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# ROTATING FLUIDS WITH SMALL VISCOSITY - THE CASE OF ILL PREPARED DATA

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ABSTRACT. In this paper, we prove the existence of a unique, global strong solution of the damped rotating fluid system in the whole space  $\mathbb{R}^3$ , that is the Navier-Stokes system with a damping force term, a large Coriolis force term and with a small anisotropic viscosity, provided that the rotation is fast enough. We also study the convergence of this solution towards the solution of the limiting system, in the case where the limiting system is not zero, and give an application in the case where the fluid rotates between two parallel infinite plates.

## 1. INTRODUCTION

In this paper, we study the following rotating fluid system in  $\mathbb{R}^3$ , which is the incompressible Navier-Stokes equations together with a large Coriolis force term and a damping term  $\gamma u^\varepsilon$ :

$$(1.1) \quad \begin{cases} \partial_t u^\varepsilon - \nu_h(\varepsilon) \Delta_h u^\varepsilon + \gamma u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \frac{e_3 \wedge u^\varepsilon}{\varepsilon} = -\nabla p^\varepsilon & \text{in } \mathbb{R}^3 \times \mathbb{R}_+ \\ \operatorname{div} u^\varepsilon = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}_+ \\ u^\varepsilon|_{t=0} = u_0 & \text{in } \mathbb{R}^3, \end{cases}$$

where the unknowns are the velocity  $u^\varepsilon$  and the pressure  $p^\varepsilon$ . The fluid rotates around the  $e_3$ -axis and  $\varepsilon$  stands for the Rossby number. We remark that this system is anisotropic in the sense that the “total” diffusion (in all directions) is replaced by an horizontal diffusion term

$$-\nu_h(\varepsilon) \Delta_h = -\nu_h(\varepsilon) (\partial_1^2 + \partial_2^2),$$

where  $\partial_i$  is the partial derivative with respect to the variable  $x_i$  and where  $\nu_h(\varepsilon) > 0$  is the viscosity in the horizontal direction, depending on  $\varepsilon$ , and goes to 0 when  $\varepsilon \rightarrow 0$ . We emphasize that all along this paper, we always use the index “h” to refer to the horizontal terms and horizontal variables, and the index “v” or “3” to the vertical ones.

We first recall that the system (1.1), with  $\gamma = 0$ , is a simplified model, usually used to describe the behavior of geophysical fluids. The anisotropic diffusion term is often considered in meteorology and oceanography as representing the influence of the Coriolis force on the fluid movements (see [36] or [21] for the details). One can also find an extended survey on the system (1.1), with  $\gamma = 0$ , in [12] or [20].

It is clear that the Coriolis force term has no contribution in the energy estimates for the system (1.1). Using the same method as for the Euler system, we can prove that the system (1.1) is locally well-posed when the initial data  $u_0$  belong to a smooth enough Sobolev space, *e.g.*,  $\mathbf{H}^{\frac{5}{2}+\eta}(\mathbb{R}^3)$ ,  $\eta > 0$ . We also remind the reader that, the vanishing of the vertical viscosity introduces additional problems and that the existence of a weak Leray solution is still an open question. In order to obtain a result close to the Fujita-Kato theorem and the existence of a unique strong solution, Chemin, Desjardins, Gallagher and Grenier ([10]) considered the system (1.1), with  $\gamma = 0$ ,  $\nu_h > 0$  independent of  $\varepsilon$ , and with the initial data in the following anisotropic Sobolev spaces<sup>1</sup>

$$\mathbf{H}^{0,s}(\mathbb{R}^3) = \left\{ u \in \mathcal{S}' : \|u\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)} \stackrel{\text{def}}{=} \left( \int_{\mathbb{R}^3} (1 + |\xi_3|^2)^s |\mathcal{F}(u)(\xi)|^2 d\xi \right)^{\frac{1}{2}} < +\infty \right\},$$

with  $s > \frac{1}{2}$ . Here,  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  denote the Fourier transform and its inverse, and  $\mathcal{S}'$  is the space of tempered distributions. We also use the notation  $\hat{u}$  to denote the Fourier transform of  $u$ . In [10], the authors proved the local existence (global for small data) of a strong solution in  $\mathbf{H}^{0,s_0}(\mathbb{R}^3)$ ,  $s_0 > \frac{1}{2}$ , and

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<sup>1</sup>For the sake of the simplicity, we use the bold character  $\mathbf{X}$  to indicate the space of vector fields, each component of which belongs to the space  $X$ .

proved the uniqueness of this solution in  $\mathbf{H}^{0,s_1}(\mathbb{R}^3)$ ,  $s_1 > \frac{3}{2}$ . The uniqueness in  $\mathbf{H}^{0,s_0}(\mathbb{R}^3)$ ,  $s_0 > \frac{1}{2}$  was proven later by Iftimie in [25]. Their results can be generalized to the system (1.1) in the following way:

**Theorem 1.1.** *Let  $\gamma = 0$ ,  $s_0 > \frac{1}{2}$  and  $u_0 \in \mathbf{H}^{0,s_0}(\mathbb{R}^3)$ . Then, there exists  $T > 0$  and a unique strong solution  $u^\varepsilon$  of the system (1.1) such that*

$$u^\varepsilon \in \mathbf{L}^\infty([0, T], \mathbf{H}^{0,s_0}(\mathbb{R}^3)) \quad \text{and} \quad \nabla_h u^\varepsilon \in \mathbf{L}^2([0, T], \mathbf{H}^{0,s_0}(\mathbb{R}^3)).$$

Moreover, there is a constant  $c > 0$  such that, if  $\|u_0\|_{\mathbf{H}^{0,s_0}} \leq c\nu_h$ , then the solution is global.

The main ideas of the proof of this theorem are the use of the Littlewood-Paley theory and a decomposition of the system (1.1) in frequencies, where the vertical variable plays a different role from the horizontal ones. The crucial point in their method is that the missing vertical viscosity can be ‘‘compensated’’ by the divergence-free property of the velocity field. Indeed, in the nonlinear term  $u^\varepsilon \cdot \nabla u^\varepsilon$ , the vertical derivative  $\partial_3$  is always multiplied by  $u^{\varepsilon,3}$  (as in  $u^{\varepsilon,3}\partial_3$ ), and the third component  $u^{\varepsilon,3}$  is smoother than expected, because of the relation

$$\partial_3 u^{\varepsilon,3} = -\partial_1 u^{\varepsilon,1} - \partial_2 u^{\varepsilon,2} = -\operatorname{div}_h u^{\varepsilon,h},$$

and because of the smoothing effect induced by the horizontal viscosity on the horizontal components. By a minor modification of this method, we can also prove the same result for the system (1.1) in the case where  $\gamma > 0$  and where  $\nu_h(\varepsilon) > 0$  depends on  $\varepsilon$ .

In the above result, the role of the Coriolis force is not mentioned. But, actually in fast rotating systems with small Rossby number, the Coriolis force plays an important role. In the experiment of G.I. Taylor (see [15]), drops of dye injected into a rapidly rotating, homogeneous fluid, within a few rotations, formed perfectly vertical sheets of dyed fluid, known as *Taylor curtains*. In large-scale atmospheric and oceanic flows, the fluid motions also have a tendency towards columnar behaviors (*Taylor columns*). For example, currents in the western North Atlantic have been observed to extend vertically over several thousands meters without significant change in amplitude and direction ([39]).

Fast rotating flows have first been considered in periodic domains. In the work of Babin, Mahalov and Nicolaenko [1], [3], the authors studied the isotropic case, where the diffusion term is  $-\nu_h \Delta_h - \nu_v \partial_3^2$ , with  $\nu_h, \nu_v > 0$  independent of  $\varepsilon$ . For any initial data in appropriate Sobolev spaces, they proved the global existence and uniqueness of a smooth solution when the rotation is fast enough (that is, the Rossby number  $\varepsilon$  is small enough). The same result was proven by Gallagher in [18], using the method of Schochet [40], for fast rotating fluid systems and also for more general parabolic systems. It was also shown in [1], [3] and [18] that, when  $\varepsilon$  goes to zero, the fast rotating system converges to a two-dimensional Navier-Stokes system with three components, if  $\nu_h > 0$  is fixed, or to a two-dimensional Euler system with three components, if  $\nu_h$  is zero or goes to zero as  $\varepsilon$  goes to zero. The convergence in the more general case, where the rotation axis is not fixed, was proven in [19] by Gallagher and Saint-Raymond. The anisotropic case, with zero vertical viscosity ( $\nu_v = 0$ ), in periodic domains, that is the system (1.1), with  $\gamma = 0$  and  $\nu_h > 0$  independent of  $\varepsilon$ , in the torus  $\mathbb{T}^3$ , was studied later by Paicu in [33].

In the case of rotating fluids in the whole space  $\mathbb{R}^3$ , it was proved in [9], [10] and [19] that, if the initial data are divergence-free vector fields belonging to  $\mathbf{H}^{0,s}(\mathbb{R}^3)$  then, when  $\varepsilon$  goes to zero, the limiting system corresponding to (1.1), with  $\gamma = 0$ , is zero. The main idea consists in proving and using Strichartz estimates for the associated linear system, which shows that, in the case of  $\mathbb{R}^3$ , the free oscillation waves of the fluid propagate to infinity and the energy of the fluid decreases. Using this property, it was also proved in [10] that the system (1.1), with  $\gamma = 0$  and  $\nu_h > 0$  fixed, has a unique, global solution when the rotation is fast enough.

In physical models, the fluid is turbulent and  $\nu_h, \nu_v$  denote in fact turbulent viscosities instead of the molecular kinematic viscosity of the fluid. In real situations, the viscosity is not fixed but depends on the speed of rotation of the fluid. The viscosities  $\nu_h, \nu_v$  go to 0 as the Rossby number goes to 0 (*i.e.*, as the rotation speed goes to infinity). Such models were studied in the papers of Grenier and Masmoudi [22] and of Masmoudi [30] for fast rotating fluids between two parallel plates. In [32], following the ideas of [22] and [30], we considered the Navier-Stokes-Coriolis system (1.1) in the case where  $\gamma = 0$ ,  $\nu_v = 0$  and  $\nu_h = \nu_h(\varepsilon)$  goes slowly to 0 as  $\varepsilon$  goes to 0 and where the initial data belong to  $\mathbf{H}^{0,s}$ ,  $s > \frac{1}{2}$ . In [32], the fact that  $\nu_h(\varepsilon)$  goes slowly to 0 was expressed by setting  $\nu_h(\varepsilon) = \varepsilon^\alpha$ , with  $\alpha \in ]0, \alpha_0]$ , for some  $\alpha_0 > 0$ . The difficulty arising in the study of this model consists in the fact that the coefficient  $\frac{1}{\nu_h(\varepsilon)}$  in the energy estimates goes to infinity when  $\varepsilon$  goes to 0, which prevents one to use Gronwall-type arguments to deduce the global existence of strong solutions. In [32], by modifying the method of [10],

we proved that the limiting system is zero and there exists a unique global strong solution if the rotation is fast enough. We emphasize the importance of a positive horizontal viscosity which does not go to zero too fast so that the dispersion due to Strichartz estimates can balance the large coefficient  $\frac{1}{\nu_h(\varepsilon)}$ .

In this paper, also following the ideas of [22] and [30], we consider the system (1.1) in the anisotropic framework, with  $\gamma > 0$ ,  $\nu_v = 0$  and  $\nu_h = \nu_h(\varepsilon) = \varepsilon^\alpha$ ,  $0 < \alpha \leq \alpha_0$ , for some  $\alpha_0 > 0$ . We remark that the choice  $\nu_h(\varepsilon) = \varepsilon^\alpha$  is to simplify the notations and the calculations. We can of course consider other forms of  $\nu_h$  which are equivalent or smaller than  $\varepsilon^{\alpha_0}$ . We also remark that the results remain true in the isotropic case, with initial data in appropriate isotropic Sobolev spaces. Here, the idea of choosing  $\gamma > 0$  is motivated by the dissipation of energy which was put in evidence in [22] and [30] when the rotation is fast. More motivations to take  $\gamma > 0$  will be given after the introduction of the systems (1.3), (1.4) and after Theorem 1.2.

The goal of the paper is to determine what we can obtain if the limiting system is not zero. Can we still prove the existence of a unique, global solution when the rotation is fast enough? In what follows, we consider initial data of the form

$$(1.2) \quad u_0 = \bar{u}_0 + v_0,$$

where

$$\bar{u}_0 = (\bar{u}_0^1(x_1, x_2), \bar{u}_0^2(x_1, x_2), \bar{u}_0^3(x_1, x_2))$$

is a divergence-free vector field, independent of  $x_3$  and

$$v_0 = (v_0^1(x_1, x_2, x_3), v_0^2(x_1, x_2, x_3), v_0^3(x_1, x_2, x_3)).$$

We recall that in [10], Chemin *et al.* proved that, for the system (1.1), with  $\gamma = 0$  and  $\nu_h > 0$  fixed, if we consider initial data of the form (1.2), with  $\bar{u}_0 \in \mathbf{L}^2(\mathbb{R}_h^2)$  and  $v_0 \in \mathbf{H}^{0,s}(\mathbb{R}^3)$ , the limiting system is not zero, but the two-dimensional Navier-Stokes system with three components

$$(1.3) \quad \begin{cases} \partial_t \bar{u}^h + \nu_h \Delta_h \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h = -\nabla_h \bar{p} \\ \partial_t \bar{u}^3 + \nu_h \Delta_h \bar{u}^3 + \bar{u}^h \cdot \nabla_h \bar{u}^3 = 0 \\ \operatorname{div}_h \bar{u} = 0 \\ \partial_3 \bar{u} = 0 \\ \bar{u}|_{t=0} = \bar{u}_0, \end{cases}$$

where  $\bar{u} = (\bar{u}^h, \bar{u}^3)$  and  $\bar{u}^h = (\bar{u}^1, \bar{u}^2)$ . When  $\nu_h = \nu_h(\varepsilon)$  goes to zero as  $\varepsilon$  goes to zero, the limiting system is the two-dimensional Euler system with three components ([19])

$$(1.4) \quad \begin{cases} \partial_t \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h = -\nabla_h \bar{p} \\ \partial_t \bar{u}^3 + \bar{u}^h \cdot \nabla_h \bar{u}^3 = 0 \\ \operatorname{div}_h \bar{u} = 0 \\ \partial_3 \bar{u} = 0 \\ \bar{u}|_{t=0} = \bar{u}_0. \end{cases}$$

We remark that the above limiting systems have very different properties. The first one, which is a 2D Navier-Stokes system, is globally well-posed in  $\mathbf{L}^2(\mathbb{R}_h^2)$ . In addition, the solution is bounded in  $\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2)) \cap \mathbf{L}^2(\mathbb{R}_+, \dot{\mathbf{H}}^1(\mathbb{R}_h^2))$ . This property allows to prove (see [10]) that, if the initial data are of the form (1.2), with  $\bar{u}_0 \in \mathbf{L}^2(\mathbb{R}_h^2)$  and  $v_0 \in \mathbf{H}^{0,s}(\mathbb{R}^3)$ ,  $s > \frac{1}{2}$ , then there exists a global solution of the system (1.1), with  $\gamma = 0$  and  $\nu_h > 0$  fixed, when  $\varepsilon$  is small enough. Furthermore, this solution of (1.1) is close to the solution of (1.3) in  $\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{H}^{0,s}(\mathbb{R}^3))$ . The second limiting system, which is the system (1.4), is still well-posed in  $\mathbf{H}^\sigma(\mathbb{R}_h^2)$ ,  $\sigma > 2$ . However, the solution of (1.4) could exponentially grow in  $\mathbf{H}^\sigma(\mathbb{R}_h^2)$ -norm,  $\sigma > 2$ , and we do not know if this solution is integrable with respect to the time variable.

For the system (1.1) with  $\gamma = 0$ ,  $\nu_v = 0$  and where  $\nu_h = \nu_h(\varepsilon)$  goes to zero with  $\varepsilon$ , the above difficulty prevents us from using the method of [10] or [32] for proving the existence of a global solution when  $\varepsilon$  is small enough, in the case where  $\nu_h = \nu_h(\varepsilon)$  goes to zero with  $\varepsilon$ . In this paper, the fact that  $\gamma > 0$  allows us to simply avoid this difficulty as explained in Section 3. The system (1.1) is thus considered with the following hypotheses

- $\gamma > 0$  is fixed,
- $\nu_h = \nu_h(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  (more precisely, we will consider  $\nu_h(\varepsilon) = \varepsilon^\alpha$ ,  $\alpha > 0$ ),
- and  $u_0 = \bar{u}_0 + v_0$  as in (1.2).

As we will explain after the introduction of Theorem 1.2, such a choice is not artificial at all. We will prove that the limiting system of (1.1) with  $\gamma > 0$ , when  $\varepsilon$  goes to zero, is the following two-dimensional damped Euler system with three components:

$$(1.5) \quad \begin{cases} \partial_t \bar{u}^h + \gamma \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h = -\nabla_h \bar{p} \\ \partial_t \bar{u}^3 + \gamma \bar{u}^3 + \bar{u}^h \cdot \nabla_h \bar{u}^3 = 0 \\ \operatorname{div}_h \bar{u} = 0 \\ \partial_3 \bar{u} = 0 \\ \bar{u}|_{t=0} = \bar{u}_0, \end{cases}$$

where  $\bar{u} = (\bar{u}^h, \bar{u}^3)$ .

We remark that, in the system (1.1) with  $\gamma > 0$ , the additional damping term  $\gamma u^\varepsilon$  does not regularize the system. But, with presence of the damping term  $\gamma \bar{u}$  in the limiting system (1.5), we can prove the exponential decay in time of the  $\mathbf{H}^\sigma(\mathbb{R}_h^2)$ -norm,  $\sigma > 2$ , of the solution of (1.5). Due to the decay property, by using the method of [10] and [32], we obtain the existence of a unique, global strong solution of the system (1.1), and the convergence of (1.1) to (1.5) as  $\varepsilon$  goes to zero.

Before stating the main result of this paper, we want to say a few words about the role of the viscosity  $\nu_h(\varepsilon)$ . The fact that  $\nu_h(\varepsilon) > 0$  is primordial, even when  $\nu_h(\varepsilon)$  is small and goes to 0 as  $\varepsilon \rightarrow 0$  (e.g.,  $\nu_h(\varepsilon) = \varepsilon^\alpha$ ,  $\alpha > 0$ ). Indeed, if we consider the fast rotating Euler system with damping term (that is, the system (1.1), with  $\gamma > 0$  and  $\nu_h(\varepsilon) = 0$ ), we cannot obtain better results than in [1], [2] or in [16], where the authors proved the existence of a unique solution on any time interval  $[0, T]$ , with  $T > 0$  arbitrarily large, provided that  $\varepsilon$  is small enough.

In order to state our main result, for any  $\sigma_1, \sigma_2 \in \mathbb{R}$ , we define the following spaces

$$\begin{aligned} \mathbf{H}^{\sigma_1, \sigma_2}(\mathbb{R}^3) &= \left\{ u \in \mathcal{S}' : \left( \int_{\mathbb{R}^3} \left(1 + |\xi_h|^2\right)^{\sigma_1} \left(1 + |\xi_3|^2\right)^{\sigma_2} |\mathcal{F}(u)(\xi)|^2 d\xi \right)^{\frac{1}{2}} < +\infty \right\} \\ \dot{\mathbf{H}}^{\sigma_1, \sigma_2}(\mathbb{R}^3) &= \left\{ u \in \mathcal{S}' : \left( \int_{\mathbb{R}^3} |\xi_h|^{2\sigma_1} \left(1 + |\xi_3|^2\right)^{\sigma_2} |\mathcal{F}(u)(\xi)|^2 d\xi \right)^{\frac{1}{2}} < +\infty \right\} \\ \mathbf{L}_h^2 \dot{\mathbf{H}}_v^{\sigma_2}(\mathbb{R}^3) &= \left\{ u \in \mathcal{S}' : \left( \int_{\mathbb{R}^3} |\xi_3|^{2\sigma_2} |\mathcal{F}(u)(\xi)|^2 d\xi \right)^{\frac{1}{2}} < +\infty \right\}, \end{aligned}$$

and for any  $\eta > 0$ ,  $s > \frac{1}{2}$ , let

$$\mathbf{Y}_{s, \eta} = \dot{\mathbf{H}}^{-\eta, s}(\mathbb{R}^3) \cap \mathbf{L}_h^2 \dot{\mathbf{H}}_v^{-\eta}(\mathbb{R}^3) \cap \mathbf{H}^{\eta, \eta+s}(\mathbb{R}^3).$$

The main result of this paper is the following theorem.

