid x_o = x, \deg(o) = m+k \right) p(k, x),
\end{aligned}$$

where

$$B_{s,k} = \{\deg(o) = m+k\} \cap \{\deg((0,s)) = M', \deg((0,j)) \leq M \forall j \neq s\}.$$

If $j \neq s$, then

$$\begin{aligned}
&\mathbb{E} \left(|T_{i,r,M}^{(0,j)}| 1(B_{s,k}) \mid x_{(0,1)}, \dots, x_{(0,m+k)} \right) \\
&\leq \mathbb{E} \left(|T_{i,r,M}^{(0,j)}| 1(\deg((0,j)) \leq M) \mid x_{(0,j)} \right) \mathbb{P}(\deg((0,s)) = M' \mid x_{(0,s)}) \\
(119) \quad &= f_i(x_{(0,j)}) p(M', x_{(0,s)}).
\end{aligned}$$

Now using this estimate and the fact that $x_{(0,1)}, \dots, x_{(0,m)}$ are uniformly distributed in $[0, x_o]$ and that $x_{(0,m+1)}, \dots, x_{(0,m+k)}$ are distributed in $[x_o, 1]$ with density $\frac{\psi y^{\psi-1} dy}{1-x_o^\psi}$, we deduce from (118) and (119) that

$$\begin{aligned}
(120) \quad h_{1,i}(x) &\leq \left(\frac{1}{x} \int_0^x f_i(y) dy \right) \alpha'' \sum_{k=0}^{M-m} m^2 p(k, x) \\
&+ \left(\frac{1}{1-x^\psi} \int_x^1 \psi y^{\psi-1} f_i(y) dy \right) \beta'' \sum_{k=0}^{M-m} k^2 p(k, x) \\
&+ \left[\left(\frac{1}{x} \int_0^x f_i(y) dy \right) \beta'' + \left(\frac{1}{1-x^\psi} \int_x^1 \psi y^{\psi-1} f_i(y) dy \right) \alpha'' \right] \sum_{k=0}^{M-m} (mk) p(k, x).
\end{aligned}$$

Then using these estimates, (94), (95), (96), we obtain

$$\begin{aligned}
h_{1,i}(x) &\lesssim (\log M)^i \left(\alpha'' F^*(M, x) \sum_{k \leq M} p(k, x) + \beta'' \sum_{k \leq M} k^2 p(k, x) + (\beta'' F^*(M, x) + \alpha'') \right) \\
&= (\log M)^i (H_1(x) + H_2(x) + H_3(x)).
\end{aligned}$$

After some computation, we get

$$\int_0^1 (H_1(x) + H_2(x) + H_3(x)) x^\psi dx = \mathcal{O}((M')^{-1-1/\psi} \log M)$$

Hence

$$\int_0^1 h_{1,i}(x) x^\psi dx \leq (C \log M)^{i+1} (M')^{-1-1/\psi}.$$

Note that in (117), we need an estimate for $h_{1,i-2}(x)$:

$$(121) \quad \int_0^1 h_{1,i-2}(x)x^\psi dx \leq (C \log M)^{i-1}(M')^{-1-1/\psi}.$$

If $j = s$, then

$$\begin{aligned} & \mathbb{E} \left(|T_{i,r,M}^{(0,j)}| 1(B_{j,k}) \mid x_{(0,1)}, \dots, x_{(0,m+k)} \right) \\ & \leq \mathbb{E} \left(|T_{i,r,M}^{(0,j)}| 1(\deg((v,j)) = M') \mid x_{(0,j)} \right) \\ & \leq \left(\frac{m}{x_{(0,j)}} \int_0^{x_{(0,j)}} f_{i-1}(y) dy + \frac{M' - m}{1 - x_{(0,j)}^\psi} \int_{x_{(0,j)}}^1 \psi x^{\psi-1} f_{i-1}(y) dy \right) p(M', x_{(0,j)}) \\ & \leq C^i (\log M)^{i-1} M' p(M', x_{(0,j)}). \end{aligned}$$

Hence

$$\begin{aligned} h_{2,i}(x) &= \mathbb{E} \left(|T_{i,r,M}^{o*}| \mid x_o = x \right) \\ &= \sum_{k=0}^{M-m} \mathbb{E} \left(\sum_{j=1}^{m+k} |T_{i,r,M}^{(0,j)}| 1(B_{j,k}) \mid x_o = x, \deg(o) = m+k \right) p(k, x) \\ &\leq C^i (\log M)^{i-1} M' \sum_{k=0}^{M-m} (m\alpha'' + k\beta'') p(k, x). \end{aligned}$$

Therefore using the same estimate as (116), we get

$$(122) \quad \int_0^1 h_{2,i}(x)x^\psi dx \leq C^i (\log M)^{i-1} M' (M')^{-1-1/\psi}.$$

