APPROXIMATE CONTROLLABILITY OF SECOND GRADE FLUIDS

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Abstract. This paper deals with the controllability of the second grade fluids, a class of non-Newtonian of differentiel type, on a two-dimensional torus. Using the method of Agrachev-Sarychev [1], [2] and of Sirikyan [25], we prove that the system of second grade fluids is approximately controllable by a finite-dimensional control force.

se:Intro

1. Introduction

The goal of this paper is to study the approximate controllability of the system of fluids of second grade, using low-mode (finite-dimensional) control forces. More precisely, we consider the following system

sys:FG2eta

$$\begin{cases} \partial_t \left(u - \alpha\Delta u\right) - \nu\Delta u + \operatorname{rot}\left(u - \alpha\Delta u\right) \times u + \nabla p = f + \eta \\ \operatorname{div} u = 0 \\ u(0) = u_0, \end{cases} \tag{1.1}$$

on the domain $\mathbb{T}^2$, which is the two-dimensional torus $]0, 2\pi q_1[\times]0, 2\pi q_2[$, with $q_1 > 0$ and $q_2 > 0$. Here $u = (u_1(t,x), u_2(t,x))$ and $p = p(t,x)$ are unknown and represent the velocity vector field and the pressure function; $f = f(t,x)$ is the extenal force field; and the control force $\eta = \eta(t,x)$ is supposed to belong to a finite-dimensional space which will be precised later.

Fluids of second grade belong to a particular class of non-Newtonian Rivlin-Ericksen fluids of differential type [24], which usually arise in petroleum industry, in polymer technology or in liquid crystal suspension problems. For these fluids, the Cauchy stress tensor σ is not linearly proportional to the local strain rate but given by

tressTensor

$$\sigma = -pI + 2\nu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \tag{1.2}$$

where ν stands for the kinematic viscosity, p is the pressure and A_1, A_2 represent the first two Rivlin-Ericksen tensors, which are

$$A_1(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right),$$

corresponding to the local strain tensor and

$$A_2(u) = \frac{DA_1}{Dt} + \left(\nabla u\right)^T A_1 + A_1\left(\nabla u\right),$$

where

$$\frac{D}{Dt} = \partial_t + u \cdot \nabla$$

is the material derivative. In [7], Dunn and Fosdick used the compatibility of (1.2) with thermodynamics to prove that

$$\alpha_1 + \alpha_2 = 0; \quad \alpha_1 \geq 0.$$

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Setting $\alpha = \alpha_1$ and writing the equation

$$\frac{Du}{Dt} = \partial_t u + u \cdot \nabla u = \operatorname{div} \sigma + f,$$

one obtain the equations of second grade fluids of the following form

sys:FG2

$$\begin{cases} \partial_t (u - \alpha \Delta u) - \nu \Delta u + \operatorname{rot} (u - \alpha \Delta u) \times u + \nabla p = f & \text{in } \mathbb{R}_+ \times \mathbb{T}^2 \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^2 \\ u(0) = u_0 & \text{in } \mathbb{T}^2. \end{cases} \tag{1.3}$$

The local existence in time and uniqueness of a strong solution to (1.3) have been proven by Cioranescu and Ouazar in [6] in the case of two-dimensional or three-dimensional domains with non-slip boundary conditions. Moreover, the solution is global in time in the two-dimensional case. Second grade fluids in these domains was also studied by Moise, Rosa and Wang in [20], where the authors proved the existence of a compact global attractor in the two-dimensional case. The existence, the uniqueness of a strong solution and the dynamics of second grade fluids in the torus $\mathbb{T}^2$ was studied in [23] by Paicu, Raugel and Rekalo, and in [22] by the first two authors, using the Lagrangian approach. For further results concerning the system (1.3), we refer the readers to [3], [4], [5], [13], [14], [15], [16], [17], [18], [19], ...

In this paper, in order to study the approximate controllability of the second grade fluid system (1.1) by a low-mode control η, we use the method introduced by Agrachev and Sarychev in [1] and [2] for Navier-Stokes and Euler systems in the two-dimensional torus $\mathbb{T}^2$. This method was extended later for the three-dimensional Navier-Stokes system by Shirikyan in [25] and [26] and for the three-dimensional Euler system by Nersisyan in [21]. The main idea consists in proving that, if the (finite-dimensional) space of controls E contains sufficiently many Fourier modes then, for any $T > 0$, the system (1.1) is approximately controllable in time T by an E-valued control η.

Before stating the main results and the main ideas of this paper, we will introduce the needed notations and function spaces. Let $H^m(\mathbb{T}^2)^2$ be the classical Sobolev space of two-dimensional vector fields, whose components belong to $H^m(\mathbb{T}^2)$. For $m = 0$, we simply have $H^0(\mathbb{T}^2)^2 = L^2(\mathbb{T}^2)^2$. As in [23], for any $m \in \mathbb{N}$, we denote $V^m(\mathbb{T}^2)^2$ the closure of the space

$$\left\{ u \in C^\infty(\mathbb{T}^2)^2 \mid u \text{ is periodic }, \operatorname{div} u = 0, \int_{\mathbb{T}^2} u dx = 0 \right\}$$

in $H^m(\mathbb{T}^2)^2$. Then $V^m(\mathbb{T}^2)^2$ is a Banach space, endowed with the classical norm of $H^m(\mathbb{T}^2)^2$. We also use $H^m_{per}(\mathbb{T}^2)^2$ to denote the space of vector fields $u \in H^m(\mathbb{T}^2)^2$, which are periodic and whose mean value is zero.

In what follows, we recall the definition of a strong solution of the system (1.3).

e:StrongSol **Definition 1.1.** *Let $T > 0$. For any $f \in L^\infty\left(0, T, H^1_{per}(\mathbb{T}^2)^2\right)$ and $u_0 \in V^3(\mathbb{T}^2)^2$, the vector field $u(t,x)$ is said to be a strong solution of the system (1.3), with data (f, u_0), on the time interval $[0,T]$ if $u \in C(0,T,V^3(\mathbb{T}^2)^2)$, $\partial_t u \in L^\infty(0,T,V^2(\mathbb{T}^2)^2)$, $u(0) = u_0$, and for any $t \in]0,T]$, for any $\phi \in V^0(\mathbb{T}^2)^2$, the following equation holds*

q:StrongSol
$$\langle \partial_t (u(t) - \alpha \Delta u(t)) - \nu \Delta u(t) + rot(u(t) - \alpha \Delta u(t)) \times u(t), \phi \rangle = \langle f(t), \phi \rangle . \tag{1.4}$$

In [23], the authors prove that

Theorem 1.2. *Let $\alpha > 0$ and $T > 0$.*

(1) *For any $f \in L^\infty(0,T,H^1_{per}(\mathbb{T}^2))$ and any $u_0 \in V^3(\mathbb{T}^2)^2$, there exists a unique strong solution*

$$u \in C(0,T,V^3(\mathbb{T}^2)^2) \cap W^{1,\infty}(0,T,V^2(\mathbb{T}^2)^2)$$

of the system (1.3). *Moreover, for any* $t \in [0,T]$, *the map*

$$V^3(\mathbb{T}^2)^2 \ni u_0 \mapsto u(t) \in V^3(\mathbb{T}^2)^2$$

is continuous.

(2) *Let* $m \geq 2$. *Assume that* $f \in L^\infty(0,T,H^m_{per}(\mathbb{T}^2)^2)$ *and* $u_0 \in V^{m+3}(\mathbb{T}^2)^2$. *Then, the solution* u *of the system* (1.3) *belongs to* $C(0,T,V^{m+3}(\mathbb{T}^2)^2)$.

For the system (1.1), we want to define the approximate controllability using low-mode controls. We will adapt the definition of approximate controllability given in [25] to the case of fluids of second grade.

:ApproxCont **Definition 1.3.** *Let* $T > 0$ *and let* E *be a finite-dimensional subspace of* $V^3(\mathbb{T}^2)^2$. *The second grade fluid system* (1.1) *is approximately controllable in time* T *by* E*-valued controls if, for any* $\varepsilon > 0$, *for any* $u_0, u_T \in V^4(\mathbb{T}^2)$, *there exist a control* $\eta \in L^\infty(0,T,E)^2$ *and a strong solution* $u \in C(0,T,V^4(\mathbb{T}^2)^2)$ *of the system* (1.1) *such that*

$$\|u(T) - u_T\|_{V^3(\mathbb{T}^2)^2} \leq \varepsilon.$$

For any $m \in \mathbb{Z}^2 \setminus \{0\}$, let

$$c_m(x) = m^{q,\perp} \cos \langle m, x \rangle_q \qquad \text{and} \qquad s_m(x) = m^{q,\perp} \sin \langle m, x \rangle_q .$$

Then, it is classical that c_m, s_m, with $m \in \mathbb{Z}^2 \setminus \{0\}$, are eigenvectors of the Stokes operator $-\mathbb{P}\Delta$, where $\mathbb{P}$ is the Leray projection, and that the family $\{c_m, s_m \mid m \in \mathbb{Z}^2 \setminus \{0\}\}$ forms an orthonormal basis of $V^k(\mathbb{T}^2_q)^2$, $k \in \mathbb{N}$. For any $N \in \mathbb{N}^*$, we set

eq:HN
$$\mathcal{H}^N_q = Span\{c_m, s_m \mid m \in \mathbb{Z} \setminus \{0\}, |m| \leq N\}. \tag{1.5}$$

The main result of this paper is the following theorem.

h:FG2approx **Theorem 1.4.** *Let* $T > 0$ *and* $f \in L^\infty(0,T,H^2_{per}(\mathbb{T}^2)^2)$. *Then the system* (1.1) *is approximately controllable in time* T *by* $\mathcal{H}^3_q$*-valued controls.*

We note that, unlike the case of Navier-Stokes equations, the system of second grade fluids is an exemple of asymptotically smooth system, which only possesses a smoothing effect in infinite time. The systems (1.1) or (1.3) also differ from the α-type models, such as the so-called α-Navier-Stokes system (see [8], [9] and the references therein). Indeed, the α-Navier-Stokes system contains the very regularizing term $-\nu\Delta(u - \alpha\Delta u)$ instead of $-\nu\Delta u$, and thus is a semilinear problem, which is easier to solve. It is different in the case of second grade fluids where the dissipation is much weaker. This weak smoothing effect explains why in our result (Definition 1.3 and Theorem 1.4) we can not obtain an approximate control in the $V^4(\mathbb{T}^2)^2$-norm (norm of the same space as the initial data) but only a control in $V^3(\mathbb{T}^2)^2$-norm. Another problem when we want to apply the method of Shirikyan [25] lies in the complexity of the nonlinear term and the appearance of the term $\partial_t(-\alpha\Delta u)$. To avoid this difficulty, let

$$\mathcal{U} = u - \alpha\Delta u \quad \text{and} \quad \mathcal{U}_0 = u_0 - \alpha\Delta u_0$$

and let us rewrite the system (1.1) in the following form

s:FG2etaCal
$$\begin{cases} \partial_t \mathcal{U} + \mathcal{L}\mathcal{U} + \mathcal{B}(\mathcal{U},\mathcal{U}) = \mathbb{P}f + \eta \\ \operatorname{div} \mathcal{U} = 0 \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \tag{1.6}$$

where

eq:LcalBcal
$$\begin{cases} \mathcal{L}\mathcal{U} = -\nu\mathbb{P}\Delta(I - \alpha\Delta)^{-1}\mathcal{U} \\ \mathcal{B}(\mathcal{U}_1,\mathcal{U}_2) = \mathbb{P}\left(\operatorname{rot}\mathcal{U}_1 \times \left((I - \alpha\Delta)^{-1}\mathcal{U}_2\right)\right). \end{cases} \tag{1.7}$$

Along with the system (1.6), we consider the following controlled system

2etazetaCal
$$\begin{cases} \partial_t \mathcal{U} + \mathcal{L}(\mathcal{U}+\zeta) + \mathcal{B}(\mathcal{U}+\zeta, \mathcal{U}+\zeta) = \mathbb{P}f + \eta \\ \operatorname{div} \mathcal{U} = 0 \\ \mathcal{U}(0) = \mathcal{U}_0. \end{cases} \tag{1.8}$$

Following the ideas of [1] and [25], we give the following definition

exification **Definition 1.5.** *For any finite-dimensional subspace E of $V^3(\mathbb{T}^2)^2$, we define $\mathcal{F}(E)$ as the largest vector subspace of $V^3(\mathbb{T}^2)^2$ such that for any $\overline{\eta} \in \mathcal{F}(E)$, there exist*

$$k \in \mathbb{N},\ \alpha_1, \dots, \alpha_k > 0,\ \eta, \rho^1, \dots, \rho^k \in E$$

satisfying

$$\overline{\eta} = \eta - \sum_{j=1}^{k} \alpha_j \mathcal{B}(\rho^j, \rho^j).$$

It is easy to verify that $\mathcal{F}(E)$ is well defined and $E \subset \mathcal{F}(E)$. Then, we can define a sequence of subspace

$$E = E_0 \subset E_1 \subset \ldots \subset E_n \subset \ldots$$

such that, for any $n \in \mathbb{N}$ we have $E_{n+1} = \mathcal{F}(E_n)$. Let

$$E_\infty = \bigcup_{n=0}^{\infty} E_n.$$

The goal of this paper is to prove that Theorem 1.4 is only a consequence of the following results.

G2extension **Theorem 1.6** (Extension)**.** *Let $T > 0$ and E a finite-dimensional subspace of $V^3(\mathbb{T}^2)^2$. Then, the system* (1.6) *is approximately controllable in time T by an E-valued control η if and only if so is the system* (1.8) *with E-valued controls η and ζ.*

h:FG2convex **Theorem 1.7** (Convexification)**.** *Let $T > 0$ and E a finite-dimensional subspace of $V^3(\mathbb{T}^2)^2$. Then, the system* (1.8) *is approximately controllable in time T by E-valued controls η and ζ if and only if so is the system* (1.8) *with $\mathcal{F}(E)$-valued controls η.*

2Saturation **Theorem 1.8.** *If $E = \mathcal{H}^3_q$ then $E_\infty \supset \mathcal{H}^N_q$, for any $N \in \mathbb{N}$, $N \geq 3$.*

The paper will be organized as follows. In Section 2, we study a pertubation of the system (1.6), which is necessary to prove our main theorem. Theorem 1.6 will be proved in Section 3. Section 4 is devoted to the demonstration of Theorem 1.7. In Section 5, we put in evidence the saturation property given in Theorem 1.8. Finally, in the last section, we wil prove the main theorem 1.4.