**Theorem 1.2.** *Let  $s > \frac{1}{2}$ ,  $\sigma > 2$ ,  $\gamma > 0$  and  $\eta > 0$ . There is a constant  $M > 0$  such that, for any  $\alpha > 0$ ,  $a > 0$ ,  $3\alpha + 2a < M$ , for any  $a_0 > 0$ ,  $b_0 > 0$ , there exists  $\varepsilon_0 > 0$ , depending on  $s, \sigma, \gamma, \eta, a_0$  and  $b_0$ , so that, if  $\nu_h(\varepsilon) = \varepsilon^\alpha$ , with  $0 < \varepsilon \leq \varepsilon_0$ , and if*

$$u_0 = \bar{u}_0 + v_0,$$

where

- (i)  $\bar{u}_0 = (\bar{u}_0^1(x_1, x_2), \bar{u}_0^2(x_1, x_2), \bar{u}_0^3(x_1, x_2))$  is a two-dimensional, divergence-free vector field belonging to  $\mathbf{H}^\sigma(\mathbb{R}_h^2)$ , satisfying  $\|\bar{u}_0\|_{\mathbf{H}^\sigma(\mathbb{R}_h^2)} \leq a_0$ ,
- (ii)  $v_0 = (v_0^1(x_1, x_2, x_3), v_0^2(x_1, x_2, x_3), v_0^3(x_1, x_2, x_3))$  is a three-dimensional, divergence-free vector field in  $\mathbf{H}^{0, s}(\mathbb{R}^3) \cap \mathbf{Y}_{s, \eta}$  such that  $\|v_0\|_{\mathbf{Y}_{s, \eta}} \leq b_0 \varepsilon^{-a}$ ,

the system (1.1) has a unique global solution

$$u^\varepsilon \in \mathbf{C}(\mathbb{R}_+, \mathbf{H}^{0, s}(\mathbb{R}^3)) \cap \mathbf{L}^\infty(\mathbb{R}_+, \mathbf{H}^{0, s}(\mathbb{R}^3)) \quad \text{with} \quad \nabla_h u^\varepsilon \in \mathbf{L}^2(\mathbb{R}_+, \mathbf{H}^{0, s}(\mathbb{R}^3)).$$

### Remark

- (1) In the above theorem, the three-dimensional part  $v_0$  of the initial data can be chosen arbitrarily large when  $\varepsilon$  is small enough.
- (2) The constant  $M$  can be chosen as follows

$$M = \frac{\eta}{6 + 4\eta}.$$

We want to emphasize that the choice of the damping term  $\gamma u^\varepsilon$ , with  $\gamma > 0$ , in the system (1.1) is not artificial. If we consider the fast rotating fluid system between two parallel plates, with Dirichlet boundary conditions, as in [22], [30] or [11]

$$(1.6) \quad \begin{cases} \partial_t u^\varepsilon - \nu_h(\varepsilon) \Delta_h u^\varepsilon - \beta \varepsilon \partial_3^2 u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \frac{e_3 \wedge u^\varepsilon}{\varepsilon} = -\nabla p^\varepsilon & \text{in } \mathbb{R}_+ \times \Omega_h \times [0, 1] \\ \operatorname{div} u^\varepsilon = 0 & \text{in } \mathbb{R}_+ \times \Omega_h \times [0, 1] \\ u^\varepsilon|_{t=0} = u_0^\varepsilon, & \text{in } \Omega_h \times [0, 1], \end{cases}$$

where  $\Omega_h = \mathbb{R}_h^2$  and where  $\nu_h = \varepsilon^\alpha$ , with  $\alpha > 0$ , then, the damping term really exists in the limiting system. Indeed, when the rotation goes to infinity, the *Taylor columns* are only formed in the interior of the domain. Near the boundary, Ekman boundary layers exist. The behaviors of the fluid become very complex and the friction slows the fluid down in a way that the velocity is zero on the boundary. As a consequence, in the limiting system, we obtain the additional damping term of the form  $\gamma \bar{u}$ ,  $\gamma > 0$  (the coefficient  $\gamma$  was proved to be  $\sqrt{2\beta}$  in [22]). This phenomenon is well known in fluid mechanics as the Ekman pumping.

Since the viscosity is positive in all three directions ( $\nu_h = \nu_h(\varepsilon) > 0$  and  $\nu_v = \beta\varepsilon > 0$ ), the system (1.6) possesses a weak Leray solution

$$u^\varepsilon \in \mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}^3)) \cap \mathbf{L}^2(\mathbb{R}_+, \dot{\mathbf{H}}^1(\mathbb{R}^3)).$$

In the case where  $\nu_h > 0$  is fixed and where the initial data are well prepared (*i.e.*  $\lim_{\varepsilon \rightarrow 0} u_0^\varepsilon = \bar{u}_0 = (\bar{u}_0^1(x_1, x_2), \bar{u}_0^2(x_1, x_2), 0)$  in  $\mathbf{L}^2(\mathbb{R}_h^2 \times [0, 1])$  and  $\bar{u}_0$  is a divergence-free two-dimensional vector field in  $\mathbf{H}^\sigma(\mathbb{R}_h^2)$ ,  $\sigma > 2$ ), it was proved by Grenier and Masmoudi in [22], in the case  $\Omega_h = \mathbb{T}_h^2$  and by Chemin *et al.* in [11], in the case  $\Omega_h = \mathbb{R}_h^2$  that, when  $\varepsilon$  goes to zero,  $u^\varepsilon$  converges to the solution of the following limiting system in  $\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}^3))$

$$(1.7) \quad \begin{cases} \partial_t \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h + \sqrt{2\beta} \bar{u}^h = -\nabla_h \bar{p} \\ \partial_t \bar{u}^3 + \bar{u}^h \cdot \nabla_h \bar{u}^3 + \sqrt{2\beta} \bar{u}^3 = 0 \\ \operatorname{div}_h \bar{u}^h = 0 \\ \partial_3 \bar{u} = 0 \\ \bar{u}|_{t=0} = \bar{u}_0. \end{cases}$$

We want to remark that in this case, as  $\bar{u}_0^3 = 0$ , the third component  $\bar{u}^3 = 0$  for any  $t > 0$ . In the case where  $\nu_h \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , however, the convergence is only local with respect to the time variable (see [22]). In this paper, applying the method of the proof of Theorem 1.2, we show the exponential decay of the solution of the system (1.7) in appropriate Sobolev norms, and we improve the result of [22], in the case where  $\nu_h(\varepsilon) = \varepsilon^\alpha$ ,  $\alpha > 0$ . More precisely, we prove the uniform convergence (with respect to the time variable) of (1.6) towards (1.7).

**Theorem 1.3.** *Let  $0 < \alpha < \frac{1}{2}$  and  $u_0^\varepsilon \in \mathbf{L}^2(\mathbb{R}_h^2 \times [0, 1])$  be a family of initial data such that*

$$\lim_{\varepsilon \rightarrow 0} u_0^\varepsilon = \bar{u}_0 = (\bar{u}_0^1(x_1, x_2), \bar{u}_0^2(x_1, x_2), 0) \quad \text{in } \mathbf{L}^2(\mathbb{R}_h^2 \times [0, 1]),$$

where  $\bar{u}_0$  is a divergence-free two-dimensional vector field in  $\mathbf{H}^\sigma(\mathbb{R}_h^2)$ ,  $\sigma > 2$ . Let  $\bar{u}$  be the solution of the limiting system (1.7) with initial data  $\bar{u}_0$  and, for each  $\varepsilon > 0$ , let  $u^\varepsilon$  be a weak solution of (1.6) with initial data  $u_0^\varepsilon$ . Then

$$\lim_{\varepsilon \rightarrow 0} \|u^\varepsilon - \bar{u}\|_{\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2 \times [0, 1]))} = 0.$$

The structure of the paper is as follows. In Section 2, we recall some elements of the Littlewood-Paley theory and the dyadic decompositions. Section 3 deals with estimates concerning the solution of the system (1.5). In Section 4, we prove Strichartz estimates for the cut-off linear system (4.2). Section 5 is devoted to the proof of Theorem 1.2. Finally, in the last section, we study the system (1.6) of rotating fluids between two parallel infinite plates and prove Theorem 1.3.

## 2. PRELIMINARIES: DYADIC DECOMPOSITIONS

In this section, we briefly recall the properties of dyadic decompositions in the Fourier space and give some elements of the Littlewood-Paley theory. Using dyadic decompositions, we define the anisotropic

Sobolev spaces  $\mathbf{H}^{0,s}$ , which will be used in this paper. At the end of the section, we recall important a priori estimates which we need in the proof of Theorem 1.2.

**2.1. Dyadic decomposition.** We recall that  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  are the Fourier transform and its inverse, and that we also write  $\widehat{u} = \mathcal{F}u$ . For any  $d \in \mathbb{N}^*$  and  $0 < r < R$ , we denote  $B_d(0, R) = \{\xi \in \mathbb{R}^d \mid |\xi| \leq R\}$ , and  $C_d(r, R) = \{\xi \in \mathbb{R}^d \mid r \leq |\xi| \leq R\}$ . The following Bernstein lemma gives important properties of a distribution  $u$  when its Fourier transform is well localized. We refer the reader to [8] for the proof of this lemma.

**Lemma 2.1.** *Let  $k \in \mathbb{N}$ ,  $d \in \mathbb{N}^*$  and  $r_1, r_2 \in \mathbb{R}$  satisfy  $0 < r_1 < r_2$ . There exists a constant  $C > 0$  such that, for any  $a, b \in \mathbb{R}$ ,  $1 \leq a \leq b \leq +\infty$ , for any  $\lambda > 0$  and for any  $u \in L^a(\mathbb{R}^d)$ , we have*

$$(2.1) \quad \text{supp}(\widehat{u}) \subset B_d(0, r_1\lambda) \implies \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^k \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a},$$

and

$$(2.2) \quad \text{supp}(\widehat{u}) \subset C_d(r_1\lambda, r_2\lambda) \implies C^{-k} \lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^k \lambda^k \|u\|_{L^a}.$$

Let  $\psi$  be an even smooth function in  $C_0^\infty(\mathbb{R})$ , whose support is contained in the ball  $B_1(0, \frac{4}{3})$ , such that  $\psi$  is equal to 1 on a neighborhood of the ball  $B_1(0, \frac{3}{4})$ . Let

$$\varphi(z) = \psi\left(\frac{z}{2}\right) - \psi(z).$$

Then, the support of  $\varphi$  is contained in the ring  $C_1(\frac{3}{4}, \frac{8}{3})$ , and  $\varphi$  is identically equal to 1 on the ring  $C_1(\frac{4}{3}, \frac{3}{2})$ . The functions  $\psi$  and  $\varphi$  allow us to define a dyadic partition of  $\mathbb{R}^d$ ,  $d \in \mathbb{N}^*$ , as follows

$$\forall z \in \mathbb{R}, \quad \psi(z) + \sum_{j \in \mathbb{N}} \varphi(2^{-j}z) = 1.$$

Moreover, this decomposition is almost orthogonal, in the sense that, if  $|j - j'| \geq 2$ , then

$$\text{supp} \varphi(2^{-j}(\cdot)) \cap \text{supp} \varphi(2^{-j'}(\cdot)) = \emptyset.$$

We introduce the following dyadic frequency cut-off operators. We refer to [5] and [8] for more details.

**Definition 2.2.** *For any  $d \in \mathbb{N}^*$  and for any tempered distribution  $u \in \mathcal{S}'(\mathbb{R}^d)$ , we set*

$$\begin{aligned} \Delta_q u &= \mathcal{F}^{-1}(\varphi(2^{-q}|\xi|)\widehat{u}(\xi)), & \forall q \in \mathbb{N}, \\ \Delta_{-1} u &= \mathcal{F}^{-1}(\psi(|\xi|)\widehat{u}(\xi)), \\ \Delta_q u &= 0, & \forall q \leq -2, \\ S_q u &= \sum_{q' \leq q-1} \Delta_{q'} u, & \forall q \geq 1. \end{aligned}$$

Using the properties of  $\psi$  and  $\varphi$ , one can prove that for any tempered distribution  $u \in \mathcal{S}'(\mathbb{R}^d)$ , we have

$$u = \sum_{q \geq -1} \Delta_q u \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d),$$

and the (isotropic) nonhomogeneous Sobolev spaces  $H^s(\mathbb{R}^d)$ , with  $s \in \mathbb{R}$ , can be characterized as follows

**Proposition 2.3.** *Let  $d \in \mathbb{N}^*$ ,  $s \in \mathbb{R}$  and  $u \in H^s(\mathbb{R}^d)$ . Then,*

$$\|u\|_{H^s} := \left( \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \sim \left( \sum_{q \geq -1} 2^{2qs} \|\Delta_q u\|_{L^2}^2 \right)^{\frac{1}{2}}$$

Moreover, there exists a square-summable sequence of positive numbers  $\{c_q(u)\}$  with  $\sum_q c_q(u)^2 = 1$ , such that

$$\|\Delta_q u\|_{L^2} \leq c_q(u) 2^{-qs} \|u\|_{H^s}.$$

The decomposition into dyadic blocks also gives a very simple characterization of Hölder spaces.

**Definition 2.4.** *Let  $d \in \mathbb{N}^*$  and  $r \in \mathbb{R}_+ \setminus \mathbb{N}$ .*

- (1) If  $r \in ]0, 1[$ , we denote  $C^r(\mathbb{R}^d)$  the set of bounded functions  $u : \mathbb{R}^d \rightarrow \mathbb{R}$  such that there exists  $C > 0$  satisfying

$$\forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad |u(x) - u(y)| \leq C |x - y|^r.$$

- (2) If  $r > 1$  is not an integer, we denote  $C^r(\mathbb{R}^d)$  the set of  $[r]$  times differentiable functions  $u$  such that  $\partial^\alpha u \in C^{r-[\alpha]}(\mathbb{R}^d)$ , for any  $\alpha \in \mathbb{N}^d$ ,  $|\alpha| \leq [r]$ , where  $[r]$  is the largest integer smaller than  $r$ .

One can prove that the set  $C^r(\mathbb{R}^d)$ , endowed with the norm

$$\|u\|_{C^r} := \sum_{|\alpha| \leq [r]} \left( \|\partial^\alpha u\|_{L^\infty} + \sup_{x \neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x - y|^{r-[\alpha]}} \right)$$

is a Banach space. Moreover, we have the following result, the proof of which can be found in [8].

**Proposition 2.5.** *There exists a constant  $C > 0$  such that, for any  $r \in \mathbb{R}_+ \setminus \mathbb{N}$  and for any  $u \in C^r(\mathbb{R}^d)$ , we have*

$$\sup_q 2^{qr} \|\Delta_q u\|_{L^\infty} \leq \frac{C^{r+1}}{[r]!} \|u\|_{C^r}.$$

Conversely, if the sequence  $(2^{qr} \|\Delta_q u\|_{L^\infty})_{q \geq -1}$  is bounded, then

$$\|u\|_{C^r} \leq C^{r+1} \left( \frac{1}{r - [r]} + \frac{1}{[r] + 1 - r} \right) \sup_q 2^{qr} \|\Delta_q u\|_{L^\infty}.$$

In what follows, we give the definition of the anisotropic Sobolev spaces on  $\mathbb{R}^3$ , using separate dyadic decompositions in the horizontal and vertical directions.

**Definition 2.6.** *For any tempered distribution  $u \in \mathcal{S}'(\mathbb{R}^3)$ , we set*

$$\begin{aligned} \Delta_q^h u &= \mathcal{F}^{-1}(\varphi(2^{-q} |\xi_h|) \widehat{u}(\xi)), & \Delta_q^v u &= \mathcal{F}^{-1}(\varphi(2^{-q} |\xi_3|) \widehat{u}(\xi)), & \forall q \in \mathbb{N}, \\ \Delta_{-1}^h u &= \mathcal{F}^{-1}(\psi(|\xi_h|) \widehat{u}(\xi)), & \Delta_{-1}^v u &= \mathcal{F}^{-1}(\psi(|\xi_3|) \widehat{u}(\xi)), \\ \Delta_q^h u &= 0, & \Delta_q^v u &= 0, & \forall q \leq -2, \\ S_q^h u &= \sum_{q' \leq q-1} \Delta_{q'}^h u, & S_q^v u &= \sum_{q' \leq q-1} \Delta_{q'}^v u, & \forall q \geq 1. \end{aligned}$$

Then, we can characterize the anisotropic Sobolev spaces  $H^{0,s}$ , defined in the introduction, by the following equivalent norm.

**Proposition 2.7.** *Let  $s \in \mathbb{R}$  and  $u \in H^{0,s}(\mathbb{R}^3)$ . Then,*

$$\|u\|_{H^{0,s}} \sim \left( \sum_{q \geq -1} 2^{2qs} \|\Delta_q^v u\|_{L^2}^2 \right)^{\frac{1}{2}}$$

Moreover, there exists a square-summable sequence of positive numbers  $\{c_q(u)\}$  with  $\sum_q c_q(u)^2 = 1$ , such that

$$\|\Delta_q^v u\|_{L^2} \leq c_q(u) 2^{-qs} \|u\|_{H^{0,s}}.$$

**2.2. A priori estimates.** In what follows, we always denote by  $(c_q)$  (respectively  $(d_q)$ ) a square-summable (respectively summable) sequence, with  $\sum_q c_q^2 = 1$  (respectively  $\sum_q d_q = 1$ ), of positive numbers (which can depend on several parameters). We also remind the reader that, in order to simplify the notations, we use the bold character  $\mathbf{X}$  to indicate the space of vector fields, each component of which belongs to the space  $X$ .

First, we need the following results (for a proof, see [35]). Let  $[\cdot, \cdot]$  denote the usual commutator.

**Lemma 2.8.** *Let  $d \in \mathbb{N}^*$ . There exists a constant  $C > 0$  such that, for any tempered distributions  $u, v$  in  $\mathcal{S}'(\mathbb{R}^d)$ , we have*

$$\|[\Delta_q, u]v\|_{L^2} := \|\Delta_q(uv) - u\Delta_q v\|_{L^2} \leq C 2^{-q} \|\nabla u\|_{L^\infty} \|v\|_{L^2}.$$

We recall that, for any  $p, q \geq 1$ , the anisotropic Lebesgue spaces are defined as

$$\mathbf{L}_v^q \mathbf{L}_h^p(\mathbb{R}^3) = \mathbf{L}^q(\mathbb{R}; \mathbf{L}^p(\mathbb{R}_h^2)).$$



**Lemma 2.9.** *For any real numbers  $m, n, l \geq 1$  such that  $\frac{1}{m} = \frac{1}{n} + \frac{1}{l}$ , there exists a constant  $C > 0$  such that, for any tempered distributions  $u, v$  in  $\mathcal{S}'(\mathbb{R}^3)$ , we have*

$$\|[\Delta_q^v, u]v\|_{L_v^2 L_h^m} = \|\Delta_q^v(uv) - u\Delta_q^v v\|_{L_v^2 L_h^m} \leq C2^{-q} \|\partial_3 u\|_{L_v^\infty L_h^n} \|v\|_{L_v^2 L_h^l}.$$

In [10], Chemin, Desjardins, Gallagher and Grenier proved the following lemma.

**Lemma 2.10.** *Let  $s > \frac{1}{2}$ . There exists a constant  $C > 0$  such that, for any divergence-free vector fields  $u$  and  $v$  in  $\mathbf{H}^{0,s}(\mathbb{R}^3)$ , the horizontal gradients of which also belong to  $\mathbf{H}^{0,s}(\mathbb{R}^3)$ , we have*

$$\begin{aligned} |\langle \Delta_q^v(u \cdot \nabla v) | \Delta_q^v v \rangle| &\leq Cd_q 2^{-2qs} \left( \|\nabla_h u\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)} \|v\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)} \|\nabla_h v\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)} \right. \\ &\quad \left. + \|u\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla_h u\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)}^{\frac{1}{2}} \|v\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla_h v\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)}^{\frac{3}{2}} \right), \end{aligned}$$

where  $\{d_q\}$  is a summable sequence of positive numbers.