From (115), (117), (121) and (122) we deduce that

$$\mathbb{E}_{\mathbb{Q}_2}(|T_{i,r,M}^*|) \leq C^i (\log M)^{i-1} M'.$$

We estimate $|S_{i,r,M}^*|$ by the same way. First, as for $h_{1,i}(x)$, we have

$$\begin{aligned} l_{1,i}(x) &= \mathbb{E} \left(\sum_{(0,j) \neq o^*} |S_{i,r,M}^{(0,j)}| 1(A)1(B) \mid x_o = x \right) \\ &\leq \left(\frac{1}{x} \int_0^x g_i(y) dy \right) \alpha'' \sum_{k=0}^{M-m} m^2 p(k, x) \\ &\quad + \left(\frac{1}{1 - x^\psi} \int_x^1 \psi y^{\psi-1} g_i(y) dy \right) \beta'' \sum_{k=0}^{M-m} k^2 p(k, x) \\ &\quad + \left[\left(\frac{1}{x} \int_0^x g_i(y) dy \right) \beta'' + \left(\frac{1}{1 - x^\psi} \int_x^1 \psi y^{\psi-1} g_i(y) dy \right) \alpha'' \right] \sum_{k=0}^{M-m} (mk) p(k, x). \end{aligned}$$

Then using (105), (106), (107) and some computations, we get

$$\int_0^1 l_{1,i-2}(x)x^\psi dx = \mathcal{O}((\log M)^{i-1} M' M^{-1} (M')^{-1-1/\psi}).$$

We now consider

$$\begin{aligned} l_{2,i}(x) &= \mathbb{E}(|S_{i,r,M}^{o*}| \mid x_o = x) \\ &= \sum_{k=0}^{M-m} \mathbb{E} \left(\sum_{j=1}^{m+k} |S_{i,r,M}^{(0,j)}| 1(B_{j,k}) \mid x_o = x, \deg(o) = m+k \right) p(k, x). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\mathbb{E} \left(|S_{i,r,M}^{(0,j)}| 1(B_{j,k}) \mid x_{(0,1)}, \dots, x_{(0,m+k)} \right) \\ &\leq \left(\frac{m}{x_{(0,j)}} \int_0^{x_{(0,j)}} g_{i-1}(y) dy + \frac{M' - m}{1 - x_{(0,j)}^\psi} \int_{x_{(0,j)}}^1 g_{i-1}(y) \psi y^{\psi-1} dy \right) p(M', x_{(0,j)}) \\ &= \mathcal{O}((\log M)^{i-1} M' M^{-1} p(M', x_{(0,j)})). \end{aligned}$$

Here, we used (106) to estimate the second term. Therefore

$$\int_0^1 l_{2,i}(x) x^\psi dx = \mathcal{O}((\log M)^{i-1} M' M^{-1} (M')^{-1-1/\psi}).$$

We then conclude that

$$\begin{aligned} \mathbb{E}(|S_{i,r,M}^*| 1(A \cap B)) &= \int_0^1 (l_{1,i-2}(x) + l_{2,i}(x)) x^\psi dx \\ &= \mathcal{O}((\log M)^{i-1} M' M^{-1} (M')^{-1-1/\psi}). \end{aligned}$$

Therefore, by (115)

$$\mathbb{E}_{\mathbb{Q}_2}(|S_{i,r,M}^*|) = \mathcal{O}((\log M)^{i-1} M' M^{-1}).$$

To estimate $\mathbb{E}_{\mathbb{Q}_1}(|T_{i,r,M}|)$, we use (109) and the same argument as for $\mathbb{E}_{\mathbb{Q}_2}(|T_{i,r,M}^*|)$. More precisely, we replace B by $\tilde{B} = \{|S_{1,r,M}| = 1\}$, replace $B_{s,k}$ by

$$\tilde{B}_{s,k} = \{\deg(o) = m+k\} \cap \{\deg((0,s)) > M, \deg((0,j)) \leq M \forall j \neq s\},$$

replace α'' and β'' by α and β respectively. We now have

$$\begin{aligned} &\mathbb{E}(|T_{i,r,M}| 1(A) 1(\tilde{B}) \mid x_o = x) \\ &= \mathbb{E} \left(\sum_{j=1}^{\deg(o)} |T_{i-1,r-1,M}^{(0,j)}| 1(A) 1(\tilde{B}) 1(\deg((0,j)) \leq M) \mid x_o = x \right) \\ &= \sum_{k=0}^{M-m} \mathbb{E} \left(\sum_{1 \leq j \neq s \leq m+k} |T_{i-1,r-1,M}^{(0,j)}| 1(\tilde{B}_{s,k}) \mid x_o = x, \deg(o) = m+k \right) p(k, x) \\ &:= \tilde{h}_{i-1}(x). \end{aligned}$$

We estimate $\int_0^1 \tilde{h}_{i-1}(x) x^\psi dx$ by the same way as for $h_{1,i-2}(x)$, and get the desired result. \square

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