2. Preliminary results on the system of fluids of second grade

se:Pre

In this section, we consider the following perturbed system of fluids of second grade

s:FG2perCal
$$\begin{cases} \partial_t \mathcal{W} + \mathcal{L}\mathcal{W} + \mathcal{B}(\mathcal{W}) + \mathcal{B}(\mathcal{W}, \mathcal{V}) + \mathcal{B}(\mathcal{V}, \mathcal{W}) = \mathbb{P}f & \text{in } \mathbb{T}^2 \times [0,T] \\ \operatorname{div} \mathcal{W} = 0 & \text{in } \mathbb{T}^2 \times [0,T] \\ \mathcal{W}(0) = \mathcal{W}_0 & \text{in } \mathbb{T}^2, \end{cases} \tag{2.1}$$

where $\mathcal{V} \in L^\infty(0,T, V^2(\mathbb{T}^2)^2)$, $f \in L^\infty(0,T, H^1_{per}(\mathbb{T}^2)^2)$. We want to prove that

:FG2pertCal **Theorem 2.1.** *Let $T > 0$ fixed.*

(1) *For any* $\widehat{\mathcal{V}} \in L^2(0,T,V^2(\mathbb{T}^2)^2)$, $\widehat{f} \in L^2(0,T,H^1_{per}(\mathbb{T}^2)^2)$ *and* $\widehat{\mathcal{W}}_0 \in V^1(\mathbb{T}^2)^2$, *the system* (2.1) *with* $\mathcal{V} = \widehat{\mathcal{V}}$, $f = \widehat{f}$ *and* $\mathcal{W}_0 = \widehat{\mathcal{W}}_0$ *has a unique solution*

$$\widehat{\mathcal{W}} \in L^\infty(0,T,V^1(\mathbb{T}^2)^2) \cap C(0,T,V^1(\mathbb{T}^2)^2).$$

Moreover, if $\widehat{\mathcal{V}} \in L^2(0,T,V^3(\mathbb{T}^2)^2)$, $\widehat{f} \in L^2(0,T,H^2_{per}(\mathbb{T}^2)^2)$ *and* $\widehat{\mathcal{W}}_0 \in V^2(\mathbb{T}^2)^2$ *then*

$$\widehat{\mathcal{W}} \in L^\infty(0,T,V^2(\mathbb{T}^2)^2) \cap C(0,T,V^2(\mathbb{T}^2)^2).$$

FG2pertCalc (2) *Suppose that*

$$\widehat{\mathcal{W}} \in L^\infty(0,T,V^2(\mathbb{T}^2)^2) \cap C(0,T,V^2(\mathbb{T}^2)^2).$$

Let $\mathcal{W}$ *be the solution of the system* (2.1) *with data* $(\mathcal{V}, f, \mathcal{W}_0)$. *Then, there exists a constant* $C > 0$ *such that, for any* $t \in [0,T]$, *we have*

ipschitzCal (2.2)
$$\left\|\mathcal{W}(t) - \widehat{\mathcal{W}}(t)\right\|_{V^1(\mathbb{T}^2)^2} \leq C\left(\left\|\mathcal{V} - \widehat{\mathcal{V}}\right\|_{L^2(0,T,V^2)^2} + \left\|\mathbb{P}f - \mathbb{P}\widehat{f}\right\|_{L^2(0,T,V^1)^2} + \left\|\mathcal{W}_0 - \widehat{\mathcal{W}}_0\right\|_{V^1(\mathbb{T}^2)^2}\right).$$

We remark that if we set

$$\begin{cases} v = (I - \alpha\Delta)^{-1}\mathcal{V} \\ w_0 = (I - \alpha\Delta)^{-1}\mathcal{W}_0 \\ w = (I - \alpha\Delta)^{-1}\mathcal{W} \end{cases}$$

then, w is solution of the following system

sys:FG2perP (2.3)
$$\begin{cases} \partial_t(w - \alpha\Delta w) - \nu\Delta w + \mathbb{P}B(w) + \mathbb{P}B(w,v) + \mathbb{P}B(v,w) = \mathbb{P}f \\ \operatorname{div} w = 0 \\ w(0) = w_0, \end{cases}$$

where

$$B(u_1,u_2) = \operatorname{rot}(u_1 - \alpha\Delta u_1) \times u_2 \qquad \text{and} \qquad B(u) = B(u,u).$$

Theorem 2.1 is in fact equivalent to the following theorem for the system (2.3)

h:FG2pert2D **Theorem 2.2.** *Let* $T > 0$ *fixed.*

(1) *For any* $\widehat{v} \in L^2(0,T,V^4(\mathbb{T}^2)^2)$, $\widehat{f} \in L^2(0,T,H^1_{per}(\mathbb{T}^2)^2)$ *and* $\widehat{w}_0 \in V^3(\mathbb{T}^2)^2$, *the system* (2.3) *with* $v = \widehat{v}$, $f = \widehat{f}$ *and* $w_0 = \widehat{w}_0$ *has a unique solution*

$$\widehat{w} \in L^\infty(0,T,V^3(\mathbb{T}^2)^2) \cap C(0,T,V^3(\mathbb{T}^2)^2).$$

Moreover, if $\widehat{v} \in L^2(0,T,V^5(\mathbb{T}^2)^2)$, $\widehat{f} \in L^2(0,T,H^2_{per}(\mathbb{T}^2)^2)$ *and* $\widehat{w}_0 \in V^4(\mathbb{T}^2)^2$ *then*

$$\widehat{w} \in L^\infty(0,T,V^4(\mathbb{T}^2)^2) \cap C(0,T,V^4(\mathbb{T}^2)^2).$$

:FG2pert2Dc (2) *Suppose that*

$$\widehat{w} \in L^\infty(0,T,V^4(\mathbb{T}^2)^2) \cap C(0,T,V^4(\mathbb{T}^2)^2).$$

Let w *be the solution of the system* (2.3) *with data* (v, f, w_0). *Then, there exists a constant* $C > 0$ *such that, for any* $t \in [0,T]$, *we have*

Lipschitz2D (2.4)
$$\|w(t) - \widehat{w}(t)\|_{V^3(\mathbb{T}^2)^2} \leq C\left(\|v - \widehat{v}\|_{L^2(0,T,V^4)^2} + \left\|\mathbb{P}f - \mathbb{P}\widehat{f}\right\|_{L^2(0,T,V^1)^2} + \|w_0 - \widehat{w}_0\|_{V^3(\mathbb{T}^2)^2}\right).$$

In what follows, using the same notations as in [6], [23] or [22], we identify a 2D vector (u_1, u_2) with the 3D vector $(u_1, u_2, 0)$ and a scalar λ with the 3D vector $(0, 0, \lambda)$. We also set

q:Notations
$$\begin{aligned} &\Omega = \mathrm{rot}\,(w - \alpha\Delta w), && \widehat{\Omega} = \mathrm{rot}\,(\widehat{w} - \alpha\Delta\widehat{w}) \\ &V = \mathrm{rot}\,(v - \alpha\Delta v), && \widehat{V} = \mathrm{rot}\,(\widehat{v} - \alpha\Delta\widehat{v}) \\ &\mathcal{O} = \Omega - \widehat{\Omega}. && W = w - \widehat{w}. \end{aligned} \tag{2.5}$$

2.1. **Propagation of the V^3-norm.** In this paragraph, we give a priori estimates of a solution of the system (2.3) in $V^3(\mathbb{T}^2)$-norm. Applying the rot operator to the first equation of (2.3), we obtain

eq:propag2D
$$\partial_t\Omega + \frac{\nu}{\alpha}\Omega + \mathbb{P}\,(w\cdot\nabla\Omega) + \mathbb{P}\,(v\cdot\nabla\Omega) + \mathbb{P}\,(w\cdot\nabla V) = \mathrm{rot}\,\mathbb{P}f + \frac{\nu}{\alpha}\mathrm{rot}\,w. \tag{2.6}$$

Since v and w are divergence-free vector fields on $\mathbb{T}^2$, integrations by parts show that

$$\langle w\cdot\nabla\Omega, \Omega\rangle_{L^2(\mathbb{T}^2)^2} = \langle v\cdot\nabla\Omega, \Omega\rangle_{L^2(\mathbb{T}^2)^2} = 0.$$

As a consequence, taking the $L^2(\mathbb{T}^2)^2$ inner product of (2.6) with Ω, we get

:propag2DL2
$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Omega\|^2_{L^2(\mathbb{T}^2)^2} + \frac{\nu}{\alpha}\|\Omega\|^2_{L^2(\mathbb{T}^2)^2} &\le \left|\langle w\cdot\nabla V, \Omega\rangle_{L^2(\mathbb{T}^2)^2}\right| + \left|\langle \mathrm{rot}\,f, \Omega\rangle_{L^2(\mathbb{T}^2)^2}\right| + \frac{\nu}{\alpha}\left|\langle \mathrm{rot}\,w, \Omega\rangle_{L^2(\mathbb{T}^2)^2}\right| \\ &\le \|\mathrm{rot}\,f\|_{L^2(\mathbb{T}^2)^2}\|\Omega\|_{L^2(\mathbb{T}^2)^2} + \left(\|\nabla V\|_{L^2(\mathbb{T}^2)^2} + \frac{\nu}{\alpha}\right)\|\Omega\|^2_{L^2(\mathbb{T}^2)^2}, \end{aligned} \tag{2.7}$$

which implies that

opag2DL2bis
$$\frac{d}{dt}\|\Omega\|^2_{L^2(\mathbb{T}^2)^2} \le \|f\|^2_{V^1(\mathbb{T}^2)^2} + \left(1 + 2\|v\|_{V^4(\mathbb{T}^2)^2}\right)\|\Omega\|^2_{L^2(\mathbb{T}^2)^2}. \tag{2.8}$$

Finally, the Gronwall lemma gives, for any $0 \le t \le T$,

pag2DL2gron
$$\|\Omega(t)\|^2_{L^2(\mathbb{T}^2)^2} \le \left(\|\Omega(0)\|^2_{L^2(\mathbb{T}^2)^2} + \|f\|^2_{L^\infty(0,T,V^1(\mathbb{T}^2)^2)}\right)\exp t\left(1 + 2\|v\|_{L^\infty(0,T,V^4(\mathbb{T}^2)^2)}\right). \tag{2.9}$$

2.2. **2D a priori estimates of the difference of two solutions in V^3-norm.** We recall the identity (given for example in [6] or [23])

$$\mathrm{rot}\,(\mathrm{rot}\,(a)\times b) = b\cdot\nabla\mathrm{rot}\,(a),$$

where a and b are 2D vector fields and $\mathrm{div}\,v = 0$. We deduce from (2.3) that Ω and $\widehat{\Omega}$ are solutions of the following equation, with data (v, f, w_0) and $(\widehat{v}, \widehat{f}, \widehat{w}_0)$ respectively.

rotFG2per2D
$$\partial_t\Omega + \frac{\nu}{\alpha}\Omega + \mathbb{P}\,(w\cdot\nabla\Omega) + \mathbb{P}\,(v\cdot\nabla\Omega) + \mathbb{P}\,(w\cdot\nabla V) = \mathrm{rot}\,\mathbb{P}f + \frac{\nu}{\alpha}\mathrm{rot}\,w. \tag{2.10}$$

The calculation of the difference between the equations corresponding to Ω and $\widehat{\Omega}$ shows that $\mathcal{O}$ satisfies the following equation

eq:Ocal2D
$$\begin{aligned} \partial_t\mathcal{O} + \frac{\nu}{\alpha}\mathcal{O} + \mathbb{P}\,(w\cdot\nabla\mathcal{O}) + \mathbb{P}\left((w-\widehat{w})\cdot\nabla\widehat{\Omega}\right) + \mathbb{P}\,(v\cdot\nabla\mathcal{O}) + \mathbb{P}\left((v-\widehat{v})\cdot\nabla\widehat{\Omega}\right) \\ + \mathbb{P}\left(w\cdot\nabla\left(V-\widehat{V}\right)\right) + \mathbb{P}\left((w-\widehat{w})\cdot\nabla\widehat{V}\right) \;=\; \mathrm{rot}\,(\mathbb{P}f - \mathbb{P}\widehat{f}) + \frac{\nu}{\alpha}\mathrm{rot}\,(w-\widehat{w}). \end{aligned} \tag{2.11}$$

Using the divergence-free property of v and w, we have

eq:Ocal2D01
$$\langle v\cdot\nabla\mathcal{O}, \mathcal{O}\rangle_{L^2(\mathbb{T}^2)^2} = \langle w\cdot\nabla\mathcal{O}, \mathcal{O}\rangle_{L^2(\mathbb{T}^2)^2} = 0. \tag{2.12}$$

Now, using Hölder's and Cauchy-Schwarz's inequalities, we get

eq:0cal2D02
$$\begin{aligned}\left|\left\langle (w-\widehat{w})\cdot\nabla\widehat{\Omega},\mathcal{O}\right\rangle_{L^2(\mathbb{T}^2)^2}\right| &\leq \left\|\nabla\widehat{\Omega}\right\|_{L^2(\mathbb{T}^2)^2}\|w-\widehat{w}\|_{L^\infty(\mathbb{T}^2)^2}\|\mathcal{O}\|_{L^2(\mathbb{T}^2)^2}\\ &\leq C\left\|\nabla\widehat{\Omega}\right\|_{L^2(\mathbb{T}^2)^2}\|w-\widehat{w}\|_{V^3(\mathbb{T}^2)^2}\|\mathcal{O}\|_{L^2(\mathbb{T}^2)^2}\\ &\leq C\|\widehat{w}\|_{V^4(\mathbb{T}^2)^2}\|\mathcal{O}\|^2_{L^2(\mathbb{T}^2)^2}.\end{aligned} \tag{2.13}$$