### 3. ESTIMATES FOR THE 2D LIMITING SYSTEM

In this section, we give useful auxiliary results concerning the 2D limiting system (1.5). Throughout this paper, for any vector field  $\bar{u} = (\bar{u}^1, \bar{u}^2, \bar{u}^3)$  independent of the vertical variable  $x_3$ , we denote by  $\bar{w}$  the associated horizontal vorticity,  $\bar{w} = \partial_1 \bar{u}^2 - \partial_2 \bar{u}^1$ . The first result of this section is the following lemma

**Lemma 3.1.** *Let  $\bar{u}_0 = (\bar{u}_0^1(x_1, x_2), \bar{u}_0^2(x_1, x_2), \bar{u}_0^3(x_1, x_2)) \in \mathbf{L}^2(\mathbb{R}_h^2)$  be a divergence-free vector field, the horizontal vorticity of which*

$$\bar{w}_0 = \partial_1 \bar{u}_0^2 - \partial_2 \bar{u}_0^1 \in L^2(\mathbb{R}_h^2) \cap L^\infty(\mathbb{R}_h^2).$$

*Then, the system (1.5), with initial data  $\bar{u}_0$ , has a unique, global solution*

$$\bar{u} \in \mathbf{C}(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2)) \cap \mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2)).$$

*Moreover,*

(i) *There exists a constant  $C > 0$  such that, for any  $p \geq 2$  and for any  $t > 0$ , we have*

$$(3.1) \quad \|\nabla_h \bar{u}^h(t)\|_{\mathbf{L}^p(\mathbb{R}_h^2)} \leq CMp e^{-\gamma t},$$

$$(3.2) \quad \|\bar{u}^h(t)\|_{\mathbf{L}^p(\mathbb{R}_h^2)} \leq CM e^{-\gamma t},$$

*where*

$$M = \max \left\{ \|\bar{u}_0^h\|_{\mathbf{L}^2(\mathbb{R}_h^2)}, \|\bar{w}_0\|_{L^2(\mathbb{R}_h^2)}, \|\bar{w}_0\|_{L^\infty(\mathbb{R}_h^2)} \right\}.$$

(ii) *For any  $p \geq 2$ , if  $\bar{u}_0^3 \in \mathbf{L}^p(\mathbb{R}_h^2)$ , then,*

$$(3.3) \quad \|\bar{u}^3(t)\|_{\mathbf{L}^p(\mathbb{R}_h^2)} \leq \|\bar{u}_0^3\|_{\mathbf{L}^p(\mathbb{R}_h^2)} e^{-\gamma t}.$$

#### Proof

We remark that in (1.5), the first two components of  $\bar{u}$  verify a two-dimensional Euler system with damping term. Then, according to the Yudovitch theorem [44] (see also [8]), this system has a unique solution  $\bar{u}^h \in \mathbf{C}(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2)) \cap \mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2))$  such that the horizontal vorticity  $\bar{w} \in \mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2)) \cap \mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^\infty(\mathbb{R}_h^2))$ . Since the third component  $\bar{u}^3$  satisfies a linear transport-type equation, we can deduce the existence and uniqueness of the solution  $\bar{u}$  of the limiting system (1.5).

By definition, the horizontal vorticity  $\bar{w}$  satisfies the following equation

$$(3.4) \quad \partial_t \bar{w} + \gamma \bar{w} + \bar{u}^h \cdot \nabla_h \bar{w} = 0$$

Taking the  $L^2$  scalar product of (3.4) with  $|\bar{w}|^{p-2} \bar{w}$ , we get

$$\frac{1}{p} \frac{d}{dt} \|\bar{w}\|_{\mathbf{L}^p(\mathbb{R}_h^2)}^p + \gamma \|\bar{w}\|_{\mathbf{L}^p(\mathbb{R}_h^2)}^p = 0.$$

By using an interpolation between  $L^2(\mathbb{R}_h^2)$  and  $L^\infty(\mathbb{R}_h^2)$ , we deduce from the above equation that

$$(3.5) \quad \|\bar{w}(t)\|_{\mathbf{L}^p(\mathbb{R}_h^2)} \leq \|\bar{w}_0\|_{\mathbf{L}^p(\mathbb{R}_h^2)} e^{-\gamma t} \leq C \|\bar{w}_0\|_{L^2(\mathbb{R}_h^2)}^{\frac{2}{p}} \|\bar{w}_0\|_{L^\infty(\mathbb{R}_h^2)}^{1-\frac{2}{p}} e^{-\gamma t} \leq CM e^{-\gamma t}.$$

We recall that  $\nabla_h \bar{u}^h = R\bar{w}$  where  $R$  is an homogeneous Calderón-Zygmund operator of order 0. According to [[8], Theorem 3.1.1], for any  $p > 1$  and for any  $t > 0$ ,

$$\|\nabla_h \bar{u}^h(t)\|_{L^p(\mathbb{R}_h^2)} \leq C \frac{p^2}{p-1} \|\bar{w}(t)\|_{L^p(\mathbb{R}_h^2)},$$

which implies (3.1), for any  $p \geq 2$ . In particular, we have

$$(3.6) \quad \|\bar{u}^h(t)\|_{\dot{\mathbf{H}}^1(\mathbb{R}_h^2)} = \|\nabla_h \bar{u}^h(t)\|_{\mathbf{L}^2(\mathbb{R}_h^2)} = \|\bar{w}(t)\|_{L^2(\mathbb{R}_h^2)} \leq CM e^{-\gamma t}.$$

Next, we remark that Inequality (3.2), in the case where  $p = 2$  is a direct consequence of the energy estimate for the damped Euler systems. So, using the Sobolev embedding  $\dot{\mathbf{H}}^{\frac{p-2}{p}}(\mathbb{R}_h^2) \hookrightarrow \mathbf{L}^p(\mathbb{R}_h^2)$  and the fact that  $\dot{\mathbf{H}}^{\frac{p-2}{p}}(\mathbb{R}_h^2)$  is an interpolated space between  $\mathbf{L}^2(\mathbb{R}_h^2)$  and  $\dot{\mathbf{H}}^1(\mathbb{R}_h^2)$ , we deduce (3.2) from Estimate (3.6), when  $p \geq 2$ .

Since the vertical component  $\bar{u}^3$  satisfies the same linear transport equation as  $\bar{w}$ , Inequality (3.3) can be proved in the same way as (3.5).  $\square$

In (3.1), we obtained a control of  $\nabla_h \bar{u}^h$  in the  $\mathbf{L}^p$ -norm, for any  $2 \leq p < +\infty$ . However, we need a  $\mathbf{L}^\infty$ -estimate of  $\nabla_h \bar{u}^h$  for the proof of Theorem 1.2. To this end, we first prove the following estimate.

**Lemma 3.2.** *Let  $r > 1$ . There is a constant  $C > 0$  such that, for any divergence-free vector fields  $\bar{u}$  in  $\mathbf{L}^2(\mathbb{R}_h^2)$ , the horizontal gradient  $\nabla_h \bar{u}^h \in \mathbf{L}^\infty(\mathbb{R}_h^2)$  and the vorticity  $\bar{w} = \partial_1 \bar{u}^2 - \partial_2 \bar{u}^1 \in H^r(\mathbb{R}_h^2)$ , we have*

$$(3.7) \quad |\langle \Delta_q^h(\bar{u}^h \cdot \nabla_h \bar{w}) \mid \Delta_q^h \bar{w} \rangle| \leq Cd_q 2^{-2qr} (\|\nabla_h \bar{u}^h\|_{\mathbf{L}^\infty} + \|\bar{w}\|_{L^\infty}) \|\bar{w}\|_{H^r}.$$

where  $(d_q)_{q \geq -1}$  is a summable sequence of positive numbers.

### Proof

First of all, using the Bony decomposition into paraproducts and remainders (see [5], [13] or [10]), we can write

$$(3.8) \quad |\langle \Delta_q^h(\bar{u}^h \cdot \nabla_h \bar{w}) \mid \Delta_q^h \bar{w} \rangle| \leq I_q^1 + I_q^2,$$

where

$$I_q^1 = \left| \left\langle \Delta_q^h \left( \sum_{q'-q \geq N} S_{q'+2}^h(\nabla_h \bar{w}) \Delta_{q'}^h \bar{u}^h \right) \mid \Delta_q^h \bar{w} \right\rangle \right|,$$

and

$$I_q^2 = \left| \left\langle \Delta_q^h \left( \sum_{|q'-q| \leq N} S_{q'-1}^h \bar{u}^h \Delta_{q'}^h(\nabla_h \bar{w}) \right) \mid \Delta_q^h \bar{w} \right\rangle \right|,$$

and where  $N > 0$  is a fixed large enough integer.

Using Cauchy-Schwarz and Hölder inequalities, we get

$$(3.9) \quad I_q^1 \leq \sum_{q'-q \geq N} \|S_{q'+2}^h(\nabla_h \bar{w})\|_{\mathbf{L}^\infty} \|\Delta_{q'}^h \bar{u}^h\|_{\mathbf{L}^2} \|\Delta_q^h \bar{w}\|_{L^2}.$$

The Bernstein lemma 2.1 implies that

$$\|S_{q'+2}^h(\nabla_h \bar{w})\|_{\mathbf{L}^\infty} \leq C2^{q'} \|S_{q'+2}^h \bar{w}\|_{L^\infty} \leq C2^{q'} \|\bar{w}\|_{L^\infty},$$

and

$$\|\Delta_{q'}^h \bar{u}^h\|_{\mathbf{L}^2} \leq C2^{-q'} \|\Delta_{q'}^h \nabla_h \bar{u}^h\|_{\mathbf{L}^2}.$$

Taking into account the fact that  $\bar{w} = \partial_1 \bar{u}^2 - \partial_2 \bar{u}^1$  and  $\operatorname{div}_h \bar{u}^h = 0$ , using an integration by parts, we easily obtain

$$\begin{aligned} \|\Delta_{q'}^h \bar{w}\|_{L^2}^2 &= \|\Delta_{q'}^h \partial_1 \bar{u}^2\|_{L^2}^2 + \|\Delta_{q'}^h \partial_2 \bar{u}^1\|_{L^2}^2 - 2 \langle \Delta_{q'}^h \partial_1 \bar{u}^2 \mid \Delta_{q'}^h \partial_2 \bar{u}^1 \rangle \\ &= \|\Delta_{q'}^h \partial_1 \bar{u}^2\|_{L^2}^2 + \|\Delta_{q'}^h \partial_2 \bar{u}^1\|_{L^2}^2 + \|\Delta_{q'}^h \partial_1 \bar{u}^1\|_{L^2}^2 + \|\Delta_{q'}^h \partial_2 \bar{u}^2\|_{L^2}^2 \\ &= \|\Delta_{q'}^h \nabla_h \bar{u}^h\|_{\mathbf{L}^2}^2. \end{aligned}$$

Therefore, using Proposition 2.3, we deduce from (3.9) that

$$\begin{aligned} I_q^1 &\leq C \sum_{q'-q \geq N} \|\bar{w}\|_{L^\infty} \|\Delta_{q'}^h \bar{w}\|_{L^2} \|\Delta_q^h \bar{w}\|_{L^2} \\ &\leq C \left( c_q \sum_{q'-q \geq N} \tilde{c}_{q'} 2^{-(q'-q)r} \right) 2^{-2qr} \|\bar{w}\|_{L^\infty} \|\bar{w}\|_{H^r}^2, \end{aligned}$$

where  $(c_q)_{q \geq -1}$  and  $(\tilde{c}_{q'})_{q' \geq -1}$  are square-summable sequences of positive numbers. Let

$$d_q = c_q \sum_{q'-q \geq N} \tilde{c}_{q'} 2^{-(q'-q)r}.$$

It is easy to prove that  $(d_q)_{q \geq -1}$  is a summable sequence of positive numbers. Thus, we have

$$(3.10) \quad I_q^1 \leq C d_q 2^{-2qr} \|\bar{w}\|_{L^\infty} \|\bar{w}\|_{H^r}^2.$$

In order to estimate  $I_q^2$ , we write

$$(3.11) \quad I_q^2 \leq I_A + I_B + I_C,$$

where

$$\begin{aligned} I_A &= |\langle S_q^h \bar{u}^h \cdot \nabla_h \Delta_q^h \bar{w} \mid \Delta_q^h \bar{w} \rangle|, \\ I_B &= \left| \left\langle \sum_{|q'-q| \leq N} (S_q^h - S_{q'-1}^h) \bar{u}_h \cdot \nabla_h \Delta_{q'}^h \bar{w} \mid \Delta_q^h \bar{w} \right\rangle \right|, \end{aligned}$$

and

$$I_C = \left| \left\langle \sum_{|q'-q| \leq N} [\Delta_q^h, S_{q'-1}^h \bar{u}_h] \nabla_h \Delta_{q'}^h \bar{w} \mid \Delta_q^h \bar{w} \right\rangle \right|.$$

Using an integration by parts, we get

$$(3.12) \quad I_A = \frac{1}{2} |\langle S_q^h (\operatorname{div}_h \bar{u}^h) \Delta_q^h \bar{w} \mid \Delta_q^h \bar{w} \rangle| = 0.$$

We remark that when  $q, q' \geq 1$ ,  $S_q^h - S_{q'-1}^h$  does not contain the low Fourier frequencies. So,  $I_B$  can be bounded in the same way as  $I_q^1$ , and we have

$$(3.13) \quad I_B \leq C d_q 2^{-2qr} \|\bar{w}\|_{L^\infty} \|\bar{w}\|_{H^r}^2.$$

Finally, using Lemma 2.8, we obtain

$$\begin{aligned} (3.14) \quad I_C &\leq C 2^{-q} \sum_{|q'-q| \leq N} \|S_{q'-1}^h \nabla_h \bar{u}^h\|_{\mathbf{L}^\infty} \|\nabla_h \Delta_{q'}^h \bar{w}\|_{\mathbf{L}^2} \|\Delta_q^h \bar{w}\|_{L^2} \\ &\leq C 2^{-2qr} \|\nabla_h \bar{u}^h\|_{\mathbf{L}^\infty} \|\bar{w}\|_{H^r}^2 \left( c_q \sum_{|q'-q| \leq N} c_{q'} 2^{-(q'-q)(r-1)} \right) \\ &\leq C d_q 2^{-2qr} \|\nabla_h \bar{u}^h\|_{\mathbf{L}^\infty} \|\bar{w}\|_{H^r}^2. \end{aligned}$$

From (3.10)-(3.14), we easily deduce (3.7) and Lemma 3.2 is proved.  $\square$

Next, we prove the following Brezis-Gallouet type inequality (see [6] and also [8]).

**Lemma 3.3.** *Let  $r > 1$ . There exists a constant  $C_r > 0$  such that, for any divergence-free vector fields  $\bar{u}$  in  $\mathbf{L}^2(\mathbb{R}_h^2)$ , the horizontal gradient  $\nabla_h \bar{u}^h \in \mathbf{L}^\infty(\mathbb{R}_h^2)$  and the vorticity  $\bar{w} = \partial_1 \bar{u}^2 - \partial_2 \bar{u}^1 \in H^r(\mathbb{R}_h^2)$ , we have*

$$(3.15) \quad \|\nabla_h \bar{u}^h\|_{\mathbf{L}^\infty(\mathbb{R}_h^2)} \leq C_r \|\bar{w}\|_{L^\infty(\mathbb{R}_h^2)} \ln \left( e + \frac{\|\bar{w}\|_{H^r(\mathbb{R}_h^2)}}{\|\bar{w}\|_{L^\infty(\mathbb{R}_h^2)}} \right).$$

**Proof**

Using Lemma 2.1, for any  $q \geq 1$  and for any  $i, j \in \{1, 2\}$ , we have

$$\|\partial_i \partial_j \Delta^{-1} \Delta_q^h \bar{w}\|_{L^\infty} \leq C \|\Delta_q^h \bar{w}\|_{L^\infty}.$$

Therefore, we can write

$$\|\partial_i \partial_j \Delta^{-1} \bar{w}\|_{L^\infty} \leq C \sum_{-1 \leq q \leq N_0 - 1} \|\Delta_q^h \bar{w}\|_{L^\infty} + C \sum_{q \geq N_0} \|\Delta_q^h \bar{w}\|_{L^\infty},$$

where  $N_0$  will be made more precise below.

Using the definition of Hölderian norms and Lemma 2.1, we obtain

$$\begin{aligned} \|\partial_i \partial_j \Delta^{-1} \bar{w}\|_{L^\infty} &\leq C(N_0 + 1) \|\bar{w}\|_{L^\infty} + C \sum_{q \geq N_0} 2^{-q(r-1)} 2^{q(r-1)} \|\Delta_q^h \bar{w}\|_{L^\infty} \\ &\leq C(N_0 + 1) \|\bar{w}\|_{L^\infty} + C \|\bar{w}\|_{C^{r-1}} \frac{2^{-(N_0-1)(r-1)}}{2^{r-1} - 1}. \end{aligned}$$

Choosing

$$N_0 = 1 + \left\lceil \frac{1}{r-1} \log_2 \frac{\|\bar{w}\|_{C^{r-1}}}{\|\bar{w}\|_{L^\infty}} \right\rceil,$$

where  $\lceil X \rceil$  denotes the largest integer smaller than or equal to  $X$ , we deduce from the above inequality that

$$\|\partial_i \partial_j \Delta^{-1} \bar{w}\|_{L^\infty} \leq C \|\bar{w}\|_{L^\infty} \left( 2 + \frac{1}{r-1} \log_2 \frac{\|\bar{w}\|_{C^{r-1}}}{\|\bar{w}\|_{L^\infty}} \right) + \frac{C}{2^{r-1} - 1} \|\bar{w}\|_{L^\infty}.$$

Thus, we can show the existence of a positive constant  $C_r$  such that

$$(3.16) \quad \|\partial_i \partial_j \Delta^{-1} \bar{w}\|_{L^\infty} \leq C_r \|\bar{w}\|_{L^\infty} \ln \left( e + \frac{\|\bar{w}\|_{C^{r-1}}}{\|\bar{w}\|_{L^\infty}} \right).$$

We recall the embedding  $H^r(\mathbb{R}_h^2) \hookrightarrow C^{r-1}(\mathbb{R}_h^2)$  and we remark that  $x \ln(e + \frac{\alpha}{x})$  is an increasing function on  $\mathbb{R}_+$ , for any  $\alpha \geq 0$ . So, using the Biot-Savart law, we finally deduce from (3.16) that

$$\|\nabla_h \bar{u}^h\|_{\mathbf{L}^\infty} \leq C_r \|\bar{w}\|_{L^\infty} \ln \left( e + \frac{\|\bar{w}\|_{H^r}}{\|\bar{w}\|_{L^\infty}} \right). \quad \square$$

**Lemma 3.4.** *Let  $r > 1$ . Under the hypotheses of Lemma 3.1 and the additional hypothesis that  $\bar{w}_0$  belongs to  $H^r(\mathbb{R}_h^2)$ , there exist positive constants  $C_1 = C_1(\gamma, \|\bar{w}_0\|_{H^r(\mathbb{R}^2)})$  and  $C_2 = C_2(\gamma, \|\bar{w}_0\|_{H^r(\mathbb{R}^2)})$  such that*

$$(3.17) \quad \|\bar{w}(t)\|_{H^r(\mathbb{R}_h^2)} \leq C_1 e^{-\gamma t},$$

and

$$(3.18) \quad \|\nabla_h \bar{u}^h(t)\|_{\mathbf{L}^\infty(\mathbb{R}_h^2)} \leq C_2 e^{-\gamma t}.$$

### Proof

For any  $r > 1$ , using Lemma 3.2, we get the following energy estimate in the Sobolev  $H^r$ -norm:

$$(3.19) \quad \frac{1}{2} \frac{d}{dt} \|\bar{w}(t)\|_{H^r}^2 + \gamma \|\bar{w}(t)\|_{H^r}^2 \leq C (\|\bar{w}(t)\|_{L^\infty} + \|\nabla_h \bar{u}^h(t)\|_{\mathbf{L}^\infty}) \|\bar{w}(t)\|_{H^r}^2.$$

Taking into account Estimate (3.15), we rewrite (3.19) as follows

$$(3.20) \quad \frac{d}{dt} \|\bar{w}(t)\|_{H^r} + \gamma \|\bar{w}(t)\|_{H^r} \leq C \|\bar{w}\|_{L^\infty} \left( 1 + \ln \left( e + \frac{\|\bar{w}\|_{H^r}}{\|\bar{w}\|_{L^\infty}} \right) \right) \|\bar{w}(t)\|_{H^r}.$$

From (3.5), we have  $\|\bar{w}(t)\|_{L^p} \leq CM e^{-\gamma t}$ , where  $C$  is a positive constant and

$$M = \max \left\{ \|\bar{u}_0^h\|_{\mathbf{L}^2(\mathbb{R}_h^2)}, \|\bar{w}_0\|_{L^2(\mathbb{R}_h^2)}, \|\bar{w}_0\|_{L^\infty(\mathbb{R}_h^2)} \right\}.$$

Since  $C$  and  $M$  do not depend on  $p$ , we have

$$\|\bar{w}(t)\|_{L^\infty} \leq CM e^{-\gamma t}.$$

Therefore, considering  $y(t) = \|\bar{w}(t)\|_{H^r} e^{\gamma t}$ , we can deduce from (3.20) that

$$(3.21) \quad \frac{d}{dt} y(t) \leq CM e^{-\gamma t} y(t) [1 + \ln(e + y(t))].$$

Integrating (3.21) with respect to  $t$ , we obtain

$$\ln(1 + \ln(e + \|\bar{w}(t)\|_{H^r} e^{\gamma t})) \leq \ln(1 + \ln(e + \|\bar{w}_0\|_{H^r})) + \frac{CM}{\gamma}.$$

Choosing

$$C_1 = (e + \|\bar{w}_0\|_{H^r})^{2e \frac{CM}{\gamma}},$$

we get

$$\|\bar{w}(t)\|_{H^r} \leq C_1 e^{-\gamma t}.$$

Combining the above estimate with (3.15) and using again the fact that  $x \ln(e + \frac{\alpha}{x})$  is an increasing function, we obtain the existence of a positive constant  $C_2$ , depending on  $\gamma$  and  $\|\bar{w}_0\|_{H^r}$ , such that

$$\|\nabla_h \bar{u}^h(t)\|_{\mathbf{L}^\infty} \leq C_2 e^{-\gamma t}. \quad \square$$

Until now, we proved estimates concerning the first two components of the solution  $\bar{u}$  of the system (1.5). In what follows, we wish to prove an estimate similar to (3.18) for the third component  $\bar{u}^3$  of  $\bar{u}$ . We first need a slightly more general version of Lemma 3.2, in the case where  $0 < r < 1$  and in the case where  $w$  is an arbitrary function in  $H^r(\mathbb{R}^2)$ .

