Metastability for the contact process on the preferential attachment graph

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METASTABILITY FOR THE CONTACT PROCESS ON THE PREFERENTIAL ATTACHMENT GRAPH

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Abstract. We consider the contact process on the preferential attachment graph. The work of Berger, Borgs, Chayes and Saberi [BBCS1] confirmed physicists predictions that the contact process starting from a typical vertex becomes endemic for an arbitrarily small infection rate λ with positive probability. More precisely, they showed that with probability $\lambda^{\Theta(1)}$, it survives for a time exponential in the largest degree. Here we obtain sharp bounds for the density of infected sites at a time close to exponential in the number of vertices (up to some logarithmic factor).

1. Introduction

The paper aims at proving a metastability result for the contact process on the preferential attachment random graph, improving [BBCS1]'s result in two aspects: obtaining a better bound on the extinction time, and estimating more accurately the density of the infected sites.

The contact process is one of the most studied interacting particle systems, see in particular Liggett's book [L], and is also often interpreted as a model to describing how a virus spreads in a network. Mathematically, it can be defined as follows: given a locally finite graph $G = (V, E)$ and $\lambda > 0$, the contact process on G with infection rate λ is a Markov process $(\xi_t)_{t\geq 0}$ on $\{0,1\}^V$. Vertices of V (also called sites) are regarded as individuals which are either infected (state 1) or healthy (state 0). By considering ξ_t as a subset of V via $\xi_t \equiv \{v : \xi_t(v) = 1\}$, the transition rates are given by

$$\xi_t \to \xi_t \setminus \{v\} \text{ for } v \in \xi_t \text{ at rate 1, and}$$
$$\xi_t \to \xi_t \cup \{v\} \text{ for } v \notin \xi_t \text{ at rate } \lambda \deg_{\xi_t}(v),$$

where $\deg_{\xi_t}(v)$ denotes the number of edges between v and other infected sites (Note that if G is a simple graph, i.e. contains no multiple edges, then $\deg_{\xi_t}(v)$ is just the number of infected neighbors of v at time t). Given that $A \subset V$, we denote by $(\xi_t^A)_{t\geq 0}$ the contact process with initial configuration A. If $A = \{v\}$ we simply write (ξ_t^v).

Originally the contact process was studied on integer lattices or homogeneous trees. More recently, probabilists started investigating this process on some families of random graphs like the Galton-Watson trees, configuration models, random regular graphs, and preferential attachment graphs, see for instance [P, CD, CS, D, MVY, MV, BBCS1].

The preferential attachment graph (a definition will be given later) is well-known as a pattern of scale-free or social networks. Indeed, it not only explains the power-law degree sequence of a host in real world networks, but also reflects a wisdom that the rich get richer - the newbies are more likely to get acquainted with more famous people rather than a relatively unknown person. Therefore there has been great interest in this random

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graph as well as the processes occuring on it, including the contact process. In [BBCS1], by introducing a new representation of the graph, the authors proved a remarkable result which validated physicists predictions that the phase transition of the contact process occurs at $\lambda = 0$. More precisely, they showed that there are positive constants θ, c and C, such that for all $\lambda > 0$

$$\lambda^c \leq \mathbb{P}_n \left(\xi^u_{\exp(\theta\lambda^2\sqrt{n})} \neq \varnothing \right) \leq \lambda^C, \tag{1}$$

where (ξ^u_t) is the contact process starting from a uniformly chosen vertex.

Recently, in [BBCS2] they used their new representation to show that the preferential attachment graph converges weakly to a limit, called the Pólya-point graph.

In this paper we will use this convergence as well as the new representation to improve (1). Here is our main result.

Theorem 1.1. *Let (G_n) be the sequential model of the preferential attachment graph with parameters $m \geq 2$ and $\alpha \in [0, 1)$. Consider the contact process (ξ_t) with infection rate $\lambda > 0$ starting from full occupancy on G_n. Then there exist positive constants c and C, such that for λ small enough,*

$$\mathbb{P}_n \left(c\lambda^{1+\frac{2}{\psi}} |\log \lambda|^{-\frac{1}{\psi}} \leq \frac{|\xi_{t_n}|}{n} \leq C\lambda^{1+\frac{2}{\psi}} |\log \lambda|^{-\frac{1}{\psi}} \right) = 1 - o(1). \tag{2}$$

where $\psi = \frac{1-\alpha}{1+\alpha}$ and (t_n) is any sequence satisfying $t_n \to \infty$ and $t_n \leq T_n = \exp\left(\frac{c\lambda^2 n}{(\log n)^{1/\psi}} \right)$.

By a well-known property of the contact process called self-duality (see [L], Section I.1) for any $t \geq 0$ we have

$$\sum_{v \in V_n} 1(\{\xi^v_t \neq \varnothing\}) \overset{(\mathcal{L})}{=} |\xi_t|. \tag{3}$$

Therefore the survival probability as in (1) is just the expected value of the density of infected sites as in Theorem 1.1, so that our result is a stronger form of the one in [BBCS1]. Additionally we get a more precise estimate of the density and we allow (t_n) to be larger. Let us comment on its proof now.

First, to obtain the time T_n we will use the maintenance mechanism as in [CD] instead of the one in [BBCS1]. In the latter the authors used that in the preferential attachment graph the maximal degree is of order $\sqrt{n}$, plus the well-known fact that for any vertex v, the process survives a time exponential in the degree of v, once it is infected, yielding (1). In the former, on the other hand, when considering the contact process on the configuration model, Chatterjee and Durrett employed many vertices with total degree of order $n^{1-\varepsilon}$, for any $\varepsilon > 0$, and derived a much better bound on the extinction time. Here, our strategy is to find vertices with degree larger than $Cd(G_n)$, where $C = C(\lambda) > 0$ is a constant and $d(G_n)$ is the diameter of G_n, which is of order $\log n$. Thanks to Proposition 1 in [CD], we can deduce that the virus propagates along these vertices for a time exponential in their total degree. Moreover, the degree distribution of the graph, denoted by $\mathbf{p}$, has a power-law with exponent $\nu = 2 + 1/\psi$. Thus the number of these vertices is of order $n(\log n)^{1-\nu}$ and their total degree is of order $n(\log n)^{-1/\psi}$, which explains the bound on t_n in Theorem 1.1.

It is worth noting that for any graph with order n edges, like G_n, the extinction time of the contact process is w.h.p. smaller than $\exp(Cn)$, for some $C > 0$. Hence our bound on t_n is nearly optimal.

Secondly, to gain the precise estimate on the density, we use ideas from [P, BBCS1, CD, MVY]: if the virus starting at a typical vertex wants to survive a long time, it has to infect a big vertex of degree significantly larger than λ^{-2}. Then the virus is likely to survive in the neighborhood of this vertex for a time which is long enough to infect another big vertex, and so on. We can see that the time required for a virus to spread from one big vertex to another is at least $\lambda^{-\Theta(1)}$ (corresponding to the case when the distance between them is constant). Besides, it was shown that if $\deg(v) \geq K/\lambda^2$, then the survival time of the contact process on the star graph formed by v and it neighbors is about $\exp(cK)$. Hence the degree of big vertices should be larger than $C\lambda^{-2}|\log\lambda|$. Then we consider Λ, the set of vertices which have a big neighbor. The probability for a vertex in Λ to infect its big neighbor is of order λ. Moreover, we will show in Section 4 that any big vertex has a positive probability to make the virus survive up to time T_n. This means that the probability for the dual process starting from any vertex in Λ to be active at time T_n is of order λ. Furthermore, we will see that these events are asymptotically independent. Therefore the density of vertices from where the dual process survives up to T_n is about λ times the density of Λ. This is of order $\lambda \times \mathbf{p}\big([\lambda^{-2}|\log\lambda|,\infty)\big)\lambda^{-2}|\log\lambda| \asymp \lambda^{1+\frac{2}{\psi}}|\log\lambda|^{-\frac{1}{\psi}}$ yielding the desired lower bound.

We notice that the density of big vertices is of order $\mathbf{p}\big([\lambda^{-2}|\log\lambda|,\infty)\big)$, which is not optimal. Hence we need to consider also their neighbors. In fact this idea of using Λ was first introduced in [CD] for the configuration model (CM).

For the upper bound, we look at the local structure of G_n. In particular, we will show that its weak limit, the Pólya-point graph, is locally dominated by Galton-Watson trees. Then the results in [MVY] will be applied to deduce our desired bound.

It is interesting to note also that if we consider the contact process on the CM with the same power-law degree distribution, the density is of order $\lambda^{1+\frac{2}{\psi}}|\log\lambda|^{-\frac{2}{\psi}}$ (see [MVY, Theorem 1.1]), which is slightly smaller than the one in (2). This difference is due to the fact that the distance between big vertices in the CM is about $|\log\lambda|$, instead of constant here.

Finally, the above strategy works properly when studying the contact process on the Pólya-point graph. In fact, proving the following result helped us pave the way to potential solutions for Theorem 1.1.

Proposition 1.2. *Let (ξ_t^o) be the contact process on the Pólya-point graph with infection rate $\lambda > 0$ starting from the root o. There exist positive constants c and C, such that for λ small enough,*

$$c\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi} \leq \mathbb{P}(\xi_t^o \neq \varnothing\ \forall t) \leq C\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi}.$$

Theorem 1.1 (resp. Proposition 1.2) implies that for all $\lambda > 0$, the contact process becomes endemic (resp. survives forever) with positive probability. We say that the critical values of the contact process on the preferential attachment graph and its weak limit are all zero. This is a new example of a more general phenomena that there is a relationship between the phase transition for the contact process on a sequence of finite graphs and the one on its weak local limit in the sense of Benjamini–Schramm's convergence. Here are some known results on this topic: the contact process on the integer lattice $\mathbb{Z}^d$ and on finite boxes $[\![1,n]\!]^d$ exhibit a phase transition at the same critical value $\lambda_c = \lambda_c(d)$, see [L, Part I] for all $d \geq 1$; the phase transition of the process on the random regular graph of degree d and its limit, the homogeneous tree $\mathbb{T}_d$, occurs at the

same constant $\lambda_1(\mathbb{T}_d)$, see [MV]; the phase transition of the contact process on $\mathbb{T}_d^\ell$ (the d-homogeneous tree of height ℓ) and its limit, the canopy tree $\mathbb{CT}_d$, happens at $\lambda_2(\mathbb{T}_d)$, see [CMMV, MV]; the critical value of the contact process on the configuration model with heavy tail degree distributions or on its limit, the Galton-Watson tree, is zero, see [P, CD, MVY, MMVY, CS].

Now the paper is organized as follows. In the next section, based on [BBCS2], we give the definition of the sequential model of the preferential attachment graph as well as its weak local limit, the Pólya-point graph. We also prove preliminary results on the graph structure and fix some notation. In Section 3, we prove Proposition 1.2. Finally, the main theorem is proved in Section 4.

2. Preliminaries

2.1. **Construction of the random graph and notation.** Let us give a definition following [BBCS2] of the sequential model of the preferential attachment graph with parameters $m \geq 2$ and $\alpha \in [0, 1)$. We construct a sequence of graphs (G_n) with vertex set $V_n = \{v_1, \dots, v_n\}$ as follows:

First G_1 contains one vertex v_1 and no edge, and G_2 contains 2 vertices v_1, v_2 and m edges connecting them. Given G_{n-1}, we define G_n the following way. Add the vertex v_n to the graph, and draw edges between v_n and m vertices $w_{n,1}, \dots, w_{n,m}$ (possibly with repetitions) from G_{n-1} as follows: with probability $\alpha_n^{(i)}$, the vertex $w_{n,i}$ is chosen uniformly at random from V_{n-1}. Otherwise, $w_{n,i} = v_k$ with probability

$$\frac{\deg_{n-1}^{(i)}(v_k)}{Z_{n-1}^{(i)}},$$

where

$$\alpha_n^{(i)} = \begin{cases} \alpha & \text{if } i = 1, \\ \alpha \frac{2m(n-1)}{2m(n-2)+2m\alpha+(1-\alpha)(i-1)} = \alpha + \mathcal{O}(n^{-1}) & \text{if } i \geq 2, \end{cases}$$

$$\deg_{n-1}^{(i)}(v_k) = \deg_{n-1}(v_k) + \#\{1 \leq j \leq i-1 : w_{n,j} = v_k\},$$

is the degree of v_k before choosing $w_{n,i}$, and

$$Z_{n-1}^{(i)} = \sum_{k=1}^{n-1} \deg_{n-1}^{(i)}(v_k) = \sum_{k=1}^{n-1} \deg_{n-1}(v_k) + i - 1 = 2m(m-2) + i - 1,$$

with $\deg_{n-1}(v_k)$ the degree of v_k in G_{n-1}.

