



HAL
open science

Quasi-real time modeling for security of a drinking water distribution network

Hervé Ung, Olivier Piller, Denis Gilbert

► **To cite this version:**

Hervé Ung, Olivier Piller, Denis Gilbert. Quasi-real time modeling for security of a drinking water distribution network. *Procedia Engineering*, 2014, 70, pp.800-809. 10.1016/j.proeng.2014.02.087 . hal-01118428

HAL Id: hal-01118428

<https://hal.science/hal-01118428>

Submitted on 19 Feb 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



12th International Conference on Computing and Control for the Water Industry, CCWI2013

Quasi-real time modeling for security of a drinking water distribution network

U. Hervé^a, O. Piller^{a*}, G. Denis^a

^a*Irstea, 50, avenue de Verdun, F-33612 Cestas (France)*

Abstract

New sensors are currently being installed in water distribution networks, which give data every few minutes on water quantity and quality. How to manage the data to produce alarms, to get reliable water quality and water quantity models and to take appropriate countermeasures are the principal questions the SMaRT-Online^{WDN} project focuses on. This paper discusses how to enhance hydraulic and transport models and adapt them to the online use and security. Firstly, the transport model is enriched by adding incomplete junction mixing at nodes, axial dispersion and diffusion. Then the hydraulic model is also improved adopting transient formulation, graph simplification and online calibration of demands. Finally, source identification algorithms adapted for real-time are developed.

© 2013 The Authors. Published by Elsevier Ltd.

Selection and peer-review under responsibility of the CCWI2013 Committee

Keywords: Water distribution network; Security; Real-time, Slow-transient; T-junction; Imperfect mixing; Source identification

1. Introduction

Water distribution networks (WDNs) are critical infrastructures that are subject to potential terrorism. The injection of poison or a disease-causing agent in the network through supply tanks, reservoirs or pipes would cause great harm to the consumers. It would also raise severe reactions from the citizens and the contracting utilities about the water project manager's ability to secure and manage this important daily resource. To maintain water

* Corresponding author.

E-mail address: olivier.piller@irstea.fr

quality and improve security of water distribution networks, water operators are installing new sensors. These permit measurements every few minutes of the current water quantity (head, velocity) and water quality (pH, conductivity, turbidity, free chlorine) from a few well-chosen different positions of the network. These new data can be used for instance to calibrate demands and physical-chemical parameters or generate contamination alarms. The aim of the SMaRT-Online^{WDN} project (SMaRT-Online^{WDN}, 2013) in which this research thesis is involved is to develop an online security management toolkit for water distribution networks (WDNs). How to manage the data to produce alarms, to get reliable water quality and water quantity models and to take appropriate countermeasures are the principal questions that need to be answered. The SMaRT-Online^{WDN} project is a Franco-German project bringing together experts of water distribution networks from both countries. The consortium consists of operators, Veolia Environnement, Veolia Eau d'Île-de-France and Berliner Wasserbetriebe, an urban community, The Communauté Urbaine de Strasbourg (CUS), research institutes, Irstea, Engees, Fraunhofer Institute (IOSB) and DVGW-Technologiezentrum Wasser and an engineering consulting firm, 3S Consult GmbH. It incorporates the two WDN simulation software solutions, *Porteau* (2013) and *Sir3S* (2013). Two networks will serve as laboratory experiments in Berlin and Dresden to help in the development and validation of the codes. Finally, the software and the method solutions will be demonstrated with application on three different real case studies. This project will cover five different aspects: the event detection and alarm generation, the optimal sensor placement, the online simulation, the online contaminant source identification and mitigation of risks and the risk analysis with impact assessment. A PhD thesis is being carried at Irstea that focuses on the modeling aspect in the project.

The objective of this PhD research is to enhance hydraulic and transport models and adapt them to the online use and security. This paper reports on the preliminary results. It is presented in three complementary parts. Firstly, the transport model is enriched by adding incomplete junction mixing at nodes, axial dispersion and diffusion. Then the hydraulic model is also improved adopting transient formulation, graph simplification and online calibration of demands. Finally, source identification that relies on online modeling is realized.

In the following, first bibliography analysis and first results for the transport of water quality indicators are given.