**Lemma 3.5.** *Let  $0 < r < 1$ . There exists a constant  $C > 0$  such that, for any divergence-free vector field  $\bar{u}$  in  $\mathbf{L}^2(\mathbb{R}_h^2)$ , the horizontal gradient  $\nabla_h \bar{u}^h \in \mathbf{L}^\infty(\mathbb{R}_h^2)$ , and for any function  $w \in H^r(\mathbb{R}^2)$ , we have*

$$(3.22) \quad |\langle \Delta_q^h(\bar{u}^h \cdot \nabla_h w) \mid \Delta_q^h w \rangle| \leq C d_q 2^{-2qr} \|\nabla_h \bar{u}^h\|_{\mathbf{L}^\infty} \|w\|_{H^r}^2,$$

where  $(d_q)_{q \geq -1}$  is a summable sequence of positive numbers.

**Proof**

As in the proof of Lemma 3.2, performing the Bony decomposition into paraproducts and remainders yields

$$(3.23) \quad |\langle \Delta_q^h(\bar{u}^h \cdot \nabla_h w) \mid \Delta_q^h w \rangle| \leq I_1 + I_2,$$

where

$$I_1 = \left| \left\langle \Delta_q^h \left( \sum_{q'-q \geq N} S_{q'+2}^h(\nabla_h w) \Delta_{q'}^h \bar{u}^h \right) \mid \Delta_q^h w \right\rangle \right|,$$

and

$$I_2 = \left| \left\langle \Delta_q^h \left( \sum_{|q'-q| \leq N} S_{q'-1}^h \bar{u}^h \Delta_{q'}^h(\nabla_h w) \right) \mid \Delta_q^h w \right\rangle \right|.$$

Since  $0 < r < 1$ , we have  $\|S_{q'+2}^h(\nabla_h w)\|_{\mathbf{L}^2} \leq C 2^{q'(1-r)} \|w\|_{H^r}$ . Then, using Bernstein Lemma 2.1, Proposition 2.3, Cauchy-Schwarz and Hölder inequalities, we get

$$\begin{aligned} I_1 &\leq \sum_{q'-q \geq N} \|S_{q'+2}^h(\nabla_h w)\|_{\mathbf{L}^2} \|\Delta_{q'}^h \bar{u}^h\|_{\mathbf{L}^\infty} \|\Delta_q^h w\|_{L^2} \\ &\leq C d_q 2^{-2qr} \|\nabla_h \bar{u}^h\|_{\mathbf{L}^\infty} \|w\|_{H^r}^2, \end{aligned}$$

where

$$d_q = c_q \sum_{q'-q \geq N} c_{q'} 2^{-(q'-q)r}.$$

Next, we decompose  $I_2$  into three parts

$$I_2 \leq I_{2A} + I_{2B} + I_{2C},$$

where

$$I_{2A} = |\langle S_q^h \bar{u}^h \cdot \nabla_h \Delta_q^h w \mid \Delta_q^h w \rangle|,$$

$$I_{2B} = \left| \left\langle \sum_{|q'-q| \leq N} (S_q^h - S_{q'-1}^h) \bar{u}^h \cdot \nabla_h \Delta_{q'}^h w \mid \Delta_q^h w \right\rangle \right|,$$

and

$$I_{2C} = \left| \left\langle \sum_{|q'-q| \leq N} [\Delta_q^h, S_{q'-1}^h \bar{u}^h] \nabla_h \Delta_{q'}^h w \mid \Delta_q^h w \right\rangle \right|.$$

Then, an integration by parts implies that  $I_{2A} = 0$  and since  $S_q^h - S_{q'-1}^h$  does not contain the low Fourier frequencies, when  $q, q' \geq 1$ ,  $I_{2B}$  can be bounded in the same way as  $I_1$ . Finally, as in the proof of Lemma 3.2, using Lemma 2.8, we obtain

$$\begin{aligned} I_{2C} &\leq C2^{-q} \sum_{|q'-q| \leq N} \|S_{q'-1}^h \nabla_h \bar{u}^h\|_{\mathbf{L}^\infty} \|\nabla_h \Delta_{q'}^h w\|_{\mathbf{L}^2} \|\Delta_q^h w\|_{L^2} \\ &\leq C2^{-2qr} \|\nabla_h \bar{u}^h\|_{\mathbf{L}^\infty} \|w\|_{H^r}^2 \left( c_q \sum_{|q'-q| \leq N} c_{q'} 2^{-(q'-q)(r-1)} \right) \\ &\leq Cd_q 2^{-2qr} \|\nabla_h \bar{u}^h\|_{\mathbf{L}^\infty} \|w\|_{H^r}^2. \end{aligned}$$

Lemma 3.5 is then proved.  $\square$

**Lemma 3.6.** *Let  $2 < r < 3$  and  $\bar{u}(t, x)$  be a solution of (1.5), with initial data  $\bar{u}_0$  in  $\mathbf{H}^r(\mathbb{R}_h^2)$ . Then, there exist a positive constant  $C_3$ , depending on  $\gamma$  and  $\|\bar{u}_0\|_{\mathbf{H}^r(\mathbb{R}_h^2)}$  such that, for any  $t \geq 0$ ,*

$$(3.24) \quad \|\bar{u}^3(t)\|_{H^r(\mathbb{R}_h^2)} \leq C_3 e^{-\gamma t}.$$

**Proof**

Differentiating two times the equation verified by  $\bar{u}^3$ , for any  $i, j \in \{1, 2\}$ , we have

$$\begin{aligned} \partial_i \partial_i \partial_j \bar{u}^3 + \gamma \partial_i \partial_j \bar{u}^3 + (\partial_i \partial_j \bar{u}^h) \cdot \nabla_h \bar{u}^3 + (\partial_i \bar{u}^h) \cdot \nabla_h \partial_j \bar{u}^3 \\ + (\partial_j \bar{u}^h) \cdot \nabla_h \partial_i \bar{u}^3 + \bar{u}^h \cdot \nabla_h \partial_i \partial_j \bar{u}^3 = 0. \end{aligned}$$

Taking the  $H^{r-2}$  scalar product of the above equation with  $\partial_i \partial_j \bar{u}^3$ , we get

$$(3.25) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_i \partial_j \bar{u}^3\|_{H^{r-2}}^2 + \gamma \|\partial_i \partial_j \bar{u}^3\|_{H^{r-2}}^2 \\ \leq \left| \langle (\partial_i \partial_j \bar{u}^h) \cdot \nabla_h \bar{u}^3 \mid \partial_i \partial_j \bar{u}^3 \rangle_{H^{r-2}} \right| + \left| \langle (\partial_i \bar{u}^h) \cdot \nabla_h \partial_j \bar{u}^3 \mid \partial_i \partial_j \bar{u}^3 \rangle_{H^{r-2}} \right| \\ + \left| \langle (\partial_j \bar{u}^h) \cdot \nabla_h \partial_i \bar{u}^3 \mid \partial_i \partial_j \bar{u}^3 \rangle_{H^{r-2}} \right| + \left| \langle \bar{u}^h \cdot \nabla_h \partial_i \partial_j \bar{u}^3 \mid \partial_i \partial_j \bar{u}^3 \rangle_{H^{r-2}} \right|. \end{aligned}$$

The divergence-free property allows us to write

$$\begin{aligned} (\partial_i \partial_j \bar{u}^h) \cdot \nabla_h \bar{u}^3 &= \partial_i ((\partial_j \bar{u}^h) \cdot \nabla_h \bar{u}^3) - (\partial_j \bar{u}^h) \cdot \nabla_h \partial_i \bar{u}^3 \\ &= \partial_i ((\partial_j \bar{u}^h) \cdot \nabla_h \bar{u}^3) - \operatorname{div}_h (\partial_i \bar{u}^3 \partial_j \bar{u}^h). \end{aligned}$$

Then, using the Cauchy-Schwarz inequality, classical estimates in Sobolev spaces (see [[8], Theorem 2.4.1]) and the Sobolev embedding  $H^{r-1}(\mathbb{R}_h^2) \hookrightarrow L^\infty(\mathbb{R}_h^2)$ , we obtain

$$\begin{aligned} \left| \langle (\partial_i \partial_j \bar{u}^h) \cdot \nabla_h \bar{u}^3 \mid \partial_i \partial_j \bar{u}^3 \rangle_{H^{r-2}} \right| \\ \leq \|(\partial_j \bar{u}^h) \cdot \nabla_h \bar{u}^3\|_{H^{r-1}} \|\partial_i \partial_j \bar{u}^3\|_{H^{r-2}} + \|\partial_i \bar{u}^3 \partial_j \bar{u}^h\|_{\mathbf{H}^{r-1}} \|\partial_i \partial_j \bar{u}^3\|_{H^{r-2}} \\ \leq (\|\partial_j \bar{u}^h\|_{\mathbf{L}^\infty} \|\bar{u}^3\|_{H^r} + \|\nabla_h \bar{u}^3\|_{\mathbf{L}^\infty} \|\partial_j \bar{u}^h\|_{\mathbf{H}^{r-1}}) \|\bar{u}^3\|_{H^r} \\ + (\|\partial_i \bar{u}^3\|_{L^\infty} \|\partial_j \bar{u}^h\|_{\mathbf{H}^{r-1}} + \|\partial_j \bar{u}^h\|_{\mathbf{L}^\infty} \|\bar{u}^3\|_{H^r}) \|\bar{u}^3\|_{H^r} \\ \leq C \|\bar{u}\|_{H^{r-1}} \|\bar{u}^3\|_{H^r}^2. \end{aligned}$$

The same arguments imply

$$\begin{aligned} \left| \langle (\partial_i \bar{u}^h) \cdot \nabla_h \partial_j \bar{u}^3 \mid \partial_i \partial_j \bar{u}^3 \rangle_{H^{r-2}} \right| \\ \leq \|\operatorname{div}_h (\partial_j \bar{u}^3 \partial_i \bar{u}^h) - \partial_j \bar{u}^3 \partial_i (\operatorname{div}_h \bar{u}^h)\|_{H^{r-2}} \|\partial_i \partial_j \bar{u}^3\|_{H^{r-2}} \\ \leq (\|\partial_j \bar{u}^3\|_{L^\infty} \|\partial_i \bar{u}^h\|_{\mathbf{H}^{r-1}} + \|\partial_i \bar{u}^h\|_{\mathbf{L}^\infty} \|\bar{u}^3\|_{H^r}) \|\bar{u}^3\|_{H^r} \\ \leq C \|\bar{u}\|_{H^{r-1}} \|\bar{u}^3\|_{H^r}^2, \end{aligned}$$

and likewise,

$$\left| \langle (\partial_j \bar{u}^h) \cdot \nabla_h \partial_i \bar{u}^3 \mid \partial_i \partial_j \bar{u}^3 \rangle_{H^{r-2}} \right| \leq C \|\bar{u}\|_{H^{r-1}} \|\bar{u}^3\|_{H^r}^2.$$

For the last term of (3.25), Lemma 3.5, with  $0 < r - 2 < 1$ , yields

$$\left| \langle \bar{u}^h \cdot \nabla_h \partial_i \partial_j \bar{u}^3 \mid \partial_i \partial_j \bar{u}^3 \rangle_{H^{r-2}} \right| \leq C \|\nabla_h \bar{u}^h\|_{\mathbf{L}^\infty} \|\partial_i \partial_j \bar{u}^3\|_{H^{r-2}}^2 \leq C \|\nabla_h \bar{u}^h\|_{\mathbf{L}^\infty} \|\bar{u}^3\|_{H^r}^2.$$

Multiplying (3.25) by  $e^{\gamma t}$ , then integrating the obtained equation with respect to the time variable and using Lemma 3.4, we get

$$\begin{aligned} \|\bar{u}^3(t)\|_{H^r} e^{\gamma t} &\leq \|\bar{u}_0^3\|_{H^r} + C \int_0^t (\|\nabla_h \bar{u}^h(\tau)\|_{\mathbf{L}^\infty} + \|\bar{w}(\tau)\|_{H^{r-1}}) \|\bar{u}^3(\tau)\|_{H^r} e^{\gamma \tau} d\tau \\ &\leq \|\bar{u}_0^3\|_{H^r} + C(C_1 + C_2) \int_0^t \|\bar{u}^3(\tau)\|_{H^r} d\tau. \end{aligned}$$

Thus, the Gronwall lemma allows us to obtain (3.24).  $\square$

In the previous section, we recalled Lemma 2.10 which allows us to control the term  $(u \cdot \nabla)v$ . However, when this term contains the solution of the limiting system (1.5), the estimate in Lemma 2.10 becomes much simpler.

**Lemma 3.7.** *Let  $s > \frac{1}{2}$  and  $v \in \mathbf{H}^{0,s}(\mathbb{R}^3)$  be a divergence-free vector field.*

(1) *For any  $q \geq -1$ , we have*

$$\langle \Delta_q^v((\bar{u} \cdot \nabla)v) | \Delta_q^v v \rangle = 0.$$

(2) *There exists a positive constant  $C$  and a summable sequence of positive numbers  $\{d_q\}$  such that*

$$|\langle \Delta_q^v((v \cdot \nabla)\bar{u}) | \Delta_q^v v \rangle| \leq C d_q 2^{-2qs} \|\nabla_h \bar{u}\|_{\mathbf{L}^\infty(\mathbb{R}_h^2)} \|v\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)}^2.$$

### Proof

We remark that since  $\bar{u} = \bar{u}(x_1, x_2)$  and  $v = v(x_1, x_2, x_3)$  belongs to  $\mathcal{S}'$ , we have  $\Delta_q^v(\bar{u}v) = \bar{u}\Delta_q^v v$ . Besides, we also have  $\operatorname{div}_h \bar{u}^h = 0$  and  $\partial_3 \bar{u} = 0$ . As a consequence, an integration by parts proves that

$$\langle \Delta_q^v((\bar{u} \cdot \nabla)v) | \Delta_q^v v \rangle = \langle (\bar{u} \cdot \nabla) \Delta_q^v v | \Delta_q^v v \rangle = 0.$$

For the second part of Lemma 3.7, using the same remark as above and using Hölder's inequality, we get

$$|\langle \Delta_q^v((v \cdot \nabla)\bar{u}) | \Delta_q^v v \rangle| = |\langle (\Delta_q^v v^h \cdot \nabla_h) \bar{u} | \Delta_q^v v \rangle| \leq \|\nabla_h \bar{u}\|_{\mathbf{L}^\infty(\mathbb{R}_h^2)} \|\Delta_q^v v\|_{\mathbf{L}^2(\mathbb{R}^3)}^2.$$

Using Proposition 2.7, we easily obtain the second estimate of Lemma 3.7.  $\square$

In the next paragraphs, we will not directly compare the system (1.1) with the limiting system (1.5) because of technical difficulties. Instead of (1.5), we consider the following system

$$(3.26) \quad \begin{cases} \partial_t \bar{u}^{\varepsilon,h} - \nu_h(\varepsilon) \Delta_h \bar{u}^{\varepsilon,h} + \gamma \bar{u}^{\varepsilon,h} + \bar{u}^{\varepsilon,h} \cdot \nabla_h \bar{u}^{\varepsilon,h} = -\nabla \bar{p}^\varepsilon \\ \partial_t \bar{u}^{\varepsilon,3} - \nu_h(\varepsilon) \Delta_h \bar{u}^{\varepsilon,3} + \gamma \bar{u}^{\varepsilon,3} + \bar{u}^{\varepsilon,h} \cdot \nabla_h \bar{u}^{\varepsilon,3} = 0 \\ \operatorname{div}_h \bar{u}^{\varepsilon,h} = 0 \\ \partial_3 \bar{u}^\varepsilon = 0 \\ \bar{u}^\varepsilon|_{t=0} = \bar{u}_0 \end{cases}$$

with  $\lim_{\varepsilon \rightarrow 0} \nu_h(\varepsilon) = 0$ , say  $\nu_h(\varepsilon) = \varepsilon^\alpha$ ,  $\alpha > 0$ .

**Proposition 3.8.** *Like the system (1.5), the system (3.26) has a unique, global solution*

$$\bar{u}^\varepsilon \in \mathbf{C}(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2)) \cap \mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2)) \cap \mathbf{L}^2(\mathbb{R}_+, \dot{\mathbf{H}}^1(\mathbb{R}_h^2)),$$

which also satisfies the properties of Lemmas 3.1, 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7.

In the following lemma, we will prove the convergence of  $\bar{u}^\varepsilon$  towards  $\bar{u}$  when  $\varepsilon$  goes to zero.