This construction might seem less natural than in the independent model where with probability α we choose $w_{n,i}$ uniformly from V_{n-1} and with probability $1-\alpha$ it is chosen according to a simpler rule: $w_{n,i} = v_k$ with probability $\deg_{n-1}(v_k)/2m(m-2)$. However the sequential model is easier to analyze because it is exchangeable, and as a consequence it admits an alternative representation which contains more independence. In [BBCS2], the authors called it the Pólya urn representation which we now recall in the following theorem. To this end, we denote by $\beta(a, b)$ the Beta distribution, whose density is proportional to $x^{a-1}(1-x)^{b-1}$ on $[0, 1]$, and by $\Gamma(a, b)$ the Gamma distribution, whose density is proportional to $x^{a-1}e^{-bx}$ on $[0, \infty)$. For any $a < b$, $\mathcal{U}([a, b])$ stands for the uniform distribution on $[a, b]$.

Theorem 2.1. [BBCS2, Theorem 2.1.] *Fix $m \geq 2$, $\alpha \in [0,1)$ and $n \geq 1$. Set $r = \alpha/(1-\alpha), \psi_1 = 1$, and let $\psi_2, \dots, \psi_n$ be independent random variables with law*

$$\psi_j \sim \beta(m + 2mr, (2j-3)m + 2mr(j-1)).$$

Define

$$\varphi_j = \psi_j \prod_{t=j+1}^{n} (1-\psi_t), \qquad S_k = \sum_{j=1}^{k} \varphi_j, \quad \text{and} \quad I_k = [S_{k-1}, S_k).$$

Conditionally on $\psi_1, \dots, \psi_n$, let $\{U_{k,i}\}_{k=1,\dots,n, i=1,\dots,m}$ be a sequence of independent random variables, with $U_{k,i} \sim \mathcal{U}([0, S_{k-1}])$. Start with the vertex set $V_n = \{v_1, \dots, v_n\}$. For $j < k$, join v_j and v_k by as many edges as the number of indices $i \in \{1, \dots, m\}$, such that $U_{k,i} \in I_j$. Denote the resulting random graph by G_n.

Then G_n has the same distribution as the sequential model of the preferential attachment graph.

From now on, we always consider the random multi-graph G_n constructed as in this theorem.

We now look at the local structure of G_n. It was shown in [BBCS2] that G_n is locally tree-like, with some subtle degree distribution that we now recall. First we fix some constants:

$$\chi = \frac{1+2r}{2+2r} \qquad \text{and} \qquad \psi = \frac{1-\chi}{\chi} = \frac{1}{1+2r}.$$

Note that $1/2 \leq \chi < 1$ and $0 < \psi \leq 1$. Let $F \sim \Gamma(m+2mr, 1)$ and $F' \sim \Gamma(m+2mr+1, 1)$.

We will construct inductively a random rooted tree (T, o) with vertices identified with elements of $\cup_{\ell \geq 1} \mathbb{N}^\ell$ (where vertices at generation ℓ are elements of $\mathbb{N}^\ell$) and a map which associates to each vertex v a position x_v in $[0,1]$. Additionally each vertex (except the root) will be assigned a type, either R or L.

- The root $o = (0)$ has position $x_o = U_0^\chi$, where $U_0 \sim \mathcal{U}([0,1])$.
- Given $v \in T$ and its position x_v, define

$$m_v = \begin{cases} m & \text{if } v \text{ is the root or of type L,} \\ m-1 & \text{if } v \text{ is of type R.} \end{cases}$$

and

$$\gamma_v \sim \begin{cases} F & \text{if } v \text{ is the root or of type R,} \\ F' & \text{if } v \text{ is of type L.} \end{cases}$$

The children of v are $(v,1), \dots, (v, m_v)$, $(v, m_v+1), \dots, (v, m_v+q_v)$, the first m_v's are of type L and the remaining ones are of type R. Conditionally on x_v, $x_{(v,1)}, \dots, x_{(v,m_v)}$ are i.i.d. uniform random variable in $[0, x_v]$, and $x_{(v,m_v+1)}, \dots, x_{(v,m_v+q_v)}$ are the points of the Poisson point process on $[x_v, 1]$ with intensity

$$\rho_v(x)dx = \gamma_v \frac{\psi x^{\psi-1}}{x_v^\psi} dx.$$

This procedure defines inductively an infinite rooted tree (T, o), which is called the Pólya-point graph and $(x_v)_{v \in T}$ is called the Pólya-point process.

For any vertex v in a graph G and any integer R, we call $B_G(v, R)$ the ball of radius R around v in G, which contains all vertices at distance smaller than or equal to R from v and all edges connecting them.

Theorem 2.2. [BBCS2, Theorem 2.2.] *Assume that the random graph G_n is constructed as in Theorem 2.1. Let u be a vertex chosen uniformly at random in G_n and let R be some fixed constant. Then $B_{G_n}(u, R)$ converges weakly to the ball $B_T(o, R)$ in the Pólya-point graph.*

Now we introduce some notation. We call $\mathbb{P}_n$ a probability measure on a space in which the random graph G_n is defined together with the contact process. Since we will fix λ, we omit it in the notation. We also call $\mathbb{P}$ a probability measure on a space in which the Pólya-point graph as well as the contact process are defined.

We denote the indicator function of a set A by $\mathbf{1}(A)$. For any vertices v and w we write $v \sim w$ if there is an edge between them (in which case we say that they are neighbors or connected), and $v \not\sim w$ otherwise. We call size of G the cardinality of its set of vertices, and we denote it by $|G|$.

A graph in which all vertices have degree one, except one which is connected to all the others is called a **star graph**. The only vertex with degree larger than one is called the center of the star graph, or central vertex.

If f and g are two real functions, we write $f = \mathcal{O}(g)$ if there exists a constant $C > 0$, such that $f(x) \leq Cg(x)$ for all x; $f \gtrsim g$ (or equivalently $g \lesssim f$) if $g = \mathcal{O}(f)$; $f \asymp g$ if $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$; $f = o(g)$ if $g(x)/f(x) \to 0$ as $x \to \infty$. Finally for a sequence of r.v.s (X_n) and a function $f : \mathbb{N} \to (0, \infty)$, we say that $X_n \asymp f(n)$ holds w.h.p. if there exist positive constants c and C, such that $\mathbb{P}_n(cf(n) \leq X_n \leq Cf(n)) \to 1$.

2.2. **Preliminary results on the random graph.** We first recall a version of the Azuma-Hoeffding inequality for martingales which we will use throughout this paper (see for instance [CL]).

Lemma 2.3. *Let $(X_i)_{i\geq 0}$ be a martingale satisfying $|X_i - X_{i-1}| \leq 1$ for all $i \geq 1$. Then for any n and $t > 0$, we have*

$$\mathbb{P}\left(|X_n - X_0| \geq t\right) \leq 2\exp(-t^2/2n).$$

From this inequality we can deduce a large deviations result. Let $(X_i)_{i\geq 1}$ be a sequence of independent Bernoulli random variables. Assume that $0 < 2p \leq \mathbb{E}_n(X_i) \leq Mp$ for all i. Then there exists $c = c(M) > 0$, such that for all n

$$\mathbb{P}\left(np \leq \sum_{i=1}^{n} X_i \leq 2Mnp\right) \geq 1 - \exp(-cnp). \tag{4}$$

Now we present some estimates on the sequences (φ_i), (ψ_j) and (S_k).

Lemma 2.4. *Let $(\varphi_i)_i$, $(\psi_j)_j$ and $(S_k)_k$ be sequences of random variables as in Theorem 2.1. Then there exist positive constants μ and θ_0, such that for all $\theta \leq \theta_0$, the following assertions hold.*

(i) $\mathbb{E}_n(\psi_j) = \frac{\chi}{j} + \mathcal{O}(\frac{1}{j^2})$, $\quad \mathbb{E}_n(\psi_j^2) \asymp \frac{1}{j^2}$.

(ii) *For any $\varepsilon > 0$, there exists $K = K(\varepsilon) < \infty$, such that*

$$\mathbb{P}_n(\mathcal{E}_\varepsilon) \geq 1 - \varepsilon,$$

where

$$\mathcal{E}_\varepsilon = \{|S_k - (k/n)^\chi| \leq \varepsilon(k/n)^\chi \quad \forall\, K(\varepsilon) \leq k \leq n\}.$$

(iii) *For any $i < j$, define* $S_i^{(j)} = \prod_{t=i+1}^{j} (1-\psi_t)$. *Then*

$$\mathbb{E}_n(S_i^{(j)}) \leq (i/j)^{\chi}.$$

(iv) $\mathbb{P}_n(\mu/j \geq \psi_j \geq \theta/j) \geq 2\theta$.

(v) $\mathbb{E}_n(\varphi_j 1(\psi_j \geq \theta/j)) \geq \theta \mathbb{E}_n(\varphi_j)$.

Proof. Let us start with Part (i). Observe that if $\psi \sim \beta(a,b)$, then

$$\mathbb{E}(\psi) = \frac{a}{a+b} \quad \text{and} \quad \mathbb{E}(\psi^2) = \frac{a(a+1)}{(a+b)(a+b+1)}.$$

Hence the result follows from the fact that $\psi_j \sim \beta(m+2mr, (2j-3)m + 2mr(j-1))$.

Part (ii) is exactly Lemma 3.1 in [BBCS2]. We now prove (iii). From the first observation in Part (i), we have $\mathbb{E}_n(\psi_t) \geq \chi/t$ for all $t \geq 1$. Moreover, the (ψ_t) are independent, hence we get that

$$\begin{aligned}\mathbb{E}_n(S_i^{(j)}) &= \prod_{t=i+1}^{j} (1 - \mathbb{E}_n(\psi_t)) \leq \prod_{t=i+1}^{j} \left(1 - \frac{\chi}{t}\right) \\ &\leq \prod_{t=i+1}^{j} \left(1 - \frac{1}{t}\right)^{\chi} = \left(\frac{i}{j}\right)^{\chi}.\end{aligned}$$

Here we have used that if $0 \leq \chi, x \leq 1$, then $(1-\chi x) \leq (1-x)^{\chi}$.

For Part (iv), Markov's inequality gives that for any $\delta \in (0,1)$

$$\mathbb{P}_n(|\psi_j - \mathbb{E}_n(\psi_j)| > (1-\delta)\mathbb{E}_n(\psi_j)) \leq \frac{\mathbb{V}\mathrm{ar}_n(\psi_j)}{(1-\delta)^2 \mathbb{E}_n(\psi_j)^2}.$$

Moreover, if $\psi \sim \beta(a,b)$, then

$$\frac{\mathbb{V}\mathrm{ar}(\psi)}{\mathbb{E}(\psi)^2} = \frac{\mathbb{E}(\psi^2)}{\mathbb{E}(\psi)^2} - 1 = \frac{(a+1)(a+b)}{a(a+b+1)} - 1 = \frac{b}{a(a+b+1)} \leq \frac{1}{a}.$$

Therefore for any j

$$\mathbb{P}_n(\psi_j \in (\delta \mathbb{E}_n(\psi_j), (2-\delta)\mathbb{E}_n(\psi_j))) \geq 1 - \frac{1}{(1-\delta)^2(m+2mr)}.$$

Hence thanks to (i) we can choose positive constants μ and θ, such that for all j

$$\mathbb{P}_n\left(\psi_j \in \left(\frac{\theta}{j}, \frac{\mu}{j}\right)\right) \geq 2\theta. \tag{5}$$

For (v), we notice that

$$\begin{aligned}\mathbb{E}_n(\varphi_j 1(\psi_j \geq \theta/j)) &= \mathbb{E}_n(\psi_j 1(\psi_j \geq \theta/j)) \mathbb{E}_n\left(\prod_{t=j+1}^{n} (1-\psi_t)\right) \\ &\geq c\mathbb{E}_n(\psi_j)\mathbb{E}_n\left(\prod_{t=j+1}^{n} (1-\psi_t)\right) \\ &= c\mathbb{E}_n(\varphi_j),\end{aligned}$$

for some $c > 0$, independent of j. Thus the result follows by taking θ small enough. □

The preferential attachment graph is known as a prototype of small world networks whose diameter and typical distance (the distance between two randomly chosen vertices) are of logarithmic order. In fact, these quantities in the independent model were well-studied, see for instance [DVH] or [V]. In the following two lemmas, we prove similar estimates for the sequential model. These estimates are in fact weaker but sufficient for our purpose.