The same calculations give

eq:0cal2D06
$$\begin{aligned}\left|\left\langle (v-\widehat{v})\cdot\nabla\widehat{\Omega},\mathcal{O}\right\rangle_{L^2(\mathbb{T}^2)^2}\right| &\leq C\left\|\nabla\widehat{\Omega}\right\|_{L^2(\mathbb{T}^2)^2}\|v-\widehat{v}\|_{L^\infty(\mathbb{T}^2)^2}\|\mathcal{O}\|_{L^2(\mathbb{T}^2)^2}\\ &\leq C\|\widehat{w}\|_{V^4(\mathbb{T}^2)^2}\left(\|v-\widehat{v}\|^2_{V^3(\mathbb{T}^2)^2}+\|\mathcal{O}\|^2_{L^2(\mathbb{T}^2)^2}\right),\end{aligned} \tag{2.14}$$

eq:0cal2D09
$$\begin{aligned}&\left|\left\langle w\cdot\nabla\left(V-\widehat{V}\right),\mathcal{O}\right\rangle_{L^2(\mathbb{T}^2)^2}\right|\\ &\leq \|w\|_{L^\infty(\mathbb{T}^2)^2}\left\|\nabla\left(V-\widehat{V}\right)\right\|_{L^2(\mathbb{T}^2)^2}\|\mathcal{O}\|_{L^2(\mathbb{T}^2)^2}\\ &\leq C\|w\|_{L^\infty(\mathbb{T}^2)^2}\|v-\widehat{v}\|_{V^4(\mathbb{T}^2)^2}\|\mathcal{O}\|_{L^2(\mathbb{T}^2)^2}\\ &\leq C\left(\|w-\widehat{w}\|_{L^\infty(\mathbb{T}^2)^2}+\|\widehat{w}\|_{L^\infty(\mathbb{T}^2)^2}\right)\|v-\widehat{v}\|_{V^4(\mathbb{T}^2)^2}\|\mathcal{O}\|_{L^2(\mathbb{T}^2)^2}\\ &\leq C\left(\|\mathcal{O}\|_{L^2(\mathbb{T}^2)^2}+\|\widehat{w}\|_{L^\infty(\mathbb{T}^2)^2}\right)\|v-\widehat{v}\|_{V^4(\mathbb{T}^2)^2}\|\mathcal{O}\|_{L^2(\mathbb{T}^2)^2}\\ &\leq C\|\widehat{w}\|_{L^\infty(\mathbb{T}^2)^2}\|v-\widehat{v}\|^2_{V^4(\mathbb{T}^2)^2}+C\left(\|\widehat{w}\|_{L^\infty(\mathbb{T}^2)^2}+\|v-\widehat{v}\|_{V^4(\mathbb{T}^2)^2}\right)\|\mathcal{O}\|^2_{L^2(\mathbb{T}^2)^2},\end{aligned} \tag{2.15}$$

eq:0cal2D10
$$\begin{aligned}\left|\left\langle (w-\widehat{w})\cdot\nabla\widehat{V},\mathcal{O}\right\rangle_{L^2(\mathbb{T}^2)^2}\right| &\leq \|w-\widehat{w}\|_{L^\infty(\mathbb{T}^2)^2}\left\|\nabla\widehat{V}\right\|_{L^2(\mathbb{T}^2)^2}\|\mathcal{O}\|_{L^2(\mathbb{T}^2)^2}\\ &\leq C\|w-\widehat{w}\|_{V^3(\mathbb{T}^2)^2}\|\widehat{v}\|_{V^4(\mathbb{T}^2)^2}\|\mathcal{O}\|_{L^2(\mathbb{T}^2)^2}\\ &\leq C\|\widehat{v}\|_{V^4(\mathbb{T}^2)^2}\|\mathcal{O}\|^2_{L^2(\mathbb{T}^2)^2}.\end{aligned} \tag{2.16}$$

For the forcing term, we have

eq:0cal2D13
$$\left|\left\langle \mathrm{rot}\,(\mathbb{P}f-\mathbb{P}\widehat{f}),\mathcal{O}\right\rangle_{L^2(\mathbb{T}^2)^2}\right| \leq C\left(\left\|\mathbb{P}f-\mathbb{P}\widehat{f}\right\|^2_{V^1(\mathbb{T}^2)^2}+\|\mathcal{O}\|^2_{L^2(\mathbb{T}^2)^2}\right), \tag{2.17}$$

and finally,

eq:0cal2D14
$$\left|\langle \mathrm{rot}\,(w-\widehat{w}),\mathcal{O}\rangle_{L^2(\mathbb{T}^2)^2}\right| \leq \|\mathrm{rot}\,(w-\widehat{w})\|_{L^2(\mathbb{T}^2)^2}\|\mathcal{O}\|_{L^2(\mathbb{T}^2)^2} \leq C\|\mathcal{O}\|^2_{L^2(\mathbb{T}^2)^2}. \tag{2.18}$$

Taking the L^2 inner product of (2.11) with $\mathcal{O}$, and using Estimates (2.12) to (2.18), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathcal{O}\|^2_{L^2(\mathbb{T}^2)^2}+\frac{\nu}{\alpha}\|\mathcal{O}\|^2_{L^2(\mathbb{T}^2)^2}\\ &\qquad\leq C\|\widehat{w}\|_{V^4(\mathbb{T}^2)^2}\|v-\widehat{v}\|^2_{V^4(\mathbb{T}^2)^2}+C\left\|\mathbb{P}f-\mathbb{P}\widehat{f}\right\|^2_{V^1(\mathbb{T}^2)^2}\\ &\qquad\quad+C\left(1+\|\widehat{w}\|_{V^4(\mathbb{T}^2)^2}+\|\widehat{v}\|_{V^4(\mathbb{T}^2)^2}+\|v-\widehat{v}\|_{V^4(\mathbb{T}^2)^2}\right)\|\mathcal{O}\|^2_{L^2(\mathbb{T}^2)^2}.\end{aligned}$$

For any $0\leq t\leq T$, the Gronwall lemma implies that

eq:0cal2DH3
$$\begin{aligned}\|\mathcal{O}(t)\|^2_{L^2(\mathbb{T}^2)^2} \leq Ce^{C_2t}\Big(&\|w(0)-\widehat{w}(0)\|^2_{V^3(\mathbb{T}^2)^2}\\ &+T\left\|\mathbb{P}f-\mathbb{P}\widehat{f}\right\|^2_{L^\infty_t V^1(\mathbb{T}^2)^2}+C_1T\|v-\widehat{v}\|^2_{L^\infty_t V^4(\mathbb{T}^2)^2}\Big),\end{aligned} \tag{2.19}$$

where C is a generic positive constant and

$$C_1 = \|\widehat{w}\|_{L^\infty(0,T,V^4(\mathbb{T}^2)^2)},$$
$$C_2 = 1 + \|\widehat{w}\|_{L^\infty(0,T,V^4(\mathbb{T}^2)^2)} + \|\widehat{v}\|_{L^\infty(0,T,V^4(\mathbb{T}^2)^2)} + \|v - \widehat{v}\|_{L^\infty(0,T,V^4(\mathbb{T}^2)^2)}.$$

The inequality (2.4) of Theorem 2.2 (2) is then proved.

3. Study of the extended controlled system

e:Extension

In this section, we want to show that, the approximate controllability of the system (1.6) is equivalent to the approximate controllability of the system (1.8) by low-mode controls. For any finite-dimensional subspace E of $V^3(\mathbb{T}^2)^2$, we remark that the approximate controllability of the system (1.6) by E-valued controls implies immediately the approximate controllability of the system (1.8) in the same space of controls. Indeed, we only need to choose $\zeta = 0$ in the system (1.8). Then, in order to prove Theorem 1.6, we only need to prove that

:ExtensionP

Theorem 3.1. *Let $T > 0$ and E be a finite-dimensional subspace of $V^3(\mathbb{T}^2)^2$. Let $\eta, \zeta \in L^\infty(0,T,E)^2$ and*

$$\mathcal{U} \in L^\infty\left(0,T,V^2(\mathbb{T}^2)^2\right) \cap C\left(0,T,V^2(\mathbb{T}^2)^2\right)$$

be a solution of (1.8). Then, for any $k \in \mathbb{N}^*$, there are a control $\eta_k \in L^\infty(0,T,E)^2$ and a solution

$$\mathcal{U}_k \in L^\infty\left(0,T,V^2(\mathbb{T}^2)^2\right) \cap C\left(0,T,V^2(\mathbb{T}^2)^2\right)$$

of the system (1.6), with $\eta = \eta_k$, such that $\mathcal{U}_k(0) = \mathcal{U}_0$, and

$$\lim_{k\to+\infty} \|\mathcal{U}_k(T) - \mathcal{U}(T)\|_{V^3(\mathbb{T}^2)^2} = 0.$$

Proof. First of all, we can rewrite the system (1.8) as

G2etazetaP1

$$\begin{cases} \partial_t \mathcal{U} + \mathcal{L}\mathcal{U} + \mathcal{B}(\mathcal{U},\mathcal{U}) + \mathcal{B}(\mathcal{U},\zeta) + \mathcal{B}(\zeta,\mathcal{U}) = \mathbb{P}f + \eta - \mathcal{L}\zeta - \mathcal{B}(\zeta) \\ \operatorname{div}\mathcal{U} = 0 \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \tag{3.1}$$

where

$$\begin{cases} \mathcal{L}\mathcal{U} = -\nu\mathbb{P}\Delta(I - \alpha\Delta)^{-1}\mathcal{U} \\ \mathcal{B}(\mathcal{U}_1,\mathcal{U}_2) = \mathbb{P}\left(\operatorname{rot}\mathcal{U}_1 \times \left((I - \alpha\Delta)^{-1}\mathcal{U}_2\right)\right). \end{cases}$$

Thus, applying Theorem 2.1 to this system, for any $\eta, \zeta \in L^\infty(0,T,E)^2$, we obtain the existence of a unique solution of the system (3.1) (or (1.8))

$$\mathcal{U} \in L^\infty\left(0,T,V^2(\mathbb{T}^2)^2\right) \cap C\left(0,T,V^2(\mathbb{T}^2)^2\right).$$

Next, we remark that we can also rewrite the system (1.8) as

G2etazetaP2

$$\begin{cases} \partial_t\left(\mathcal{U} + \zeta\right) + \mathcal{L}\left(\mathcal{U} + \zeta\right) + \mathcal{B}\left(\mathcal{U} + \zeta\right) = \mathbb{P}f + \tilde{\eta} \\ \operatorname{div}\mathcal{U} = 0 \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \tag{3.2}$$

where

$$\tilde{\eta} = \eta + \partial_t\zeta,$$

which means that, if $\mathcal{U}$ is a solution of the system (1.8) and if $\tilde{\eta}$ is well defined and belongs to $L^\infty(0,T,E)^2$, then $\mathcal{U} + \zeta$ is a solution of the system (1.6), with η replaced by $\tilde{\eta}$ and $\mathcal{U}_0$ by $\mathcal{U}_0 + \zeta(0)$. So, if we want to construct a solution of the controlled system (1.6) satisfying the

conditions of Theorem 3.1, we only need to check whether the conditions at time $t = 0$ and $t = T$ are satisfied. To this end, we will consider a sequence of controls

$$\zeta_k \in C^1(0,T,E), \qquad \forall\, k \in \mathbb{N}^*$$

such that

$$\zeta_k(0) = \zeta_k(T) = 0$$

and

$$\lim_{k\to+\infty} \|\zeta_k - \zeta\|_{L^2(0,T,V^3(\mathbb{T}^2)^2)} = 0.$$

Applying Theorem 2.1 to the system (3.1), for any $\varepsilon > 0$, for k_0 large enough and for any $k \geq k_0$, there exists a unique solution

$$\mathcal{W}_k \in L^\infty\left(0,T,V^2(\mathbb{T}^2)^2\right) \cap C\left(0,T,V^2(\mathbb{T}^2)^2\right)$$

of the system (3.1) (or (1.8)), with ζ replaced by ζ_k, such that

$$\sup_{t\in[0,T]} \|\mathcal{W}_k(t) - \mathcal{U}(t)\|_{V^1(\mathbb{T}^2)^2} \leq \varepsilon.$$

Now, setting

$$\mathcal{U}_k = \mathcal{W}_k + \zeta_k.$$

Then, $\mathcal{U}_k$ is the solution of the system (1.6), with η replaced by $\eta + \partial_t\zeta_k \in L^\infty(0,T,E)^2$. Moreover, we have

$$\mathcal{U}_k(0) = \mathcal{W}_k(0) = \mathcal{U}(0) = \mathcal{U}_0,$$

and

$$\|\mathcal{U}_k(T) - \mathcal{U}(T)\|_{V^3(\mathbb{T}^2)^2} = \|\mathcal{W}_k(T) - \mathcal{U}(T)\|_{V^3(\mathbb{T}^2)^2} \leq \varepsilon.$$

Theorem 3.1 is proved. ■

4. Convexification of the controlled system

exification

This section is devoted the to prove of Theorem 1.7. From the definition 1.5 of $\mathcal{F}(E)$, we remark that $E \subset \mathcal{F}(E)$ ans so, the approximate controllability of the system (1.1) (or (1.6)) by E-valued controls evidently implies the approximate controllability of the system (1.1) (or (1.6)) by $\mathcal{F}(E)$-valued controls. In order to prove Theorem 1.7, we only need to prove that

th:ConvexP

Theorem 4.1. *Let $T > 0$, E be a finite-dimensional subspace of $V^3(\mathbb{T}^2)^2$. Let $\overline{\eta}$ be a control in $L^\infty(0,T,\mathcal{F}(E))^2$ and*

$$\overline{\mathcal{U}} \in L^\infty\left(0,T,V^2(\mathbb{T}^2)^2\right) \cap C\left(0,T,V^2(\mathbb{T}^2)^2\right)$$

be a solution of (1.6), with η replaced by $\overline{\eta}$. Then, for any $k \in \mathbb{N}^*$, there are controls $\eta_k, \zeta_k \in L^\infty(0,T,E)^2$ and a solution

$$\mathcal{U}_k \in L^\infty\left(0,T,V^2(\mathbb{T}^2)^2\right) \cap C\left(0,T,V^2(\mathbb{T}^2)^2\right)$$

of the system (1.8), with η replaced by η_k and ζ by ζ_k, such that $\mathcal{U}_k(0) = \mathcal{U}_0$, and

$$\lim_{k\to+\infty} \left\|\mathcal{U}_k(T) - \overline{\mathcal{U}}(T)\right\|_{V^1(\mathbb{T}^2)^2} = 0.$$

The proof of Theorem 4.1 is rather long and complex. In order to introduce this proof in a legible way, we will divide it into several steps.