**Lemma 3.9.** *Suppose that  $\nu_h(\varepsilon)$  converges to 0 when  $\varepsilon$  goes to 0 and that  $\bar{u}_0 \in \mathbf{H}^\sigma(\mathbb{R}_h^2)$ ,  $\sigma > 2$ . Then,  $\bar{u}^\varepsilon$  converges towards the solution  $\bar{u}$  of (1.5) in  $\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2))$ , as  $\varepsilon$  goes to 0.*

### Proof

Using Lemma 3.1, for any  $t > 0$ , we have

$$(3.27) \quad \|\bar{u}(t)\|_{\mathbf{L}^2(\mathbb{R}_h^2)} \leq M e^{-\gamma t} \quad \text{and} \quad \|\bar{u}^\varepsilon(t)\|_{\mathbf{L}^2(\mathbb{R}_h^2)} \leq M e^{-\gamma t}.$$

Thus, for fixed  $\mu > 0$ , there exists  $T_\mu > 0$  such that, for any  $t \geq T_\mu$ ,

$$\|\bar{u}^\varepsilon(t)\|_{\mathbf{L}^2(\mathbb{R}_h^2)} + \|\bar{u}(t)\|_{\mathbf{L}^2(\mathbb{R}_h^2)} \leq \frac{\mu}{2}.$$

On the interval  $[0, T_\mu]$ , let  $v^\varepsilon = \bar{u}^\varepsilon - \bar{u}$ . Then,  $v^\varepsilon$  is a solution of the following system

$$\begin{cases} \partial_t v^{\varepsilon,h} - \nu_h(\varepsilon) \Delta_h v^{\varepsilon,h} + \gamma v^{\varepsilon,h} + \bar{u}^{\varepsilon,h} \cdot \nabla_h v^{\varepsilon,h} + v^{\varepsilon,h} \cdot \nabla_h \bar{u}^h = \nu_h(\varepsilon) \Delta_h \bar{u}^h - \nabla \tilde{p}, \\ \partial_t v^{\varepsilon,3} - \nu_h(\varepsilon) \Delta_h v^{\varepsilon,3} + \gamma v^{\varepsilon,3} + \bar{u}^{\varepsilon,h} \cdot \nabla_h v^{\varepsilon,3} + v^{\varepsilon,h} \cdot \nabla_h \bar{u}^3 = \nu_h(\varepsilon) \Delta_h \bar{u}^3, \\ \operatorname{div}_h v^{\varepsilon,h} = 0, \\ \partial_3 v^\varepsilon = 0, \\ v^\varepsilon|_{t=0} = 0. \end{cases}$$

Taking the  $\mathbf{L}^2$ -scalar product of the first two equations of the above system with  $v^{\varepsilon,h}$  and  $v^{\varepsilon,3}$  respectively, we get

$$\frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|_{\mathbf{L}^2}^2 + \nu_h(\varepsilon) \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2}^2 + \gamma \|v^\varepsilon\|_{\mathbf{L}^2}^2 \leq \nu_h(\varepsilon) \|\nabla_h \bar{u}\|_{\mathbf{L}^2} \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2} + |\langle v^{\varepsilon,h} \cdot \nabla_h \bar{u} | v^\varepsilon \rangle|.$$

Hence,

$$\frac{d}{dt} \|v^\varepsilon\|_{\mathbf{L}^2}^2 + \gamma \|v^\varepsilon\|_{\mathbf{L}^2}^2 \leq \nu_h(\varepsilon) \|\nabla_h \bar{u}\|_{\mathbf{L}^2}^2 + 2 \|\nabla_h \bar{u}\|_{\mathbf{L}^\infty} \|v^\varepsilon\|_{\mathbf{L}^2}^2.$$

Integrating the obtained inequality, we obtain

$$\|v^\varepsilon(t)\|_{\mathbf{L}^2}^2 \leq \frac{\nu_h(\varepsilon)}{\gamma} \|\nabla_h \bar{u}\|_{\mathbf{L}^\infty([0, T_\mu], \mathbf{L}^2)}^2 + 2 \int_0^t \|\nabla_h \bar{u}(\tau)\|_{\mathbf{L}^\infty} e^{-\gamma(t-\tau)} \|v^\varepsilon(\tau)\|_{\mathbf{L}^2}^2 d\tau.$$

Then, the Gronwall Lemma proves that, for any  $0 < t < T_\mu$ ,

$$\|v^\varepsilon(t)\|_{\mathbf{L}^2}^2 \leq C \nu_h(\varepsilon) M^2 T_\mu \exp \left\{ 2 \int_0^t \|\nabla_h \bar{u}(\tau)\|_{\mathbf{L}^\infty} d\tau \right\}.$$

Combining with (3.27), we deduce from the above estimate that

$$\lim_{\varepsilon \rightarrow 0} \|\bar{u}^\varepsilon - \bar{u}\|_{\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2))} = 0. \quad \square$$

#### 4. STRICHARTZ ESTIMATES

We next introduce the following frequency cut-off function:

$$(4.1) \quad \Psi(\xi) = \chi \left( \frac{|\xi|}{R} \right) \left[ 1 - \chi \left( \frac{2|\xi_h|}{r} \right) \right] \left[ 1 - \chi \left( \frac{2|\xi_3|}{r} \right) \right],$$

where  $\chi$  is a  $C^\infty$ -function from  $\mathbb{R}$  into  $\mathbb{R}$ , defined as follows

$$\chi(x) = \begin{cases} 1 & \text{si } 0 \leq |x| \leq 1 \\ 0 & \text{si } |x| \geq 2. \end{cases}$$

We remark that  $\Psi \in C^\infty(\mathbb{R}^3)$ ,  $\operatorname{supp} \Psi \subset \mathcal{C}_{\frac{r}{2}, 2R}$  and  $\Psi \equiv 1$  in  $\mathcal{C}_{r, R}$ , where

$$\mathcal{C}_{r_1, r_2} = \{ \xi \in \mathbb{R}^3 \mid |\xi_h| \geq r_1; |\xi_3| \geq r_1 \text{ and } |\xi| \leq r_2 \},$$

for any  $0 < r_1 < r_2$ .

In this section, we provide important dispersion results concerning the linear cut-off system

$$(4.2) \quad \begin{cases} \partial_t v_L - \nu_h(\varepsilon) \Delta_h v_L + \gamma v_L + \frac{\bar{v}_L \wedge e_3}{\varepsilon} = -\nabla p_L \\ \operatorname{div} v_L = 0 \\ v_L|_{t=0} = v_{0,L} = \mathcal{F}^{-1}(\Psi(\xi) \mathcal{F}(v_0)), \end{cases}$$

where  $\nu_h(\varepsilon) = \varepsilon^\alpha$ ,  $\alpha > 0$ , and where  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  denote the Fourier transform and its inverse. The main result of this section is the following

**Theorem 4.1.** *Let  $p \geq 1$ ,  $\beta \geq 1$ , the cut-off radii  $0 < r < R$  with  $r = R^{-\beta}$ . There exists a constant  $C = C(\gamma, p, \beta) > 0$  such that, for any initial data  $v_{0,L} \in \mathbf{L}^2(\mathbb{R}^3)$ , the solution  $v_L$  of the system (4.2) satisfies*

$$\|v_L\|_{\mathbf{L}^p((0, +\infty), \mathbf{L}_h^\infty \mathbf{L}_v^2)} \leq CR^{\frac{2+2p+3\beta}{2p}} \varepsilon^{\frac{1}{4p}} \|v_{0,L}\|_{\mathbf{L}^2}.$$



In order to prove this theorem, we follow the method of [10] and [32], and we write the system (4.2) in Fourier variables. Applying the Leray projection  $\mathbb{P}$  of  $\mathbf{L}^2(\mathbb{R}^3)$  onto the subspace of divergence-free vector fields and then the Fourier transformation, we have

$$\partial_t \widehat{v}_L = \mathbb{B}(\xi, \varepsilon) \widehat{v}_L,$$

where

$$\mathbb{B}(\xi, \varepsilon) = \begin{bmatrix} -\gamma - \nu_h(\varepsilon) |\xi_h|^2 + \frac{\xi_1 \xi_2}{\varepsilon |\xi|^2} & \frac{\xi_2^2 + \xi_3^2}{\varepsilon |\xi|^2} & 0 \\ -\frac{\xi_1^2 + \xi_2^2}{\varepsilon |\xi|^2} & -\gamma - \nu_h(\varepsilon) |\xi_h|^2 - \frac{\xi_1 \xi_2}{\varepsilon |\xi|^2} & 0 \\ \frac{\xi_2 \xi_3}{\varepsilon |\xi|^2} & -\frac{\xi_1 \xi_3}{\varepsilon |\xi|^2} & -\gamma - \nu_h(\varepsilon) |\xi_h|^2 \end{bmatrix}.$$

It is well known that  $\mathbb{B}(\xi, \varepsilon)$  possesses the following eigenvalues and their corresponding eigenvectors:

$$\begin{aligned} \lambda_0 &= -\gamma - \nu_h(\varepsilon) |\xi_h|^2 & \text{and} & & V_0 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\ \lambda_+ &= -\gamma - \nu_h(\varepsilon) |\xi_h|^2 + \frac{i\xi_3}{\varepsilon |\xi|} & \text{and} & & V_+ &= \frac{1}{|\xi|^2} \begin{pmatrix} -i\xi_3 |\xi| - \xi_1 \xi_2 \\ \xi_1^2 + \xi_3^2 \\ i\xi_1 |\xi| - \xi_2 \xi_3 \end{pmatrix}, \\ \lambda_- &= -\gamma - \nu_h(\varepsilon) |\xi_h|^2 - \frac{i\xi_3}{\varepsilon |\xi|} & \text{and} & & V_- &= \frac{1}{|\xi|^2} \begin{pmatrix} i\xi_3 |\xi| - \xi_1 \xi_2 \\ \xi_1^2 + \xi_3^2 \\ -i\xi_1 |\xi| - \xi_2 \xi_3 \end{pmatrix}. \end{aligned}$$

We recall that  $\{V_+, V_-\}$  forms an orthogonal basis of the space of the Fourier transforms of all divergence-free vector fields in  $\mathbf{L}^2(\mathbb{R}^3)$ . Thus, we can write

$$\widehat{v}_L(t) = C_+ e^{t\lambda_+} V_+ + C_- e^{t\lambda_-} V_-,$$

where  $C_i = \langle \widehat{v}_{0,L}, V_i \rangle$ , for any  $i \in \{\pm\}$ . As in [10] and [32], we consider the operators  $\mathcal{G}_\pm^\varepsilon(t)$ , which associate to any tempered distribution  $f$ :

$$\begin{aligned} \mathcal{G}_\pm^\varepsilon(t) f(x) &= \mathcal{F}^{-1} \left( e^{t\lambda_\mp} \widehat{f}(\xi) \right) (x) \\ &= \int_{\mathbb{R}_{y_h}^2} \mathcal{F}_{\xi_3}^{-1} \left[ \int_{\mathbb{R}_{\xi_h}^2} e^{t\lambda_\mp + i(x_h - y_h) \cdot \xi_h} \mathcal{F}_{x_3}(f)(y_h, \xi_3) d\xi_h \right] dy_h, \end{aligned}$$

Let  $a(\xi) = \frac{\xi_3}{|\xi|}$ . Then, we have

$$\Psi(D) \mathcal{G}_\pm^\varepsilon(t) f(x) = \int_{\mathbb{R}_{y_h}^2} \mathcal{F}_{\xi_3}^{-1} \left[ K_\pm \left( \frac{t}{\varepsilon}, t, \cdot, \xi_3 \right) *_h \mathcal{F}_{x_3}(f)(\cdot, \xi_3)(x_h) \right] dy_h,$$

where  $*_h$  is the convolution product in the horizontal direction, and where

$$(4.3) \quad K_\pm(\theta, t, x_h, \xi_3) = \int_{\mathbb{R}_{\xi_3}^2} \Psi(\xi) \exp \left\{ \mp i\theta a(\xi) + ix_h \cdot \xi_h - \gamma t - t\nu_h(\varepsilon) |\xi_h|^2 \right\} d\xi_3.$$

Therefore, Theorem 4.1 can be directly deduced from the following lemma, using the method of duality, called  $TT^*$ -method (see [10] or [32]).

**Lemma 4.2.** *Let  $r, R, \beta$  be as in Theorem 4.1 and let  $K_\pm$  be defined as in (4.3). Then*

$$\|K_\pm(\theta, t, \cdot, \cdot)\|_{L_{x_h}^\infty L_{\xi_3}^\infty} \leq CR^{4+3\beta} e^{-\frac{\gamma t}{2}} \theta^{-\frac{1}{2}}.$$

To prove Lemma 4.2, we use the method of nonstationary phase and we follow the lines of the proof of [[32], Lemma 3.2]. We also remark that unlike the case of [32], here, the real part of the eigenvalues of  $\mathbb{B}(\xi, \varepsilon)$  is  $-\gamma - \nu_h(\varepsilon) |\xi_h|^2$ , which is always bounded from above by  $-\gamma$ . That is the reason why we obtain a slightly better result than in [32]. We leave the details to the reader.

Finally, in the following lemma, we give a different version of Lemma 2.10 when the nonlinear term contains the solution  $v_L$  of the system (4.2). One can find a proof of it in [[32], Lemmas 4.4 and 4.5].

**Lemma 4.3.** *Let  $s > \frac{1}{2}$  and  $r, R, \beta$  be as in Theorem 4.1. There exists a constant  $C > 0$  such that, for any divergence-free vector fields  $v_1$  and  $v_2$  in  $\mathbf{H}^{0,s}(\mathbb{R}^3)$ , the horizontal gradients of which also belong to  $\mathbf{H}^{0,s}(\mathbb{R}^3)$ , we have*

$$(4.4) \quad \left| \langle \Delta_q^v(v_2 \cdot \nabla v_L) | \Delta_q^v v_1 \rangle \right| \leq C d_q 2^{-2qs} R^{s+1} \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \|v_1\|_{\mathbf{H}^{0,s}} \|v_2\|_{\mathbf{H}^{0,s}},$$

$$(4.5) \quad \left| \langle \Delta_q^v(v_L \cdot \nabla v_1) | \Delta_q^v v_1 \rangle \right| \leq C d_q 2^{-2qs} R^s \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \|v_1\|_{\mathbf{H}^{0,s}} \|\nabla_h v_1\|_{\mathbf{H}^{0,s}} \\ + C d_q 2^{-2qs} R^{s+1} \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \|v_1\|_{\mathbf{H}^{0,s}}^2,$$

where  $\{d_q\}$  is a summable sequence of positive numbers.

## 5. GLOBAL EXISTENCE IN THE CASE OF ILL PREPARED DATA

This section is devoted to the proof of Theorem 1.2, that is, to the proof of the global existence of a unique solution of the system (1.1), in the case where the horizontal viscosity is  $\varepsilon^\alpha$ ,  $\alpha > 0$ .

$$(1.1) \quad \begin{cases} \partial_t u^\varepsilon - \varepsilon^\alpha \Delta_h u^\varepsilon + \gamma u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \frac{u^\varepsilon \wedge e_3}{\varepsilon} = -\nabla p^\varepsilon & \text{dans } \mathbb{R}^3 \times \mathbb{R}_+ \\ \operatorname{div} u^\varepsilon = 0 & \text{dans } \mathbb{R}^3 \times \mathbb{R}_+ \\ u^\varepsilon|_{t=0} = u_0 & \text{dans } \mathbb{R}^3. \end{cases}$$

In the introduction, we defined the spaces

$$\mathbf{Y}_{s,\eta} = \dot{\mathbf{H}}^{-\eta,s}(\mathbb{R}^3) \cap \mathbf{L}_h^2 \dot{\mathbf{H}}_v^{-\eta}(\mathbb{R}^3) \cap \mathbf{H}^{\eta,\eta+s}(\mathbb{R}^3),$$

where  $\eta > 0$ ,  $s > \frac{1}{2}$ . According to the hypothesis on the initial data  $u_0$ , we have

$$u_0 = \bar{u}_0 + v_0,$$

where

- (i)  $\bar{u}_0 = (\bar{u}_0^1(x_1, x_2), \bar{u}_0^2(x_1, x_2), \bar{u}_0^3(x_1, x_2))$  is a divergence-free vector field belonging to  $\mathbf{H}^\sigma(\mathbb{R}_h^2)$ ,  $\sigma > 2$ , such that  $\|\bar{u}_0\|_{\mathbf{H}^\sigma(\mathbb{R}_h^2)} \leq a_0$ ,
- (ii)  $v_0 = (v_0^1(x_1, x_2, x_3), v_0^2(x_1, x_2, x_3), v_0^3(x_1, x_2, x_3))$  is a divergence-free vector field belonging to  $\mathbf{H}^{0,s}(\mathbb{R}^3) \cap \mathbf{Y}_{s,\eta}$  such that  $\|v_0\|_{\mathbf{Y}_{s,\eta}} \leq b_0 \varepsilon^{-a}$ .

For the 3D part of the initial data, we write

$$v_0 = v_{0,L} + w_0,$$

where  $v_{0,L} = \mathcal{F}^{-1}(\Psi(\xi)\mathcal{F}v_0)$ , with  $\Psi$  being defined in (4.1), and where  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  stand for the Fourier transform and its inverse. Since  $v_0 \in \mathbf{H}^{0,s}(\mathbb{R}^3) \cap \mathbf{Y}_{s,\eta}$ , one can easily prove the following estimate.

**Lemma 5.1.** *Let  $r = R^{-\beta}$ ,  $\beta \geq 1$ ,  $s > \frac{1}{2}$  and  $\eta > 0$ . Then there exists a positive constant  $\bar{C}$  such that,*

$$\|w_0\|_{\mathbf{H}^{0,s}} \leq \bar{C} \|v_0\|_{\mathbf{Y}_{s,\eta}} R^{-\beta\eta}.$$

In the next step, we decompose the system (1.1) as follows

$$u^\varepsilon = \bar{u}^\varepsilon + v_L + w,$$

where  $\bar{u}^\varepsilon$  is the solution of the 2D system (3.26) and  $v_L$  is the solution of the cut-off linear system (4.2). Then,  $w$  satisfies the following nonlinear system

$$(5.1) \quad \begin{cases} \partial_t w - \varepsilon^\alpha \Delta_h w + \gamma w + Q(w) + \frac{w \wedge e_3}{\varepsilon} = -\nabla \tilde{p} \\ \operatorname{div} w = 0 \\ w|_{t=0} = w_0 = v_0 - v_{0,L}, \end{cases}$$

where

$$Q(w) = w \cdot \nabla w + w \cdot \nabla v_L + v_L \cdot \nabla w + v_L \cdot \nabla v_L \\ + \bar{u}^\varepsilon \cdot \nabla v_L + v_L \cdot \nabla \bar{u}^\varepsilon + \bar{u}^\varepsilon \cdot \nabla w + w \cdot \nabla \bar{u}^\varepsilon.$$

We already know (see [10] for instance) that there exists a unique local solution of (1.1) (and thus a unique local solution of (5.1)) in  $\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{H}^{0,s}(\mathbb{R}^3)) \cap \mathbf{C}(\mathbb{R}_+, \mathbf{H}^{0,s}(\mathbb{R}^3))$ , the horizontal gradient of which belongs to  $\mathbf{L}^2(\mathbb{R}_+, \mathbf{H}^{0,s}(\mathbb{R}^3))$ . Our goal is to prove that this solution exists globally in time.

### Proof of Theorem 1.2

Applying the operator  $\Delta_q^v$  to the first equation of the system (5.1) and then taking the  $\mathbf{L}^2$  scalar product of the obtained equation with  $\Delta_q^v w$ , we get

$$(5.2) \quad \frac{1}{2} \frac{d}{dt} \|\Delta_q^v w\|_{\mathbf{L}^2}^2 + \gamma \|\Delta_q^v w\|_{\mathbf{L}^2}^2 + \varepsilon^\alpha \|\Delta_q^v \nabla_h w\|_{\mathbf{L}^2}^2 \leq |\langle \Delta_q^v Q(w) | \Delta_q^v w \rangle|.$$

The definition of  $Q(w)$  implies that

$$(5.3) \quad |\langle \Delta_q^v Q(w) | \Delta_q^v w \rangle| \leq F_1 + F_2 + F_3 + F_4 + F_5,$$

where

$$\begin{aligned} F_1 &= |\langle \Delta_q^v (w \cdot \nabla w) | \Delta_q^v w \rangle|, \\ F_2 &= |\langle \Delta_q^v (v_L \cdot \nabla w) | \Delta_q^v w \rangle| + |\langle \Delta_q^v (w \cdot \nabla v_L) | \Delta_q^v w \rangle|, \\ F_3 &= |\langle \Delta_q^v (v_L \cdot \nabla v_L) | \Delta_q^v w \rangle|, \\ F_4 &= |\langle \Delta_q^v (\bar{u}^\varepsilon \cdot \nabla v_L) | \Delta_q^v w \rangle| + |\langle \Delta_q^v (v_L \cdot \nabla \bar{u}^\varepsilon) | \Delta_q^v w \rangle|, \\ F_5 &= |\langle \Delta_q^v (\bar{u}^\varepsilon \cdot \nabla w) | \Delta_q^v w \rangle| + |\langle \Delta_q^v (w \cdot \nabla \bar{u}^\varepsilon) | \Delta_q^v w \rangle|. \end{aligned}$$

According to Lemma 2.10, we have

$$(5.4) \quad F_1 \leq C d_q 2^{-2qs} \|w\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)} \|\nabla_h w\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)}^2.$$

Then, Lemma 4.3 gives

$$(5.5) \quad F_2 \leq C d_q 2^{-2qs} R^{s+1} \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \|w\|_{\mathbf{H}^{0,s}}^2 + C d_q 2^{-2qs} R^s \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \|w\|_{\mathbf{H}^{0,s}} \|\nabla_h w\|_{\mathbf{H}^{0,s}}.$$

Next, the energy estimate for the linear system (4.2) shows that,

$$\|v_L(t)\|_{\mathbf{H}^{0,s}} \leq \|v_0\|_{\mathbf{H}^{0,s}} \leq b_0 \varepsilon^{-a}, \quad \text{for any } t \geq 0.$$

Using Inequality (4.4) of Lemma 4.3, we get

$$(5.6) \quad F_3 = |\langle \Delta_q^v (v_L \cdot \nabla v_L) | \Delta_q^v w \rangle| \leq C d_q 2^{-2qs} R^{s+1} b_0 \varepsilon^{-a} \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \|w\|_{\mathbf{H}^{0,s}}.$$

As to evaluate  $F_4$ , we can not apply Lemma 4.3, because  $\bar{u}^\varepsilon$  is a two-dimensional vector fields. For the first term of  $F_4$ , since  $\partial_3 \bar{u}^{\varepsilon,3} = 0$ , we write

$$\langle \Delta_q^v (\bar{u}^\varepsilon \cdot \nabla v_L) | \Delta_q^v w \rangle = \langle \Delta_q^v \operatorname{div} (\bar{u}^\varepsilon \otimes v_L) | \Delta_q^v w \rangle + \langle \Delta_q^v ((\operatorname{div}_h \bar{u}^{\varepsilon,h}) v_L) | \Delta_q^v w \rangle.$$