Lemma 2.5. *Let $d(G_n)$ be the diameter of the random graph G_n, i.e. the maximal distance between pair of vertices in G_n. Then there exists a positive constant b_1, such that*

$$\mathbb{P}_n(d(G_n) \leq b_1 \log n) = 1 - o(1).$$

Proof. Let $\varepsilon \in (0, 1/2)$ be given, and recall the definitions of $K(\varepsilon)$ and $\mathcal{E}_\varepsilon$ given in Lemma 2.4 (ii). We first bound $d(v_1, v_n)$. Define a decreasing sequence $(n_i)_{i\geq 0}$ by $n_0 = n$, and the condition that $v_{n_{i+1}}$ and v_{n_i} are neighbors (note that the choice of (n_i) is not unique in general). Define

$$X_i = 1(\{n_i \leq n_{i-1}/2\}) \text{ and } \mathcal{F}_i = \sigma(n_j : j \leq i) \vee \sigma((\varphi_t)).$$

Denote by

$$\sigma_n = \inf\{i : n_{i+1} \leq \log n\}.$$

If $i \leq \sigma_n$, then $n_i > \log n \geq 2K(\varepsilon)$. Therefore due to the construction of the graph, for such i we have on $\mathcal{E}_\varepsilon$

$$\mathbb{P}_n(n_{i+1} \leq n_i/2 \mid \mathcal{F}_i) = \frac{S_{[n_i/2]}}{S_{n_i-1}} \geq \frac{1-\varepsilon}{(1+\varepsilon)2^\chi} \geq \frac{1}{2^{\chi+2}} = p > 0.$$

In other words, on $\mathcal{E}_\varepsilon$ we have

$$\mathbb{E}_n(X_{i+1} \mid \mathcal{F}_i)1(i \leq \sigma_n) \geq p1(i \leq \sigma_n). \tag{6}$$

Let

$$Y_k = \sum_{i=1}^{k} \left(X_i - \mathbb{E}_n(X_i \mid \mathcal{F}_{i-1})\right).$$

Then (Y_k) is a martingale with respect to the filtration $(\mathcal{F}_k)$ and $|Y_k - Y_{k-1}| \leq 1$. By using Lemma 2.3 we get that

$$\mathbb{P}_n\left(Y_k \leq -kp/2 \mid (\varphi_t)\right) \leq 2\exp(-kp^2/8),$$

which implies by using (6) that on $\mathcal{E}_\varepsilon$

$$\mathbb{P}_n\left(\sum_{i=1}^{k} X_i \leq kp/2, \sigma_n \geq k \mid (\varphi_t)\right) \leq 2\exp(-kp^2/8). \tag{7}$$

Moreover, if $\sum_1^k X_i \geq (\log_2 n - \log_2 \log n)$, then $n_k \leq \log n$ (or equivalently $\sigma_n \leq k$). Hence it follows from (7) that on $\mathcal{E}_\varepsilon$

$$\mathbb{P}_n(\sigma_n \geq C \log n \mid (\varphi_t)) \leq \mathbb{P}_n\left(\sum_{i=1}^{C\log n} X_i \leq \log_2 n - \log_2 \log n \mid (\varphi_t)\right) = \mathcal{O}(n^{-2}),$$

for some $C = C(p) > 0$.
On the other hand, if $\sigma_n \leq C \log n$, then $d(v_1, v_n) \leq (C+1)\log n$. We thus obtain that on $\mathcal{E}_\varepsilon$

$$\mathbb{P}_n(d(v_1, v_n) \geq (C+1)\log n \mid (\varphi_t)) = \mathcal{O}(n^{-2}). \tag{8}$$

Let $d_{G_k}(v_i, v_j)$ be the distance between v_i and v_j in G_k for $i, j \leq k \leq n$. Note that

$$d_{G_k}(v_i, v_j) \geq d(v_i, v_j) = d_{G_n}(v_i, v_j).$$

Similarly to (8), we deduce that on $\mathcal{E}_\varepsilon$, for all $i \geq C \log n$,

$$\mathbb{P}_n(d(v_1, v_i) \geq (C+1)\log i \mid (\varphi_t)) \leq \mathbb{P}_i(d_{G_i}(v_1, v_i) \geq (C+1)\log i \mid (\varphi_t)) = \mathcal{O}(i^{-2}).$$

Hence on $\mathcal{E}_\varepsilon$

$$\mathbb{P}_n(d(v_1, v_i) \leq (C+1)\log n \ \forall\, i \geq C \log n \mid (\varphi_t)) = 1 - o(1).$$

Therefore by taking expectation with respect to (φ_t) and using Lemma 2.4 (ii), we get

$$\mathbb{P}_n(d(G_n) \leq 2(C+1)\log n) \geq 1 - 2\varepsilon,$$

which proves the result by letting ε tend to 0. □

Before proving the lower bound on the typical distance, we make a remark which will be used frequently in this paper. It follows from the definition of G_n that for all $i < j$,

$$\mathbb{P}_n(v_i \not\sim v_j \mid (\varphi_t)) = \left(1 - \frac{\varphi_i}{S_{j-1}}\right)^m.$$

Hence

$$\frac{\varphi_i}{S_{j-1}} \leq \mathbb{P}_n(v_i \sim v_j \mid (\varphi_t)) \leq \frac{m\varphi_i}{S_{j-1}}.$$

Then by using the following identities

$$S_{j-1} = \sum_{t=1}^{j-1} \varphi_t = \prod_{t=j}^{n}(1-\psi_t) \quad \text{and} \quad \varphi_i = \psi_i \prod_{t=i+1}^{n}(1-\psi_t),$$

we obtain that

$$\psi_i S_i^{(j-1)} \leq \mathbb{P}_n(v_i \sim v_j \mid (\varphi_t)) \leq m\psi_i S_i^{(j-1)}, \tag{9}$$

where we recall that

$$S_i^{(j)} = \prod_{t=i+1}^{j}(1-\psi_t).$$

Lemma 2.6. *Let w_1 and w_2 be two uniformly chosen vertices from V_n. Then there is a positive constant b_2, such that w.h.p.*

$$d(w_1, w_2) \geq \frac{b_2 \log n}{\log\log n}.$$

Proof. We will use an argument from [V, Lemma 7.16]. We call a sequence of distinct vertices $\pi = (\pi_1, \ldots, \pi_k)$ a self-avoiding path. We write $\pi \subset G_n$ if π_i and π_{i+1} are neighbors for all $1 \leq i \leq k-1$. Let $\Pi(i, j, k)$ be the set of all self-avoiding paths of length k starting from v_i and finishing at v_j. We then claim that for all $i, j, k \geq 1$,

(i) $\mathbb{P}_n(d(v_i, v_j) = k) \leq g_k(i, j) := \sum_{\pi \in \Pi(i,j,k)} \mathbb{P}_n(\pi \subset G_n)$,

(ii) $g_{k+1}(i, j) \leq \sum_{s \neq i,j} g_1(i, s) g_k(s, j)$.

The first claim is clear, because if $d(v_i, v_j) = k$ then there exists a self-avoiding path in $\Pi(i, j, k)$ which is in G_n. For the second one, we note that for any self-avoiding path $\pi = (\pi_1, \dots, \pi_k)$,

$$\mathbb{P}_n(\pi \subset G_n) = \mathbb{P}_n(\pi_1 \sim \pi_2)\mathbb{P}_n(\bar{\pi} \subset G_n),$$

where $\bar{\pi} = (\pi_2, \dots, \pi_k)$. Indeed, if $j < k$, then the event that $v_j \sim v_k$ depends only on the $(U_{k,i})_{i \le m}$. Hence this result follows from the facts that the vertices in π are distinct and that the $\{(U_{k,i})_{i \le m}\}_k$ are independent. We are now in position to prove (ii):

$$\begin{aligned} g_{k+1}(i, j) &= \sum_{s \neq i,j} \sum_{\substack{v_i \notin \bar{\pi} \\ \bar{\pi} \in \Pi(s,j,k)}} \mathbb{P}_n(v_i \sim v_s, \bar{\pi} \subset G_n) \\ &\le \sum_{s \neq i,j} \sum_{\bar{\pi} \in \Pi(s,j,k)} \mathbb{P}_n(v_i \sim v_s)\mathbb{P}_n(\bar{\pi} \subset G_n) \\ &= \sum_{s \neq i,j} g_1(i, s) g_k(s, j). \end{aligned}$$

We prove by induction on k that there is a positive constant C, such that

$$g_k(i, j) \le \frac{(C \log n)^k}{\sqrt{ij}}. \tag{10}$$

For $k = 1$, thanks to (9) and Lemma 2.4 (iii), for all $i < j$ we have

$$\begin{aligned} g_1(i, j) &= \mathbb{P}_n(v_i \sim v_j) \\ &\le m\mathbb{E}_n(\psi_i)\mathbb{E}_n(S_i^{(j-1)}) \\ &\le m\mathbb{E}_n(\psi_i) \left(\frac{i}{j-1}\right)^{\chi} \\ &\le \frac{C}{\sqrt{ij}}, \end{aligned} \tag{11}$$

for some $C > 0$. The existence of C follows from the facts that $\mathbb{E}_n(\psi_i) \asymp 1/i$ and $\chi \ge 1/2$.

Assume now that the result is true for some k, and let us prove it for $k + 1$. By using the induction hypothesis, (ii) and (11) we get that

$$\begin{aligned} g_{k+1}(i, j) &\le \sum_{s \neq i,j} g_1(i, s) g_k(s, j) \\ &\le \sum_{s \neq i,j} \frac{C}{\sqrt{is}} \frac{(C \log n)^k}{\sqrt{sj}} \\ &\le \frac{(C \log n)^{k+1}}{\sqrt{ij}}, \end{aligned}$$

which proves the induction step. Now it follows from (i) and (10) that

$$\begin{aligned}\mathbb{P}_n(d(w_1,w_2)\le K) &\le \frac{1}{n^2}\sum_{k=1}^{K}\sum_{1\le i,j\le n} g_k(i,j)\\ &\le \frac{1}{n^2}\sum_{k=1}^{K}\sum_{1\le i,j\le n}\frac{(C\log n)^k}{\sqrt{ij}}\\ &\le \frac{(C\log n)^{K+1}}{n}.\end{aligned}$$

Therefore if $K = \log n/(2C\log\log n)$, then $\mathbb{P}_n(d(w_1,w_2)\le K) = o(1)$. This gives the desired lower bound with $b_2 = 1/(2C)$. □

Remark 2.7. The bound in the above lemma is probably not sharp. Indeed for the independent model, it is proved in [DVH] or [V] that if $\alpha > 0$ (or equivalent $\chi > 1/2$), then w.h.p. $d(w_1,w_2)\ge c\log n$; and otherwise, w.h.p. $d(w_1,w_2)\ge \log n/(C+\log\log n)$, for some positive constants c and C.

2.3. **Contact process on star graphs.** We will see that star graphs play a crucial role in the conservation of the virus on the preferential attachment graph. Hence, it is important to understand the behavior of the contact process on a single star graph as well as the transmission between them. These have been studied for a long time by many authors, for instance in [P, BBCS1, CD, MVY]. The results we need will be summarized in Lemma 2.8 and 2.9 below, but first a definition: we say that a vertex v is **lit** (the term is taken from [CD]) at some time t if the proportion of its infected neighbors at time t is larger than $\lambda/(16e)$.

Lemma 2.8. *Let (ξ_t) be the contact process on a star graph S with center v. There exists a positive constant c^*, such that the following assertions hold.*

(i) $\mathbb{P}\,(v \text{ is lit at time } 1 \mid \xi_0(v)=1) \ge c^*(1-\exp(-c^*\lambda|S|))$.

(ii) $\mathbb{P}\,(\exists t>0 : v \text{ is lit at time } t \mid \xi_0(v)=1) \to 1 \quad \text{as } |S|\to\infty$.

(iii) *If $\lambda^2|S| \ge 64e^2$, and v is lit at time 0, then v is lit during the time interval $[\exp(c^*\lambda^2|S|), 2\exp(c^*\lambda^2|S|)]$ with probability larger than $1-2\exp(-c^*\lambda^2|S|)$.*

Proof. Parts (i) and (ii) are exactly Lemma 3.1 (i), (iii) in [MVY]. For (iii) we need an additional definition: a vertex v is said to be **hot** at some time t if the proportion of its infected neighbors at time t is larger than $\lambda/(8e)$. Then in [CD] the authors proved (with different constants in the defintion of lit and hot vertices, but this does not effect the proof) that

- if v is lit at some time t, then it becomes hot before $t+\exp(c^*\lambda^2\deg(v))$ with probability larger than $1-\exp(-c^*\lambda^2\deg(v))$,
- if v is hot at some time t, then it remains lit until $t+2\exp(c^*\lambda^2\deg(v))$ with probability larger than $1-\exp(-c^*\lambda^2\deg(v))$.