Step 1. *Limit the proof of Theorem 4.1 to the case of $\mathcal{F}(E)$-valued piecewise constant controls (with respect to the time variable).*

We suppose that Theorem 4.1 is true for $\mathcal{F}(E)$-valued piecewise constant controls. We want to prove that Theorem 4.1 is also true in the general case. Let $\overline{\eta} \in L^\infty(0,T,\mathcal{F}(E))$ and let

$$\overline{\mathcal{U}} \in L^\infty\left(0,T,V^2(\mathbb{T}^2)^2\right) \cap C\left(0,T,V^2(\mathbb{T}^2)^2\right)$$

be a solution of (1.6), with η replaced by $\overline{\eta}$. We consider an approximation of $\overline{\eta}$ by a sequence $\{\eta^m\}$ of $\mathcal{F}(E)$-valued piecewise constant controls such that

$$\lim_{m\to+\infty} \|\eta^m - \overline{\eta}\|_{L^2(0,T,V^2(\mathbb{T}^2)^2)} = 0.$$

Applying Theorem 2.1 (while taking $\mathcal{V}=0$ and replacing f by $f+\eta^m$ and $f+\overline{\eta}$), we deduce the existence of a solution

$$\mathcal{U}^m \in L^\infty\left(0,T,V^2(\mathbb{T}^2)^2\right) \cap C\left(0,T,V^2(\mathbb{T}^2)^2\right)$$

of the system (1.6), with η replaced by η^m, such that

$$\sup_{t\in[0,T]} \left\|\mathcal{U}^m(t) - \overline{\mathcal{U}}(t)\right\|_{V^1(\mathbb{T}^2)^2} \leq C\left(\left\|\overline{\mathcal{U}}\right\|_{L^2(0,T,V^2(\mathbb{T}^2)^2)}\right) \|\eta^m - \overline{\eta}\|_{L^2(0,T,V^2(\mathbb{T}^2)^2)} \leq \frac{\varepsilon}{2},$$

for any m larger than a certain $m_0 \in \mathbb{N}^*$.

Now, Theorem 4.1 being true for $\mathcal{F}(E)$-valued piecewise constant controls, we deduce the existence of η and ζ in $L^\infty(0,T,E)^2$, and a solution

$$\mathcal{U} \in L^\infty\left(0,T,V^2(\mathbb{T}^2)^2\right) \cap C\left(0,T,V^2(\mathbb{T}^2)^2\right)$$

of the system (1.8) such that

$$\mathcal{U}(0) = \mathcal{U}^{m_0}(0) = \overline{\mathcal{U}}(0),$$

and

$$\sup_{t\in[0,T]} \|\mathcal{U}(t) - \mathcal{U}^{m_0}(t)\|_{V^1(\mathbb{T}^2)^2} \leq \frac{\varepsilon}{2}.$$

Therefore,

$$\sup_{t\in[0,T]} \left\|\mathcal{U}(t) - \overline{\mathcal{U}}(t)\right\|_{V^1(\mathbb{T}^2)^2} \leq \sup_{t\in[0,T]} \|\mathcal{U}(t) - \mathcal{U}^{m_0}(t)\|_{V^1(\mathbb{T}^2)^2} + \sup_{t\in[0,T]} \left\|\mathcal{U}^{m_0}(t) - \overline{\mathcal{U}}(t)\right\|_{V^1(\mathbb{T}^2)^2} \leq \varepsilon.$$

Remark We can even reduce the study to the case where the control $\overline{\eta}$ accepts only one interval of constancy, with respect to the time variable. For that, we suppose that the intervals of constancy of $\overline{\eta}$ is $]0,t_1[$, $]t_1,t_2[$, $\ldots]t_n,T[$ and then we perform an argument par iteration (as in [25]). So from now on, we will consider $\overline{\eta} \in \mathcal{F}(E)$ independent of the time variable.

Step 2. *Construction of solutions of the extended controlled system* (1.8).

The construction of controls η_k, ζ_k and a solution $\mathcal{U}_k$ of the controlled system (1.8), with (η,ζ) replaced by (η_k,ζ_k), follows the lines of the construction in [25] (see also [1] and [2]). The main difficulty consists in the treatment of the relaxation norm and the convergence to zero of the solution of the system (4.14) described later.

Let

$$\overline{\mathcal{U}} \in L^\infty\left(0,T,V^2(\mathbb{T}^2)^2\right) \cap C\left(0,T,V^2(\mathbb{T}^2)^2\right)$$

be the solution of the controlled system (1.6), with η replaced by $\overline{\eta} \in \mathcal{F}(E)$. Let $\varepsilon > 0$ and $\delta > 0$ which will be precised later and choose $N > 0$ large enough such that

$$\|f - P_N f\|_{L^2(0,T,V^1(\mathbb{T}^2)^2)} + \|\mathcal{U}_0 - P_N\mathcal{U}_0\|_{V^1(\mathbb{T}^2)^2} \leq \delta,$$

where P_N is the projection onto the space of the first N eigenvectors of the Stokes operator $-\mathbb{P}\Delta$ and where $\mathbb{P}$ is the Leray projection onto the subspace of divergence-free vector fields of $L^2(\mathbb{T}^2)^2$. Let

$$\mathcal{V}_0 = P_N\mathcal{U}_0$$

and $\mathcal{V}_N$ be the solution of the system

sys:FG2etaN

$$\begin{cases} \partial_t \mathcal{V}_N + \mathcal{L}\mathcal{V}_N + \mathcal{B}\left(\mathcal{V}_N\right) = \mathbb{P}P_N f + \overline{\eta} \\ \operatorname{div} \mathcal{V}_N = 0 \\ \mathcal{V}_N(0) = \mathcal{V}_0 = P_N \mathcal{U}_0. \end{cases} \tag{4.1}$$

Using the definition in (1.7), we remark that $(I-\alpha\Delta)^{-1}\mathcal{V}_N$ is the solution of the system (1.1), with $u_0 = (I-\alpha\Delta)^{-1}P_N\mathcal{U}_0$ and η replaced by $\overline{\eta}$. Then, applying [[23], Theorems 2.1 and 2.4], we obtain the existence of a unique solution

$$\mathcal{V}_N \in L^\infty\left(0,T,V^3(\mathbb{T}^2)^2\right) \cap C\left(0,T,V^3(\mathbb{T}^2)^2\right)$$

of the system (4.1). The following lemma (see [25]) allows us to have a "good decomposition" of $\mathcal{F}(E)$-valued controls in terms of E-valued controls.

le:newdefE1

Lemma 4.2. *Let E be a finite-dimensional subspace of $V^3(\mathbb{T}^2)^2$. Then, for any $\overline{\eta} \in \mathcal{F}(E)$, there exist $m \in \mathbb{N}^*$; $\eta, \rho^1, \ldots, \rho^m \in E$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}_+^*$, with $\sum \lambda_j = 1$, such that, for any $\mathcal{U} \in V^2(\mathbb{T}^2)^2$, we have*

$$\mathcal{B}\left(\mathcal{U}\right) - \overline{\eta} = \sum_{j=1}^{m} \lambda_j \left(\mathcal{B}\left(u+\rho^j\right) + \mathcal{L}\rho^j\right) - \eta.$$

Proof. Since $\overline{\eta} \in \mathcal{F}(E)$, Definition 1.5 implies that there exist

$$k \in \mathbb{N}^*;\ \alpha_1, \ldots, \alpha_k > 0;\ \text{and } \tilde{\eta}, \tilde{\rho}^1, \ldots, \tilde{\rho}^k \in E$$

such that

$$\overline{\eta} = \tilde{\eta} - \sum_{j=1}^{k} \alpha_j \mathcal{B}(\tilde{\zeta}^j).$$

Let $m = 2k$, $\alpha = \alpha_1 + \ldots + \alpha_k$ and

$$\begin{cases} \lambda_j = \dfrac{\alpha_j}{2\alpha},\ \rho^j = \sqrt{\alpha}\tilde{\rho}^j, \quad \forall\, j \in \{1, \ldots, k\} \\ \lambda_j = \dfrac{\alpha_{j-k}}{2\alpha},\ \rho^j = -\sqrt{\alpha}\tilde{\rho}^{j-k}, \quad \forall\, j \in \{k+1, \ldots, m\}. \end{cases}$$

We remark that, for any $j \in \{1, \ldots, k\}$, we have

$$\lambda_j = \lambda_{j+k} \quad \text{and} \quad \rho^j = -\rho^{j+k}.$$

Then, for any $\mathcal{U} \in V^2(\mathbb{T}^2)^2$, direct calculations give,

$$\mathcal{B}(\mathcal{U},\mathcal{U}) - \overline{\eta} = \sum_{j=1}^{m} \lambda_j \left(\mathcal{B}(\mathcal{U}+\rho^j) + \mathcal{L}\rho^j\right) - \eta. \qquad \blacksquare$$

Lemma 4.2 allows us to rewrite the system (4.1) as follows

ys:FG2etaN1

$$\begin{cases} \partial_t \mathcal{V}_N + \mathcal{L}\mathcal{V}_N + \displaystyle\sum_{j=1}^{m} \lambda_j \left(\mathcal{B}(\mathcal{V}_N+\rho^j) + \mathcal{L}\rho^j\right) = \mathbb{P}P_N f + \eta \\ \operatorname{div} \mathcal{V}_N = 0 \\ \mathcal{V}_N(0) = \mathcal{V}_0 = P_N \mathcal{U}_0. \end{cases} \tag{4.2}$$

Now, we use the same construction as explained in [25] to build the needed additional control ζ in the system (1.8). To this end, we introduce the following 1-periodic function $\varphi : \mathbb{R}_+ \to E$

eq:phi
$$\begin{cases} \varphi(s) = \varphi(s+1) & \text{for any } s \in \mathbb{R}_+ \\ \varphi(s) = \rho^1 & \text{si } 0 \le s < \lambda_1 \\ \varphi(s) = \rho^j & \text{si } \lambda_1 + \ldots + \lambda_{j-1} \le s < \lambda_1 + \ldots + \lambda_j, \quad \text{for any } 2 \le j \le m \end{cases} \tag{4.3}$$

For any $k \ge 1$, let

eq:psik
$$\psi_k(t) = \varphi\left(\frac{kt}{T}\right). \tag{4.4}$$

Then, the system (4.2) can be rewritten as follows

ys:FG2etaN2
$$\begin{cases} \partial_t \mathcal{V}_N + \mathcal{L}(\mathcal{V}_N + \psi_k) + \mathcal{B}(\mathcal{V}_N + \psi_k) = \mathbb{P}P_N f + \eta + f_k \\ \operatorname{div} \mathcal{V}_N = 0 \\ \mathcal{V}_N(0) = \mathcal{V}_0 = P_N \mathcal{U}_0, \end{cases} \tag{4.5}$$

where

eq:3fk
$$f_k(t) = g_k(t) + h_k(t), \tag{4.6}$$

with

eq:3ghk
$$\begin{cases} g_k(t) = \mathcal{L}\psi_k(t) - \displaystyle\sum_{j=1}^{m} \lambda_j \mathcal{L}\rho^j \\ h_k(t) = \mathcal{B}(\mathcal{V}_N + \psi_k(t)) - \displaystyle\sum_{j=1}^{m} \lambda_j \mathcal{B}(\mathcal{V}_N + \rho^j). \end{cases} \tag{4.7}$$

We remark that, for any $s \in \{1, 2\}$ and for any $\mathcal{U} \in V^s(\mathbb{T}^2)^2$, we have

$$\|\mathcal{L}\mathcal{U}\|_{V^s(\mathbb{T}^2)^2} \le \frac{\nu}{\alpha} \|\mathcal{U}\|_{V^s(\mathbb{T}^2)^2}.$$

Then, for $s \in \{1, 2\}$ and for any $t \ge 0$, simple calculations give,

$$\|g_k(t)\|_{V^s(\mathbb{T}^2)^2} \le \|\mathcal{L}\psi_k(t)\|_{V^s(\mathbb{T}^2)^2} + \sum_{j=1}^{m} \lambda_j \left\|\mathcal{L}\rho^j\right\|_{V^s(\mathbb{T}^2)^2} \le 2\nu \max_{1\le j\le m} \left\|\rho^j\right\|_{V^s(\mathbb{T}^2)^2} \tag{4.8}$$

$$\begin{aligned} \|h_k(t)\|_{V^s(\mathbb{T}^2)^2} &\le \|B(\mathcal{V}_N + \psi_k(t))\|_{V^s(\mathbb{T}^2)^2} + \sum_{j=1}^{m} \lambda_j \left\|B(\mathcal{V}_N + \rho^j)\right\|_{V^s(\mathbb{T}^2)^2} \\ &\le 2 \max_{1\le j\le m} \left\|B(\mathcal{V}_N + \rho^j)\right\|_{V^s(\mathbb{T}^2)^2}. \end{aligned} \tag{4.9}$$

