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Generic Hahn-Banach results

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Abstract

Given $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ a convex and lower semi-continuous function defined on a reflexive Banach space X , and L , a closed linear manifold of X over which f takes at least a real value, the aim of this note is to prove the following Baire category result: in the Euclidean setting, the set of affine functions dominated by f on L for which there is no dominated extension to X is always of first Baire category, but this set can be as large as a residual set, provided that X is a reflexive Banach space of infinite dimension.

Keywords: affine Hahn-Banach theorem, Baire category, basic sequence

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1. Introduction

The dominated extension form of the Hahn-Banach theorem ("The Crown Jewel of Functional Analysis" as it is called in [7, Section 1]) proves that,

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given X a linear vector space, L one of its linear subspaces, and $p : X \rightarrow \mathbb{R}$ a sublinear function (that is a convex function which is also positively homogeneous of degree one), any linear function $\ell : L \rightarrow \mathbb{R}$ dominated on L by p may be extended to a linear function $\bar{\ell} : X \rightarrow \mathbb{R}$ dominated by p on the whole space X .

It is well-known¹ that the the conclusion of this theorem ceases to hold true when, instead of a sublinear, we address a hypolinear dominating function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ (that is a convex function which is positively homogeneous of degree one, and which is allowed to take the value $+\infty$); this fact goes back at least to 1968, when Simons ([9, pages 113-114]) showed that, over any vector space X of infinite dimension, there is a hypolinear functional which dominates no linear functional.

In order to overcome this difficulty, some regularity conditions have to be imposed on the pair formed by the hypolinear function f and the linear subspace L . The qualification condition provided by the seminal duality theorem by Fenchel (originally published as result number 48, at page 109 in [5]; see also [8, Theorem 31.1, §31]) is at the origin of a very fruitful line of research, and an overview, even a concise one, of the large number of works providing generalizations of Fenchel theorem lies outside the scope of the present paper. The interested reader is referred to Corollary 2.8.5

¹Maybe not as widely known as it should be, judging by the unusually large number of flawed Hahn-Banach type theorems for convex functions allowed to take the value $+\infty$ which can be found in the mathematical literature; in the articles [1], [12], [13], and [14], the reader can find examples and criticism of as much as ten such incorrect results published between 1969 and 2012.

and Theorem 2.8.3 from [11], results which gather some of the most useful regularity condition ensuring the validity of the Fenchel - Rockafellar duality formula, and hence of the dominated extension property.

Our article addresses the domination form of the Hahn-Banach theorem from a different angle. Let X be a real reflexive Banach space, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex and lower semi-continuous function, and L be a closed linear manifold of X over which f takes at least a real value. Let $\mathcal{A}(f, L)$, and respectively $\mathcal{A}(f, X)$, denote the set of all the affine continuous functions defined on L , and respectively on X , which are dominated by f ; $\mathcal{N}(f, L)$ is the set of all the functions from $\mathcal{A}(f, L)$ which cannot be extended to functions from $\mathcal{A}(f, X)$ (roughly speaking, $\mathcal{N}(f, L)$ is the set on which the Hahn-Banach dominated extension theorem fails).

Our main concern is the Baire category of $\mathcal{N}(f, L)$ as a subset of the complete metric space obtained by endowing the set $\mathcal{A}(f, L)$ with the standard metric. In point of fact, it is easy to see that the convex set $\mathcal{A}(f, L) \setminus \mathcal{N}(f, L)$ is dense in $\mathcal{A}(f, L)$. When X is of finite dimension, the set $\mathcal{A}(f, L) \setminus \mathcal{N}(f, L)$ is then automatically a residual subset of $\mathcal{A}(f, L)$, so $\mathcal{N}(f, L)$ is always a meagre, and therefore small, subset of $\mathcal{A}(f, L)$.

The case of a reflexive Banach space X of infinite dimension is however less obvious, since in this setting, although there still exists convex sets which are residual subsets of their closure, it is not a very difficult task to construct a convex set which is of first Baire category in its closure (take for instance the linear span of a compact set containing the origin of X), or even a convex set which is neither residual, nor of first Baire category with respect of its closure (this is the case of a dense hyperplane).