We recall that  $\bar{u}^\varepsilon$  does not depend on  $x_3$ . So, in the vertical direction, the support of the Fourier transform of  $\bar{u}^\varepsilon \otimes v_L$  is included in  $[-R, R]$ . Considering that  $\operatorname{div}_h \bar{u}^{\varepsilon,h} = 0$ , Bernstein lemma 2.1, Inequalities of Cauchy-Schwarz and Hölder and Proposition 2.7 imply that

$$\begin{aligned} |\langle \Delta_q^v (\bar{u}^\varepsilon \cdot \nabla v_L) | \Delta_q^v w \rangle| &\leq C R \|\Delta_q^v (\bar{u}^\varepsilon \otimes v_L)\|_{\mathbf{L}^2} \|\Delta_q^v \nabla_h w\|_{\mathbf{L}^2} \\ &\leq C R d_q 2^{-2qs} \|\bar{u}^\varepsilon \otimes v_L\|_{\mathbf{H}^{0,s}} \|\nabla_h w\|_{\mathbf{H}^{0,s}} \\ &\leq C R^{s+1} d_q 2^{-2qs} \|\bar{u}^\varepsilon\|_{\mathbf{L}^2(\mathbb{R}_h^2)} \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \|\nabla_h w\|_{\mathbf{H}^{0,s}} \\ &\leq C R^{s+1} d_q 2^{-2qs} \|\bar{u}_0\|_{\mathbf{L}^2(\mathbb{R}_h^2)} \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \|\nabla_h w\|_{\mathbf{H}^{0,s}}. \end{aligned}$$

In the same way, we have

$$\begin{aligned} |\langle \Delta_q^v (v_L^h \cdot \nabla_h \bar{u}^\varepsilon) | \Delta_q^v w \rangle| &\leq C d_q 2^{-2qs} \|v_L^h \cdot \nabla_h \bar{u}^\varepsilon\|_{\mathbf{H}^{0,s}} \|w\|_{\mathbf{H}^{0,s}} \\ &\leq C R^s d_q 2^{-2qs} \|v_L^h \cdot \nabla_h \bar{u}^\varepsilon\|_{\mathbf{L}^2} \|w\|_{\mathbf{H}^{0,s}} \\ &\leq C R^s d_q 2^{-2qs} \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \|\nabla_h \bar{u}^\varepsilon\|_{\mathbf{L}^2(\mathbb{R}_h^2)} \|w\|_{\mathbf{H}^{0,s}} \\ &\leq C R^s d_q 2^{-2qs} \|\bar{u}_0\|_{\mathbf{H}^\sigma(\mathbb{R}_h^2)} \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \|w\|_{\mathbf{H}^{0,s}}. \end{aligned}$$

So,

$$(5.7) \quad \begin{aligned} F_4 &= |\langle \Delta_q^v (\bar{u}^\varepsilon \cdot \nabla v_L) | \Delta_q^v w \rangle| + |\langle \Delta_q^v (v_L \cdot \nabla \bar{u}^\varepsilon) | \Delta_q^v w \rangle| \\ &\leq C a_0 d_q 2^{-2qs} \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \left[ R^s \|w\|_{\mathbf{H}^{0,s}} + R^{s+1} \|\nabla_h w\|_{\mathbf{H}^{0,s}} \right]. \end{aligned}$$

Finally, we deduce from Lemma 3.7 and Proposition 3.8 that

$$(5.8) \quad F_5 \leq C d_q 2^{-2qs} \|\nabla_h \bar{u}^\varepsilon\|_{\mathbf{L}^\infty(\mathbb{R}_h^2)} \|w\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)}^2.$$

We now go back to the inequality (5.3). Replacing  $F_1, F_2, F_3, F_4$  and  $F_5$  by their upper bounds given in the inequalities (5.4) to (5.8), we obtain an inequality for  $|\langle \Delta_q^v Q(w) | \Delta_q^v w \rangle|$ . Multiplying the obtained inequality by  $2^{2qs}$ , then summing with respect to  $q$ , and combining that with (5.2), we get

$$(5.9) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|w\|_{\mathbf{H}^{0,s}}^2 + \gamma \|w\|_{\mathbf{H}^{0,s}}^2 + \varepsilon^\alpha \|\nabla_h w\|_{\mathbf{H}^{0,s}}^2 \\ & \leq C \|w\|_{\mathbf{H}^{0,s}} \| \nabla_h w \|_{\mathbf{H}^{0,s}}^2 + CR^{s+1} b_0 \varepsilon^{-a} \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \|w\|_{\mathbf{H}^{0,s}} \\ & \quad + CR^{s+1} \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \|w\|_{\mathbf{H}^{0,s}}^2 + CR^s \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \|w\|_{\mathbf{H}^{0,s}} \|\nabla_h w\|_{\mathbf{H}^{0,s}} \\ & \quad + Ca_0 \|v_L\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2} \left[ R^s \|w\|_{\mathbf{H}^{0,s}} + R^{s+1} \|\nabla_h w\|_{\mathbf{H}^{0,s}} \right] + C \|\nabla_h \bar{w}^\varepsilon\|_{\mathbf{L}^\infty(\mathbb{R}_h^2)} \|w\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)}. \end{aligned}$$

Without loss of generality, we can suppose that  $\varepsilon < 1$  and  $R > 1$ . Using Young's inequality and Lemmas 3.4 and 3.6, we obtain the existence of a constant  $C_0 = C_0(\gamma, a_0, b_0)$  such that

$$(5.10) \quad \begin{aligned} & \frac{d}{dt} \|w\|_{\mathbf{H}^{0,s}}^2 + 2\gamma \|w\|_{\mathbf{H}^{0,s}}^2 + \varepsilon^\alpha \|\nabla_h w\|_{\mathbf{H}^{0,s}}^2 \\ & \leq C \|w\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)} \|\nabla_h w\|_{\mathbf{H}^{0,s}(\mathbb{R}^3)}^2 + C_0 \Gamma(v_L) + C_0 (e^{-\gamma t} + \Gamma(v_L)) \|w\|_{\mathbf{H}^{0,s}}^2, \end{aligned}$$

where for any  $t \geq 0$ , we have set

$$\Gamma(v_L)(t) = R^{2s+2} \varepsilon^{-\alpha} \|v_L(t)\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2}^2 + R^{s+1} \varepsilon^{-a} \|v_L(t)\|_{\mathbf{L}_h^\infty \mathbf{L}_v^2}.$$

From now on, we choose

$$R = \left( 4C\bar{C}b_0 e^{\frac{C_0}{\gamma}} \right)^{\frac{1}{\beta\eta}} \varepsilon^{-\frac{\alpha+a}{\beta\eta}}.$$

We immediately deduce from Lemma 5.1 that

$$(5.11) \quad \|w_0\|_{\mathbf{H}^{0,s}} \leq \bar{C}b_0 \varepsilon^{-a} R^{-\beta\eta} = \frac{\varepsilon^\alpha e^{-\frac{C_0}{\gamma}}}{4C}.$$

Let

$$T^* = \sup \left\{ T \geq 0 \mid \|w(t)\|_{\mathbf{H}^{0,s}} \leq \frac{\varepsilon^\alpha}{2C}, \forall 0 \leq t \leq T \right\}.$$

Then, the continuity of the application  $t \mapsto \|w(t)\|_{\mathbf{H}^{0,s}}$  and Inequality (5.11) imply that  $T^* > 0$ . Integrating Equation (5.10) with respect to the time variable, for any  $0 \leq t < T^*$ , we get

$$\|w(t)\|_{\mathbf{H}^{0,s}}^2 \leq \|w_0\|_{\mathbf{H}^{0,s}}^2 + C_0 \int_0^t \Gamma(v_L)(\tau) d\tau + C_0 \int_0^t (e^{-\gamma\tau} + \Gamma(v_L)(\tau)) \|w(\tau)\|_{\mathbf{H}^{0,s}}^2 d\tau.$$

Using Gronwall lemma, we obtain

$$(5.12) \quad \|w(t)\|_{\mathbf{H}^{0,s}}^2 \leq \left( \|w_0\|_{\mathbf{H}^{0,s}}^2 + C_0 \int_0^{+\infty} \Gamma(v_L)(\tau) d\tau \right) \exp \left\{ \frac{C_0}{\gamma} + C_0 \int_0^{+\infty} \Gamma(v_L)(\tau) d\tau \right\}.$$

The choice of  $R$  and the Strichartz estimates proved in Theorem 4.1 allow us to write

$$\begin{aligned} \int_0^{+\infty} \Gamma(v_L)(\tau) d\tau &= R^{2s+2} \varepsilon^{-\alpha} \|v_L\|_{\mathbf{L}^2(\mathbb{R}_+, \mathbf{L}_h^\infty \mathbf{L}_v^2)}^2 + R^{s+1} \varepsilon^{-a} \|v_L\|_{\mathbf{L}^1(\mathbb{R}_+, \mathbf{L}_h^\infty \mathbf{L}_v^2)} \\ &\leq CR^{2s+2} \varepsilon^{-\alpha} R^{\frac{6+3\beta}{2}} \varepsilon^{\frac{1}{4}} \|v_{0,L}\|_{\mathbf{L}^2}^2 + CR^{s+1} \varepsilon^{-a} R^{\frac{4+3\beta}{2}} \varepsilon^{\frac{1}{4}} \|v_{0,L}\|_{\mathbf{L}^2} \\ &< \tilde{C} \varepsilon^{\frac{1}{4} - \alpha - 2a - \frac{(\alpha+a)(10+4s+3\beta)}{2\beta\eta}}, \end{aligned}$$

where we can choose

$$\tilde{C} = C(b_0 + 1)^2 \left( 4C\bar{C}b_0 e^{\frac{C_0}{\gamma}} \right)^{\frac{10+4s+3\beta}{2\beta\eta}}.$$

The hypothesis

$$3\alpha + 2a < \frac{\eta}{6 + 4\eta}$$

allows us to choose  $\beta \geq 1$  such that

$$(5.13) \quad 3\alpha + 2a < \frac{\beta\eta}{20 + 8s + 6\beta + 4\beta\eta}.$$

Then, the inequality (5.13) implies that,

$$\frac{1}{4} - \alpha - 2a - \frac{(\alpha+a)(10+4s+3\beta)}{2\beta\eta} > 0.$$

Thus, there exists  $\varepsilon_1 > 0$  such that, for any  $0 \leq \varepsilon \leq \varepsilon_1$ , we have

$$\int_0^{+\infty} \Gamma(v_L)(\tau) d\tau \leq \frac{1}{\gamma}.$$

We rewrite (5.12) as follows

$$(5.14) \quad \|w(t)\|_{\mathbf{H}^{0,s}}^2 \leq \left( \|w_0\|_{\mathbf{H}^{0,s}}^2 + C_0 \int_0^{+\infty} \Gamma(v_L)(\tau) d\tau \right) e^{\frac{2C_0}{\gamma}}.$$

From (5.13), we also deduce that

$$\frac{1}{4} - \alpha - 2a - \frac{(\alpha + a)(10 + 4s + 3\beta)}{2\beta\eta} > 2\alpha,$$

which means that there exists  $\varepsilon_2 > 0$  such that, for any  $0 \leq \varepsilon \leq \varepsilon_2$ ,

$$(5.15) \quad \int_0^{+\infty} \Gamma(v_L)(\tau) d\tau \leq \frac{\varepsilon^{2\alpha} e^{-\frac{2C_0}{\gamma}}}{16C_0 C^2}.$$

Let  $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$ . By gathering (5.11), (5.14) and (5.15), we finally obtain that, for any  $0 < \varepsilon \leq \varepsilon_0$  and for any  $0 \leq t < T^*$ ,

$$\|w(t)\|_{\mathbf{H}^{0,s}} \leq \sqrt{\frac{\varepsilon^{2\alpha}}{8C^2}} < \frac{\varepsilon^\alpha}{2C}.$$

Hence,  $T^* = +\infty$  and Theorem 1.2 is proved.  $\square$

## 6. ROTATING FLUIDS BETWEEN TWO INFINITE PARALLEL PLATES: PROOF OF THEOREM 1.3

**6.1. Ekman boundary layers.** As mentioned in the introduction, when  $\varepsilon$  goes to 0, the fluid has the tendency to have a two-dimensional behavior. In the interior part of the domain, far from the boundary, the fluid moves in vertical columns, according to the Taylor-Proudman theorem. Near the boundary, the Taylor columns are destroyed and thin boundary layers are formed. The movements of the fluid inside the layers are very complex and the friction stops the fluid on the boundary. The goal of this paragraph is to briefly recall the mathematical construction of these boundary layers. More precisely, we will “correct” the solution of the limiting system (1.7) (which is a divergence-free vector field, independent of  $x_3$ ) by adding a “boundary layer term”  $\mathcal{B}$  such that  $\bar{u} + \mathcal{B}$  is a divergence-free vector field which vanishes on the boundary.

In order to construct such boundary layers, a typical approach consists in looking for the approximate solutions of the system (1.6) in the following form, that is, to make the Ansatz

$$(6.1) \quad \begin{aligned} u_{app}^\varepsilon &= u_{0,int} + u_{0,BL} + \varepsilon u_{1,int} + \varepsilon u_{1,BL} + \dots \\ p_{app}^\varepsilon &= \frac{1}{\varepsilon} p_{-1,int} + \frac{1}{\varepsilon} p_{-1,BL} + p_{0,int} + p_{0,BL} + \dots \end{aligned}$$

The terms with the index “int” stand for the “interior” part, which consists in smooth functions of  $(x_h, x_3)$  and the index “BL” refers to the boundary layer part, which consists in smooth functions of the form

$$(x_h, x_3) \rightarrow F_0(t, x_h, \frac{x_3}{\delta}) + F_1(t, x_h, \frac{1-x_3}{\delta}),$$

where  $F_0(x_h, \zeta)$  and  $F_1(x_h, \zeta)$  rapidly decrease in  $\zeta$  at infinity. The quantity  $\delta > 0$ , which goes to zero as  $\varepsilon$  goes to zero, represents the size of the boundary layers. It is proved that  $\delta$  is of the same order as  $\varepsilon$  (see [22], [30], [11] and [12]). In this paper, we simply choose  $\delta = \varepsilon$ .

We recall that  $\bar{u}$  is the solution of the limiting system (1.7), the third component  $\bar{u}^3 = 0$  and we set  $\text{curl } \bar{u} = \partial_1 \bar{u}^2 - \partial_2 \bar{u}^1$ . In [22], [30] and [11], by a careful study of the Ansatz (6.1), the authors proved that we can write the boundary layer part in the following form

$$(6.2) \quad \mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4,$$

where  $\mathcal{B}_i$ ,  $i \in \{1, 2, 3, 4\}$ , are defined as described below:

1. The term  $\mathcal{B}_1$  is defined by

$$(6.3) \quad \mathcal{B}_1 = \begin{pmatrix} \tilde{w}_1 + \check{w}_1 \\ \tilde{w}_2 + \check{w}_2 \\ \varepsilon \sqrt{\beta} G(x_3) \text{curl } \bar{u} \end{pmatrix}$$

where

$$\begin{aligned}\tilde{w}_1 &= -\exp\left(-\frac{x_3}{\varepsilon\sqrt{2\beta}}\right)\left(\bar{u}^1 \cos\left(\frac{x_3}{\varepsilon\sqrt{2\beta}}\right) + \bar{u}^2 \sin\left(\frac{x_3}{\varepsilon\sqrt{2\beta}}\right)\right), \\ \tilde{w}_2 &= -\exp\left(-\frac{x_3}{\varepsilon\sqrt{2\beta}}\right)\left(\bar{u}^2 \cos\left(\frac{x_3}{\varepsilon\sqrt{2\beta}}\right) - \bar{u}^1 \sin\left(\frac{x_3}{\varepsilon\sqrt{2\beta}}\right)\right), \\ \check{w}_1 &= -\exp\left(-\frac{1-x_3}{\varepsilon\sqrt{2\beta}}\right)\left(\bar{u}^1 \cos\left(\frac{1-x_3}{\varepsilon\sqrt{2\beta}}\right) + \bar{u}^2 \sin\left(\frac{1-x_3}{\varepsilon\sqrt{2\beta}}\right)\right), \\ \check{w}_2 &= -\exp\left(-\frac{1-x_3}{\varepsilon\sqrt{2\beta}}\right)\left(\bar{u}^2 \cos\left(\frac{1-x_3}{\varepsilon\sqrt{2\beta}}\right) - \bar{u}^1 \sin\left(\frac{1-x_3}{\varepsilon\sqrt{2\beta}}\right)\right), \\ G(x_3) &= -\exp\left(-\frac{x_3}{\varepsilon\sqrt{2\beta}}\right)\sin\left(\frac{x_3}{\varepsilon\sqrt{2\beta}} + \frac{\pi}{4}\right) + \exp\left(-\frac{1-x_3}{\varepsilon\sqrt{2\beta}}\right)\sin\left(\frac{1-x_3}{\varepsilon\sqrt{2\beta}} + \frac{\pi}{4}\right).\end{aligned}$$

2. The terms  $\mathcal{B}_2$  and  $\mathcal{B}_3$  are defined by

$$(6.4) \quad \mathcal{B}_2 = \begin{pmatrix} \varepsilon\sqrt{2\beta} \bar{u}^2 \\ -\varepsilon\sqrt{2\beta} \bar{u}^1 \\ \varepsilon\sqrt{2\beta} \left(\frac{1}{2} - x_3\right) \operatorname{curl} \bar{u} \end{pmatrix}$$

$$(6.5) \quad \mathcal{B}_3 = \exp\left(-\frac{1}{\varepsilon\sqrt{2\beta}}\right) \cos\left(\frac{1}{\varepsilon\sqrt{2\beta}}\right) \begin{pmatrix} \bar{u}^1 \\ \bar{u}^2 \\ 0 \end{pmatrix}.$$

3. Finally,

$$(6.6) \quad \mathcal{B}_4 = f(x_3) \begin{pmatrix} \bar{u}^2 \\ -\bar{u}^1 \\ 0 \end{pmatrix} + g(x_3) \begin{pmatrix} 0 \\ 0 \\ \operatorname{curl} \bar{u} \end{pmatrix},$$

where

$$(6.7) \quad \begin{aligned}f(x_3) &= a \left[ \exp\left(-\frac{x_3}{\varepsilon\sqrt{2\beta}}\right) + \exp\left(-\frac{1-x_3}{\varepsilon\sqrt{2\beta}}\right) \right] + b, \\ g(x_3) &= -\varepsilon\sqrt{\beta} \exp\left(-\frac{1}{\varepsilon\sqrt{2\beta}}\right) \sin\left(\frac{1}{\varepsilon\sqrt{2\beta}} + \frac{\pi}{4}\right) - \int_0^{x_3} f(s) ds,\end{aligned}$$

and where  $(a, b)$  is the solution of the linear system

$$(6.8) \quad \begin{cases} a \left[ 1 + \exp\left(-\frac{1}{\varepsilon\sqrt{2\beta}}\right) \right] + b = -\varepsilon\sqrt{2\beta} + \exp\left(-\frac{1}{\varepsilon\sqrt{2\beta}}\right) \sin\left(\frac{1}{\varepsilon\sqrt{2\beta}}\right) \\ 2a\varepsilon\sqrt{2\beta} \left[ 1 - \exp\left(-\frac{1}{\varepsilon\sqrt{2\beta}}\right) \right] + b = 2\varepsilon\sqrt{\beta} \exp\left(-\frac{1}{\varepsilon\sqrt{2\beta}}\right) \sin\left(\frac{1}{\varepsilon\sqrt{2\beta}} + \frac{\pi}{4}\right). \end{cases}$$

We remark that the determinant of the system (6.8) is

$$D = 1 + \exp\left(-\frac{1}{\varepsilon\sqrt{2\beta}}\right) - 2\varepsilon\sqrt{2\beta} \left[ 1 - \exp\left(-\frac{1}{\varepsilon\sqrt{2\beta}}\right) \right].$$

Thus, for  $\varepsilon > 0$  small enough, we have  $D > \frac{1}{2}$  and (6.8) always has the following solution

$$a = \frac{J_\varepsilon - K_\varepsilon}{D} \quad \text{and} \quad b = \frac{K_\varepsilon \left[ 1 + \exp\left(-\frac{1}{\varepsilon\sqrt{2\beta}}\right) \right] - 2J_\varepsilon\varepsilon\sqrt{2\beta} \left[ 1 - \exp\left(-\frac{1}{\varepsilon\sqrt{2\beta}}\right) \right]}{D},$$

where

$$\begin{aligned}J_\varepsilon &= -\varepsilon\sqrt{2\beta} + \exp\left(-\frac{1}{\varepsilon\sqrt{2\beta}}\right) \sin\left(\frac{1}{\varepsilon\sqrt{2\beta}}\right), \\ K_\varepsilon &= 2\varepsilon\sqrt{\beta} \exp\left(-\frac{1}{\varepsilon\sqrt{2\beta}}\right) \sin\left(\frac{1}{\varepsilon\sqrt{2\beta}} + \frac{\pi}{4}\right).\end{aligned}$$