Concerning the bilinear operator $\mathcal{B}$, classical results imply, for any $\mathcal{U} \in V^3(\mathbb{T}^2)^2$ and $\mathcal{V} \in V^2(\mathbb{T}^2)^2$,

eq:bilin01
$$\begin{aligned} \|\mathcal{B}(\mathcal{U}, \mathcal{V})\|_{V^1(\mathbb{T}^2)^2} &\le \left\|\operatorname{rot}\left(\operatorname{rot}\mathcal{U} \times \left((I - \alpha\Delta)^{-1}\mathcal{V}\right)\right)\right\|_{L^2(\mathbb{T}^2)^2} \\ &= \left\|\left((I - \alpha\Delta)^{-1}\mathcal{V}\right) \cdot \nabla(\operatorname{rot}\mathcal{U})\right\|_{L^2(\mathbb{T}^2)^2} \le C \|\mathcal{V}\|_{V^2(\mathbb{T}^2)^2} \|\mathcal{U}\|_{V^2(\mathbb{T}^2)^2} \end{aligned} \tag{4.10}$$

and

eq:bilin02
$$\begin{aligned} \|\mathcal{B}(\mathcal{U}, \mathcal{V})\|_{V^2(\mathbb{T}^2)^2} &\le C\Big(\|\operatorname{rot}\mathcal{U}\|_{V^2(\mathbb{T}^2)^2} \left\|(I - \alpha\Delta)^{-1}\mathcal{V}\right\|_{L^\infty(\mathbb{T}^2)^2} \\ &\qquad + \|\operatorname{rot}\mathcal{U}\|_{L^\infty(\mathbb{T}^2)^2} \left\|(I - \alpha\Delta)^{-1}\mathcal{V}\right\|_{V^2(\mathbb{T}^2)^2} \Big) \\ &\le C \|\mathcal{V}\|_{V^2(\mathbb{T}^2)^2} \|\mathcal{U}\|_{V^3(\mathbb{T}^2)^2}. \end{aligned} \tag{4.11}$$

Then, we obtain, for $s \in \{1, 2\}$ and for any $t \ge 0$,

eq:3gk
$$\|g_k(t)\|_{V^s(\mathbb{T}^2)^2} \le 2\nu \max_{1\le j\le m} \left\|\rho^j\right\|_{V^s(\mathbb{T}^2)^2} \tag{4.12}$$

and

eq:3hk
$$\|h_k(t)\|_{V^s(\mathbb{T}^2)^2} \leq 2C \max_{1\leq j\leq m} \left\|\mathcal{V}_N + \rho^j\right\|_{V^2(\mathbb{T}^2)^2} \left\|\mathcal{V}_N + \rho^j\right\|_{V^{s+1}(\mathbb{T}^2)^2}. \tag{4.13}$$

Next, following the ideas of [25], for any $f \in L^\infty(0,T,H^1_{per}(\mathbb{T}^2)^2)$, let $\mathcal{K}f$ be the solution of the system

sys:FG2lin
$$\begin{cases} \partial_t \mathcal{Z} + \mathcal{L}\mathcal{Z} = \mathbb{P}f \\ \operatorname{div} \mathcal{Z} = 0 \\ \mathcal{Z}(0) = 0. \end{cases} \tag{4.14}$$

By considering $z = (I - \alpha\Delta)^{-1}\mathcal{Z}$ and applying [[23], Theorem 2.4] (with $u^* = 0$), we have the following

le:Kerf
Lemma 4.3. *Let $s \in \mathbb{N}^*$. If $f \in L^\infty(0,T,V^s(\mathbb{T}^2)^2)$ then*
$$\mathcal{K}f \in L^\infty(0,T,V^s(\mathbb{T}^2)^2) \cap C(0,T,V^s(\mathbb{T}^2)^2).$$

For any $k \in \mathbb{N}^*$, we set
$$\widehat{\mathcal{W}}_k = \mathcal{V}_N - \mathcal{K}f_k.$$
Then, the system (4.5) becomes

sys:WKF
$$\begin{cases} \partial_t \widehat{\mathcal{W}}_k + \mathcal{L}\widehat{\mathcal{W}}_k + \mathcal{B}(\widehat{\mathcal{W}}_k) + \mathcal{B}(\widehat{\mathcal{W}}_k, \psi_k + \mathcal{K}f_k) + \mathcal{B}(\psi_k + \mathcal{K}f_k, \widehat{\mathcal{W}}_k) \\ \qquad\qquad = \mathbb{P}P_N f + \eta - \mathbb{P}B(\psi_k + \mathcal{K}f_k) - \mathcal{L}\psi_k \\ \operatorname{div} \widehat{\mathcal{W}}_k = 0 \\ \widehat{\mathcal{W}}_k(0) = \mathcal{V}_N(0) = P_N\mathcal{U}_0. \end{cases} \tag{4.15}$$

In other words, $\widehat{\mathcal{W}}_k$ is the solution of the system (2.1), with

eq:hatw
$$\widehat{\mathcal{V}} = \psi_k + \mathcal{K}f_k, \quad \widehat{f} = P_N f + \eta - \mathcal{B}(\psi_k + \mathcal{K}f_k) - \mathcal{L}\psi_k, \quad \widehat{\mathcal{W}}_0 = P_N\mathcal{U}_0. \tag{4.16}$$

Let

eq:barw
$$\overline{\mathcal{V}} = \psi_k, \quad \overline{f} = f + \eta - \mathcal{B}(\psi_k) - \mathcal{L}\psi_k, \quad \overline{\mathcal{W}}_0 = \mathcal{U}_0, \tag{4.17}$$

and let $\mathcal{U}_k$ be the solution of the system (2.1) with data $(\overline{\mathcal{V}}, \overline{f}, \overline{\mathcal{W}}_0)$. It is then easy to show that $\mathcal{U}_k$ is the solution of the controlled system (1.8), with controls η and $\zeta = \psi_k$
$$\begin{cases} \partial_t \mathcal{U}_k + \mathcal{L}\left(\mathcal{U}_k + \psi_k\right) + \mathcal{B}\left(\mathcal{U}_k + \psi_k\right) = \mathbb{P}f + \eta \\ \operatorname{div} \mathcal{U}_k = 0 \\ \mathcal{U}_k(0) = \mathcal{U}_0. \end{cases}$$

So, all we need to do now is to prove that, for any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}^*$ such that, for any $k \geq k_0$, we have
$$\left\|\mathcal{U}_k(T) - \overline{\mathcal{U}}(T)\right\|_{V^1(\mathbb{T}^2)^2} \leq \varepsilon,$$
where $\overline{\mathcal{U}}$ is the solution of the controlled system (1.6), with η replaced by $\overline{\eta} \in \mathcal{F}(E)$.

Recall that $\widehat{\mathcal{W}}_k = \mathcal{V}_N - \mathcal{K}f_k$. For any $t \geq 0$, we have

eq:UKUBar
$$\begin{aligned} \left\|\mathcal{U}_k(t) - \overline{\mathcal{U}}(t)\right\|_{V^1(\mathbb{T}^2)^2} &\leq \|\mathcal{U}_k(t) - \mathcal{V}_N(t)\|_{V^1(\mathbb{T}^2)^2} + \left\|\mathcal{V}_N(t) - \overline{\mathcal{U}}(t)\right\|_{V^1(\mathbb{T}^2)^2} \\ &\leq \left\|\mathcal{U}_k(t) - \widehat{\mathcal{W}}_k(t)\right\|_{V^1(\mathbb{T}^2)^2} + \|\mathcal{K}f_k(t)\|_{V^1(\mathbb{T}^2)^2} + \left\|\mathcal{V}_N(t) - \overline{\mathcal{U}}(t)\right\|_{V^1(\mathbb{T}^2)^2}. \end{aligned} \tag{4.18}$$

Applying Theorem 2.1 with $\mathcal{V} = \widehat{\mathcal{V}} = 0$, we obtain

$$\left\|\mathcal{V}_N(t) - \overline{\mathcal{U}}(t)\right\|_{V^1(\mathbb{T}^2)^2} \leq C\left(\|f - P_N f\|_{L^2(0,T,V^1(\mathbb{T}^2)^2)} + \|\mathcal{U}_0 - P_N\mathcal{U}_0\|_{V^1(\mathbb{T}^2)^2}\right) \leq C\delta.$$

Applying Theorem 2.1 once again with

$$\widehat{\mathcal{V}} = \psi_k + \mathcal{K}f_k, \quad \widehat{f} = P_N f + \eta - \mathcal{B}(\psi_k + \mathcal{K}f_k) - \mathcal{L}\psi_k, \quad \widehat{\mathcal{W}_0} = P_N\mathcal{U}_0.$$

and

$$\overline{\mathcal{V}} = \psi_k, \quad \overline{f} = f + \eta - \mathcal{B}(\psi_k) - \mathcal{L}\psi_k, \quad \overline{\mathcal{W}_0} = \mathcal{U}_0,$$

we get

$$\begin{aligned}
&\left\|\mathcal{U}_k(t) - \widehat{\mathcal{W}_k}(t)\right\|_{V^1(\mathbb{T}^2)^2} \\
&\quad\leq C\Big(\left\|\overline{\mathcal{V}} - \widehat{\mathcal{V}}\right\|_{L^2(0,T,V^2(\mathbb{T}^2)^2)} + \left\|\overline{f} - \widehat{f}\right\|_{L^2(0,T,V^1(\mathbb{T}^2)^2)} + \left\|\overline{\mathcal{W}_0} - \widehat{\mathcal{W}_0}\right\|_{V^1(\mathbb{T}^2)^2}\Big) \\
&\quad\leq C\Big(\|\mathcal{K}f_k\|_{L^2(0,T,V^2(\mathbb{T}^2)^2)} + \|\mathcal{B}(\psi_k + \mathcal{K}f_k) - \mathcal{B}(\psi_k)\|_{L^2(0,T,V^1(\mathbb{T}^2)^2)} \\
&\qquad\qquad + \|f - P_N f\|_{L^2(0,T,V^1(\mathbb{T}^2)^2)} + \|\mathcal{U}_0 - P_N\mathcal{U}_0\|_{V^1(\mathbb{T}^2)^2}\Big).
\end{aligned}$$

Next, Estimate (4.10) implies that

$$\begin{aligned}
&\|\mathcal{B}(\psi_k + \mathcal{K}f_k) - \mathcal{B}(\psi_k)\|_{L^2(0,T,V^1(\mathbb{T}^2)^2)} \\
&\quad\leq \|\mathcal{B}(\psi_k, \mathcal{K}f_k)\|_{L^2(0,T,V^1(\mathbb{T}^2)^2)} + \|\mathcal{B}(\mathcal{K}f_k, \psi_k)\|_{L^2(0,T,V^1(\mathbb{T}^2)^2)} + \|\mathcal{B}(\mathcal{K}f_k)\|_{L^2(0,T,V^1(\mathbb{T}^2)^2)} \\
&\quad\leq CT^{\frac{1}{2}}\|\mathcal{K}f_k\|_{L^\infty(0,T,V^2(\mathbb{T}^2)^2)}\left(\|\mathcal{K}f_k\|_{L^\infty(0,T,V^2(\mathbb{T}^2)^2)} + \max_{1\leq j\leq m}\left\|\rho^j\right\|_{V^2(\mathbb{T}^2)^2}\right)
\end{aligned}$$

Then, we deduce from (4.18) that

$$\begin{aligned}
&\left\|\mathcal{U}_k(t) - \overline{\mathcal{U}}(t)\right\|_{V^1(\mathbb{T}^2)^2} \\
&\quad\leq C\delta + CT^{\frac{1}{2}}\|\mathcal{K}f_k\|_{L^\infty(0,T,V^2(\mathbb{T}^2)^2)}\left(\|\mathcal{K}f_k\|_{L^\infty(0,T,V^2(\mathbb{T}^2)^2)} + \max_{1\leq j\leq m}\left\|\rho^j\right\|_{V^2(\mathbb{T}^2)^2} + 1\right).
\end{aligned}$$

Now, we fix $N \in \mathbb{N}^*$ so large that $C\delta \leq \frac{\varepsilon}{2}$. In order to prove Theorem 4.1, we only need to prove the following lemma.

le:Kernel **Lemma 4.4.** *We have*

$$\lim_{k\to+\infty}\|\mathcal{K}f_k\|_{L^\infty(0,T,V^2(\mathbb{T}^2)^2)} = 0.$$

In order to prove Lemma 4.4, we need to prove the following result concerning a relaxation norm of f_k (see [1] and [2] for more details on the definition of relaxation norm).

le:fkrelax **Lemma 4.5.** *Let $T > 0$ and for any $k \in \mathbb{N}^*$, let f_k be defined as in (4.6). Then, we have*

$$\lim_{k\to+\infty}\sup_{t\in[0,T]}\left\|\int_0^t f_k(s)ds\right\|_{V^2(\mathbb{T}^2)^2} = 0.$$

Step 3. *Proof of Lemma 4.5.*

For any $k \in \mathbb{N}^*$ and for any $t > 0$, let

$$F_k(t) = \int_0^t f_k(s)ds.$$

So, our goal is to prove that

eq:fkprim (4.19)
$$\lim_{k\to+\infty}\|F_k\|_{C(0,T,V^2(\mathbb{T}^2)^2)} = 0.$$

We remark that Estimates (4.10), (4.11) and the definition of f_k imply that if (4.19) is true for all piecewise constant (with respect to the time variable) functions $\mathcal{V}_N$, then (4.19) is true for all functions $\mathcal{V}_N$ (by using an approximation of $\mathcal{V}_N$ by piecewise constant functions). For this reason, we suppose that there exist $L \in \mathbb{N}^*$, $t_0, \ldots, t_L \in \mathbb{R}_+$ such that

$$0 = t_0 < \ldots < t_L = T,$$

and that

$$\mathcal{V}_N(t) = v_q, \qquad \forall\, t \in]t_{q-1}, t_q[, \quad \forall\, q \in \{1, \ldots, L\}\,.$$

Step 3.1. We will proof that

$$\lim_{k \to +\infty} F_k(t) = 0.$$

First of all, we suppose that $\mathcal{V}_N$ is a constant function with respect to the time variable, *i.e.*, $\mathcal{V}_N(t) = v$, for any $t \in [0, T]$. Then, recalling the definition of g_k, h_k in (4.7) and of ψ_k in (4.3) and (4.4), for any $t \in [0, T]$ we have

$$\begin{aligned}\int_0^t g_k(s)ds &= \int_0^t \mathcal{L}\psi_k(s)ds - \sum_{j=1}^m \lambda_j \int_0^t \mathcal{L}\rho^j ds = \left(\int_0^t \mathcal{L}\varphi\left(\frac{ks}{T}\right)ds\right) - t\sum_{j=1}^m \lambda_j \mathcal{L}\rho^j \\ &= \frac{T}{k}\int_0^{\frac{kt}{T}} \mathcal{L}\varphi\,(\tau)\,d\tau - t\sum_{j=1}^m \lambda_j \mathcal{L}\rho^j,\end{aligned}$$

and similarly,

$$\int_0^t h_k(s)ds = \frac{T}{k}\int_0^{\frac{kt}{T}} \mathcal{B}\,(v + \varphi\,(\tau))\,d\tau - t\sum_{j=1}^m \lambda_j \mathcal{B}(v + \rho^j).$$