2. Transport Model

2.1. Axial dispersion and diffusion

The simplification from a 3D to 1D model causes dispersion due to the use of plug flow instead of the flow profile. The phenomenon of advection-dispersion is well known and the equation, the Initial Conditions and the Boundary Conditions are the following:

$$\begin{aligned} \frac{\partial C}{\partial t} &= E(t, x) \frac{\partial^2 C}{\partial x^2} - U \frac{\partial C}{\partial x} \\ IC: C(x, 0) &= 0 \\ BC_u: C(0, t) &= C_0(t), BC_d: \partial_x C(L, t) = 0 \end{aligned} \quad (1)$$

Taylor (1953) studied the problem and proposed an approximation of the dispersion coefficient for very long travel time and laminar flow:

$$E^* = \frac{R^2 U_0^2}{48D} \quad (2)$$

Where R is the pipe radius, U_0 is the mean flow velocity and D is the molecular diffusivity [m^2/s].

To complete this formula Gill and Sankarasubramanian (1970) developed an exact but complex formula in steady laminar flow. A simplified one was given by Lee (2004):

$$\begin{aligned} E(t) &= E^* \beta(T) + E(0) \exp(-16T) \\ \beta(T) &= 1 - \exp(-16T) \\ T &= Dt/R^2 \end{aligned} \quad (3)$$

Where $\beta(T)$ is a correction for the transitional effect; $E(0)$ is the initial value of the dispersion rate; T is a dimensionless time.

Romero-Gomez and Choi (2011) have proposed to use a direction-dependent dispersion rate: with two backward and forward $E(0)$ and two different betas.

In turbulent regimes the velocity profile becomes flat and the hydrodynamic dispersion effect disappears. Taylor (1954) found that for turbulent regimes the coefficient for long travel time is proportional to U rather than U^2 and another formula than (2) should be used. The dimensionless advection-dispersion equation does not depend on U . When the dispersion coefficient E is small, *i.e.* when the dispersion term is negligible compared to the advection term, the problem becomes stiff and common practice is to not consider the dispersion term and only solve an advection equation.

When the velocity is almost zero (the water is at rest) the diffusion effect can't be neglected anymore. The phenomena equation is the following:

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} - U \frac{\partial C}{\partial x} \quad (4)$$

The diffusive coefficient needs to be estimated via tables and calibration *in situ*. It depends on the property of the chemical agent and the water quality. The diffusive coefficient is thermally activated, therefore it depends on the temperature. It follows the Arrhenius law.

$$D_i^m(T) = D_{i0}^m * e^{-\frac{E}{kT}} \quad (5)$$

Where E is the activating energy, k the Boltzmann constant and T the temperature.

2.2. Double T-junction

In recent studies, Ho and Khalsa, (2008), Choi *et al.* (2008), showed that mixing at junction is important regarding security and its assessment by calculation of the hydraulic quality state of the network. Ho and Khalsa, (2008) developed and adjusted with experiments, the Bulk-Advective Mixing model BAM for cross-junction imperfect mixing depending on the Reynolds number at the inlets and outlets.

In the cross-junction configuration, there is a principal flow where the inlet goes into the two-outlet direction and the secondary flow where the inlet is only going to the nearest outlet. For this configuration, the pure BAM model reads:

$$\begin{aligned} C4 &= C1 \\ C3 &= \frac{Q2C2 + (Q1 - Q4)C1}{Q3} \end{aligned} \quad (6)$$

The complete BAM model (Ho and Khalsa, 2008) consists of combining the perfect mixing and the pure BAM model with the following equation:

$$C_{scaled} = C_{bulk} + s (C_{perfect} - C_{bulk}) \quad (7)$$

Where s is a coefficient that depends on real state of the cross-junction; and C_{bulk} is calculated with Eq (6). They proposed to calibrate the s coefficient from experiments. Their results for X and N-junctions (or double T junctions) is implemented in the Epanet-BAM (2013).

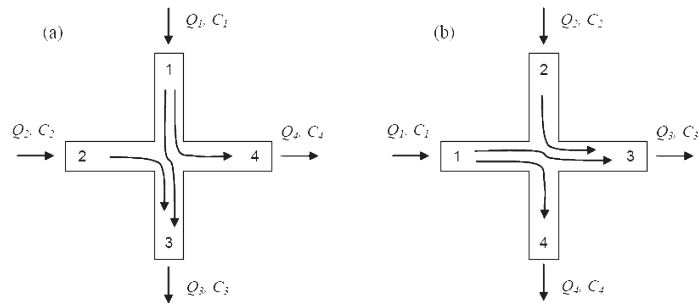


Figure 1: Cross-junction configuration

$$C4 = C1$$

$$C3 = \frac{Q2C2 + (Q1 - Q4)C1}{Q3}$$

The AZRED model (Choi *et al.*, 2008) proposes the use of a lookup table with interpolation or extrapolation for entries not in the table. It was validated on experimental values with a large range of flows. One limitation is the need for extrapolation for Reynolds ratios approaching zero or infinity. What is proposed in the SMaRT-Online^{W^{DN}} project is to complete the two previous models to be applicable for a one-dimensional model (Braun *et al.*, 2013). The approach is to develop a method with comparison and validation with CFD model and laboratory experiments.