Now (iii) follows from these results. □

The following result is Lemma 3.2 in [MVY].

Lemma 2.9. *Let us consider the contact process on a graph $G=(V,E)$. There exist positive constants c^* and λ_0, such that if $0<\lambda<\lambda_0$, the following holds. Let v and w be*

two vertices satisfying $\deg(v) \geq \frac{7}{c^*}\frac{1}{\lambda^2}\log\left(\frac{1}{\lambda}\right) d(v,w)$. *Assume that* v *is lit at time* 0. *Then* w *is lit before* $\exp(c^*\lambda^2 \deg(v))$ *with probability larger than* $1 - 2\exp(-c^*\lambda^2 \deg(v))$.

3. Proof of Proposition 1.2

In this section we study the contact process on the Pólya-point graph (T, o). In fact like with other examples mentioned in the introduction, we will see in the next section that the results and proofs on this graph will give us some insight in dealing with the original finite graph's problem.

We first make some observation on the degrees of the neighbors of the root (0). We denote by $w_0 = (0)$, and $x_0 = x_{w_0}$. For any $i \geq 1$, let

$$w_i = (0, 1, ..., 1) \text{ and } x_i = x_{w_i}.$$

Then w_i's degree conditioned on x_i is $m+1$ plus a Poisson random variable with parameter

$$\frac{\gamma}{x_i^\psi}\int_{x_i}^1 \psi x^{\psi-1}dx = \gamma\frac{1-x_i^\psi}{x_i^\psi},$$

where γ is a Gamma random variable with parameters $a = m + 2mr + 1$ and 1. Therefore letting $\kappa = (1 - x_i^\psi)/x_i^\psi$, we have

$$\begin{aligned} q(k \mid x_i) := \mathbb{P}(\deg(w_i) = m + 1 + k \mid x_i) &= \mathbb{E}\left(\frac{e^{-\gamma\kappa}}{k!}(\gamma\kappa)^k \,\middle|\, x_i\right) = \frac{\kappa^k\Gamma(k+a)}{(\kappa+1)^{k+a}\Gamma(a)k!} \\ &= \frac{\Gamma(k+a)}{\Gamma(a)k!}(1-x_i^\psi)^k x_i^{a\psi}, \end{aligned} \tag{12}$$

where $\Gamma(b) = \int_0^\infty x^{b-1}e^{-x}dx$.

3.1. **Proof of the upper bound.** The idea of this part is to show that locally the Pólya-point graph can be viewed as a subgraph of a certain Galton-Watson tree and then apply some proofs from [MVY]. For all i and $k \geq 0$, we denote by

$$q_i(k) = \mathbb{P}(\deg(w_i) = m + 1 + k).$$

The following lemma gives estimates on the tail distributions of these laws.

Lemma 3.1. *There exist positive constants* c *and* C, *such that for all* $k \geq 0$

$$ck^{-2-1/\psi} \leq q_0(k) \leq Ck^{-2-1/\psi} \qquad \text{and} \tag{13}$$

$$ck^{-1-1/\psi} \leq q_i(k) \leq C(\log k)^{i-1}k^{-1-1/\psi} \text{ for all } i \geq 1. \tag{14}$$

Moreover, the sequence (q_i) *is stochastically increasing i.e.* $q_i \preceq q_{i+1}$ *for all* $i \geq 0$.

Proof. Assume that $i \geq 1$. Due to the construction of the Pólya-point process, x_i is distributed as $U_i \dots U_1 U_0^{1/(1+\psi)}$, where $(U_j)_{j\geq 0}$ is a sequence of i.i.d. uniform random variables in $[0, 1]$. Hence by changing the order of integration,

$$\begin{aligned} q_i(k) &= \mathbb{E}(\mathbb{P}(\deg(w_i) = m + 1 + k \mid x_i)) \\ &= \int_0^1 (\psi+1)x_0^\psi dx_0 \int_0^{x_0} \frac{dx_1}{x_0} \dots \int_0^{x_{i-2}} \frac{dx_{i-1}}{x_{i-2}} \int_0^{x_{i-1}} q(k \mid x_i)\frac{dx_i}{x_{i-1}} \\ &= (\psi+1)\int_0^1 q(k \mid x_i)dx_i \int_{x_i}^1 \frac{dx_{i-1}}{x_{i-1}} \dots \int_{x_2}^1 \frac{dx_1}{x_1}\int_{x_1}^1 x_0^{\psi-1}dx_0. \end{aligned} \tag{15}$$

By induction we can show that

$$g(x_i) := \int_{x_i}^1 \frac{dx_{i-1}}{x_{i-1}} .... \int_{x_2}^1 \frac{dx_1}{x_1} \int_{x_1}^1 x_0^{\psi-1} dx_0 = (-1)^{i-1} \left[\frac{1-x_i^\psi}{\psi^i} + \sum_{j=1}^{i-1} \frac{(\log x_i)^j}{\psi^{i-j} j!} \right]$$

$$= \frac{(-1)^{i-1}}{\psi^i} \left[1 - x_i^\psi + \sum_{j=1}^{i-1} \frac{(\log x_i^\psi)^j}{j!} \right]. \tag{16}$$

By combining (12), (15), (16) and using the change variables $y = x_i^\psi$, we obtain

$$q_i(k) = \frac{(\psi+1)\Gamma(k+a)}{\psi\Gamma(a)k!} \int_0^1 (1-y)^k y^{a+\frac{1}{\psi}-1} h(y) dy, \tag{17}$$

where

$$h(y) = \frac{(-1)^{i-1}}{\psi^i} \left(1 - y + \sum_{j=1}^{i-1} \frac{(\log y)^j}{j!} \right).$$

Since $g(\cdot)$ is decreasing, so is the function $h(\cdot)$. Therefore

$$\int_0^1 (1-y)^k y^{a+\frac{1}{\psi}-1} h(y) dy \geq h(1/2) \int_0^{1/2} (1-y)^k y^{a+\frac{1}{\psi}-1} dy. \tag{18}$$

On the other hand for $k \geq a - 1 + 1/\psi$, we have

$$\int_{1/2}^1 (1-y)^k y^{a+\frac{1}{\psi}-1} dy = \int_0^{1/2} y^k (1-y)^{a+\frac{1}{\psi}-1} dy \leq \int_0^{1/2} (1-y)^k y^{a+\frac{1}{\psi}-1} dy.$$

Hence,

$$\int_0^{1/2} (1-y)^k y^{a+\frac{1}{\psi}-1} dy \gtrsim \int_0^1 (1-y)^k y^{a+\frac{1}{\psi}-1} dy.$$

Combining this with (18) yields that

$$\int_0^1 (1-y)^k y^{a+\frac{1}{\psi}-1} h(y) dy \gtrsim h(1/2) B(k+1, a+1/\psi), \tag{19}$$

where $B(a,b) = \int_0^1 (1-y)^{a-1} y^{b-1} dy = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. Note that

$$\frac{\Gamma(k+a)}{\Gamma(k)} \asymp k^a. \tag{20}$$

By using this in (17) and (19) we deduce that

$$q_i(k) \gtrsim k^{-1-1/\psi},$$

which implies the lower bound in (14). For the upper bound, observe that

$$\int_0^1 |\log y|^j (1-y)^k y^{a+\frac{1}{\psi}-1} dy \leq \int_0^{1/k} |\log y|^j y^{a+\frac{1}{\psi}-1} dy + \int_{1/k}^1 |\log y|^j (1-y)^k y^{a+\frac{1}{\psi}-1} dy$$

$$\leq C (\log k)^j k^{-a-1/\psi},$$

for some $C = C(j) > 0$, where we have used that the function $|\log y| y^{a+\frac{1}{\psi}-1}$ is increasing in a neighborhood of the origin to bound the first term and the approximation (20) for the second term. On the other hand by using (20) again, we get

$$\int_0^1 (1-y)^{k+1} y^{a+\frac{1}{\psi}-1} dy \lesssim k^{-a-1/\psi}.$$

Now the result follows from these estimates and by using (20) again for the term in front of the integral in (17).

When $i = 0$, the proof is simpler, as the distribution of x_0 is just U_0^χ. We safely leave it to the reader.

For the last assertion, we note that $\deg(w_i)$ is $m+1$ plus a Poisson random variable with parameter $\gamma(1-x_i^\psi)/x_i^\psi$, with $\gamma \sim \Gamma(m+2mr+1,1)$. On the other hand, $x_{i+1} \le x_i$ for all $i \ge 0$ (as $x_{i+1} \sim \mathcal{U}([0,x_i])$). The result now follows from the well-known monotonicity property of Poisson random variables. □

Remark 3.2. The logarithmic correction in (14) can not be droped, at least not completely. For instance, it follows from (17) and (20) that

$$\begin{aligned} q_2(k) &\gtrsim k^{a-1} \int_{1/2k}^{1/k} (|\log y| + y - 1)(1-y)^k y^{a+\frac{1}{\psi}-1} dy \\ &\gtrsim k^{-1-1/\psi} \log k. \end{aligned}$$

We are now ready to show the relation between the Pólya-point graph and Galton-Watson trees. To do that we introduce some new notation and definition. Given distributions $\mathbf{p}, \mathbf{p}', \mathbf{p}''$, we denote by $\mathrm{GW}(\mathbf{p}, \mathbf{p}', \mathbf{p}'')$ the Galton-Watson tree defined as follows: the root o has degree distriution $\mathbf{p}$, all children of o have degree distribution $\mathbf{p}'$ and all other vertices have degree distribution $\mathbf{p}''$. Then for $i \ge 2$, we let

$$(\mathbb{T}_i, o) = GW(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_{i-1}),$$

where we recall that $\mathbf{q}_i$ is the degree distribution of w_i.

Let (T_1, o_1) and (T_2, o_2) be two random rooted trees. We say that (T_1, o_1) is stochastically dominated by (T_2, o_2), and write it $(T_1, o_1) \preceq (T_2, o_2)$, if there exists a coupling of the two trees such that a.s. T_1 is a subgraph of T_2 and $o_1 = o_2$.

Lemma 3.3. *For any integer $i \ge 2$, the ball in the Pólya-point graph of radius i around the root is stochastically dominated by the corresponding ball in the Galton-Watson tree $GW(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_{i-1})$, i.e. $B_T(w_0, i) \preceq B_{\mathbb{T}_i}(o, i)$.*

Proof. We first observe that

(i) By construction of the Pólya-point graph, for any $n \ge 0$, conditionally on the positions of the vertices at generation n (i.e. at distance n from w_0), their degree distributions are independent.

(ii) If v and w have the same type and $x_v \preceq x_w$, then $\deg(w) \preceq \deg(v)$. Indeed, this follows from the construction of the graph and the monotonicity property of Poisson random variables.

(iii) If $w = (v, \ell)$ with $\ell > m_v$, then $\deg(w) \preceq \deg(v)$. Indeed, since w is of type R, we have $x_w \ge x_v$ and $\gamma_w \preceq \gamma_v$ (recall that $\Gamma(\alpha, 1)$ is stochastically increasing in α,

and thus $F \preceq F'$). Hence the claim follows as well from the monotonicity property of Poisson random variables.

We now prove the lemma by induction on i. Let us start with $i = 2$. The claim follows from the following facts:

- The degree distribution of the root is $\mathbf{q}_0$.
- If $v = (0, \ell)$ is some child of the root of type L (i.e. with $\ell \leq m$), then v has the same degree distribution as $(0, 1)$, which is $\mathbf{q}_1$. Otherwise by using (iii) above, we get that $\deg(v) \preceq \deg((0)) \preceq \mathbf{q}_1$ (recall that $\mathbf{q}_0 \preceq \mathbf{q}_1$).
- Conditionally on (x_v), the random variables $\{\deg(v) : d(w_0, v) = 1\}$ are independent.

Suppose now that the result holds for some i, and let us prove it for $i+1$. Let $S(i) = \{v \in B_T(w_0, i) : d(w_0, v) = i\}$. Then as above the induction step follows from the following three facts:

- $B_T(w_0, i) \preceq B_{\mathbb{T}_i}(o, i) \preceq B_{\mathbb{T}_{i+1}}(o, i)$ by using the induction hypothesis and that $\mathbf{q}_i \preceq \mathbf{q}_{i+1}$.
- Conditionally on (x_v), the random variables $\{\deg(v) : v \in S(i)\}$ are independent.
- If $v \in S(i)$, then $\deg(v) \preceq \mathbf{q}_i$. Indeed,
  – if v is of type R, then $v = (w, \ell)$, with $\ell > m_w$. By using (iii) above and the induction hypothesis, we obtain that $\deg(v) \preceq \deg(w) \preceq \mathbf{q}_{i-1}$.