Using the definition of φ, we deduce that, for any $t \geq 0$,

$$\int_t^{t+1} \mathcal{L}\varphi\,(\tau)\,d\tau = \sum_{j=1}^m \lambda_j \mathcal{L}\rho^j,$$

and thus,

$$\frac{T}{k}\int_0^{\frac{kt}{T}} \mathcal{L}\varphi\,(\tau)\,d\tau = \frac{T}{k}\left[\frac{kt}{T}\right]\sum_{j=1}^m \lambda_j \mathcal{L}\rho^j + \frac{T}{k}\int_{\left[\frac{kt}{T}\right]}^{\frac{kt}{T}} \mathcal{L}\varphi\,(\tau)\,d\tau,$$

where $[x]$ is the largest integer smaller than or equal to the real number x. Since, $\displaystyle\int_{\left[\frac{kt}{T}\right]}^{\frac{kt}{T}} \mathcal{L}\varphi\,(\tau)\,d\tau$ is bounded for all $k \in \mathbb{N}^*$, and since

$$\lim_{k \to +\infty} \frac{T}{k}\left[\frac{kt}{T}\right] = t,$$

for any $t \geq 0$, we obtain

$$\lim_{k \to +\infty} \int_0^t g_k(s)ds = 0.$$

The same argument gives

$$\lim_{k \to +\infty} \int_0^t h_k(s)ds = 0,$$

and thus, we have

$$\lim_{k \to +\infty} F_k(t) = \lim_{k \to +\infty} \int_0^t f_k(s)ds = 0.$$

Now, if $\mathcal{V}_N$ is a piecewise constant (with respect to the time variable) function, then there exist $L \in \mathbb{N}^*$, t_0, ..., t_L such that

$$0 = t_0 < \ldots < t_L = T,$$

and that

$$\mathcal{V}_N(t) = v_q, \qquad \forall\, t \in]t_{q-1}, t_q[, \quad \forall\, q \in \{1, \ldots, L\}.$$

For any $t \in [0, T[$, there exist $q \in \{1, \ldots, L\}$ such that $t_{q-1} \leq t < t_q$, and we write

Fkpiecewise

$$F_k(t) = \sum_{p=1}^{q-1} \int_{t_{p-1}}^{t_p} f_k(s)ds + \int_{t_{q-1}}^{t} f_k(s)ds. \tag{4.20}$$

Since $\mathcal{V}_N$ is constant on every time interval $]t_{q-1}, t_q[$, for any $q \in \{1, \ldots, L\}$, by a change of time variable, we deduce from the previous paragraph that

$$\lim_{k \to +\infty} \int_{t_{q-1}}^{t} f_k(s)ds = 0,$$

for any $t \in [t_{q-1}, t_q]$ and for any $q \in \{1, \ldots, L\}$. Then, (4.20) implies that

$$\lim_{k \to +\infty} F_k(t) = \lim_{k \to +\infty} \int_0^t f_k(s)ds = 0.$$

Step 3.2. We remark that, for any $t \in [0, T]$, the set $\{F_k(t)\}_k$ is relatively compact in $V^2(\mathbb{T}^2)^2$. Indeed, the set $\{f_k(t)\}$ only takes a finite number of value, independently of k. Let M be the set of value of $\{f_k(t)\}$ and we suppose that

$$M = \{M_1, \ldots, M_K\}, \qquad K \in \mathbb{N}^*.$$

Then, there exist positive constant a_1, ..., a_K such that

$$a_1 + \ldots + a_K = t$$

and

$$F_k(t) = \sum_{i=1}^{K} a_i M_i.$$

Thus, $\{F_k(t)\}_k$ is relatively compact in $V^2(\mathbb{T}^2)^2$. Moreover, from (4.12) and (4.13), there exists a positive constant C_0 such that

$$\sup_{t \in [0,T]} \|f_k(t)\|_{V^2(\mathbb{T}^2)^2} \leq C_0,$$

which means that $\{F_k(\cdot)\}_k$ is equicontinuous. Then, the Ascoli's theorem implies that $\{F_k(\cdot)\}_k$ is relatively compact in $C(0, T, V^2(\mathbb{T}^2)^2)$.

Step 3.3. We have

- $\{F_k(\cdot)\}_k$ is relatively compact in $C(0, T, V^2(\mathbb{T}^2)^2)$.
- $F_k(t) \to 0$ in $V^2(\mathbb{T}^2)^2$, for any $t \in [0, T]$.

Then, it is clear that

$$\lim_{k \to +\infty} \sup_{t \in [0,T]} \|F_k(t)\|_{V^2(\mathbb{T}^2)^2} = \lim_{k \to +\infty} \sup_{t \in [0,T]} \left\| \int_0^t f_k(s)ds \right\|_{V^2(\mathbb{T}^2)^2} = 0. \qquad \blacksquare$$

Step 4. *Proof of Lemma 4.4.*

We recall that $\mathcal{Z} = \mathcal{K} f_k$ is the solution of the system

:FG2linKCal (4.21)
$$\begin{cases} \partial_t \mathcal{Z} + \mathcal{L}\mathcal{Z} = \mathbb{P} f_k \\ \operatorname{div} \mathcal{Z} = 0 \\ \mathcal{Z}(0) = 0. \end{cases}$$

or equivalently, $z = (I - \alpha\Delta)^{-1}\mathcal{Z}$ is solution of the system

sys:FG2linK (4.22)
$$\begin{cases} \partial_t (z - \alpha\Delta z) - \nu\Delta z = \mathbb{P} f_k \\ \operatorname{div} z = 0 \\ z(0) = 0. \end{cases}$$

We also recall that *a priori* estimates in this paragraph can be justified by applying an approximation by a Galerkin scheme. Following the method presented in [23], we apply the *rot* operator to the first equation of (4.22) and then we take the L^2 scalar product of the obtained equation with $-\operatorname{rot}\left(\Delta z - \alpha\Delta^2 z\right)$. We get

eq:SKH4a (4.23)
$$\frac{1}{2}\frac{d}{dt}\left\|\nabla\left(\operatorname{rot} z - \alpha \operatorname{rot}\Delta z\right)\right\|^2_{L^2(\mathbb{T}^2)^2} + \nu\left(\left\|\Delta \operatorname{rot} z\right\|^2_{L^2(\mathbb{T}^2)^2} + \alpha\left\|\Delta^2 z\right\|^2_{L^2(\mathbb{T}^2)^2}\right) = -\left\langle \operatorname{rot} f_k \,,\, \operatorname{rot}\left(\Delta z - \alpha\Delta^2 z\right)\right\rangle_{L^2(\mathbb{T}^2)^2}.$$

Integrating over $[0,t]$ and then, performing multiple integrations by parts (with respect to the space variable x and then with respect to the time variable t), we have

eq:SKH4b (4.24)
$$\begin{aligned} &\frac{1}{2}\left\|\nabla\left(\operatorname{rot} z(t) - \alpha \operatorname{rot}\Delta z(t)\right)\right\|^2_{L^2(\mathbb{T}^2)^2} + \nu\left(\int_0^t \left\|\Delta \operatorname{rot} z(s)\right\|^2_{L^2(\mathbb{T}^2)^2} ds + \alpha\int_0^t \left\|\Delta^2 z(s)\right\|^2_{L^2(\mathbb{T}^2)^2} ds\right) \\ &\quad = -\int_0^t\int_{\mathbb{T}^2} \operatorname{rot} f_k(s,x)\cdot \operatorname{rot}\left(\Delta z - \alpha\Delta^2 z\right)(s,x)dxds \\ &\quad = \int_0^t\int_{\mathbb{T}^2} \nabla \operatorname{rot} f_k(s,x) : \nabla(\operatorname{rot} z - \alpha \operatorname{rot}\Delta z)(s,x)dxdt \\ &\quad = \int_{\mathbb{T}^2}\left(\int_0^t \nabla \operatorname{rot} f_k(s,x)ds\right) : \nabla\left(\operatorname{rot} z - \alpha \operatorname{rot}\Delta z\right)(t,x)dx \\ &\qquad\qquad - \int_{\mathbb{T}^2}\int_0^t \left(\int_0^s \nabla \operatorname{rot} f_k(\tau,x)d\tau\right) : \left(\frac{\partial}{\partial s}\nabla\left(\operatorname{rot} z - \alpha \operatorname{rot}\Delta z\right)(s,x)\right) dsdx \\ &\quad = J_1(t,k) + J_2(t,k). \end{aligned}$$

For the first term on the right-hand side, Cauchy-Schwarz inequality implies that

eq:J1 (4.25)
$$\begin{aligned} |J_1(t,k)| &\le \left\|\int_0^t \nabla \operatorname{rot} f_k(s,x)ds\right\|_{L^2(\mathbb{T}^2)^2} \left\|\nabla\left(\operatorname{rot} z(t) - \alpha \operatorname{rot}\Delta z(t)\right)\right\|_{L^2(\mathbb{T}^2)^2} \\ &\le C\left\|\int_0^t f_k(s,x)ds\right\|_{V^2(\mathbb{T}^2)^2} \left\|\nabla\left(\operatorname{rot} z(t) - \alpha \operatorname{rot}\Delta z(t)\right)\right\|_{L^2(\mathbb{T}^2)^2}. \end{aligned}$$

For the second term, also using Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
|J_2(t,k)| &\leq \int_0^t \left(\left\| \int_0^s \nabla \mathrm{rot} f_k(\tau) d\tau \right\|_{L^2(\mathbb{T}^2)^2} \left\| \frac{\partial}{\partial s} \nabla \left(\mathrm{rot}\, z - \alpha\, \mathrm{rot}\, \Delta z \right)(s) \right\|_{L^2(\mathbb{T}^2)^2} \right) ds \\
&\leq \left(\int_0^t \left\| \int_0^s f_k(\tau) d\tau \right\|^2_{V^2(\mathbb{T}^2)^2} ds \right)^{\frac{1}{2}} \left(\int_0^t \left\| \frac{\partial}{\partial s} \nabla \left(\mathrm{rot}\, z - \alpha\, \mathrm{rot}\, \Delta z \right)(s) \right\|^2_{L^2(\mathbb{T}^2)^2} ds \right)^{\frac{1}{2}} \\
&\leq T^{\frac{1}{2}} \sup_{t\in[0,T]} \left\| \int_0^t f_k(s) ds \right\|_{V^2(\mathbb{T}^2)^2} \left(\int_0^t \left\| \frac{\partial}{\partial s} \nabla \left(\mathrm{rot}\, z - \alpha\, \mathrm{rot}\, \Delta z \right)(s) \right\|^2_{L^2(\mathbb{T}^2)^2} ds \right)^{\frac{1}{2}}.
\end{aligned}
$$

Now, we come back the first equation of (4.22). Applying the *rot* operator to this equation and then, taking the L^2 scalar product of the obtained equation with $-\frac{\partial}{\partial t}\mathrm{rot}\left(\Delta z - \alpha\Delta^2 z\right)$, we get

$$
\begin{aligned}
\|\partial_t \nabla \mathrm{rot}\,(z-\alpha\Delta z)\|^2_{L^2(\mathbb{T}^2)^2} + \frac{\nu}{2}\frac{d}{dt}\left(\|\mathrm{rot}\,\Delta z\|^2_{L^2(\mathbb{T}^2)^2} + \alpha \left\|\Delta^2 z\right\|^2_{L^2(\mathbb{T}^2)^2} \right) \\
= \left\langle \nabla \mathrm{rot}\, f_k \,,\, \partial_t \nabla \mathrm{rot}\,(z-\alpha\Delta z) \right\rangle_{L^2(\mathbb{T}^2)^2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_0^t \|\partial_s \nabla \mathrm{rot}\,(z(s)-\alpha\Delta z(s))\|^2_{L^2(\mathbb{T}^2)^2} ds + \frac{\nu}{2}\left(\|\mathrm{rot}\,\Delta z(t)\|^2_{L^2(\mathbb{T}^2)^2} + \alpha \left\|\Delta^2 z(t)\right\|^2_{L^2(\mathbb{T}^2)^2} \right) \\
\leq \int_0^t \|\nabla \mathrm{rot}\, f_k(s)\|_{L^2(\mathbb{T}^2)^2} \|\partial_s \nabla \mathrm{rot}\,(z(s)-\alpha\Delta z(s))\|_{L^2(\mathbb{T}^2)^2} ds \\
\leq \|f_k\|_{L^2(0,T,V^2(\mathbb{T}^2)^2)} \left(\int_0^t \|\partial_s \nabla \mathrm{rot}\,(z(s)-\alpha\Delta z(s))\|^2_{L^2(\mathbb{T}^2)^2} ds \right)^{\frac{1}{2}}
\end{aligned}
$$

Come back to $J_2(t,k)$, we have

eq:J2

$$
|J_2(t,k)| \leq CT \sup_{t\in[0,T]} \left\| \int_0^t f_k(s) ds \right\|_{V^2(\mathbb{T}^2)^2} \|f_k\|_{L^\infty(0,T,V^2(\mathbb{T}^2)^2)}. \tag{4.26}
$$

Combining (4.24) with (4.25), (4.26) and the fact that

$$
\lim_{k\to+\infty} \sup_{t\in[0,T]} \left\| \int_0^t f_k(s) ds \right\|_{V^2(\mathbb{T}^2)^2} = 0,
$$

we conclude that

$$
0 \leq \lim_{k\to+\infty} \|\mathcal{K} f_k\|_{L^\infty(0,T,V^2(\mathbb{T}^2)^2)} \leq C \lim_{k\to+\infty} \|\nabla\left(\mathrm{rot}\, z(t) - \alpha\, \mathrm{rot}\,\Delta z(t)\right)\|_{L^\infty(0,T,L^2(\mathbb{T}^2)^2)} = 0. \qquad \blacksquare
$$