For the case of incomplete mixing of a double-T junction or N junction, one configuration (Fig. 2) is particularly interesting: that is when there are two inputs followed by an intermediate section then two outputs. In all other cases the mixing is either complete or close to be complete. The different velocities, diameters and length of the intermediate pipe will influence the mixing. At the first step, the model is chosen with equal diameter for all inputs and outputs. A statistical study showed that this represents most cases in the network. The whole phenomenon is driven by four variables: the average Reynolds number inside the connecting pipe, the Reynolds inlet fraction of one of the entries, the Reynolds outlet fraction of one of the outputs and the distance between the T-junction. For pipes with unequal diameter the flow rate will be used in place of the Reynolds.

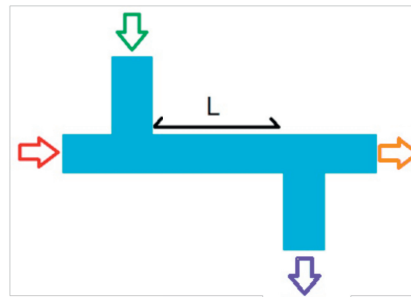


Figure 2: Double T-junctions configuration

To understand the importance of the problem case, a statistic has been made for the CUS model (12,800 links, 10,500 nodes, 970 km) on the number of T-junctions and their configurations. The first result is that 90% of the T-junction in the network are represented by 12 geometries of T-junctions and 33% with the diameters 100mm-100mm-100mm. Here after in Figs 3 and 4, statistics are shown for the more representative T-junctions (i.e., the latter case). There are six configuration possibilities depending on the flow directions that may change for the same T during the day, but this could be reduced to four configurations as shown in Figure 3. For each configuration their percentage of apparition in the network is given at the bottom right. The result is that the configurations 1 and 3 are the most represented in the network. Moreover, imperfect mixing may occur for N-junctions or U-junctions when geometries 1 then 3 are in series.

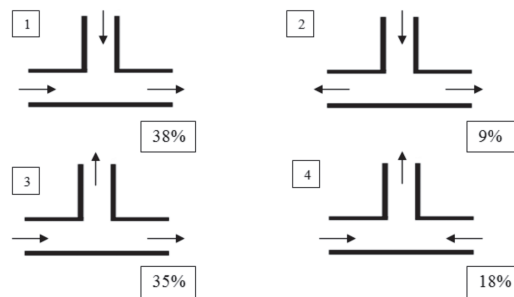


Figure 3. The 4 principal configurations of the T-junctions and their repartitions over the network

Additionally, a study of the Reynolds number at each input/output was made. The Porteau software has been used for the hydraulic simulation over a day duration with a hydraulic time step of 5 minutes to work out the algebraic Reynolds number shown in Fig. 4. It is positive if the flow goes into the T-junction. For a T-junction the branch side has been given the number 3 and the two main branches given a random number 1 or 2. The following graph in Fig. 4 presents a statistical repartition of the Reynolds numbers. It can be seen that laminar regime occurs for 21% of cases and transitional regime for the other 19% of cases. Then, only 60% of cases correspond to the turbulent flow.

3. Adaptation to real time monitoring

3.1. Slow transient formulation

Until now the flow in water distribution networks is usually calculated through the use of extended period simulation. The slow-varying boundary condition and the use of more and more hydraulics equipment, such as valves and pumps encourage the development of models to better estimate the transient phenomena that actually occur. A valve closure for instance takes a few seconds to some minutes to be carried out; hence if the time period

of the extended period simulation is bigger than this time interval, the model doesn't fit the overall working network well. In current modeling, levels of water in tanks is instantaneously changing from a steady state to another and causes miscalculations. The slow transient phenomena may be modeled by the rigid water column model (RWCM). The RWCM equations can be retrieved from the Barré de Saint-Venant equations for incompressible fluid while neglecting the compressibility and the elasticity effects. Then integrating from the two end nodes of a pipe for the momentum equation and using the divergence theorem for the mass balance at nodes leads to:

$$\begin{cases} AQ + d = 0_{nu} \\ L\dot{Q} + h(Q, r(t)) - A^T H^* - A_f^T H_f = 0_{np} \\ S\dot{H}_f + A_f Q^* + d_f(t) = 0_{nf} \end{cases} \quad (8)$$