  – if v is of type L, there are two possibilities. First, if $v = (0, \ell_1, \dots, \ell_i)$, with $\ell_j \leq m((0, \ell_1, \dots, \ell_{j-1}))$ for all $1 \leq j \leq i$, then x_v has the same distribution as x_i (note that for simplicity, here we use the notation $m(v)$ for m_v). Therefore, $\deg(v) \sim \mathbf{q}_i$. Otherwise, there are indices $1 \leq k \leq i-1$ such that $\ell_k > m((0, \ell_1, \dots, \ell_{k-1}))$. Let j be the largest such index, and let $w = (0, \dots, \ell_{j-1})$ and $w' = (w, 1, \dots, 1)$ (with $(i-j)$'s 1). We now see that x_v has the same law as $x_{(w,\ell_j)} U_1 \dots U_{i-j}$ and $x_{w'}$ has the same law as $x_w U_1 \dots U_{i-j}$, where (U_i) is a sequence of i.i.d. uniform random variables in $[0, 1]$. As $\ell_j > m(w)$, we get $x_{(w,\ell_j)} \geq x_w$ and hence, $x_{w'} \preceq x_v$. Since both v and w' are of type L, it follows from (ii) that $\deg(v) \preceq \deg(w') \preceq \mathbf{q}_{i-1}$ (note that $w' \in B_T(w_0, i)$).

This concludes the proof of the lemma. □

Proof of the upper bound. We first recall a key estimate on the survial probability of the contact process on $(\mathbb{T}_2, o) = \mathrm{GW}(\mathbf{q}_0, \mathbf{q}_1)$. Note that $\mathbf{q}_0(k) \asymp k^{-\nu}$ and $\mathbf{q}_1(k) \asymp k^{-\nu+1}$ with $\nu = 2 + 1/\psi$. In [MVY], more precisely in Sections 6.1, 6.2 and 6.3, the authors proved that when the exponent ν is larger than $5/2$, there exist positive constants $C = C(\nu)$ and $R = R(\nu)$, such that

$$\mathbb{P}(\mathcal{L}_{\mathbb{T}_2}(o, R)) \leq C\lambda^{2\nu-3} |\log \lambda|^{2-\nu}, \tag{21}$$

where

$$\mathcal{L}_{\mathbb{T}_2}(o, R) = \{(o, 0) \leftrightarrow B_{\mathbb{T}_2}(o, R)^c \times \mathbb{R}_+\},$$

is the event that the contact process starting from o infects vertices outside $B_{\mathbb{T}_2}(o, R)$.

Importantly, all their proofs involve only what happens inside the ball $B_{\mathbb{T}_2}(o,R)$ with $R=R(\nu)$ fixed. On the other hand, it follows from Lemma 3.1 that

$$c\mathbf{q}_1(k) \le \mathbf{q}_{R-1}(k) \le C(\log k)^{R-2}\mathbf{q}_1(k), \tag{22}$$

with c and C independent of λ. In fact the logarithmic term on the right-hand side does not make any difference in all the proofs in [MVY] (we refer to their Sections 6.2 and 6.3 for more details) [1] . Hence one can prove exactly as (21) that

$$\mathbb{P}(\mathcal{L}_{\mathbb{T}_R}(o,R)) \le C\lambda^{2\nu-3}|\log\lambda|^{2-\nu}, \tag{23}$$

with the same constant $R=R(\nu)$ and a possibly different $C=C(\nu)$, and where

$$\mathcal{L}_{\mathbb{T}_R}(o,R) = \{(o,0) \leftrightarrow B_{\mathbb{T}_R}(o,R)^c \times \mathbb{R}_+\}.$$

Thanks to Lemma 3.3, $B_T(w_0,R) \preceq B_{\mathbb{T}_R}(o,R)$. Hence it follows from (23) and the monotonicity property of the contact process that

$$\mathbb{P}(\mathcal{L}_T(w_0,R)) \le \mathbb{P}(\mathcal{L}_{\mathbb{T}_R}(o,R)) \le C\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi}, \tag{24}$$

where

$$\mathcal{L}_T(w_0,R) = \{(w_0,0) \leftrightarrow B_T(w_0,R)^c \times \mathbb{R}_+\}.$$

On the other hand, the contact process starting from w_0 survives forever only if the virus infects vertices outside the balls $B_T(w_0,R)$ for all R. Therefore

$$\mathbb{P}(\xi_t^{w_0} \neq \varnothing\ \forall\, t\ge 0) \le \mathbb{P}(\mathcal{L}_T(w_0,R)) \le C\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi},$$

which proves the desired upper bound. □

Remark 3.4. In the case $\nu>3$, in [MVY], the authors improved (21) as follows

$$\mathbb{P}(\mathcal{L}_{\mathbb{T}_2}(o,R)) \le C\lambda^{2\nu-3}|\log\lambda|^{2(2-\nu)},$$

for some positive constants $C=C(\nu)$ and $R=R(\nu,\lambda)$. Although in the case of $(\mathbb{T}_R,o)$ the exponent $\nu=2+1/\psi$ is larger than 3, we can not apply this result, since here $R=R(\lambda)$ depends on λ. Therefore the distribution $\mathbf{q}_{R-1}$ also depends on λ and (22) does not hold anymore, so the proof in [MVY] can not be used here.

3.2. **Proof of the lower bound.** In this part, we first estimate the probability that there is an infinite sequence of vertices, including w_1, with larger and larger degree and a small enough distance between any two consecutive elements of the sequence. We then repeatedly apply Lemma 2.8 and 2.9 to bound from below the probability that the virus propagates along these vertices, and like this survives forever. To this end, we denote by

$$\varphi(\lambda) = \frac{7}{c^*}\frac{1}{\lambda^2}\log\left(\frac{1}{\lambda}\right), \tag{25}$$

with c^* as in Lemma 2.8 and 2.9.

[1]

- In their proof, we need to replace the event $\{\xi_t^o \neq \varnothing\ \forall\, t\ge 0\}$ by $\{(o,0) \leftrightarrow B(o,R)^c \times \mathbb{R}_+\}$.
- In Section 6.3 of their proof, in order to estimate the probability of B_2^5 and B_4^5, take $\epsilon_1' = \min(2\nu-5,2)/2$ instead of $\epsilon_1' = (2\nu-5)/2$.

Lemma 3.5. *There is a positive constant c, such that for λ small enough,*

$$\mathbb{P}(\mathcal{N}) \geq c\varphi(\lambda)^{-1/\psi},$$

where

$$\mathcal{N} = \{\exists (j_\ell)_{\ell\geq 1} : j_1 = 1, \deg(w_{j_\ell}) \geq 2^{\ell+1}\varphi(\lambda)/\psi \geq \varphi(\lambda) d(w_{j_\ell}, w_{j_{\ell+1}}) \ \forall \ell \geq 1\}.$$

Proof. It follows from Markov's inequality that for any $k \geq 1$,

$$\mathbb{P}(U_1 \dots U_k > 2^{-(k+1)/2}) \leq 2^{-(k-1)/2}, \tag{26}$$

where (U_i) is a sequence of i.i.d. uniform random variables in $[0,1]$.
Now recall that for any i,

$$\mathbb{P}(\deg(w_i) = m + 1 + k \mid x_i) = \frac{\Gamma(k+a)}{\Gamma(a)k!}(1 - x_i^\psi)^k x_i^{a\psi},$$

with $a = m + 2mr + 1$. Let C be some constant, such that for all $k \geq 1$,

$$\frac{\Gamma(k+a)}{\Gamma(a)k!} \leq Ck^{a-1}.$$

The existence of C is guaranteed by the fact that $\Gamma(k+b)/\Gamma(k) \asymp k^b$ when b is fixed. Then

$$\mathbb{P}(\deg(w_i) \leq m + 1 + (c/x_i^\psi) \mid x_i) \leq \sum_{k=0}^{c/x_i^\psi} Ck^{a-1} x_i^{a\psi} \leq Cc^a, \tag{27}$$

for any $c > 0$. Set $j_1 = 1$ and $j_\ell = 4\ell/\psi$ for $\ell \geq 2$. Then define

$$\mathcal{N}_\ell = \left\{x_{j_\ell} \leq (4^\ell \varphi(\lambda)/(c\psi))^{-1/\psi}, \deg(w_{j_\ell}) \geq 2^{\ell+1}\varphi(\lambda)/\psi\right\}$$

for all $\ell \geq 1$, where c is a positive constant to be chosen later.

As $2^{\ell+1}\varphi(\lambda)/\psi \geq 4\varphi(\lambda)/\psi = \varphi(\lambda) d(w_{j_\ell}, w_{j_{\ell+1}})$, we have

$$\mathcal{N} \supset \bigcap_{\ell=1}^{\infty} \mathcal{N}_\ell. \tag{28}$$

Since x_{j_ℓ} is distributed as $x_1 U_1 \dots U_{j_\ell - 1}$, applying (26) gives that

$$\mathbb{P}(x_{j_\ell} \leq x_1 4^{-\ell/\psi}) \geq 1 - 2(4^{-\ell/\psi}).$$

Therefore

$$\mathbb{P}\left(x_{j_\ell} \leq (4^\ell \varphi(\lambda)/(c\psi))^{-1/\psi} \mid \mathcal{N}_1\right) \geq 1 - 2(4^{-\ell/\psi}). \tag{29}$$

By using (27) with $(2^{1-\ell}c)$ instead of c we obtain that

$$\mathbb{P}\left(\deg(w_{j_\ell}) \geq 2^{\ell+1}\varphi(\lambda)/\psi \mid x_{j_\ell} \leq (4^\ell \varphi(\lambda)/(c\psi))^{-1/\psi}\right) \geq 1 - C(2^{1-\ell}c)^a. \tag{30}$$

Then it follows from (29) and (30) that

$$\mathbb{P}\left(\bigcap_{\ell=2}^{\infty} \mathcal{N}_\ell \mid \mathcal{N}_1\right) \geq 1 - 2\sum_{\ell=2}^{\infty} 4^{-\ell/\psi} - C\sum_{\ell=2}^{\infty} (2^{1-\ell}c)^a \geq 1/4, \tag{31}$$

provided c is small enough. We now estimate $\mathbb{P}(\mathcal{N}_1)$. Let

$$\gamma(\lambda) = (4\varphi(\lambda)/(c\psi))^{-1/\psi}.$$

Recall that x_1 is uniformly distributed on $[0, x_0]$, with $x_0 \sim U_0^\chi$ and $U_0 \sim \mathcal{U}([0,1])$. Therefore

$$\begin{aligned}\mathbb{P}(x_1 \le \gamma(\lambda)) &= \mathbb{E}\left(\frac{\min\{\gamma(\lambda), x_0\}}{x_0}\right)\\ &\ge \gamma(\lambda)\mathbb{P}(x_0 \ge \gamma(\lambda))\\ &= \gamma(\lambda)(1-\gamma(\lambda)^{1/\chi})\\ &\ge \gamma(\lambda)/2, \qquad (32)\end{aligned}$$

for λ small enough. On the other hand, (27) gives that for c small enough

$$\mathbb{P}(\mathcal{N}_1 \mid x_1 \le \gamma(\lambda)) \ge 1 - Cc^a \ge 1/2. \qquad (33)$$

We thus can choose c such that the two inequalities in (31) and (33) are satisfied. Now it follows from (28), (31), (32) and (33) that

$$\mathbb{P}(\mathcal{N}) \gtrsim \gamma(\lambda),$$

which implies the result. □

Proof of the lower bound. By repeatedly applying Lemma 2.9 to the couple of vertices $(w_{i_\ell}, w_{i_{\ell+1}})$, we obtain that

$$\begin{aligned}\mathbb{P}(\xi_t \ne \varnothing\, \forall t \ge T \mid \mathcal{N}, w_1 \text{ is lit at some time } T) &\ge 1 - 2\sum_{\ell=1}^{\infty} \exp(-c^*\lambda^2 2^{\ell+1}\varphi(\lambda)/\psi)\\ &\ge 1 - 2\sum_{\ell=1}^{\infty} \exp(-7(2^{\ell+1})|\log\lambda|/\psi)\\ &\ge 1/2. \qquad (34)\end{aligned}$$

On the other hand, by using Lemma 2.8 (i), we have

$$\begin{aligned}&\mathbb{P}(w_1 \text{ is lit at some time } T \mid \mathcal{N}, o \text{ is infected at time } 0)\\ &\ge c\lambda\mathbb{E}(1 - \exp(-c^*\lambda \deg(w_1)) \mid \mathcal{N}) \ge c\lambda/2, \qquad (35)\end{aligned}$$

for some $c > 0$ (note that on $\mathcal{N}$, we have $c^*\lambda\deg(w_1) \ge 7$). Now the result follows from (34), (35) and Lemma 3.5. □

4. Proof of Theorem 1.1.

By using the self-duality of the contact process (3), we see that to prove (2), it is sufficient to show that

$$\mathbb{P}_n\left(\frac{1}{n}\sum_{v\in V_n} 1(\{\xi_{t_n}^v \ne \varnothing\}) \le C\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi}\right) = 1 - o(1), \qquad (36)$$

and

$$\mathbb{P}_n\left(\frac{1}{n}\sum_{v\in V_n} 1(\{\xi_{t_n}^v \ne \varnothing\}) \ge c\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi}\right) = 1 - o(1), \qquad (37)$$

for some positive constants c and C. We will prove these two statements in the next two subsections.