It is easy to prove that if  $\varepsilon > 0$  is small enough, then

$$|a| < 4(\beta + \sqrt{\beta})\varepsilon \quad \text{and} \quad |b| < 32\beta\varepsilon^2.$$

With the previously defined boundary layer term  $\mathcal{B}$ , we can verify that

$$\operatorname{div}(\bar{u} + \mathcal{B}) = 0 \quad \text{and} \quad (\bar{u} + \mathcal{B})|_{\{x_3=0\}} = (\bar{u} + \mathcal{B})|_{\{x_3=1\}} = 0.$$

Now, let

$$B_0(x_3) = \begin{bmatrix} -e^{-\frac{x_3}{\varepsilon\sqrt{2\beta}}} \cos \frac{x_3}{\varepsilon\sqrt{2\beta}} - e^{-\frac{1-x_3}{\varepsilon\sqrt{2\beta}}} \cos \frac{1-x_3}{\varepsilon\sqrt{2\beta}} & -e^{-\frac{x_3}{\varepsilon\sqrt{2\beta}}} \sin \frac{x_3}{\varepsilon\sqrt{2\beta}} - e^{-\frac{1-x_3}{\varepsilon\sqrt{2\beta}}} \sin \frac{1-x_3}{\varepsilon\sqrt{2\beta}} \\ e^{-\frac{x_3}{\varepsilon\sqrt{2\beta}}} \sin \frac{x_3}{\varepsilon\sqrt{2\beta}} + e^{-\frac{1-x_3}{\varepsilon\sqrt{2\beta}}} \sin \frac{1-x_3}{\varepsilon\sqrt{2\beta}} & -e^{-\frac{x_3}{\varepsilon\sqrt{2\beta}}} \cos \frac{x_3}{\varepsilon\sqrt{2\beta}} - e^{-\frac{1-x_3}{\varepsilon\sqrt{2\beta}}} \cos \frac{1-x_3}{\varepsilon\sqrt{2\beta}} \end{bmatrix}$$

Then, we can write  $\mathcal{B}$  in the following form

$$\mathcal{B} = \mathcal{M}(x_3) \begin{pmatrix} \bar{u}^1 \\ \bar{u}^2 \\ \text{curl } \bar{u} \end{pmatrix},$$

where

$$\mathcal{M}(x_3) = \begin{bmatrix} M(x_3) & 0 \\ 0 & m(x_3) \end{bmatrix}$$

with  $M(x_3)$  and  $m(x_3)$  defined by

$$\begin{aligned} M(x_3) &= B_0(x_3) + \left( \varepsilon\sqrt{2\beta} + f(x_3) \right) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \exp\left(-\frac{1}{\varepsilon\sqrt{2\beta}}\right) \cos \frac{1}{\varepsilon\sqrt{2\beta}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ m(x_3) &= \varepsilon\sqrt{\beta} G(x_3) + \varepsilon\sqrt{2\beta} \left( \frac{1}{2} - x_3 \right) + g(x_3). \end{aligned}$$

We can also prove the existence of a constant  $C > 0$  such that, for any  $p \geq 1$ , we have

$$(6.9) \quad \begin{cases} \|\mathcal{M}(\cdot)\|_{\mathbf{L}_{x_3}^p} \leq C\varepsilon^{\frac{1}{p}}, & \|\mathcal{M}(\cdot)\|_{\mathbf{L}_{x_3}^\infty} \leq C, & \|\mathcal{M}'(\cdot)\|_{\mathbf{L}_{x_3}^p} \leq C\varepsilon^{\frac{1}{p}-1}, \\ \|\mathcal{M}(\cdot)\|_{L_\infty} \leq C\varepsilon, & \|m(\cdot)\|_{L^p_{x_3}} \leq C\varepsilon \\ \sup_{x_3 \in [0, \frac{1}{2}]} |x_3^2 M'(x_3)| \leq C\varepsilon & \text{and} & \sup_{x_3 \in [\frac{1}{2}, 1]} |(1-x_3)^2 M'(x_3)| \leq C\varepsilon. \end{cases}$$

**6.2. Convergence to the limiting system.** In this paragraph, we provide the needed a priori estimates and a sketch of the proof of Theorem 1.3. These a priori estimates can be justified by a classical approximation by smooth functions (see for instance [12]). Following the idea of the proof of Theorem 1.2, for any  $\varepsilon > 0$ , we consider the 2D damped Navier-Stokes system with three components

$$(6.10) \quad \begin{cases} \partial_t \bar{u}^{\varepsilon, h} - \varepsilon^\alpha \Delta_h \bar{u}^{\varepsilon, h} + \sqrt{2\beta} \bar{u}^{\varepsilon, h} + \bar{u}^{\varepsilon, h} \cdot \nabla_h \bar{u}^{\varepsilon, h} = -\nabla_h \bar{p}^\varepsilon \\ \partial_t \bar{u}^{\varepsilon, 3} - \varepsilon^\alpha \Delta_h \bar{u}^{\varepsilon, 3} + \sqrt{2\beta} \bar{u}^{\varepsilon, 3} + \bar{u}^{\varepsilon, h} \cdot \nabla_h \bar{u}^{\varepsilon, 3} = 0 \\ \text{div}_h \bar{u}^{\varepsilon, h} = 0 \\ \partial_3 \bar{u}^\varepsilon = 0 \\ \bar{u}^\varepsilon|_{t=0} = \bar{u}_0. \end{cases}$$

Then, Lemma 3.9 implies that Theorem 1.3 is a corollary of the following theorem

**Theorem 6.1.** *Let  $0 < \alpha < \frac{1}{2}$  and let  $u_0^\varepsilon \in \mathbf{L}^2(\Omega)$  be a family of initial data such that*

$$\lim_{\varepsilon \rightarrow 0} u_0^\varepsilon = \bar{u}_0 = (\bar{u}_0^1(x_1, x_2), \bar{u}_0^2(x_1, x_2), 0) \quad \text{in} \quad \mathbf{L}^2(\mathbb{R}_h^2 \times [0, 1]),$$

where  $\bar{u}_0$  is a two-dimensional divergence-free vector field in  $\mathbf{H}^\sigma(\mathbb{R}_h^2)$ ,  $\sigma > 2$ . Let  $\bar{u}^\varepsilon$  be the solution of the system (6.10) with initial data  $\bar{u}_0$  and for each  $\varepsilon > 0$ , let  $u^\varepsilon$  be a weak solution of (1.6) with initial data  $u_0^\varepsilon$ . Then

$$\lim_{\varepsilon \rightarrow 0} \|u^\varepsilon - \bar{u}^\varepsilon\|_{\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2 \times [0, 1]))} = 0.$$

**Proof**

We first remark that we can construct the ‘‘boundary layers’’  $\mathcal{B}^\varepsilon$  for the system (6.10) in the same way as the construction of  $\mathcal{B}$ , with  $\bar{u}$  being replaced by  $\bar{u}^\varepsilon$ . More precisely, we also write

$$(6.11) \quad \mathcal{B}^\varepsilon = \mathcal{B}_1^\varepsilon + \mathcal{B}_2^\varepsilon + \mathcal{B}_3^\varepsilon + \mathcal{B}_4^\varepsilon,$$

where  $\mathcal{B}_i^\varepsilon$ ,  $i \in \{1, 2, 3, 4\}$  are defined by the same formulas as in (6.3) to (6.6), with  $\bar{u}$  being replaced by  $\bar{u}^\varepsilon$ . We obtain

$$(6.12) \quad \mathcal{B}^\varepsilon = \mathcal{M}(x_3) \begin{pmatrix} \bar{u}^{\varepsilon, 1} \\ \bar{u}^{\varepsilon, 2} \\ \text{curl } \bar{u}^\varepsilon \end{pmatrix},$$

where

- $\bar{u}^\varepsilon = (\bar{u}^{\varepsilon,1}(x_1, x_2), \bar{u}^{\varepsilon,2}(x_1, x_2), 0)$  is a two-dimensional divergence-free vector field, which is the solution of (6.10),
- $\text{curl } \bar{u}^\varepsilon = \partial_1 \bar{u}^{\varepsilon,2} - \partial_2 \bar{u}^{\varepsilon,1}$ ,
- and the matrix  $\mathcal{M}(x_3)$  is given in the previous paragraph.

It is easy to prove that  $\mathcal{B}^\varepsilon$  is small, *i.e.*,  $\mathcal{B}^\varepsilon$  goes to 0 in  $\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2 \times [0, 1]))$  as  $\varepsilon$  goes to 0. Then, our goal is to prove that  $v^\varepsilon = u^\varepsilon - \bar{u}^\varepsilon - \mathcal{B}^\varepsilon$  converges to 0 in  $\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2 \times [0, 1]))$  as  $\varepsilon$  goes to 0.

We recall that a two-dimensional divergence-free vector field (independent of  $x_3$ ) belongs to the kernel of the operator  $\mathbb{P}(e_3 \wedge \cdot)$ , where  $\mathbb{P}$  is the Leray projection of  $\mathbf{L}^2(\mathbb{R}^3)$  onto the subspace of divergence-free vector fields. As a consequence,  $e_3 \wedge \bar{u}^\varepsilon$  is a gradient term. Replacing  $u^\varepsilon$  by  $v^\varepsilon + \bar{u}^\varepsilon + \mathcal{B}^\varepsilon$  in the system (1.6), we deduce that  $v^\varepsilon$  satisfies the following equation

$$(6.13) \quad \partial_t v^\varepsilon - \varepsilon^\alpha \Delta_h v^\varepsilon - \beta \varepsilon \partial_3^2 v^\varepsilon + L_1 + u^\varepsilon \cdot \nabla v^\varepsilon + \mathcal{B}^\varepsilon \cdot \nabla \mathcal{B}^\varepsilon \\ + \mathcal{B}^\varepsilon \cdot \nabla \bar{u}^\varepsilon + v^\varepsilon \cdot \nabla \mathcal{B}^\varepsilon + v^\varepsilon \cdot \nabla \bar{u}^\varepsilon - L_2 + \frac{e_3 \wedge v^\varepsilon}{\varepsilon} = -\nabla \bar{p}^\varepsilon,$$

where

$$L_1 = \partial_t \mathcal{B}^\varepsilon - \varepsilon^\alpha \Delta_h \mathcal{B}^\varepsilon + \bar{u}^\varepsilon \cdot \nabla \mathcal{B}^\varepsilon \\ L_2 = \beta \varepsilon \partial_3^2 \mathcal{B}^\varepsilon - \frac{e_3 \wedge \mathcal{B}^\varepsilon}{\varepsilon} + \sqrt{2\beta} \bar{u}^\varepsilon.$$

Taking the  $\mathbf{L}^2$  scalar product of (6.13) with  $v^\varepsilon$ , then integrating by parts the obtained equation and taking into account the fact that  $v^\varepsilon$  satisfies the Dirichlet boundary condition, we get

$$(6.14) \quad \frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|_{\mathbf{L}^2}^2 + \varepsilon^\alpha \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2}^2 + \beta \varepsilon \|\partial_3 v^\varepsilon\|_{\mathbf{L}^2}^2 \\ = -\langle L_1, v^\varepsilon \rangle - \langle u^\varepsilon \cdot \nabla v^\varepsilon, v^\varepsilon \rangle - \langle \mathcal{B}^\varepsilon \cdot \nabla \mathcal{B}^\varepsilon, v^\varepsilon \rangle - \langle \mathcal{B}^\varepsilon \cdot \nabla \bar{u}^\varepsilon, v^\varepsilon \rangle \\ - \langle v^\varepsilon \cdot \nabla \mathcal{B}^\varepsilon, v^\varepsilon \rangle - \langle v^\varepsilon \cdot \nabla \bar{u}^\varepsilon, v^\varepsilon \rangle + \langle L_2, v^\varepsilon \rangle.$$

In what follows, we will separately estimate the seven terms on the right-hand side of (6.14). We will always denote  $\mathcal{B}^{\varepsilon,1}$ ,  $\mathcal{B}^{\varepsilon,2}$  and  $\mathcal{B}^{\varepsilon,3}$  ( $v^{\varepsilon,1}$ ,  $v^{\varepsilon,2}$  and  $v^{\varepsilon,3}$  respectively) the three components of  $\mathcal{B}^\varepsilon$  ( $v^\varepsilon$  respectively) and we also write  $\mathcal{B}^\varepsilon = (\mathcal{B}^{\varepsilon,h}, \mathcal{B}^{\varepsilon,3})$  and  $v^\varepsilon = (v^{\varepsilon,h}, v^{\varepsilon,3})$ .

1. Applying the operator  $\text{curl}$  to the first two equations of the system (6.10) (we recall that in this paper,  $\text{curl}$  only acts on the horizontal components and we already defined  $\text{curl } \bar{u} = \partial_1 \bar{u}^2 - \partial_2 \bar{u}^1$ ), we obtain

$$\partial_t (\text{curl } \bar{u}^\varepsilon) - \varepsilon^\alpha \Delta_h (\text{curl } \bar{u}^\varepsilon) + \sqrt{2\beta} (\text{curl } \bar{u}^\varepsilon) + \bar{u}^\varepsilon \cdot \nabla (\text{curl } \bar{u}^\varepsilon) = 0.$$

Let  $A = (\bar{u}_1^\varepsilon, \bar{u}_2^\varepsilon, \text{curl } \bar{u}^\varepsilon)$ . So combining the above equation with the first two equations of (6.10), we deduce that

$$\partial_t A - \varepsilon^\alpha \Delta_h A + \sqrt{2\beta} A + \bar{u}^\varepsilon \cdot \nabla A = -(\partial_1 \bar{p}^\varepsilon, \partial_2 \bar{p}^\varepsilon, 0).$$

Since  $\bar{u}^{\varepsilon,3} = 0$ ,  $\text{div } v^\varepsilon = 0$ ,  $\partial_3 \bar{p}^\varepsilon = 0$  and

$$\mathcal{B}^\varepsilon = \mathcal{M}(x_3) {}^t A = \mathcal{M}(x_3) \begin{pmatrix} \bar{u}_1^\varepsilon \\ \bar{u}_2^\varepsilon \\ \text{curl } \bar{u}^\varepsilon \end{pmatrix},$$

we can write

$$|\langle L_1, v^\varepsilon \rangle| = |\langle \mathcal{M}(x_3) (\partial_t A - \varepsilon^\alpha \Delta_h A + \bar{u}^\varepsilon \cdot \nabla A), v^\varepsilon \rangle| \\ \leq \sqrt{2\beta} \|\mathcal{M}(\cdot)\|_{\mathbf{L}_{x_3}^2} \|A\|_{\mathbf{L}_{x_h}^2} \|v^\varepsilon\|_{\mathbf{L}^2}.$$

Then, Estimate (6.9), Lemma 3.1 and Young's inequality imply

$$(6.15) \quad |\langle L_1, v^\varepsilon \rangle| \leq C(\bar{u}_0) \varepsilon^{\frac{1}{2}} e^{-t\sqrt{2\beta}} \left(1 + \|v^\varepsilon\|_{\mathbf{L}^2}^2\right).$$

2. For the second term, using the divergence-free property of  $u^\varepsilon$ , we simply have

$$(6.16) \quad \langle u^\varepsilon \cdot \nabla v^\varepsilon, v^\varepsilon \rangle = 0.$$

3. We decompose the third term into two parts:

$$\langle \mathcal{B}^\varepsilon \cdot \nabla \mathcal{B}^\varepsilon, v^\varepsilon \rangle = \langle \mathcal{B}^{\varepsilon,h} \cdot \nabla_h \mathcal{B}^\varepsilon, v^\varepsilon \rangle + \langle \mathcal{B}^{\varepsilon,3} \partial_3 \mathcal{B}^\varepsilon, v^\varepsilon \rangle.$$



Due to an integration by parts, the ‘‘horizontal’’ part can be bounded as follows

$$|\langle \mathcal{B}^{\varepsilon,h} \cdot \nabla_h \mathcal{B}^\varepsilon, v^\varepsilon \rangle| \leq |\langle (\operatorname{div}_h \mathcal{B}^{\varepsilon,h}) \mathcal{B}^\varepsilon, v^\varepsilon \rangle| + \sum_{i=1}^2 |\langle \mathcal{B}^{\varepsilon,i} \mathcal{B}^\varepsilon, \partial_i v^\varepsilon \rangle|.$$

Hence, the decomposition (6.12) of  $\mathcal{B}^\varepsilon$ , Hölder’s inequality, Estimates (6.9), Lemma 3.1 and Young’s inequality yield

$$(6.17) \quad \begin{aligned} |\langle (\operatorname{div}_h \mathcal{B}^{\varepsilon,h}) \mathcal{B}^\varepsilon, v^\varepsilon \rangle| &\leq \|\operatorname{div}_h \mathcal{B}^{\varepsilon,h}\|_{\mathbf{L}^4} \|\mathcal{B}^\varepsilon\|_{\mathbf{L}^4} \|v^\varepsilon\|_{\mathbf{L}^2} \\ &\leq \|\mathcal{M}(\cdot)\|_{\mathbf{L}^4_{x_3}}^2 \left( \|\bar{u}^\varepsilon\|_{\mathbf{L}^4_{x_h}}^2 + \|\nabla_h \bar{u}^\varepsilon\|_{\mathbf{L}^4_{x_h}}^2 \right) \|v^\varepsilon\|_{\mathbf{L}^2} \\ &\leq C(\bar{u}_0) \varepsilon^{\frac{1}{2}} e^{-t\sqrt{2\beta}} \left( 1 + \|v^\varepsilon\|_{\mathbf{L}^2}^2 \right), \end{aligned}$$

and

$$(6.18) \quad \begin{aligned} \sum_{i=1}^2 |\langle \mathcal{B}^{\varepsilon,i} \mathcal{B}^\varepsilon, \partial_i v^\varepsilon \rangle| &\leq \|\mathcal{B}^{\varepsilon,h}\|_{\mathbf{L}^4} \|\mathcal{B}^\varepsilon\|_{\mathbf{L}^4} \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2} \\ &\leq \|\mathcal{M}(\cdot)\|_{\mathbf{L}^4_{x_3}}^2 \left( \|\bar{u}^\varepsilon\|_{\mathbf{L}^4_{x_h}}^2 + \|\nabla_h \bar{u}^\varepsilon\|_{\mathbf{L}^4_{x_h}}^2 \right) \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2} \\ &\leq C(\bar{u}_0) \varepsilon^{\frac{1-2\alpha}{2}} e^{-t\sqrt{2\beta}} + \frac{\varepsilon^\alpha}{16} \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2}^2. \end{aligned}$$

Likewise, if  $0 < \alpha < \frac{1}{2}$ , we have the following estimate for the ‘‘vertical’’ part:

$$(6.19) \quad \begin{aligned} |\langle \mathcal{B}^{\varepsilon,3} \partial_3 \mathcal{B}^\varepsilon, v^\varepsilon \rangle| &\leq \|\mathcal{B}^{\varepsilon,3}\|_{L^{\frac{1}{\alpha}}} \|\partial_3 \mathcal{B}^\varepsilon\|_{\mathbf{L}^{\frac{2}{1-2\alpha}}} \|v^\varepsilon\|_{\mathbf{L}^2} \\ &\leq C(\bar{u}_0) e^{-t\sqrt{2\beta}} \|m(\cdot)\|_{L^{\frac{1}{\alpha}}_{x_3}} \|\mathcal{M}'(\cdot)\|_{\mathbf{L}^{\frac{2}{1-2\alpha}}_{x_3}} \|v^\varepsilon\|_{\mathbf{L}^2} \\ &\leq C(\bar{u}_0) \varepsilon^{\frac{1-2\alpha}{2}} e^{-t\sqrt{2\beta}} \left( 1 + \|v^\varepsilon\|_{\mathbf{L}^2}^2 \right). \end{aligned}$$