Different solution methods can be proposed to solve this system of equations (Onizuka, 1986; Shimida 1989; Rogalla and Wolters, 1994a and 1994b; Piller and Propato, 2006; Jaumouillé *et al.* (2007). Semi-implicit Runge-Kutta methods are most of the time used to solve Eq. (7) when the system becomes stiff (e.g., a valve closing). Onizuka (1986) reduces Eq.(7) to obtain the loop flow rate and the tank level as minimum-size state variable. Shimida (1989) transforms the system (7) to an ODE with state variables: the flow rate and the tank level variable. Piller and Propato (2006) solve the system (7) directly. They propose an upgrade formulation to pressure-driven model. Jaumouillé *et al.* (2007) extends system (7) for incorporation of background leakages. Rogalla and Wolters (1994a) derivated another slow-transient model from the water-hammer equation. They use a combination of a Method of Characteristics (MOC) with the implicit finite difference Method. In the SMART-Online^WDN project comparisons will be made between the method by Rogalla and Wolters (1994a) and that of Piller and Propato (2006).

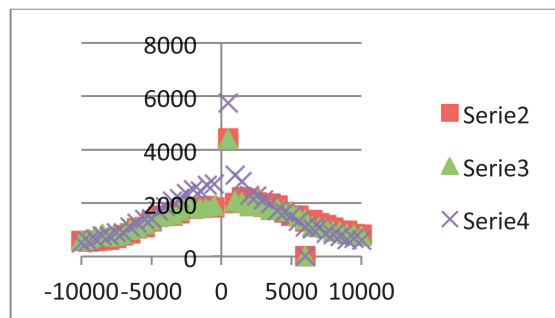


Figure 4. 3x100 Te Reynolds frequency histogram for the network

3.2. Graph Simplification

Graph simplification is useful to reduce calculation time. Quicker computation is needed in real time simulations. Some authors such as Constans *et al.* (2003), Deuerlein (2008), Deuerlein and Simpson (2012) and Perelman and Ostfeld (2012) propose different alternative solutions of simplification for hydraulic solving.

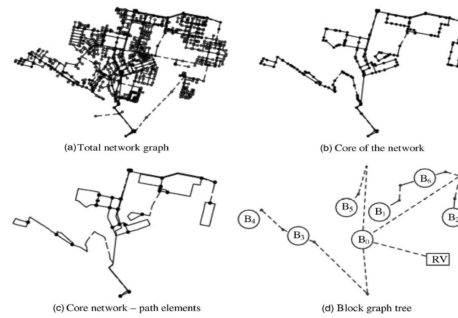


Figure 5: Simplification example by Deuerlein (2008)

Figure 5 shows the tree/core decomposition model of a general water supply network graph as defined by Jochen Deuerlein (2008). A graph is composed of forest and core parts. By gathering the information of the trees to the root nodes, the graph can be simplified to a core network (Figure 5b). The only elements left are the blocks that can be reduced to super nodes (Figure 5d). Next by backward and forward crossing, the hydraulic state can be calculated on the simplified graph (the block graph tree). The results are used to update boundary conditions at block scale. Nonlinear equations are solved for each block. Finally, it is used to work out the hydraulic outputs for the entire detailed network. Deuerlein and Simpson (2012) improved the solving method. Constans (2003) and Perelman and Ostfeld (2012) propose similar simplifications that lead to identifying different clusters that are strong or weak connected components. In this PhD research parallelization is planned to be used.

3.3. Water quantity and water quality parameter calibration

The previous parts were done to improve the direct model for water quantity and water quality to be as accurate and fast as possible. After that, an inverse problem can be formulated, the more calibrated the direct formulation is, the better the model prediction will be. These parameters are roughness, demands and transport-reaction-dispersion parameters. The use of real time sensors will permit a large quantity of data to calibrate these parameters and will also give real time varying parameters such as the demand. Least squares method coupled with a Tikhonov regularization is proposed. The method consists to minimize with a Gauss-Newton method the following quantity:

$$\frac{1}{2} \|y(D) - y^{mes}\|^2 + \lambda \|D - D_{prod}\|^2 \quad (9)$$

Lambda should be large enough for the criterion to be strongly convex but not too large for the measurements to correct the system. The choice is made to minimize the number of iterations by using information from the derivatives to update the solution.