Using Proposition 4.2, a functional analysis result which states that any reflexive Banach space X with a basis contains two closed subspaces whose Minkowski sum is both dense, and of first Baire category in X , we are able to prove the main result of our study, as provided by Theorem 4.3: given X a reflexive Banach space, there are always some f and L such that $\mathcal{N}(f, L)$ is a residual, and thus very large, subset of $\mathcal{A}(f, L)$.

The article ends with a short section stating several open problems.

2. Definitions and notation

Throughout the paper, $(X, \|\cdot\|_X)$ will be a real reflexive Banach space, with a topological dual denoted by X^* , and a duality pairing denoted by $\langle \cdot, \cdot \rangle_X : X^* \times X \rightarrow \mathbb{R}$. Let us also pick $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$, a convex and lower semi-continuous function which is proper, meaning that $\text{dom}(f) = \{x \in X : f(x) < +\infty\}$, its effective domain, is a non empty set (as customary, the class of all these functions is called $\Gamma_0(X)$), together with L , a closed linear subspace of X over which f takes at least one finite value. Our notation is completed by setting L^* for the topological dual, and $\langle \cdot, \cdot \rangle_L : L^* \times L \rightarrow \mathbb{R}$ for the duality pairing of the locally convex space obtained by endowing L with the induced topology from X .

Our study is concerned with affine continuous functions defined on L , that is functions $\alpha : L \rightarrow \mathbb{R}$ such that relation

$$\alpha(x) = \langle v_\alpha^*, x \rangle_L - r_\alpha, \quad \forall x \in L,$$

holds true for some $v_\alpha^* \in L^*$ and $r_\alpha \in \mathbb{R}$. We systematically identify an affine continuous function α defined on L with the element (v_α^*, r_α) from $L^* \times \mathbb{R}$.

Accordingly, the set $\mathcal{A}(f, L)$, of all the affine continuous functions dominated on L by f , will be identified with the set

$$\{(v^*, r) \in L^* \times \mathbb{R} : \langle v^*, x \rangle_L - r \leq f(x) \quad \forall x \in L\}.$$

Similarly, the set

$$\{(x^*, r) \in X^* \times \mathbb{R} : \langle x^*, x \rangle_X - r \leq f(x) \quad \forall x \in X\}$$

will be identified with $\mathcal{A}(f, X)$, the class of all the affine continuous functions defined on X and dominated by f .

Moreover, all the standard convex analysis notations are in order; in particular, $\text{epi } a = \{(x^*, r) \in X^* \times \mathbb{R} : a(x^*) \leq r\}$ denotes the epigraph of a function $a : X^* \rightarrow \mathbb{R} \cup \{+\infty\}$, while, for any closed subspaces $V \subset X$ and respectively $W \subset X^*$, the notation V^\perp and W^\perp is used for the sets

$$V^\perp = \{x^* \in X^* : \langle x^*, x \rangle_X = 0 \quad \forall x \in V\},$$

and respectively

$$W^\perp = \{x \in X : \langle x^*, x \rangle_X = 0 \quad \forall x^* \in W\}.$$

We also use the classical conjugacy operation from convex analysis. Let us recall from [6, §4], that relation

$$a^*(x^*) = \sup_{x \in X} (\langle x^*, x \rangle_X - a(x)) \quad \forall x^* \in X^*$$

defines the Legendre-Fenchel conjugate of a function $a : X \rightarrow \mathbb{R} \cup \{+\infty\}$.

Finally, a subset of a complete metric space is called of first Baire category, or meagre, if it is included in a countable union of closed sets of empty interior; the complement of a meagre set is called residual.

3. A dual characterization of $\mathcal{N}(f, L)$

As a first step in establishing our main result, this section is concerned with the characterization of the set $\mathcal{N}(f, L)$ in terms of the Fenchel-Legendre conjugates of f and ι_L , the indicator function of L ,

$$\iota_L(x) = \begin{cases} 0 & x \in L \\ +\infty & x \notin L \end{cases} \quad \forall x \in X.$$

We define the linear and continuous operator

$$\Pi : X^* \times \mathbb{R} \rightarrow L^* \times \mathbb{R} \quad \Pi(x^*, r) = (\pi(x^*), r) \quad \forall (x^*, r) \in X^* \times \mathbb{R},$$

where $\pi(x^*)$ is the restriction of x^* to L , or, equivalently, the only element from L^* satisfying the relation $\langle x^*, x \rangle_X = \langle \pi(x^*), x \rangle_L$ for all $x \in L$. The very simple proof of the following lemma will be skipped.