:Saturation

5. Saturation property for the controlled system of fluids of second grade

Let $q = (q_1, q_2) \in]0, +\infty[^2$ be fixed and

$$
\mathbb{T}^2_q = \mathbb{R}^2/\mathbb{Z}^2_q \qquad \text{with} \quad \mathbb{Z}^2_q = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \frac{x_i}{q_i} \in \mathbb{Z},\ i = 1, 2 \right\}.
$$

For any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, let

$$
\langle x, y \rangle = x_1 y_1 + x_2 y_2, \quad |x| = |x_1| + |x_2|, \quad \|x\| = \sqrt{\langle x, x \rangle},
$$

and

$$
\langle x, y \rangle_q = \frac{x_1 y_1}{q_1} + \frac{x_2 y_2}{q_2}, \quad \|x\|_q = \sqrt{\frac{x_1^2}{q_1^2} + \frac{x_2^2}{q_2^2}}.
$$

For any $a=(a_1,a_2)\in\mathbb{R}^2\setminus\{0\}$, let $a^\perp=(-a_2,a_1)$. We will denote $a^{q,\perp}$ the unit vector which satisfied $\langle a,a^{q,\perp}\rangle_q=0$ and $\left\|a^{q,\perp}\right\|=1$ and we denote P_a the orthogonal of $\mathbb{R}^2$ onto the subspace $Span\left\{a^{q,\perp}\right\}$ generated by $a^{q,\perp}$. Direct calculations also give

:SaturaProj **Lemma 5.1.** *Let $a,l\in\mathbb{R}^2\setminus\{0\}$. Then,*

q:SaturaCos
$$\mathbb{P}\left(a\cos\langle l,x\rangle_q\right)=(P_l a)\cos\langle l,x\rangle_q \tag{5.1}$$

q:SaturaSin
$$\mathbb{P}\left(a\sin\langle l,x\rangle_q\right)=(P_l a)\sin\langle l,x\rangle_q \tag{5.2}$$

We recall that for any $m\in\mathbb{Z}^2\setminus\{0\}$, we set

$$c_m(x)=m^{q,\perp}\cos\langle m,x\rangle_q \qquad\text{and}\qquad s_m(x)=m^{q,\perp}\sin\langle m,x\rangle_q\,.$$

These vector fields c_m, s_m, with $m\in\mathbb{Z}^2\setminus\{0\}$, are eigenvectors of the Stokes operator $-\mathbb{P}\Delta$ and the family $\left\{c_m,s_m\mid m\in\mathbb{Z}^2\setminus\{0\}\right\}$ forms an orthonormal basis of $V^k(\mathbb{T}^2_q)^2$, $k\in\mathbb{N}$. In Section 1, for any $N\in\mathbb{N}^*$, we already set

$$\mathcal{H}^N_q=Span\left\{c_m,s_m\mid m\in\mathbb{Z}\setminus\{0\}\,,|m|\le N\right\}. \tag{1.5}$$

For any $m\in\mathbb{Z}^2\setminus\{0\}$, let

$$\mathcal{C}_m=Span\left\{c_m,c_{-m}\right\}\qquad\text{and}\qquad\mathcal{S}_m=Span\left\{s_m,s_{-m}\right\}.$$

aturaDecomp **Lemma 5.2.** *Let $m,n\in\mathbb{Z}^2\setminus\{0\}$. For any $f_m\in\mathcal{C}_m$ and $g_n\in\mathcal{S}_n$, there exist $\tilde f_m,\tilde g_n\in\mathbb{R}^2$ such that*

$$\begin{aligned}\left\langle\tilde f_m,m\right\rangle_q&=\langle\tilde g_n,n\rangle_q=0,\\ f_m(x)&=\tilde f_m\cos\langle m,x\rangle_q\,,\\ g_n(x)&=\tilde g_n\sin\langle n,x\rangle_q\,.\end{aligned}$$

For any $m,n\in\mathbb{Z}^2\setminus\{0\}$ and for any $f_m\in\mathcal{C}_m$ and $g_n\in\mathcal{S}_n$, Lemma 5.2 allows to calculate

$$\begin{aligned}&\mathbb{P}\left(\operatorname{rot}f_m\times\left((I-\alpha\Delta)^{-1}g_n\right)\right)\\&\quad=\mathbb{P}\left\{\left(\nabla\times\left(\tilde f_m\cos\langle m,x\rangle_q\right)\right)\times\left((Id-\alpha\Delta)^{-1}\left(\tilde g_n\sin\langle n,x\rangle_q\right)\right)\right\}\\&\quad=\mathbb{P}\left\{\left(1+\alpha\|n\|_q^2\right)^{-1}\left[\nabla\times\left(\tilde f_m\cos\langle m,x\rangle_q\right)\right]\times\left(\tilde g_n\sin\langle n,x\rangle_q\right)\right\}\\&\quad=\left(1+\alpha\|n\|_q^2\right)^{-1}\mathbb{P}\left\{\left(\left\langle\tilde f_m^\perp,m\right\rangle_q\sin\langle m,x\rangle_q\right)\times\left(\tilde g_n\sin\langle n,x\rangle_q\right)\right\}\\&\quad=\left(1+\alpha\|n\|_q^2\right)^{-1}\left\langle\tilde f_m^\perp,m\right\rangle_q\mathbb{P}\left(\tilde g_n^\perp\sin\langle m,x\rangle_q\sin\langle n,x\rangle_q\right)\\&\quad=\frac{\left(1+\alpha\|n\|_q^2\right)^{-1}}{2}\left\langle\tilde f_m^\perp,m\right\rangle_q\mathbb{P}\left[\tilde g_n^\perp\left(\cos\langle m-n,x\rangle_q-\cos\langle m+n,x\rangle_q\right)\right].\end{aligned}$$

Using Lemma 5.1, we obtain

eq:fg
$$\begin{aligned}\mathcal{B}(f_m,g_n)&=\mathbb{P}\left(\operatorname{rot}f_m\times\left((I-\alpha\Delta)^{-1}g_n\right)\right)\\&=\frac{\left(1+\alpha\|n\|_q^2\right)^{-1}}{2}\left\langle\tilde f_m^\perp,m\right\rangle_q\left(\cos\langle m-n,x\rangle_q P_{m-n}-\cos\langle m+n,x\rangle_q P_{m+n}\right)\tilde g_n^\perp.\end{aligned} \tag{5.3}$$

Similar calculations give

eq:gf (5.4)
$$\mathcal{B}(g_m, f_n) = -\frac{\left(1+\alpha\|n\|_q^2\right)^{-1}}{2}\left\langle \tilde{g}_m^{\perp}, m\right\rangle_q \left(\cos\langle m+n, x\rangle_q P_{m+n} + \cos\langle m-n, x\rangle_q P_{m-n}\right)\tilde{f}_n^{\perp},$$

eq:ff (5.5)
$$\mathcal{B}(f_m, f_n) = \frac{\left(1+\alpha\|n\|_q^2\right)^{-1}}{2}\left\langle \tilde{f}_m^{\perp}, m\right\rangle_q \left(\sin\langle m+n, x\rangle_q P_{m+n} + \sin\langle m-n, x\rangle_q P_{m-n}\right)\tilde{f}_n^{\perp},$$

and

eq:gg (5.6)
$$\mathcal{B}(g_m, g_n) = \frac{\left(1+\alpha\|n\|_q^2\right)^{-1}}{2}\left\langle \tilde{g}_m^{\perp}, m\right\rangle_q \left(\sin\langle m-n, x\rangle_q P_{m-n} - \sin\langle m+n, x\rangle_q P_{m+n}\right)\tilde{g}_n^{\perp}.$$

The next lemma is the most important result of this section, which allows us to prove the saturation property of the space of controls (Theorem 1.8).

:Saturation **Lemma 5.3.** *Let $q=(q_1,q_2)$, $q_1,q_2>0$. For any $m,n\in\mathbb{Z}^2\setminus\{0\}$ satisfying*

- $\|m\|_q \neq \|n\|_q$,
- *m, n are not parallel,*

and for any $f\in\mathcal{C}_{m+n}$, $g\in\mathcal{S}_{m+n}$, there exist

$$a, b \in Span\left\{\mathcal{C}_m, \mathcal{C}_n, \mathcal{S}_m, \mathcal{S}_n\right\}$$

such that

aturation00 (5.7)
$$\mathcal{B}(a)+f,\ \mathcal{B}(b)+g \in Span\left\{\mathcal{C}_{m-n}, \mathcal{S}_{m-n}\right\},$$

where $\mathcal{B}$ is defined in (1.7).

Proof. Taking $m=n$ in Estimates (5.5) and (5.6), we have

eq:ffbis (5.8)
$$\mathcal{B}\left(f_m, f_m\right) = \frac{1}{2}\left(1+\alpha\|m\|_q^2\right)^{-1}\left\langle \tilde{f}_m^{\perp}, m\right\rangle_q \sin\langle 2m, x\rangle_q P_{2m}\tilde{f}_m^{\perp},$$

eq:ggbis (5.9)
$$\mathcal{B}\left(g_m, g_m\right) = -\frac{1}{2}\left(1+\alpha\|m\|_q^2\right)^{-1}\left\langle \tilde{g}_m^{\perp}, m\right\rangle_q \sin\langle 2m, x\rangle_q P_{2m}\tilde{g}_m^{\perp}.$$

Since $\left\langle \tilde{f}_m, m\right\rangle_q = \langle \tilde{g}_m, m\rangle_q = 0$, we deduce that $\tilde{f}_m, \tilde{g}_m \in Span\left\{m_q^{\perp}\right\}$. By definition, P_{2m} is the projection onto the subspace $Span\left\{(2m)_q^{\perp}\right\} = Span\left\{m_q^{\perp}\right\}$. So, we have

$$P_{2m}\left(\tilde{f}_m^{\perp}\right) = P_{2m}\left(\tilde{g}_m^{\perp}\right) = 0,$$

which means that, for any $m\in\mathbb{Z}^2\setminus\{0\}$,

eq:BffBgg (5.10)
$$\mathcal{B}(f_m, f_m) = \mathcal{B}(g_m, g_m) = 0.$$

Now, following the idea of [25], for any $f\in\mathcal{C}_{m+n}$, we look for $a\in Span\left\{\mathcal{C}_m, \mathcal{S}_n\right\}$ under the form

eq:formA (5.11)
$$a = f_m + g_n, \quad f_m\in\mathcal{C}_m, \quad g_n\in\mathcal{S}_n,$$

such that

$$\mathcal{B}(a)+f \in Span\left\{\mathcal{C}_{m-n}, \mathcal{S}_{m-n}\right\}.$$

Since $a = f_m + g_n$, taking into account (5.3), (5.4) and (5.10), we have

$$\mathcal{B}(a) = \frac{1}{2}\cos\langle m-n,x\rangle_q P_{m-n}\left\{\left(1+\alpha\|n\|_q^2\right)^{-1}\left\langle \tilde{f}_m^{\perp},m\right\rangle_q \tilde{g}_n^{\perp} - \left(1+\alpha\|m\|_q^2\right)^{-1}\left\langle \tilde{g}_n^{\perp},n\right\rangle_q \tilde{f}_m^{\perp}\right\}$$
$$-\frac{1}{2}\cos\langle m+n,x\rangle_q P_{m+n}\left\{\left(1+\alpha\|n\|_q^2\right)^{-1}\left\langle \tilde{f}_m^{\perp},m\right\rangle_q \tilde{g}_n^{\perp} + \left(1+\alpha\|m\|_q^2\right)^{-1}\left\langle \tilde{g}_n^{\perp},n\right\rangle_q \tilde{f}_m^{\perp}\right\}.$$

We remark that

$$\cos\langle m-n,x\rangle_q P_{m-n}\left\{\left(1+\alpha\|n\|_q^2\right)^{-1}\left\langle \tilde{f}_m^{\perp},m\right\rangle_q \tilde{g}_n^{\perp} - \left(1+\alpha\|m\|_q^2\right)^{-1}\left\langle \tilde{g}_n^{\perp},n\right\rangle_q \tilde{f}_m^{\perp}\right\}$$

belongs to $Span\{\mathcal{C}_{m-n},\mathcal{S}_{m-n}\}$. So, we only need to prove that, for any $f\in\mathcal{C}_{m+n}$, there are $f_m\in\mathcal{C}_m$ and $g_n\in\mathcal{S}_n$ such that

eq:ProjF

(5.12)
$$f = \frac{1}{2}\cos\langle m+n,x\rangle_q P_{m+n}\left\{\left(1+\alpha\|n\|_q^2\right)^{-1}\left\langle \tilde{f}_m^{\perp},m\right\rangle_q \tilde{g}_n^{\perp} + \left(1+\alpha\|m\|_q^2\right)^{-1}\left\langle \tilde{g}_n^{\perp},n\right\rangle_q \tilde{f}_m^{\perp}\right\}.$$

Let

$$F = \left(1+\alpha\|n\|_q^2\right)^{-1}\left\langle \tilde{f}_m^{\perp},m\right\rangle_q \tilde{g}_n^{\perp} + \left(1+\alpha\|m\|_q^2\right)^{-1}\left\langle \tilde{g}_n^{\perp},n\right\rangle_q \tilde{f}_m^{\perp}.$$

Then, we will prove (5.12) if we can find $f_m\in\mathcal{C}_m$ and $g_n\in\mathcal{S}_n$ such that $P_{m+n}F\neq 0$, or equivalently, $\langle F,G\rangle\neq 0$, for some vector $G\neq 0$ and $G\in Span\left\{(m+n)_q^{\perp}\right\}$. For the sake of simplicity, we choose

$$G = ((m_2+n_2)q_1, -(m_1+n_1)q_2).$$

For any $f_m\in\mathcal{C}_m$ and $g_n\in\mathcal{S}_n$, recall that $\left\langle \tilde{f}_m,m\right\rangle_q = \langle \tilde{g}_n,n\rangle_q = 0$. Thus, there exist $C_f, C_g\in\mathbb{R}$ such that

$$\tilde{f}_m = C_f(m_2q_1,-m_1q_2) \qquad \text{and} \qquad \tilde{g}_n = C_g(n_2q_1,-n_1q_2)$$

We get

$$\tilde{f}_m^{\perp} = C_f(m_1q_2,m_2q_1) \qquad \text{and} \qquad \tilde{g}_n^{\perp} = C_g(n_1q_2,n_2q_1),$$

and

$$\left\langle \tilde{f}_m^{\perp},m\right\rangle_q = \frac{q_2}{q_1}m_1^2 + \frac{q_1}{q_2}m_2^2 = q_1q_2\|m\|_q^2$$
$$\left\langle \tilde{g}_n^{\perp},n\right\rangle_q = \frac{q_2}{q_1}n_1^2 + \frac{q_1}{q_2}n_2^2 = q_1q_2\|n\|_q^2.$$

Let

$$C = \left(1+\alpha\|m\|_q^2\right)\left(1+\alpha\|n\|_q^2\right).$$

Then,

$$F = CC_fC_gq_1q_2\left[\left(1+\alpha\|m\|_q^2\right)\|m\|_q^2(n_1q_2,n_2q_1) + \left(1+\alpha\|n\|_q^2\right)\|n\|_q^2(m_1q_2,m_2q_1)\right],$$

Let

$$M_q = \left(1+\alpha\|m\|_q^2\right)\|m\|_q^2 \qquad \text{and} \qquad N_q = \left(1+\alpha\|n\|_q^2\right)\|n\|_q^2.$$

Since $\|m\|_q\neq\|n\|_q$ and m, n are not parallel in $\mathbb{R}^2$, we finally obtain

$$\langle F,G\rangle = CC_fC_gq_1q_2\left[(M_qn_1q_2+N_qm_1q_2)(m_2q_1+n_2q_1) - (M_qn_2q_1+N_qm_2q_1)(m_1q_2+n_1q_2)\right]$$
$$= CC_fC_gq_1^2q_2^2(M_q-N_q)(n_1m_2-n_2m_1)\neq 0,$$

if f_m and g_n are not zero.