$$D^{k+1} = D^k - (J_k J_k^T)^{-1} J_k^T residual_k \quad (10)$$

4. Source identification

The problem of the contaminant source identification is an inverse problem: knowing detections of contaminations at some locations (sensor responses), the source location of these contaminations needs to be found. The SecurEau project (2013) is a European project that is finishing this year. The main goals were the development of methods to improving CBRN contamination detection and decontamination. Only non-specific sensors were used for detection of abnormal state of water quantity and water quality indicators. Different methods of source identification have been developed with the only prerequisite the binary responses from sensors: FEUP

(Faculty of Engineering of the Technical University of Porto) used an artificial neural network model, RTU (Riga Technical University) a flow direction based algorithm and Irstea, stochastic methods (minimum relative entropy or Least-squares). The Irstea formulation is composed of two parts. The first part consists of the enumeration of every pair of nodes and times that are susceptible of being source of contamination.

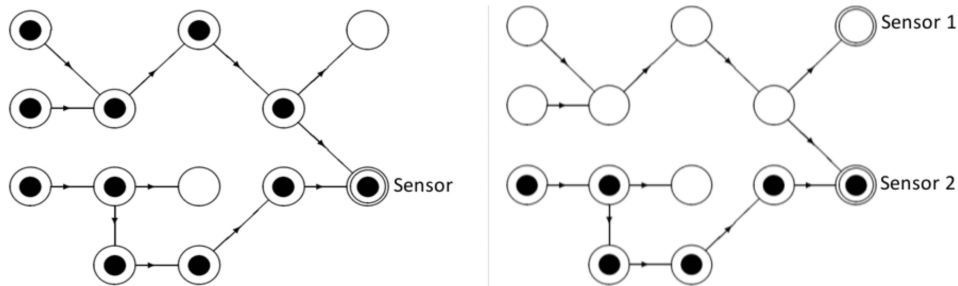


Figure 6: Backtracking scheme for enumeration of potential contamination sources

Figure 6 shows the core of the backtracking algorithm. An inverse transport equation is used from the sensor detections. Every node that can explain these detections is listed with their specific times. The same is done for each sensor that doesn't detect, which permits to eliminate nodes. Indeed, if these nodes were sources, the sensor would have been activated. A transport matrix of contamination is then assembled. The solution of the minimum relative entropy (MRE) problem permits to give weights to the nodes listed previously based on the probability of the node to be the source of contamination. The minimum relative entropy problem is the following optimization problem:

$$\begin{cases} \min REnt(q) = \int_{\Omega} q(X) \ln \left(\frac{q(X)}{p(X)} \right) dX \\ \int_{\Omega} q(X) dX = 1 \\ AE_q(X) = \mathbb{1}_n \end{cases} \quad (11)$$

Where q is the probability density function (pdf) of interest; p is the prior pdf; A is the transport matrix; and E_q is the expectation or first moment operator with pdf q . Another method has been developed using a least squares method coupled with Tikhonov regularization that may give different results. The problem is to minimize the following quantity:

$$\frac{1}{2} \|AX - \mathbb{1}_n\|^2 + \alpha \|X - X_0\|^2 \quad (12)$$

Where alpha is the Tikhonov regularization coefficient that is estimated by an L-curve method. This coefficient shouldn't be too low, to regularize the system, but not too high to give as much possible weight for the first term.

Figure 7 gives an example of the result of the methods described previously. The contamination is at the root node and the yellow nodes are the sensor node locations. The contamination matrix (Fig 7b) and its time simplification are given. The results of the MRE method are shown on the left and those of Tikhonov regularization on the right. The first method indicated three nodes with the same probability of being the source of contamination. The second method identifies five possible sources, with the real source of contamination having the highest probability to be the source of contamination. Anyway in both cases the root note was founded. The methods have also been tested on larger networks and in most cases the source of contamination is inside the pool of possible contamination sources. Further research should be done to improve the precision of the methods concerning the probabilities and range given.