Lemma 3.1. *Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function, and L be a closed subspace of X . Then:*

$$\mathcal{A}(f, L) = \Pi(\text{epi}(f + \iota_L)^*), \tag{1}$$

$$\mathcal{A}(f, L) \setminus \mathcal{N}(f, L) = \Pi(\text{epi } f^* + \text{epi } \iota_L^*), \tag{2}$$

whence

$$\mathcal{N}(f, L) = \Pi(\text{epi}(f + \iota_L)^* \setminus (\text{epi } f^* + \text{epi } \iota_L^*)). \tag{3}$$

The following statement addresses standard geometrical and topological properties of the nested pair of sets $\mathcal{A}(f, L)$ and $\mathcal{N}(f, L)$.

Proposition 3.2. *Let f be a function from $\Gamma_0(X)$, and L be a closed linear subspace of X which meets $\text{dom } f$. Then $\mathcal{A}(f, L)$ is a closed and convex subset of $L^* \times \mathbb{R}$, and $\mathcal{A}(f, L) \setminus \mathcal{N}(f, L)$ is a convex set dense in $\mathcal{A}(f, L)$.*

Proof of Proposition 3.2: Both the sets $\mathcal{A}(f, L)$ and $\mathcal{A}(f, L) \setminus \mathcal{N}(f, L)$ are convex, as they are the images of convex sets through a linear operator (see relations (1) and (2)).

In order to prove that $\mathcal{A}(f, L) \setminus \mathcal{N}(f, L)$ is dense in $\mathcal{A}(f, L)$, we need the following well-known fact (for this and other properties related to the notion of infimal convolution, see [6] and the survey [10]): given g and h two functions from $\Gamma_0(X)$ such that $\text{dom } g \cap \text{dom } h \neq \emptyset$, then the set $\text{epi } (h + g)^*$ amounts to $\overline{\text{epi } g^* + \text{epi } h^*}$, the closure of the set $\text{epi } g^* + \text{epi } h^*$. Applied to the functions f and ι_L , the previous theorem implies that

$$\overline{\text{epi } f^* + \text{epi } \iota_L^*} = \text{epi } (f + \iota_L)^*. \quad (4)$$

Moreover, the mapping Π is continuous, so the closure of the image through Π of any set always contains the image through Π of the closure of the same set; in particular,

$$\Pi(\overline{\text{epi } f^* + \text{epi } \iota_L^*}) \subset \overline{\Pi(\text{epi } f^* + \text{epi } \iota_L^*)}. \quad (5)$$

By combining relations (1), (2), (4) and (5), we infer that

$$\mathcal{A}(f, L) \setminus \mathcal{N}(f, L) \subset \mathcal{A}(f, L), \quad \mathcal{A}(f, L) \subset \overline{\mathcal{A}(f, L) \setminus \mathcal{N}(f, L)};$$

thus $\mathcal{A}(f, L) \setminus \mathcal{N}(f, L)$ is dense in $\mathcal{A}(f, L)$.

Finally, let us show that the set $\mathcal{A}(f, L)$ is closed. Since that for every $y^* \in L^\perp$ it holds that

$$\begin{aligned} (f + \iota_L)^*(x^* + y^*) &= \sup_{x \in X} (\langle x^* + y^*, x \rangle_X - (f + \iota_L)(x)) \\ &= \sup_{x \in L} (\langle x^* + y^*, x \rangle_X - f(x)) \\ &= \sup_{x \in L} (\langle x^*, x \rangle_X - f(x)) = (f + \iota_L)^*(x^*) \quad \forall x^* \in X^*, \end{aligned}$$

we deduce that

$$\text{epi}(f + \iota_L)^* + L^\perp \times \{0\} = \text{epi}(f + \iota_L)^*. \quad (6)$$

But $\text{Ker } \Pi = L^\perp \times \{0\}$, so relation (6) may be rephrased as

$$\text{epi}(f + \iota_L)^* + \text{Ker } \Pi = \text{epi}(f + \iota_L)^*. \quad (7)$$