4.1. **Proof of** (36)**.** For any $v \in V_n$ and R, we define

$$\mathcal{L}_n(v,R) = \{(v,0) \leftrightarrow B_{G_n}(v,R)^c \times \mathbb{R}_+\}$$

and

$$X_v = 1(\mathcal{L}_n(v,R)).$$

Theorem 2.2 yields that for any $R \geq 1$

$$\lim_{n\to\infty} \mathbb{P}_n(\mathcal{L}_n(u,R)) = \mathbb{P}(\mathcal{L}_T(w_0,R)), \tag{38}$$

where u is a uniformly chosen vertex from V_n. By combing this with (24) we obtain that

$$\lim_{n\to\infty} \mathbb{P}_n(X_u = 1) \leq C\lambda^{1+2/\psi} |\log\lambda|^{-1/\psi},$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{n} \sum_{v\in V_n} \mathbb{P}_n(X_v = 1) \leq C\lambda^{1+2/\psi} |\log\lambda|^{-1/\psi}. \tag{39}$$

Now, let us consider the set

$$W_n = \{(v,v') \in V_n \times V_n : d(v,v') \geq 2R+3\},$$

with $R = R(\nu)$ as in (23). Since $R+1 \leq b_2 \log n/(\log\log n)$ for n large enough, Lemma 2.6 implies that

$$\sum_{v,v'\in V_n} \mathbb{P}_n((v,v') \notin W_n) = o(n^2).$$

On the other hand, if $(v,v') \in W_n$ then X_v and $X_{v'}$ are independent. Therefore

$$\sum_{v,v'\in V_n} \mathrm{Cov}(X_v, X_{v'}) = o(n^2). \tag{40}$$

Thanks to (39) and (40) by using Chebyshev's inequality we get that

$$\mathbb{P}_n\left(\frac{1}{n}\sum_{v\in V_n} X_v \leq 2C\lambda^{1+2/\psi} |\log\lambda|^{-1/\psi}\right) = 1 - o(1). \tag{41}$$

Since the contact process on a finite ball in the Pólya-point graph a.s. dies out,

$$\lim_{t\to\infty} \mathbb{P}(\mathcal{L}_T(w_0,R)^c \cap \{\xi_t^{w_0} \neq \varnothing\}) = 0.$$

Hence for any $\varepsilon > 0$, there exists t_ε, such that

$$\mathbb{P}(\mathcal{L}_T(w_0,R)^c \cap \{\xi_{t_\varepsilon}^{w_0} \neq \varnothing\}) \leq \varepsilon. \tag{42}$$

For any $v \in V_n$, define

$$X_{v,\varepsilon} = 1(\mathcal{L}_n(v,R)^c \cap \{\xi_{t_\varepsilon}^{v} \neq \varnothing\}).$$

Then for n large enough such that $t_n \geq t_\varepsilon$, we have

$$1(\{\xi_{t_n}^{v} \neq \varnothing\}) \leq X_v + X_{v,\varepsilon}. \tag{43}$$

It follows from Theorem 2.2 and (42) that

$$\lim_{n\to\infty} \mathbb{E}_n(X_{u,\varepsilon}) = \mathbb{P}(\mathcal{L}_T(w_0,R)^c \cap \{\xi_{t_\varepsilon}^{w_0} \neq \varnothing\}) \leq \varepsilon.$$

By using this and Markov's inequality we get that for n large enough, and for any $\eta > 0$,

$$\mathbb{P}_n\left(\frac{1}{n}\sum_{v\in V_n} X_{v,\varepsilon} > \eta\right) \leq \frac{\sum_{v\in V_n} \mathbb{E}_n(X_{v,\varepsilon})}{n\eta} = \frac{\mathbb{E}_n(X_{u,\varepsilon})}{\eta} \leq \frac{2\varepsilon}{\eta}. \tag{44}$$

By combining (41), (43) and (44), then letting ε tend to 0, we obtain that

$$\mathbb{P}_n\left(\frac{1}{n}\sum_{v\in V_n} 1(\{\xi^v_{t_n}\neq\varnothing\}) \le 3C\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi}\right) = 1-o(1),$$

which proves (36). □

4.2. **Proof of** (37)**.** This subsection is divided into three parts. In the first one, we will show that w.h.p. there are many vertices with large degree (larger than $\varkappa^*\log n$). By using on the other hand that the diameter of the graph is smaller than $b_1\log n$, we can deduce that if one of these large degree vertices is infected, then the virus survives w.h.p. for a time $\exp(cn/(\log n)^{1/\psi})$, see Proposition 4.2. In the second part, we measure the density of *potential* vertices which are promising for spreading the virus to some of these large degree vertices. In the last part, we estimate the proportion of potential vertices which really send the virus to large degree vertices, getting this way (37).

4.2.1. *Lower bound on the extinction time.* Our aim in this part is to find large degree vertices as mentioned above. We then prove that if one of them is infected, the virus is likely to survive a long time.

Lemma 4.1. *Let $\varkappa>0$ be given. Then there exists a positive constant $\bar c=\bar c(\varkappa)$, such that $\mathcal{A}_n$ holds w.h.p. with*

$$\mathcal{A}_n=\{G_n \text{ contains } \bar c n/(\log n)^{1/(1-\chi)} \text{ disjoint star graphs of size larger than } \varkappa\log n\}.$$

Proof. Let $\varepsilon\in(0,1/3)$ be given, and let $K=K(\varepsilon)$ and $\mathcal{E}_\varepsilon$ be as in Lemma 2.4. Set $a_n=(M\log n)^{1/(1-\chi)}$, with M to be chosen later. Denote by

$$A=\{v_i : i\in[n/a_n, 2n/a_n] \text{ and } \psi_i\in(\theta/i,\mu/i)\}$$

and

$$\mathcal{E}=\mathcal{E}_\varepsilon\cap\{|A|\ge\theta n/a_n\},$$

with θ, μ as in Lemma 2.4. Recall that the events $\{\psi_i\in(\theta/i,\mu/i)\}$ are independent and have probability larger than 2θ. Therefore (4) implies that w.h.p. $|A|\ge\theta n/a_n$. Hence for n large enough

$$\mathbb{P}_n(\mathcal{E})\ge 1-2\varepsilon.$$

We now suppose that $\mathcal{E}$ happens. In particular, $|S_j^{(k)}-(j/k)^\chi|\le\varepsilon(j/k)^\chi$ for all $K(\varepsilon)\le j\le k$. Denote the elements of A as $\{v_{j_1},...,v_{j_\ell}\}$ with $\ell\in[\theta n/a_n, n/a_n]$. Then define

$$A_1=\{v_j : n/2\le j\le n\}.$$

We will show that all vertices in A have a large number of neighbors in A_1. First, it follows from (9) that if $j<k$, then v_j and v_k are neighbors with probability of order $\psi_j S_j^{(k-1)}$. Therefore there are positive constants c and C depending on θ, μ such that for all $v_j\in A$ and $v_k\in A_1$,

$$\frac{ca_n^{1-\chi}}{n}\le\mathbb{P}_n(v_j\sim v_k\mid\mathcal{E})\le\frac{Ca_n^{1-\chi}}{n}. \tag{45}$$

Conditionally on (ψ_j), the events $\{\{v_{j_1}\sim v_k\}\}_{k\in A_1}$ are independent. Hence thanks to (4) we get that there are positive constants θ_1, c_1, C_1 (depending on c and C), such that

$$\mathbb{P}_n\left(c_1a_n^{1-\chi}\le\sum_{v_k\in A_1}1(v_{j_1}\sim v_k)\le C_1a_n^{1-\chi}\mid\mathcal{E}\right)\ge 1-\exp(-\theta_1a_n^{1-\chi}),$$

or equivalently

$$\mathbb{P}_n(\mathcal{E}_1 \mid \mathcal{E}) \geq 1 - \exp(-\theta_1 a_n^{1-\chi}), \tag{46}$$

where

$$\mathcal{E}_1 = \{c_1 a_n^{1-\chi} \leq |B_1| \leq C_1 a_n^{1-\chi}\}$$

and

$$B_1 = \{v_k \in A_1 : v_{j_1} \sim v_k\}.$$

Note that in this proof, the values of the constants θ_1, c_1 and C_1 may change from line to line. Now let us consider

$$A_2 = A_1 \setminus B_1 \quad \text{and} \quad B_2 = \{v_k \in A_2 : v_{j_2} \sim v_k\}.$$

We notice that on $\mathcal{E}_1 \cap \mathcal{E}$, the cardinality of A_2 is larger than $n/2 - C_1 a_n^{1-\chi} \geq n/4$. Thus, similarly to (46) we can show that

$$\mathbb{P}_n(\mathcal{E}_2 \mid \mathcal{E}_1 \cap \mathcal{E}) \geq 1 - \exp(-\theta_1 a_n^{1-\chi}),$$

where

$$\mathcal{E}_2 = \{c_1 a_n^{1-\chi} \leq |B_2| \leq C_1 a_n^{1-\chi}\}.$$

Likewise for all $2 \leq s \leq \ell$, define recursively

$$A_s = A_{s-1} \setminus B_{s-1}, \quad B_s = \{v_k \in A_s : v_{j_s} \sim v_k\},$$

$$\mathcal{E}_s = \{c_1 a_n^{1-\chi} \leq |B_s| \leq C_1 a_n^{1-\chi}\}.$$

On $\mathcal{E} \cap \bigcap_{i=1}^{s-1} \mathcal{E}_i$, we have $|A_s| \geq n/2 - sC_1 a_n^{1-\chi} \geq n/4$, and

$$\mathbb{P}_n\left(\mathcal{E}_s \mid \mathcal{E} \cap \bigcap_{i=1}^{s-1} \mathcal{E}_i\right) \geq 1 - \exp -(\theta_1 a_n^{1-\chi}).$$

Hence

$$\mathbb{P}_n\left(\bigcap_{i=1}^{\ell} \mathcal{E}_i \mid \mathcal{E}\right) \geq 1 - n \exp(-\theta_1 a_n^{1-\chi})/a_n.$$

Taking M large enough such that $c_1 a_n^{1-\chi} \geq \varkappa \log n$ and $n \exp(-\theta_1 a_n^{1-\chi}) \leq 1$ yields that

$$\mathbb{P}_n(|B_s| \geq \varkappa \log n \ \forall\, 1 \leq s \leq \ell \mid \mathcal{E}) \geq 1 - a_n^{-1}. \tag{47}$$

Moreover, by definition $B_s \cap B_t = \varnothing$ for all $s \neq t$. Hence, all vertices in A have more than $\varkappa \log n$ distinct neighbors. Finally, take $\bar{c}$ such that $\bar{c} n/(\log n)^{1/(1-\chi)} \leq \theta n/a_n$, for instance $\bar{c} \leq \theta M^{-1/(1-\chi)}$. In conclusion, we have shown that for any given $\varepsilon \in (0, 1/3)$,

$$\mathbb{P}_n(\mathcal{A}_n) \geq 1 - 2\varepsilon - a_n^{-1} \geq 1 - 3\varepsilon,$$

for n large enough. Since this holds for any $\varepsilon > 0$, the result follows. □

To determine the constant $\varkappa$ in the definition of $\mathcal{A}_n$, we first recall that

$$\mathbb{P}_n(\mathcal{B}_n) = 1 - o(1),$$

where

$$\mathcal{B}_n = \{d(G_n) \leq b_1 \log n\}.$$

Hence to apply Lemma 2.9 to the large degree vertices exhibited in the previous lemma, we need

$$\varkappa \log n \geq \frac{7}{c^*} \frac{1}{\lambda^2} \log\left(\frac{1}{\lambda}\right) b_1 \log n.$$

Moreover, at some point later it will be convenient to have also $\varkappa \geq 3/(c^*\lambda^2)$. So we let

$$\varkappa^* = \max\Big\{\frac{7}{c^*}\frac{1}{\lambda^2}\log\Big(\frac{1}{\lambda}\Big)\, b_1, \frac{3}{c^*\lambda^2}\Big\}. \tag{48}$$

Then we let $\bar{c}^* = \bar{c}^*(\varkappa^*)$ and $\mathcal{A}_n$ be defined accordingly as in Lemma 4.1.