To prove the second part of the lemma, for any $g\in\mathcal{S}_{m+n}$, we can look for b under the form

$$b = f_m + f_n \qquad \text{or} \qquad b = g_m + g_n,$$

where $f_m \in \mathcal{C}_m$, $f_n \in \mathcal{C}_n$, $g_m \in \mathcal{S}_m$ and $g_n \in \mathcal{S}_n$. Indeed, taking into account (5.5), (5.6) and (5.10), we have

$$\begin{aligned}
&\mathcal{B}(f_m + f_n)\\
&= \frac{1}{2}\sin\langle m-n,x\rangle_q P_{m-n}\left\{\left(1+\alpha\|n\|_q^2\right)^{-1}\left\langle \tilde{f}_m^{\perp}, m\right\rangle_q \tilde{f}_n^{\perp} - \left(1+\alpha\|m\|_q^2\right)^{-1}\left\langle \tilde{f}_n^{\perp}, n\right\rangle_q \tilde{f}_m^{\perp}\right\}\\
&\quad + \frac{1}{2}\sin\langle m+n,x\rangle_q P_{m+n}\left\{\left(1+\alpha\|n\|_q^2\right)^{-1}\left\langle \tilde{f}_m^{\perp}, m\right\rangle_q \tilde{f}_n^{\perp} + \left(1+\alpha\|m\|_q^2\right)^{-1}\left\langle \tilde{f}_n^{\perp}, n\right\rangle_q \tilde{f}_m^{\perp}\right\},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{B}(g_m + g_n)\\
&= \frac{1}{2}\sin\langle m-n,x\rangle_q P_{m-n}\left\{\left(1+\alpha\|n\|_q^2\right)^{-1}\left\langle \tilde{g}_m^{\perp}, m\right\rangle_q \tilde{g}_n^{\perp} - \left(1+\alpha\|m\|_q^2\right)^{-1}\left\langle \tilde{g}_n^{\perp}, n\right\rangle_q \tilde{g}_m^{\perp}\right\}\\
&\quad - \frac{1}{2}\sin\langle m+n,x\rangle_q P_{m+n}\left\{\left(1+\alpha\|n\|_q^2\right)^{-1}\left\langle \tilde{g}_m^{\perp}, m\right\rangle_q \tilde{g}_n^{\perp} + \left(1+\alpha\|m\|_q^2\right)^{-1}\left\langle \tilde{g}_n^{\perp}, n\right\rangle_q \tilde{g}_m^{\perp}\right\}.
\end{aligned}$$

Then, we can repeat the above argument for a to find b. Lemma 5.3 is proved. ■

In what follows, we recall that, for any finite-dimensional subspace E of $V^3(\mathbb{T}^2)^2$, we have defined $\mathcal{F}(E)$ as the largest vector subspace of $V^3(\mathbb{T}^2)^2$ such that, for any $\overline{\eta} \in \mathcal{F}(E)$, there exist

$$k \in \mathbb{N}^*;\ \alpha_1, \dots, \alpha_k > 0;\ \eta, \rho^1, \dots, \rho^k \in E$$

satisfying

$$\overline{\eta} = \eta - \sum_{j=1}^{k} \alpha_j \mathcal{B}(\rho^j).$$

We have also defined a sequence of subspace

$$E = E_0 \subset E_1 \subset \ldots \subset E_n \subset \ldots$$

such that, for any $n \in \mathbb{N}$ we have $E_{n+1} = \mathcal{F}(E_n)$ and we set

$$E_\infty = \bigcup_{n=0}^{\infty} E_n.$$

The saturation property in Theorem 1.8 can be precised as follows

:Saturation **Theorem 5.4** (Saturation Property)**.** *Let $q = (q_1, q_2)$, $q_1, q_2 > 0$, let E be a finite-dimensional subspace of $V^3(\mathbb{T}^2)^2$ and for any $N \in \mathbb{N}^*$, let $\mathcal{H}_q^N$ be defined as in* (1.5)*. If $E \supset \mathcal{H}_q^3$, then $E_\infty \supset \mathcal{H}_q^N$, for any $N \in \mathbb{N}$, $N \geq 3$.*

Proof. Inspired by the argument of [[25], Theorem 2.5], we will prove Theorem 5.4 by recurrence that, if $E = E_0 \supset \mathcal{H}_q^3$ then, for any $k \geq 0$, we have

eq:E2k (5.13) $$E_{2k} \supset \mathcal{H}_q^{k+3}.$$

It is evident that for $k = 0$, (5.13) is true. Let $k \geq 1$. We suppose that, for any $k' \in \mathbb{N}$, $0 \leq k' < k$, we have $E_{2k'} \supset \mathcal{H}_q^{k'+3}$. In order to prove that (5.13) is true for k, we only need to prove that, for any $l \in \mathbb{Z}^2 \setminus \{0\}$, $|l| = k+3$, we have $c_l, s_l \in E_{2k}$.

1. First case: If $l = (l_1, l_2)$ with $l_1 \neq 0$ and $l_2 \neq 0$. In this case, since $|l_1| + |l_2| > 3$, without loss of generality, we can suppose that $l_1 \geq 2$. We choose $m = (l_1 - 1, l_2)$ and $n = (1, 0)$. Then, we have

$$m + n = l, \quad \|m\|_q > \|n\|_q, \quad |m| = k+2, \quad |n| = 1, \quad |m-n| = k+1$$

and m, n are not parallel in $\mathbb{R}^2$. Applying Lemma 5.3, we obtain the existence of

$$a, b \in Span\left\{\mathcal{C}_m, \mathcal{C}_n, \mathcal{S}_m, \mathcal{S}_n\right\} \subset \mathcal{H}_q^{k+2} \subset E_{2k-2}$$

such that

$$\mathcal{B}(a) + c_l,\ \mathcal{B}(b) + s_l \in Span\left\{\mathcal{C}_{m-n}, \mathcal{S}_{m-n}\right\}.$$

Thus, there exist

$$f, g \in Span\left\{\mathcal{C}_{m-n}, \mathcal{S}_{m-n}\right\} \subset \mathcal{H}_q^{k+1} \subset E_{2k-2}$$

such that

$$c_l = f - \mathcal{B}(a), \quad s_l = g - \mathcal{B}(b).$$

Then, using the definition of $E_{2k-1} = \mathcal{F}(E_{2k-2})$, we deduce that

$$c_l, s_l \in E_{2k-1} \subset E_{2k}.$$

2. Second case: If $l = (l_1, l_2)$ with $l_1 = |l| = 3$ and $l_2 = 0$. In this case, we choose $m = (l_1 - 1, 1)$ and $n = (1, -1)$. Then, we have

$$m + n = l, \quad \|m\|_q > \|n\|_q, \quad |m| = k + 3, \quad |n| = 2, \quad |m - n| = k + 3$$

and m, n are not parallel in $\mathbb{R}^2$. Since all the components of the vectors m, n, $m - n$ are not zero, we can apply the result of the first case and we deduce that

$$Span\left\{\mathcal{C}_m, \mathcal{C}_n, \mathcal{S}_m, \mathcal{S}_n\right\} \subset E_{2k-1}$$

and

$$Span\left\{\mathcal{C}_{m-n}, \mathcal{S}_{m-n}\right\} \subset E_{2k-1}.$$

Now, applying Lemma 5.3, we obtain the existence of

$$a, b \in Span\left\{\mathcal{C}_m, \mathcal{C}_n, \mathcal{S}_m, \mathcal{S}_n\right\} \subset E_{2k-1}$$

such that

$$\mathcal{B}(a) + c_l,\ \mathcal{B}(b) + s_l \in Span\left\{\mathcal{C}_{m-n}, \mathcal{S}_{m-n}\right\} \subset E_{2k-1}.$$

As in the first case, we can deduce that

$$c_l, s_l \in \mathcal{F}(E_{2k-1}) = E_{2k}.$$

Theorem 5.4 is proved. ■

6. Approximate controllability by high-mode controls reduced to approximate controllability by low-mode controls

se:Dense

The goal of this section is to prove the main theorem 1.4 by proving that we can reduce the control of the system (1.1) by high-mode controls to controls in $\mathcal{H}_q^3$. Let $T > 0$, $\varepsilon > 0$ and $u_0, u_T \in V^4(\mathbb{T}^2)^2$. We set

$$\mathcal{U}_0 = u_0 - \alpha\Delta u_0 \quad \text{and} \quad \mathcal{U}_T = u_T - \alpha\Delta u_T.$$

For any $t \in [0, T]$, let

$$\overline{\mathcal{U}}(t) = \frac{1}{T}\left(I - \alpha\Delta\right)\left((T - t)u_0 + tu_T\right).$$

Then, $\overline{\mathcal{U}}$ is solution of the system (1.6) with

$$\overline{\mathcal{U}}(0) = \mathcal{U}_0 \quad \text{and} \quad \eta = \partial_t\overline{\mathcal{U}} + \mathcal{L}\overline{\mathcal{U}} + \mathcal{B}(\overline{\mathcal{U}}) - \mathbb{P}f.$$

It is clear that

$$\overline{\mathcal{U}} \in C(0, T, V^2(\mathbb{T}^2)^2) \quad \text{and} \quad \eta \in L^\infty(0, T, V^1(\mathbb{T}^2)^2).$$

Let $k \in \mathbb{N}^*$, $k \geq 3$ and let

$$\eta_k = P_k\left(\partial_t\overline{\mathcal{U}} + \mathcal{L}\overline{\mathcal{U}} + \mathcal{B}(\overline{\mathcal{U}}) - \mathbb{P}f\right),$$

where P_k is the projection onto $\mathcal{H}_q^k$. Let $\overline{\mathcal{U}}_k$ be the solution of the system (1.6) with

$$\overline{\mathcal{U}}_k(0) = \mathcal{U}_0 \quad \text{and} \quad \eta = \eta_k.$$

Applying Theorem 2.1, we can choose k so large that

$$\left\|\overline{\mathcal{U}}_k(T) - \overline{\mathcal{U}}(T)\right\|_{V^1(\mathbb{T}^2)^2} \leq C \left\|\eta_k - \left(\partial_t \overline{\mathcal{U}} + \mathcal{L}\overline{\mathcal{U}} + \mathcal{B}(\overline{\mathcal{U}}) - \mathbb{P}f\right)\right\|_{L^2(0,T,V^1(\mathbb{T}^2)^2)} \leq \varepsilon.$$

Now, Theorem 5.4 implies that $\mathcal{H}_q^k \subset E_N$, where $N = 2(k-3)$. Now, we set $\mathcal{U}^N = \overline{\mathcal{U}}_k$. Applying Theorems 4.1 and 3.1, we can contruct a sequence of controls $\eta_j \in E_j$, $j \in \{0, \ldots, N\}$, and a sequence of solutions $\mathcal{U}^j$ of the system (1.6) with

$$\mathcal{U}^j(0) = \mathcal{U}_0 \quad \text{and} \quad \eta = \eta_j,$$

such that

$$\left\|\mathcal{U}^{j-1}(T) - \mathcal{U}^j(T)\right\|_{V^1(\mathbb{T}^2)^2} \leq \frac{\varepsilon}{2^{N-j}},$$

for any $j \in \{1, \ldots, N\}$. Thus, $\mathcal{U}^0$ is the solution of the system (1.6) with

$$\mathcal{U}^0(0) = \mathcal{U}_0 \quad \text{and} \quad \eta = \eta_0 \in E_0 = \mathcal{H}_q^3,$$

and moreover, we have

$$\left\|\mathcal{U}^0(T) - \mathcal{U}_T\right\|_{V^1(\mathbb{T}^2)^2} \leq \varepsilon + \sum_{j=1}^{N} \frac{\varepsilon}{2^{N-j}} < 3\varepsilon.$$

Finally, we set $u = (I - \alpha\Delta)^{-1}\mathcal{U}^0$. Then u is the solution of the system (1.1), with

$$u(0) = u_0 \quad \text{and} \quad \eta = \eta_0 \in \mathcal{H}_q^3.$$

Moreover, we have

$$\|u(T) - u_T\|_{V^3(\mathbb{T}^2)^2} \leq \frac{1}{\min\{1, \alpha\}} \left\|\mathcal{U}^0(T) - \mathcal{U}_T\right\|_{V^1(\mathbb{T}^2)^2} \leq \frac{3\varepsilon}{\min\{1, \alpha\}}.$$

Theorem 1.4 is proved. ■

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