The continuous linear operator Π between the Banach spaces $X^* \times \mathbb{R}$ and $L^* \times \mathbb{R}$ is onto; an obvious consequence of Banach-Schauder theorem says that $\Pi(S)$ is closed for any closed subset S of $X^* \times \mathbb{R}$ such that $S + \text{Ker } \Pi = S$. From relation (7) it follows that the previous theorem applies to the visibly closed set $\text{epi}(f + \iota_L)^*$, so $\mathcal{A}(f, L) = \Pi(\text{epi}(f + \iota_L)^*)$ is closed. \square

4. Baire category results

Proposition 3.2 proves that the set $\mathcal{A}(f, L)$ is a closed subset of the reflexive Banach space $L^* \times \mathbb{R}$; when endowed with the distance $d_{L^* \times \mathbb{R}}$, induced by the box norm from $L^* \times \mathbb{R}$, the set $\mathcal{A}(f, L)$ became a complete metric space. Theorems 4.1 and 4.3 are concerned with the Baire category of the subset $\mathcal{N}(f, L)$ of the complete metric space $(\mathcal{A}(f, L), d_{L^* \times \mathbb{R}})$.

Theorem 4.1. *Let X an Euclidean space, f be a function from $\Gamma_0(X)$, and L be a closed linear subspace of X which meets $\text{dom } f$. Then the subset $\mathcal{N}(f, L)$ of the complete metric space $(\mathcal{A}(f, L), d_{L^* \times \mathbb{R}})$ is of first Baire category.*

Proof of Theorem 4.1: As the convex set $\mathcal{A}(f, L) \setminus \mathcal{N}(f, L)$ is dense in $\mathcal{A}(f, L)$, Corollary 6.3.1 from [8] proves that $\mathcal{A}(f, L) \setminus \mathcal{N}(f, L)$ contains the relative interior of $\mathcal{A}(f, L)$.

Accordingly, the set $\mathcal{N}(f, L)$ is contained in the relative boundary of $\mathcal{A}(f, L)$, which is a closed subset with empty interior of the complete metric space $(\mathcal{A}(f, L), d_{L^* \times \mathbb{R}})$. In particular, $\mathcal{N}(f, L)$ is of first Baire category. \square

The following result is a key point in achieving our main theorem.

Proposition 4.2. *Any infinite dimensional reflexive Banach space X with a Schauder basis contains two closed subspaces such that their Minkowski sum is both dense, and of first Baire category in X .*

Proof of Proposition 4.2: The proof is done in two steps. First, we prove that any infinite dimensional Banach space X with a Schauder basis contains two closed subspaces whose sum is dense and proper in X . To this respect, let the sequence $(x_n)_{n \in \mathbb{N}} \subset X$ of norm one vectors be a Schauder basis of X , meaning that for each vector $x \in X$ there exists a unique sequence $(\lambda_n)_{n \in \mathbb{N}}$ of scalars such that

$$x = \lim_{n \rightarrow +\infty} \sum_{k=1}^n \lambda_k x_k,$$

set V_1 for the closed linear span of the vectors $\{v_n = x_{2n-1} : n \in \mathbb{N}\}$, while the closed linear span of the vectors $\{w_n = (x_{2n-1} + \frac{x_{2n}}{8^n}) : n \in \mathbb{N}\}$ is denoted by V_2 .

As $\{x_n : n \in \mathbb{N}\} \subset V_1 + V_2$, the set $V_1 + V_2$ is clearly dense in X .

Let us consider the sequence $z_n = \sum_{k=1}^n \frac{x_{2k}}{2^k}$; for any two positive integers $m < n$, it holds that

$$\begin{aligned} \|z_n - z_m\|_X &= \left\| \sum_{k=m+1}^n \frac{x_{2k}}{2^k} \right\|_X \leq \sum_{k=m+1}^n \frac{\|x_{2k}\|_X}{2^k} \\ &= \sum_{k=m+1}^n \frac{1}{2^k} \leq \frac{1}{2^m}, \end{aligned}$$

so $(z_n)_n$ is a Cauchy sequence. We claim that its limit,

$$\bar{z} = \lim_{n \rightarrow +\infty} \sum_{k=1}^n \frac{1}{2^k} x_{2k}, \quad (8)$$

does not belong to $V_1 + V_2$.