A set of vertices $V = \{w_1, \ldots, w_k\} \subset V_n$ is called **good** if $|S_{w_i} \setminus \cup_{j\neq i} S_{w_j}| \geq \varkappa^* \log n$ for all $1 \leq i \leq k$, where S_v denotes the star graph formed by v and its neighbors.

Let V_n^* be a maximal good set i.e. $|V_n^*| = \max\{|V| : V \subset V_n \text{ is good}\}$.

Proposition 4.2. *There exists a positive constant c, such that*

$$\mathbb{P}_n\left(\xi_{T_n} \neq \varnothing \mid \xi_0 \cap V_n^* \neq \varnothing\right) = 1 - o(1),$$

where $T_n = \exp(c\lambda^2 n/(\log n)^{1/\psi})$.

Proof. Thanks to Lemma 2.5 and 4.1, we can assume that $d(G_n) \leq b_1 \log n$ and $|V_n^*| \geq \bar{c}^* n/(\log n)^{1/(1-\chi)}$. Assume also that at time 0 a vertex in V_n^*, say v, is infected.

Due to the definiton of V_n^*, for any $w \in V_n^*$, we can select from the set of w's neighbors a subset $D(w)$ of size $\varkappa^* \log n$, such that $D(w) \cap D(w') = \varnothing$ for all $w \neq w'$.
We say that a vertex w in V_n^* is *infested* at some time t if the proportion of infected sites in $D(w)$ at time t is larger than $\lambda/(16e)$ (the term is taken from [MMVY]).

It follows from Lemma 2.8 (ii) that v becomes infested with probability tending to 1, as $n \to \infty$. Using Lemma 2.8 (iii) and 2.9 (note that $|D(w)| \geq (7/(c^*\lambda^2))|\log\lambda| d(w, w'))$, we deduce that for any $t \geq 0$ and $w \in V_n^*$,

$$\begin{aligned}&\mathbb{P}_n(v \text{ makes } w \text{ infested at } t + 2\exp(c^*\lambda^2\varkappa^*\log n) \mid v \text{ is infested at } t)\\ &\geq 1 - 4\exp(-c^*\lambda^2\varkappa^*\log n).\end{aligned}$$

Therefore

$$\begin{aligned}&\mathbb{P}_n(v \text{ makes all vertices in } V_n^* \text{ infested at } t + 2\exp(c^*\lambda^2\varkappa^*\log n) \mid v \text{ is infested at } t)\\ &\geq 1 - 4n\exp(-c^*\lambda^2\varkappa^*\log n)\\ &\geq 1 - n^{-1},\end{aligned}$$

where for the last inequality we have used that $c^*\lambda^2\varkappa^* \geq 3$. Now if all vertices in V_n^* are infested at the same time, then the proof of Proposition 1 in [CD] shows that the virus survives a time exponential in $\sum_{v\in V_n^*}\deg(v)$. More precisely, let $I_{n,t}$ be the number of infested vertices in V_n^* at time t. Then there is a positive constant η, such that for all $k \leq |V_n^*|$,

$$\mathbb{P}_n\left(I_{n,s_k} \geq k/2 \mid I_{n,0} \geq k\right) \geq 1 - s_k^{-1},$$

where $s_k = \exp(\eta\lambda^2 k\varkappa^*\log n)$. The result follows by taking $k = \bar{c}^* n/(\log n)^{1/(1-\chi)}$. □

4.2.2. *Density of potential vertices.* In this part we will estimate the proprotion of *potential* sites from where the virus can be sent with positive probability to a vertex at distance quite small (of order $(\log\log n)^2$) and with large degree (larger than $\varkappa^*\log n$).

This proportion approximates the probability that there is an infection path from the uniformly chosen vertex u to a vertex with degree larger than $\varkappa^*\log n$. To bound from below this probability, based on the idea of Section 3.2, in particular Lemma 3.5, we find a sequence of vertices starting from a neighbor of u and finishing at a large degree vertex, satisfying the hypothesis of Lemma 2.9 for spreading the virus from u to the ending vertex, see Lemma 4.4.

Here are just some comments on the order of magnitude above. First, if a vertex with degree larger than $\varkappa^* \log n$ is infected, then w.h.p. it will infect a site in V_n^* , and then we can conclude with Proposition 4.2. Secondly, $(\log\log n)^2$ is the distance from a potential vertex to a large degree vertex and is much smaller than the typical distance between two different potential vertices. Hence the propagation of the virus from these potential vertices to their closest large degree vertex are approximately independent events.

Set

$$R_n = (\log\log n)^2.$$

For $w \in V_n$, define $k_0(w)$ by $w = v_{k_0(w)}$, and for $i \geq 1$ define $k_i(w)$ by the conditions $k_i(w) < k_{i-1}(w)$ and $v_{k_i(w)} \sim v_{k_{i-1}(w)}$ (note that the choice of $k_i(w)$ is not necessarily unique). We define also

$$\mathcal{H}_n(w) = \{k_0(w) \geq n/\log n\} \cap \{k_{i+1}(w) \geq k_i(w)/\log k_i(w) \geq n^{1/2} \ \forall\, 0 \leq i \leq R_n\}.$$

Lemma 4.3. *There is a positive constant θ_0, such that for all $\theta \leq \theta_0$, for all $\varepsilon \in (0, 1/2)$, and for any vertex w, we have*

$$\mathbb{P}_n\left(\max_{v \in B_{G_n}(w,i)} \deg(v) \geq \theta e^{\theta i}(n/k_0(w))^{1-\chi} \mid \mathcal{E}_\varepsilon \cap \mathcal{H}_n(w)\right) \geq 1 - e^{-\theta i},$$

for all $i \leq R_n$, and

$$\mathbb{P}_n\left(\mathcal{H}_n(w) \mid k_0(w) \geq n/\log n, \mathcal{E}_\varepsilon\right) = 1 - o(1/\log\log n).$$

Proof. Let us begin with the second assertion. Due to the construction of G_n, if $k_i(w)/\log k_i(w) \geq K(\varepsilon)$, then

$$\begin{aligned}\mathbb{P}_n(k_{i+1}(w) \leq k_i(w)/\log k_i(w) \mid \mathcal{E}_\varepsilon, k_i(w), (\varphi_t)) &= \frac{S_{[k_i(w)/\log k_i(w)]}}{S_{k_i(w)-1}} \\ &\leq \frac{1+\varepsilon}{1-\varepsilon}\left(\frac{1}{\log k_i(w)}\right)^{\chi}.\end{aligned}$$

Hence for all $i \leq R_n = (\log\log n)^2$,

$$\mathbb{P}_n\left(k_{i+1}(w) \geq n/(\log n)^{i+2} \mid \mathcal{E}_\varepsilon, k_i(w) \geq n/(\log n)^{i+1}\right) = 1 - o((\log n)^{-\chi/2}),$$

which proves the result by using a union bound.

For the first inequality, we claim that there is a positive constant c_0, such that for any $c < c_0$, there exists $c' = c'(c) > 0$, such that for all $i \leq R_n$

(i) $\mathbb{P}_n\left(k_{[i/2]}(w) \leq e^{-ci} k_0(w) \mid \mathcal{E}_\varepsilon \cap \mathcal{H}_n(w)\right) \geq 1 - e^{-c'i}$,

(ii) $\mathbb{P}_n\left(\exists j \in (i/2, i) : \psi_{k_j(w)} \geq c/k_j(w) \mid \mathcal{E}_\varepsilon \cap \mathcal{H}_n(w)\right) \geq 1 - e^{-c'i}$,

(iii) $\mathbb{P}_n(\deg(v_k) \geq c'(n/k)^{1-\chi} \mid \psi_k \geq c/k, \mathcal{E}_\varepsilon) \geq 1 - \exp(-c'(n/k)^{1-\chi})$, for any $v_k \in V_n$.

From these claims we can deduce the result. Indeed, (i) and (ii) imply that with probability larger than $1 - 2\exp(-c'i)$ there is an integer $j \in (i/2, i)$ such that $k_j(w) \leq e^{-ci} k_0(w)$ and $\psi_{k_j(w)} \geq c/k_j(w)$. Then (iii) gives that $\deg(v_{k_j(w)}) \geq c' e^{c(1-\chi)i}(n/k_0(w))^{1-\chi}$ with probability larger than $1 - \exp(-c' e^{c(1-\chi)i})$. Hence the result follows by taking θ small enough.

To prove (i), similarly to Lemma 2.6, we consider

$$X_j(w) = 1(\{k_j(w) \leq k_{j-1}(w)/2\}) \text{ and } \mathcal{F}_j(w) = \sigma(k_t(w) : t \leq j) \vee \sigma((\varphi_t)).$$

On $\mathcal{H}_n(w)$, we have $K(\varepsilon) \le \sqrt{n} \le k_j(w)$ for all $j \le R_n$. Then by using the same argument as in Lemma 2.6 we obtain that on $\mathcal{H}_n(w) \cap \mathcal{E}_\varepsilon$,

$$\mathbb{E}_n(X_j(w) \mid \mathcal{F}_{j-1}(w)) \ge \frac{S_{[k_j(w)/2]} - S_{[k_j(w)/\log k_j(w)]}}{S_{k_j(w)-1}} \ge p$$

and

$$\mathbb{P}_n\left(\sum_{j=1}^{[i/2]} X_j(w) \ge ip/4\right) \ge 1 - 2\exp(-ip^2/16), \tag{49}$$

for some constant $p > 0$. Since $k_{[i/2]}(w) \le 2^{-ip/4} k_0(w)$ as soon as $\sum_{j=1}^{[i/2]} X_j(w) \ge ip/4$, the result follows from (49).

We now prove (ii). Let θ be the constant as in Lemma 2.4 (v). Fix some $j \in (i/2, i)$ and set $k = k_j(w) - 1$ and $\ell = [k_j(w)/\log k_j(w)]$. On $\mathcal{H}_n(w) \cap \mathcal{E}_\varepsilon$, we have $k \ge k_{j+1}(w) \ge \ell \ge \sqrt{n} \ge K(\varepsilon)$, and

$$\begin{aligned}
\mathbb{P}_n\left(\psi_{k_{j+1}(w)} \ge \theta/k_{j+1}(w) \mid k, \ell\right) &= \mathbb{E}_n\left(\frac{1}{S_k - S_\ell} \sum_{t=\ell+1}^{k} \varphi_t 1(\psi_t \ge \theta/t) \mid k, \ell\right) \\
&\ge \mathbb{E}_n\left(\frac{n^\chi}{(1+\varepsilon)k^\chi} \sum_{t=\ell+1}^{k} \varphi_t 1(\psi_t \ge \theta/t) \mid k, \ell\right) \\
&\ge \frac{n^\chi}{(1+\varepsilon)k^\chi} \mathbb{E}_n\left(\theta \sum_{t=\ell+1}^{k} \varphi_t \mid k, \ell\right) \\
&\ge \frac{\theta n^\chi}{(1+\varepsilon)k^\chi} \mathbb{E}_n\left(S_k - S_\ell \mid k, \ell\right) \\
&\ge \frac{\theta n^\chi}{(1+\varepsilon)k^\chi} \left[(1-\varepsilon)\,(k/n)^\chi - (1+\varepsilon)\,(\ell/n)^\chi\right] \\
&\ge \theta/4, \qquad (50)
\end{aligned}$$

for $\varepsilon \in (0, 1/2)$, where for the second inequality we have used Lemma 2.4 (v). Now (ii) follows from the same argument as (i). Finally, (iii) can be proved as (46). □

Lemma 4.4. *Let u be a uniformly chosen vertex from V_n. Then there exists a positive constant c, such that*

$$\mathbb{P}_n(\mathcal{M}) \ge c\lambda\varphi(\lambda)^{-1/\psi},$$

where

$$\mathcal{M} = \{\exists w \in B_{G_n}(u, R_n) : \deg(w) \ge \varkappa^* \log n\} \cap \{(\xi_\cdot^u) \text{ makes } w \text{ lit inside } B_{G_n}(u, R_n)\}.$$

Proof. Define k_0 by $v_{k_0} = u$ and for $i \ge 1$ define k_i by the conditions $k_i < k_{i-1}$ and $v_{k_i} \sim v_{k_{i-1}}$. Let us denote $u_1 = v_{k_1}$ and define also

$$\mathcal{H}_n := \mathcal{H}_n(u_1) = \{k_1 \ge n/\log n\} \cap \{k_{i+1} \ge k_i/\log k_i \ge n^{1/2}\ \forall\, 1 \le i \le R_n + 1\}.$$

In this proof, we assume that $\varepsilon = o(\lambda\varphi(\lambda)^{-1/\psi})$. Similarly to Lemma 4.3 by using that k_0 is chosen uniformly from $\{1, \dots, n\}$, we have $\mathbb{P}_n(\mathcal{H}_n \mid \mathcal{E}_\varepsilon) = 1 - o(1/\log\log n)$ and hence $\mathbb{P}_n(\mathcal{E}_\varepsilon \cap \mathcal{H}_n) = 1 - o(\lambda\varphi(\lambda)^{-1/\psi})$. We assume now that these two events happen.