4. For the fourth term, taking into account the fact that  $\bar{u}^\varepsilon$  is independent of  $x_3$ , Estimates (6.9) and Lemma 3.1 imply

$$(6.20) \quad \begin{aligned} |\langle \mathcal{B}^\varepsilon \cdot \nabla \bar{u}^\varepsilon, v^\varepsilon \rangle| &\leq \|\mathcal{B}^\varepsilon\|_{\mathbf{L}^4_{x_h} \mathbf{L}^2_{x_3}} \|\nabla_h \bar{u}^\varepsilon\|_{\mathbf{L}^4_{x_h}} \|v^\varepsilon\|_{\mathbf{L}^2} \\ &\leq \|\mathcal{M}(\cdot)\|_{\mathbf{L}^2_{x_3}} \left( \|\bar{u}^\varepsilon\|_{\mathbf{L}^4}^2 + \|\nabla_h \bar{u}^\varepsilon\|_{\mathbf{L}^4}^2 \right) \|v^\varepsilon\|_{\mathbf{L}^2} \\ &\leq C(\bar{u}_0) \varepsilon^{\frac{1}{2}} e^{-t\sqrt{2\beta}} \left( 1 + \|v^\varepsilon\|_{\mathbf{L}^2}^2 \right). \end{aligned}$$

5. The fifth term is the most difficult to treat. First, we decompose this term as follows

$$\langle v^\varepsilon \cdot \nabla \mathcal{B}^\varepsilon, v^\varepsilon \rangle = \langle v^{\varepsilon,h} \cdot \nabla_h \mathcal{B}^{\varepsilon,h}, v^{\varepsilon,h} \rangle + \langle v^{\varepsilon,h} \cdot \nabla_h \mathcal{B}^{\varepsilon,3}, v^{\varepsilon,3} \rangle + \langle v^{\varepsilon,3} \partial_3 \mathcal{B}^{\varepsilon,3}, v^{\varepsilon,3} \rangle + \langle v^{\varepsilon,3} \partial_3 \mathcal{B}^{\varepsilon,h}, v^{\varepsilon,h} \rangle.$$

For the first term on the right-hand side, Hölder’s inequality implies that

$$|\langle v^{\varepsilon,h} \cdot \nabla_h \mathcal{B}^{\varepsilon,h}, v^{\varepsilon,h} \rangle| \leq C \|v^{\varepsilon,h}\|_{\mathbf{L}^2} \|\nabla_h \mathcal{B}^{\varepsilon,h}\|_{\mathbf{L}^\infty} \|v^{\varepsilon,h}\|_{\mathbf{L}^2} \leq C \|\mathcal{M}(\cdot)\|_{\mathbf{L}^\infty_{x_3}} \|\nabla_h \bar{u}^\varepsilon\|_{\mathbf{L}^\infty_{x_h}} \|v^\varepsilon\|_{\mathbf{L}^2}^2.$$

Then, using Estimates (6.9) and Lemma 3.4, we obtain

$$(6.21) \quad |\langle v^{\varepsilon,h} \cdot \nabla_h \mathcal{B}^{\varepsilon,h}, v^{\varepsilon,h} \rangle| \leq C(\bar{u}_0) e^{-t\sqrt{2\beta}} \|v^\varepsilon\|_{\mathbf{L}^2}^2.$$

Next, by integrating by parts and using Hölder’s inequality, we deduce that

$$|\langle v^{\varepsilon,h} \cdot \nabla_h \mathcal{B}^{\varepsilon,3}, v^{\varepsilon,3} \rangle| \leq C \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2} \|\mathcal{B}^{\varepsilon,3}\|_{\mathbf{L}^\infty} \|v^\varepsilon\|_{\mathbf{L}^2}.$$

So, Estimates (6.9), Lemmas 3.1 and 3.4 and Young’s inequality imply

$$(6.22) \quad \begin{aligned} |\langle v^{\varepsilon,h} \cdot \nabla_h \mathcal{B}^{\varepsilon,3}, v^{\varepsilon,3} \rangle| &\leq C \|m(\cdot)\|_{\mathbf{L}^\infty_{x_3}} \|\operatorname{curl} \bar{u}^\varepsilon\|_{\mathbf{L}^\infty_{x_h}} \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2} \|v^\varepsilon\|_{\mathbf{L}^2} \\ &\leq C(\bar{u}_0) \varepsilon^{2-\alpha} e^{-t\sqrt{2\beta}} \|v^\varepsilon\|_{\mathbf{L}^2}^2 + \frac{\varepsilon^\alpha}{16} \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2}^2. \end{aligned}$$

Performing an integration by parts, we can control the third term in the same way as the second one:

$$(6.23) \quad \begin{aligned} |\langle v^{\varepsilon,3} \partial_3 \mathcal{B}^{\varepsilon,3}, v^{\varepsilon,3} \rangle| &= 2 |\langle \mathcal{B}^{\varepsilon,3} v^{\varepsilon,3}, \partial_3 v^{\varepsilon,3} \rangle| = 2 |\langle \mathcal{B}^{\varepsilon,3} v^{\varepsilon,3}, \operatorname{div}_h v^{\varepsilon,h} \rangle| \\ &\leq C(\bar{u}_0) \varepsilon^{2-\alpha} e^{-t\sqrt{2\beta}} \|v^\varepsilon\|_{\mathbf{L}^2}^2 + \frac{\varepsilon^\alpha}{16} \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2}^2. \end{aligned}$$

In order to estimate the last term of the right-hand side, we decompose it into two parts, the first part corresponding to the boundary layer near  $\{x_3 = 0\}$  and the other part corresponding to the one near  $\{x_3 = 1\}$ :

$$\langle v^{\varepsilon,3} \partial_3 \mathcal{B}^{\varepsilon,h}, v^{\varepsilon,h} \rangle = \int_{\mathbb{R}_h^2 \times [0, \frac{1}{2}]} (v^{\varepsilon,3} \partial_3 \mathcal{B}^{\varepsilon,h}) \cdot v^{\varepsilon,h} dx + \int_{\mathbb{R}_h^2 \times [\frac{1}{2}, 1]} (v^{\varepsilon,3} \partial_3 \mathcal{B}^{\varepsilon,h}) \cdot v^{\varepsilon,h} dx$$

For the first part, since  $v^\varepsilon$  vanishes on  $\{x_3 = 0\}$ , using Hölder's inequality and Hardy's inequality, we get

$$\begin{aligned} \left| \int_{\mathbb{R}_h^2 \times [0, \frac{1}{2}]} (v^{\varepsilon,3} \partial_3 \mathcal{B}^{\varepsilon,h}) \cdot v^{\varepsilon,h} dx \right| &\leq \sup_{x_3 \in [0, \frac{1}{2}]} |x_3^2 M'(x_3)| \|\bar{u}^\varepsilon\|_{\mathbf{L}^\infty_{x_h}} \left\| \frac{v^{\varepsilon,3}}{x_3} \right\|_{L^2} \left\| \frac{v^{\varepsilon,h}}{x_3} \right\|_{L^2} \\ &\leq \sup_{x_3 \in [0, \frac{1}{2}]} |x_3^2 M'(x_3)| \|\bar{u}^\varepsilon\|_{\mathbf{L}^\infty_{x_h}} \|\partial_3 v^{\varepsilon,3}\|_{L^2} \|\partial_3 v^{\varepsilon,h}\|_{L^2}. \end{aligned}$$

We recall that  $\partial_3 v^{\varepsilon,3} = -\operatorname{div}_h v^{\varepsilon,h}$ . Then, Lemmas 3.1 and 3.4, Estimates (6.9) and Young's inequality imply

$$(6.24) \quad \begin{aligned} \left| \int_{\mathbb{R}_h^2 \times [0, \frac{1}{2}]} (v^{\varepsilon,3} \partial_3 \mathcal{B}^{\varepsilon,h}) \cdot v^{\varepsilon,h} dx \right| &\leq C(\bar{u}_0) \varepsilon e^{-t\sqrt{2\beta}} \|\operatorname{div}_h v^{\varepsilon,3}\|_{L^2} \|\partial_3 v^{\varepsilon,h}\|_{L^2} \\ &\leq C(\bar{u}_0) \varepsilon \|\nabla_h v^\varepsilon\|_{L^2}^2 + \frac{\beta\varepsilon}{4} \|\partial_3 v^\varepsilon\|_{L^2}^2. \end{aligned}$$

For the second part concerning the boundary layer near  $\{x_3 = 1\}$ , since  $v^\varepsilon = (v^{\varepsilon,h}, v^{\varepsilon,3})$  vanishes on  $\{x_3 = 1\}$ , Hardy's inequality implies that

$$\begin{aligned} I_v &= \int_{\mathbb{R}_h^2} \left( \int_{\frac{1}{2}}^1 \left| \frac{v^{\varepsilon,3}(x_h, x_3)}{1-x_3} \right|^2 dx_3 \right) dx_h = \int_{\mathbb{R}_h^2} \left( \int_0^{\frac{1}{2}} \left| \frac{v^{\varepsilon,3}(x_h, 1-x_3)}{x_3} \right|^2 dx_3 \right) dx_h \\ &\leq C \int_{\mathbb{R}_h^2} \left( \int_0^{\frac{1}{2}} |\partial_3 v^{\varepsilon,3}(x_h, 1-x_3)|^2 dx_3 \right) dx_h \\ &\leq C \|\partial_3 v^{\varepsilon,3}\|_{L^2}^2 = C \|\operatorname{div}_h v^{\varepsilon,h}\|_{L^2}^2. \end{aligned}$$

Likewise,

$$I_h = \int_{\mathbb{R}_h^2} \left( \int_{\frac{1}{2}}^1 \left| \frac{v^{\varepsilon,h}(x_h, x_3)}{1-x_3} \right|^2 dx_3 \right) dx_h \leq C \|\partial_3 v^{\varepsilon,h}\|_{L^2}^2.$$

Thus, using Hölder's inequality, we get

$$(6.25) \quad \begin{aligned} \left| \int_{\mathbb{R}_h^2 \times [\frac{1}{2}, 1]} (v^{\varepsilon,3} \partial_3 \mathcal{B}^{\varepsilon,h}) \cdot v^{\varepsilon,h} dx \right| &\leq \sup_{x_3 \in [\frac{1}{2}, 1]} |(1-x_3)^2 M'(x_3)| \|\bar{u}^\varepsilon\|_{\mathbf{L}^\infty_{x_h}} \sqrt{I_v} \sqrt{I_h} \\ &\leq C(\bar{u}_0) \varepsilon e^{-t\sqrt{2\beta}} \|\operatorname{div}_h v^{\varepsilon,3}\|_{L^2} \|\partial_3 v^{\varepsilon,h}\|_{L^2} \\ &\leq C(\bar{u}_0) \varepsilon \|\nabla_h v^\varepsilon\|_{L^2}^2 + \frac{\beta\varepsilon}{4} \|\partial_3 v^\varepsilon\|_{L^2}^2. \end{aligned}$$

6. The sixth term on the right-hand side of (6.14) can be treated using Hölder's inequality and Lemma 3.4. We have

$$(6.26) \quad |\langle v^\varepsilon \cdot \nabla \bar{u}^\varepsilon, v^\varepsilon \rangle| \leq C \|\nabla_h \bar{u}^\varepsilon\|_{\mathbf{L}^\infty(\mathbb{R}_h^2)} \|v^\varepsilon\|_{L^2}^2 \leq C(\bar{u}_0) e^{-t\sqrt{2\beta}} \|v^\varepsilon\|_{L^2}^2$$

7. We will evaluate the seventh term as in [22] or [30]. We have

$$\langle L_2, v^\varepsilon \rangle = \langle \beta\varepsilon \partial_3^2 \mathcal{B}^\varepsilon, v^\varepsilon \rangle - \left\langle \frac{e_3 \wedge \mathcal{B}^\varepsilon}{\varepsilon}, v^\varepsilon \right\rangle + \left\langle \sqrt{2\beta} \bar{u}^\varepsilon, v^\varepsilon \right\rangle.$$

We recall the definition (6.11) of  $\mathcal{B}^\varepsilon$

$$\mathcal{B}^\varepsilon = \mathcal{B}_1^\varepsilon + \mathcal{B}_2^\varepsilon + \mathcal{B}_3^\varepsilon + \mathcal{B}_4^\varepsilon,$$

and for any  $i \in \{1, 2, 3, 4\}$ , we set  $\mathcal{B}_i^\varepsilon = (\mathcal{B}_i^{\varepsilon,h}, \mathcal{B}_i^{\varepsilon,3})$ , where  $\mathcal{B}_i^{\varepsilon,h}$  and  $\mathcal{B}_i^{\varepsilon,3}$  denote the horizontal and vertical components of  $\mathcal{B}_i^\varepsilon$  respectively. We also recall that  $\mathcal{B}^\varepsilon = (\mathcal{B}^{\varepsilon,h}, \mathcal{B}^{\varepsilon,3})$ . Then, the following identities are

immediate

$$\begin{aligned}\partial_3^2 \mathcal{B}_3^{\varepsilon,h} &= 0, \\ \beta\varepsilon \partial_3^2 \mathcal{B}_1^{\varepsilon,h} - \frac{e_3 \wedge \mathcal{B}_1^\varepsilon}{\varepsilon} &= 0, \\ \beta\varepsilon \partial_3^2 \mathcal{B}_2^{\varepsilon,h} - \frac{e_3 \wedge \mathcal{B}_2^\varepsilon}{\varepsilon} + \sqrt{2\beta} \bar{u}^\varepsilon &= 0.\end{aligned}$$

For the remaining terms, we have

$$\begin{aligned}(6.27) \quad \beta\varepsilon \left| \langle \partial_3^2 \mathcal{B}^{\varepsilon,3}, v^{\varepsilon,3} \rangle \right| &\leq \beta\varepsilon \|\partial_3 \mathcal{B}^{\varepsilon,3}\|_{\mathbf{L}^2} \|\partial_3 v^{\varepsilon,3}\| \\ &\leq \beta\varepsilon \|\nabla_h \mathcal{B}^{\varepsilon,h}\|_{\mathbf{L}^2} \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2} \\ &\leq \beta\varepsilon \|M(x_3)\|_{\mathbf{L}^2_{x_3}} \|\nabla_h \bar{u}^\varepsilon\|_{\mathbf{L}^2_{x_h}} \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2} \\ &\leq C(\bar{u}_0) \varepsilon^{3-\alpha} e^{-t\sqrt{2\beta}} + \frac{\varepsilon^\alpha}{16} \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2}^2;\end{aligned}$$

$$(6.28) \quad \left| \left\langle \frac{e_3 \wedge \mathcal{B}_3^\varepsilon}{\varepsilon}, v^{\varepsilon,h} \right\rangle \right| \leq C\varepsilon^{-1} e^{-\frac{1}{\varepsilon}} \|\bar{u}^\varepsilon\|_{\mathbf{L}^2} \|v^\varepsilon\|_{\mathbf{L}^2} \leq C(\bar{u}_0) \varepsilon e^{-t\sqrt{2\beta}} \left(1 + \|v^\varepsilon\|_{\mathbf{L}^2}^2\right).$$

We recall that

$$\mathcal{B}_4^{\varepsilon,h} = f(x_3) \begin{pmatrix} \bar{u}^{\varepsilon,2} \\ -\bar{u}^{\varepsilon,1} \end{pmatrix},$$

where  $f$  is defined in (6.7),

$$f(x_3) = a \left[ \exp\left(-\frac{x_3}{\varepsilon\sqrt{2\beta}}\right) + \exp\left(-\frac{1-x_3}{\varepsilon\sqrt{2\beta}}\right) \right] + b,$$

and that, if  $\varepsilon > 0$  is small enough, we have

$$|a| < 4(\beta + \sqrt{\beta})\varepsilon \quad \text{and} \quad |b| < 32\beta\varepsilon^2.$$

Then,

$$\begin{aligned}(6.29) \quad \beta\varepsilon \left| \langle \partial_3^2 \mathcal{B}_4^{\varepsilon,h}, v^{\varepsilon,h} \rangle \right| &= \beta\varepsilon \left| \langle f''(x_3) \bar{u}^\varepsilon, v^{\varepsilon,h} \rangle \right| \\ &\leq C\varepsilon^{\frac{1}{2}} \|\bar{u}^\varepsilon\|_{\mathbf{L}^2} \|v^\varepsilon\|_{\mathbf{L}^2} \\ &\leq C(\bar{u}_0) \varepsilon^{\frac{1}{2}} e^{-t\sqrt{2\beta}} \left(1 + \|v^\varepsilon\|_{\mathbf{L}^2}^2\right).\end{aligned}$$

Finally, we have

$$\begin{aligned}(6.30) \quad \left| \left\langle \frac{e_3 \wedge \mathcal{B}_4^\varepsilon}{\varepsilon}, v^{\varepsilon,h} \right\rangle \right| &\leq C \left[ \left( \int_0^1 \left| e^{-\frac{x_3}{\varepsilon\sqrt{2\beta}}} + e^{-\frac{1-x_3}{\varepsilon\sqrt{2\beta}}} \right|^2 dx_3 \right)^{\frac{1}{2}} + \beta\varepsilon \right] \|\bar{u}^\varepsilon\|_{\mathbf{L}^2} \|v^{\varepsilon,h}\|_{\mathbf{L}^2} \\ &\leq C(\bar{u}_0) \varepsilon^{\frac{1}{2}} e^{-t\sqrt{2\beta}} \left(1 + \|v^\varepsilon\|_{\mathbf{L}^2}^2\right).\end{aligned}$$

**End of the proof:** Summing all the inequalities from (6.15) to (6.30), we deduce from (6.14) that

$$\begin{aligned}\frac{d}{dt} \|v^\varepsilon\|_{\mathbf{L}^2}^2 + \varepsilon^\alpha \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2}^2 + \beta\varepsilon \|\partial_3 v^\varepsilon\|_{\mathbf{L}^2}^2 \\ \leq C(\bar{u}_0) \varepsilon^{\frac{1-2\alpha}{2}} e^{-t\sqrt{2\beta}} + C(\bar{u}_0) e^{-t\sqrt{2\beta}} \|v^\varepsilon\|_{\mathbf{L}^2}^2 + C(\bar{u}_0) \varepsilon \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2}^2.\end{aligned}$$

Since  $\alpha < \frac{1}{2}$ , there exists  $\varepsilon_0 = \varepsilon_0(\bar{u}_0) \in ]0, 1[$  such that, for any  $0 < \varepsilon < \varepsilon_0$ , we have  $C(\bar{u}_0) \varepsilon < \varepsilon^\alpha$ . Therefore, for any  $\varepsilon \in ]0, \varepsilon_0[$  small enough, by integrating the above inequality with respect to the time variable, we get

$$(6.31) \quad \|v^\varepsilon(t)\|_{\mathbf{L}^2}^2 \leq \|v^\varepsilon(0)\|_{\mathbf{L}^2}^2 + \bar{C}(\bar{u}_0) \varepsilon^{\frac{1-2\alpha}{2}} + C(\bar{u}_0) \int_0^t e^{-s\sqrt{2\beta}} \|v^\varepsilon(s)\|_{\mathbf{L}^2}^2 ds.$$

We recall that  $v^\varepsilon = u^\varepsilon - \bar{u}^\varepsilon - \mathcal{B}^\varepsilon$ . Thus,

$$\begin{aligned}\|v^\varepsilon(0)\|_{\mathbf{L}^2}^2 &\leq \|u^\varepsilon(0) - \bar{u}^\varepsilon(0)\|_{\mathbf{L}^2}^2 + \|\mathcal{B}^\varepsilon(0)\|_{\mathbf{L}^2}^2 \\ &\leq \|u_0^\varepsilon - \bar{u}_0\|_{\mathbf{L}^2}^2 + \|\mathcal{M}(\cdot)\|_{\mathbf{L}^2_{x_3}}^2 \|\bar{u}_0\|_{\mathbf{L}^2}^2 \leq \|u_0^\varepsilon - \bar{u}_0\|_{\mathbf{L}^2}^2 + C\varepsilon^{\frac{1}{2}} \|\bar{u}_0\|_{\mathbf{L}^2}^2.\end{aligned}$$

According to the Gronwall lemma, it follows from (6.31) that

$$\|v^\varepsilon(t)\|_{\mathbf{L}^2}^2 \leq \left( \|u_0^\varepsilon - \bar{u}_0\|_{\mathbf{L}^2}^2 + C\varepsilon^{\frac{1}{2}} \|\bar{u}_0\|_{\mathbf{L}^2}^2 + \bar{C}(\bar{u}_0) \varepsilon^{\frac{1-2\alpha}{2}} \right) \exp \left\{ \frac{\bar{C}(\bar{u}_0)}{\sqrt{2\beta}} \right\}.$$

Since  $\alpha < \frac{1}{2}$  and

$$\lim_{\varepsilon \rightarrow 0} \|u_0^\varepsilon - \bar{u}_0\|_{\mathbf{L}^2} = 0,$$

we obtain

$$\lim_{\varepsilon \rightarrow 0} \|v^\varepsilon\|_{\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2)} = 0,$$

and Theorem 6.1 is proved.  $\square$

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