To the end of achieving a contradiction, we assume that there exists $v_1 \in V_1$ and $v_2 \in V_2$ such that $v_1 + v_2 = \bar{z}$. Hence,

$$v_1 = \lim_{n \rightarrow +\infty} \sum_{k=1}^n \lambda_k v_k, \quad v_2 = \lim_{n \rightarrow +\infty} \sum_{k=1}^n \mu_k w_k, \quad (9)$$

for some sequences $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ and $(\mu_n)_{n \in \mathbb{N}} \subset \mathbb{R}$. By computing the sum $v_1 + v_2$, one obtains that

$$\bar{z} = v_1 + v_2 = \lim_{n \rightarrow +\infty} \sum_{k=1}^n \left((\lambda_k + \mu_k) x_{2k-1} + \frac{\mu_k}{8^k} x_{2k} \right); \quad (10)$$

relation

$$\lambda_n + \mu_n = 0, \quad \frac{\mu_n}{8^n} = \frac{1}{2^n} \quad \forall n \in \mathbb{N} \quad (11)$$

easily results when comparing statements (8) and (10).

It is an immediate consequence of relation (11) that $\lambda_n = -4^n$. Keeping in mind that the sequence $\zeta_n = \sum_{k=1}^n \lambda_k v_k$ converges to v_1 , let us compute its norm:

$$\begin{aligned} \|\zeta_n\|_X &= \left\| \sum_{k=1}^n \lambda_k v_k \right\|_X = \left\| \sum_{k=1}^n -4^k x_{2k-1} \right\|_X \\ &\geq \left\| -4^n x_n \right\|_X - \left(\sum_{k=1}^{n-1} \left\| -4^k x_k \right\|_X \right) \\ &= 4^n - \sum_{k=1}^{n-1} 4^k > 4^n - \frac{4^n - 4}{3} = \frac{2^{2n+1} - 2^2}{3}. \end{aligned} \quad (12)$$

Inequality (12) implies that the norm of the convergent sequence ζ_n goes to infinity, a clear contradiction. Our initial assumption is therefore false; that is, the subspace $V_1 + V_2$ does not contain the vector \bar{z} .

In the second step of our proof, we show that, in the setting of a reflexive Banach space X , the sum of two closed subspaces is of first Baire category in X , provided that this sum is proper. Indeed, given any two closed subspaces of X , say V_1 and V_2 , it is obvious that

$$V_1 + V_2 = \bigcup_{n \in \mathbb{N}} (nB + V_2), \quad (13)$$

where $B = \{x \in V_1 : \|x\|_X \leq 1\}$.

On one hand, for any $n \in \mathbb{N}$, the set $nB + V_2$ is a subset of $V_1 + V_2$, which is a proper subspace of X ; therefore, the interior of $nB + V_2$ is empty. On the other hand, being closed and convex, the set V_2 is automatically weakly closed. For its part, the set B is a bounded closed and convex subset of a reflexive Banach space, thus it is weakly compact. Accordingly, their sum, $B + V_2$, as well as any set of form $nB + V_2$, with $n \in \mathbb{N}$, is a weakly closed, and, a fortiori, closed subset of X .

Relation (13) reads now that $V_1 + V_2$ is the union of a countable family of closed sets, each one of them having an empty interior. \square

We are now in a position to prove the main result of this article.

Theorem 4.3. *Let X be an infinite dimensional reflexive Banach space. Then there exists f , a function from $\Gamma_0(X)$, and L , a closed linear subspace of X which meets $\text{dom } f$, such that $\mathcal{N}(f, L)$ is a residual subset of the complete metric space $\mathcal{A}(f, L)$.*

Proof of Theorem 4.3: A well-known theorem (originally stated by Banach without any proof at page 238 of his foundational treatise [2], this result was proved in [3, Corollary 3, page 157]; the reader may found a complete account

of this topic in the chapter V of the textbook [4]), says that any infinite dimensional Banach space contains an infinite dimensional closed subspace with a Schauder basis.