We recall the claim (iii) in the proof of Lemma 4.3: there is a positive constant c_0, such that for any $c < c_0$, there exists $c' = c'(c) > 0$, such that

$$\mathbb{P}_n(\deg(v_k) \geq c'(n/k)^{1-\chi} \mid \psi_k \geq c/k) \geq 1 - \exp(-c'(n/k)^{1-\chi}), \tag{51}$$

for any $v_k \in V_n$. Let us consider

$$\mathcal{M}_1 = \big\{k_1 \leq n/\tilde{\gamma}(\lambda)\big\},$$

where $\tilde{\gamma}(\lambda) = (4\varphi(\lambda)/c'\theta^2)^{1/1-\chi}$, with θ a small enough constant (smaller than θ_0 as in Lemma 2.4 and 4.3 and than c_0), and $c' = c'(\theta)$. Define

$$\mathcal{M}_2 = \mathcal{M}_1 \cap \big\{\forall\, 1 \leq \ell \leq R'_n \;\; \exists w_\ell : d(u_1, w_\ell) \leq r_\ell, \deg(w_\ell) \geq \varphi(\lambda)\exp(\theta r_\ell)\big\},$$

where $r_\ell = 4\ell/\theta^2$ for $1 \leq \ell \leq R'_n := \theta^2 R_n/8$.
By using Lemma 4.3 for u_1 we get that for any $\ell \leq R'_n$

$$\mathbb{P}_n\left(\max_{v \in B_{G_n}(u_1, r_\ell)} \deg(v) \geq \theta e^{\theta r_\ell}(n/k_1)^{1-\chi}\right) \geq 1 - e^{-\theta r_\ell}.$$

If $k_1 \leq n/\tilde{\gamma}(\lambda)$, then $\theta\exp(\theta r_\ell)(n/k_1)^{1-\chi} \geq \theta\exp(\theta r_\ell)\tilde{\gamma}(\lambda)^{1-\chi} \geq \varphi(\lambda)\exp(\theta r_\ell)$. Thus

$$\mathbb{P}_n\left(\exists v : d(v, u_1) \leq r_\ell, \deg(v) \geq \varphi(\lambda)\exp(\theta r_\ell) \mid \mathcal{M}_1\right) \geq 1 - e^{-4\ell/\theta}.$$

Hence

$$\mathbb{P}_n\left(\mathcal{M}_2 \mid \mathcal{M}_1\right) \geq 1 - \sum_{\ell=1}^{R'_n} \exp(-4\ell/\theta) \geq 1 - 2\exp(-4/\theta). \tag{52}$$

Define

$$\mathcal{M}_3 = \mathcal{M}_1 \cap \{\deg(u_1) \geq 4\varphi(\lambda)/\theta^2\}.$$

Similarly to (50), we can show that

$$\mathbb{P}_n\left(\psi_{k_1} \geq \theta/k_1 \mid k_1 \leq n/\tilde{\gamma}(\lambda)\right) \geq \theta/4.$$

It follows from (51) and the fact that $c'\tilde{\gamma}(\lambda)^{1-\chi} = 4\varphi(\lambda)/\theta^2$, that

$$\mathbb{P}_n\left(\deg(u_1) \geq c'(n/k_1)^{1-\chi} \mid k_1 \leq n/\tilde{\gamma}(\lambda), \psi_{k_1} \geq \theta/k_1\right) \geq 1 - \exp(-4\varphi(\lambda)/\theta^2) \geq 1/2.$$

From the last two inequalities we deduce that

$$\mathbb{P}_n(\mathcal{M}_3 \mid \mathcal{M}_1) \geq \theta/8.$$

Combining this with (52) we obatin that

$$\mathbb{P}_n(\mathcal{M}_2 \cap \mathcal{M}_3 \mid \mathcal{M}_1) \geq \theta/8 - 2\exp(-4/\theta) \geq \theta/16. \tag{53}$$

We now bound from below $\mathbb{P}_n(\mathcal{M}_1)$. Observe that

$$\begin{aligned}\mathbb{P}_n\left(k_1 \leq n/\tilde{\gamma}(\lambda) \,\middle|\, k_0, (\varphi_j)\right) &\geq \frac{S_{[n/\tilde{\gamma}(\lambda)]}1(\{k_0 > n/\tilde{\gamma}(\lambda)\})}{S_{k_0-1}} \\ &\gtrsim \tilde{\gamma}(\lambda)^{-\chi}\left(\frac{k_0}{n}\right)^{\chi} 1(\{k_0 > n/\tilde{\gamma}(\lambda)\}).\end{aligned}$$

Since k_0 is distributed uniformly on $\{1, \ldots, n\}$, we get

$$\mathbb{E}_n\left((k_0/n)^{\chi}1(\{k_0 > n/\tilde{\gamma}(\lambda)\})\right) = \frac{1}{n}\sum_{k=n/\tilde{\gamma}(\lambda)}^{n}(k/n)^{\chi} \asymp 1.$$

Therefore

$$\mathbb{P}_n(\mathcal{M}_1) \gtrsim \tilde{\gamma}(\lambda)^{-\chi} \gtrsim \varphi(\lambda)^{-1/\psi}.$$

This and (53) give that

$$\mathbb{P}_n(\mathcal{M}_2 \cap \mathcal{M}_3) \gtrsim \varphi(\lambda)^{-1/\psi}. \tag{54}$$

Observe that on $\mathcal{M}_2 \cap \mathcal{M}_3$, we have $\deg(u_1) \geq \varphi(\lambda) r_1 \geq \varphi(\lambda) d(u_1, w_1)$ and

$$\deg(w_\ell) \geq \varphi(\lambda) \exp(\theta r_\ell) \geq 2\varphi(\lambda) r_{\ell+1} \geq \varphi(\lambda) d(w_\ell, w_{\ell+1})$$

for all $1 \leq \ell \leq R'_n$. In other words, u_1 and the vertices (w_ℓ) satisfy the condition in Lemma 2.9, and thus applying this lemma inductively yields that

$$\begin{aligned}
&\mathbb{P}_n(w_{R'_n} \text{ is lit inside } B_{G_n}(u_1, R_n) \mid \mathcal{M}_2 \cap \mathcal{M}_3, u_1 \text{ is lit}) \\
&\geq 1 - \sum_{\ell=1}^{R'_n} \exp(-c^* \lambda^2 \varphi(\lambda) e^{\theta r_\ell}) \\
&\gtrsim 1. \qquad (55)
\end{aligned}$$

Similarly to (35), the probability that $(\xi^u_\cdot)$ makes u_1 lit is of order λ. It follows from this and (55) that

$$\mathbb{P}_n((\xi^u_\cdot) \text{ makes } w_{R'_n} \text{ lit inside } B_{G_n}(u, R_n) \mid \mathcal{M}_2 \cap \mathcal{M}_3) \gtrsim \lambda. \tag{56}$$

In addition, $\deg(w_{R'_n}) \geq \varkappa^* \log n$. Therefore

$$\mathcal{M} \supset \mathcal{M}_2 \cap \mathcal{M}_3 \cap \{(\xi^u_\cdot) \text{ makes } w_{R'_n} \text{ lit inside } B_{G_n}(u, R_n)\}.$$

Combining this with (54) and (56) gives the result. □

4.2.3. *Proof of* (37). For any $v \in V_n$, we define

$$Y_v = 1(\{\exists w \in B_{G_n}(v, R_n) : \deg(w) \geq \varkappa^* \log n\} \cap \{(\xi^v_\cdot) \text{ makes } w \text{ lit inside } B_{G_n}(v, R_n)\})$$

and

$$Z_v = Y_v 1(\{\xi^v_{T_n} \neq \varnothing\}),$$

where T_n is as in Proposition 4.2. Then

$$\sum_{v \in V_n} Z_v \leq \sum_{v \in V_n} 1(\{\xi^v_{T_n} \neq \varnothing\}),$$

and thus (37) follows from Lemma 4.5 below and an application of Markov's inequality.

Lemma 4.5. *The following assertions hold:*

(i) $\mathbb{P}_n\left(\frac{1}{n}\sum_{v \in V_n} Y_v \geq c\lambda\varphi(\lambda)^{-1/\psi}\right) = 1 - o(1)$, *for some* $c > 0$, *independent of* λ.

(ii) $\mathbb{P}_n(Z_v = 1 \mid Y_v = 1) \to 1$, *as* $n \to \infty$ *uniformly in* $v \in V_n$.

Proof. For (i), let $\varepsilon \in (0, 1/2)$ be given. We have to show that the probability in the left-hand side is larger than $1 - 2\varepsilon$ for n large enough. First, Lemma 4.4 implies that

$$\mathbb{P}_n(Y_u = 1) \gtrsim \lambda\varphi(\lambda)^{-1/\psi},$$

or equivalently

$$\frac{1}{n}\sum_{v \in V_n} \mathbb{P}_n(Y_v = 1) \gtrsim \lambda\varphi(\lambda)^{-1/\psi}.$$

Using Chebyshev's inequality, the result follows from this and the following claim: on $\mathcal{E}_\varepsilon$

$$\sum_{v,v' \in V_n} \mathrm{Cov}(Y_v, Y_{v'}) = o(n^2). \tag{57}$$

To prove it, we consider

$$\mathcal{V}_n = \{(v_i, v_j) : i, j \geq n/\log n, d(v_i, v_j) \geq 2R_n + 3\}.$$

Since $R_n + 1 \leq b_2 \log n/(\log\log n)$ for n large enough, it follows from Lemma 2.6 that

$$\sum_{v,v' \in V_n} \mathbb{P}_n((v, v') \notin \mathcal{V}_n) = o(n^2). \tag{58}$$

On the other hand, Lemma 4.3 gives that if $i \geq n/\log n$, then on $\mathcal{E}_\varepsilon$

$$\mathbb{P}_n\left(\exists w \in B_{G_n}(v_i, R_n) : \deg(w) \geq \varkappa^* \log n\right) = 1 - o(1/\log\log n).$$

Moreover, given the graph G_n, Y_v and $Y_{v'}$ only depend on the Poisson processes defined on the vertices and edges on the balls $B_{G_n}(v, R_n)$ and $B_{G_n}(v', R_n)$ respectively. Hence on $\mathcal{E}_\varepsilon$ for all $(v, v') \in \mathcal{V}_n$,

$$\mathrm{Cov}(Y_v, Y_{v'}) = o(1/\log\log n). \tag{59}$$

Now (57) follows from (58) and (59).

We now prove (ii). If $Y_v = 1$, then there exists a vertex w such that $\deg(w) \geq \varkappa^* \log n$ and w is lit at some time. Besides, on $\mathcal{B}_n$ the diameter of the graph is bounded by $b_1 \log n$ w.h.p. Hence similarly to Lemma 2.9, we can show that on $\mathcal{B}_n$

$$\mathbb{P}_n(w \text{ infects a vertex in } V_n^* \,) \geq 1 - \exp(-c^* \varkappa^* \lambda^2 \log n).$$

If one of the vertices in V_n^* is infected, it follows from Proposition 4.2 that w.h.p. the virus survives up to time T_n. Hence we obtain (ii) by using that $\mathcal{B}_n$ holds w.h.p. □

Remark 4.6. Using Proposition 6.2 in [CS] and the facts that G_n is connected and $d(G_n) = \mathcal{O}(\log n)$, we can obtain another metastability result. Let τ_n be the extinction time of the contact process with infection rate $\lambda > 0$ starting from full occupancy. Then the following convergence in law holds

$$\frac{\tau_n}{\mathbb{E}_n(\tau_n)} \xrightarrow[n\to\infty]{(\mathcal{L})} \mathcal{E}(1),$$

with $\mathcal{E}(1)$ an exponential random variable with mean one.

As mentioned in the introduction, the bound on t_n in Theorem 1.1 is nearly optimal, but by Remark 6.4 in [CS] one could improve it if one could prove that w.h.p. $\tau_n \geq \exp(cn)$, for some $c > 0$. We expect the last assertion to hold like in the case of the configuration model [MMVY, CS] for instance.

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