In particular, this means that there exists V , an infinite dimensional closed subspace of X^* , which possess such a basis. Being a closed subspace of a reflexive space, V is itself a reflexive Banach space, so Proposition 4.2 applies to V : there are two closed subspaces of V , say V_1 et V_2 , such that the set $V_1 + V_2$ is both dense, and of first Baire category in V .

Let us set $f(x) = \iota_{V_1^\perp}$, and $L = V_2^\perp$; we claim that $\mathcal{N}(f, L)$ is a residual subset of the complete metric space $\mathcal{A}(f, L)$. Since X is a reflexive Banach space, it is easy to see that $\text{epi } f^* = V_1 \times \mathbb{R}_+$, and $\text{epi } \iota_L^* = V_2 \times \mathbb{R}_+$, so

$$\text{epi } f^* + \text{epi } \iota_L^* = (V_1 + V_2) \times \mathbb{R}_+, \quad (14)$$

and from relations (2) and (14) it follows that

$$\mathcal{A}(f, L) \setminus \mathcal{N}(f, L) = \Pi((V_1 + V_2) \times \mathbb{R}_+).$$

Moreover, the closure of the set $V_1 + V_2$ is V . Hence

$$\overline{\text{epi } f^* + \text{epi } \iota_L^*} = V \times \mathbb{R}_+, \quad (15)$$

and combining relations (1) and (15) we infer that

$$\mathcal{A}(f, L) = \Pi(V \times \mathbb{R}_+) = \pi(V) \times \mathbb{R}_+.$$

In order to prove that $\Pi((V_1 + V_2) \times \mathbb{R}_+)$ is of first Baire category in $\pi(V) \times \mathbb{R}_+$, let us notice the following obvious fact.

Remark 4.4. Let Y be a Banach space, A be a subset of X , and B be a subset of $X \times \mathbb{R}_+$. Then:

- i)* $A \times \mathbb{R}_+$ is of first Baire category in $X \times \mathbb{R}_+$ if and only if A is of first Baire category in X , and
- ii)* B is of first Baire category in $X \times \mathbb{R}_+$ if and only if B is of first Baire category in $X \times \mathbb{R}$.

Thus $(V_1 + V_2) \times \mathbb{R}_+$ is of first Baire category in its closure $V \times \mathbb{R}$ (apply point *i*) in the previous statement for $V = Y$ and $V_1 + V_2 = A$, followed by point *ii*) of the same statement for $V = Y$ and $B = (V_1 + V_2) \times \mathbb{R}_+$).

A standard application of the Banach-Schauder theorem for the surjective continuous linear operator Π says that the set $\Pi(S)$ is meagre in $\Pi(\bar{S})$ provided that S is itself meagre in \bar{S} , and $S + \ker \Pi = S$. In particular, this means that the set $\Pi((V_1 + V_2) \times \mathbb{R}_+)$ is of first Baire category in $\pi(V) \times \mathbb{R}$.

It remains to apply point *ii*) from Remark 4.4 for $\pi(V) \times \mathbb{R} = Y$ and $\Pi((V_1 + V_2) \times \mathbb{R}_+) = B$, to deduce that $\Pi((V_1 + V_2) \times \mathbb{R}_+)$ is meagre in $\Pi(V \times \mathbb{R}_+)$, completing in this way the proof of Theorem 4.3. \square

5. Open problems

A natural question raised by our study concerns the validity of Proposition 4.2 and respectively Theorem 4.3 when the underlying space X is a Banach space, and respectively a real normed space. Another direction in which the result of Theorem 4.3 may be generalized is by addressing the more versatile sandwich theorem form of the Hahn-Banach theorem. In this setting, the main issue is the existence of functions $f \in \Gamma_0(X)$ such that the

set

$$\mathcal{N}(f) = \{g \in \mathcal{A}(f) : L(f, g) = \emptyset\}$$

is a residual subset of the completed metrizable space obtained when endowing the set

$$\mathcal{A}(f) = \{g \in \Gamma_0(X) : -g \leq f\}$$

with the Attouch-Wets topology; here

$$L(f, g) = \{(x^*; a) \in X^* \times \mathbb{R} : -g \leq x^* - a \leq f\}$$

denotes the set of all the continuous affine functions sandwiched between the concave function $-g$ and the convex function f .

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