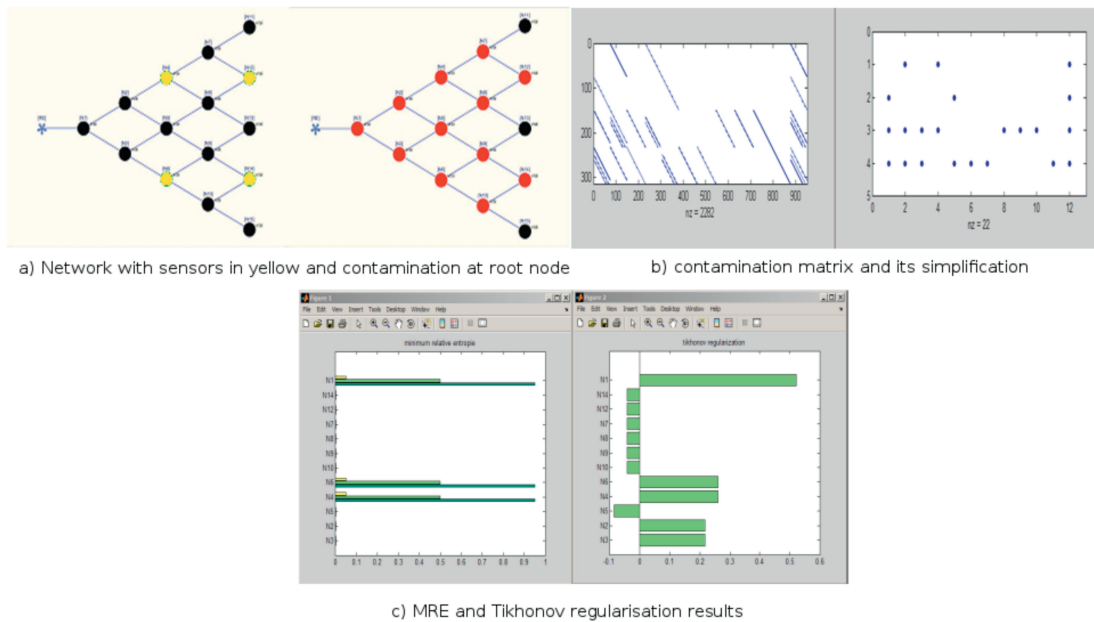


Figure 7: Minimum relative entropy and Tikhonov regularization source identification result on a rooted contaminated network

5. Conclusion

This paper discusses how to enhance hydraulic and transport models and adapt them for online observation and for improving security. The research is part of the SMaRT-Online^WDN project and the first author is conducting a PhD research on this topic.

The first step is to improve transport models to reproduce the main phenomena observable at this scale of time. For the quality model, imperfect junction mixing, axial dispersion and diffusion effects should be considered. The imperfect mixing solution completes previous research work by Choi *et al.* (the AZRED model). A lookup table is built on the full range of Reynolds with CFD simulations. The model is validated on several real cases (*e.g.* the CUS Strasbourg water utility). Hydrodynamics dispersion for transport along pipe of laminar and transitional flow is necessary as molecular diffusion. Importance of these phenomena for continuously changing boundary conditions will be assessed in the project.

Slow-transient flow must replace extended period simulation models to better assess the regulation of equipments and the slow-varying boundary conditions. For acceleration of computation speed, different scales of the network graph need to be created with different simplification levels with automatic aggregation/disaggregation process. Then, this hydraulic transport model should be calibrated online for the demand. A least-squares problem is formulated and a prediction-correction method is applied. It is necessary to develop a gradient-based algorithm that is able to give prediction in pseudo-real time.

In case of intrusion threats, binary responses at non-specific water quality sensors can permit finding of the possible sources of contamination and its progression. It consists of a backtracking algorithm to enumerate all potential sources of contamination and times. Then the minimum relative entropy and Tikhonov regularization methods are used to explore among the potential solutions and give probabilities of being one of the source of contamination to each possible nodes.

This development should benefit the Irstea freeware solution named Porteau (Porteau, 2013) and the 3SConsult software Sir3S (Sir3s, 2013). Source identification algorithms that use reliable previously-calibrated velocity for inverse transport will be developed.

Acknowledgements

The project is supported by the German Federal Ministry of Education and Research (BMBF; project: 13N12180) and by the French Agence Nationale de la Recherche (ANR; project: ANR-11-SECU-006).

References

- Braun M., Bernard T., Ung H., Piller O. and Gilbert D. 2013. Model based Investigation of Transport Phenomena in Water Distribution Networks for Contamination Scenarios. 12th International Conference on Computing and Control for the Water Industry, CCWI2013, Perugia, 10 pages.
- Choi, C. Y., Shen, J. Y., and Austin, R. G. 2008. Development of a Comprehensive Solute Mixing Model (AZRED) for Double-Tee, Cross, and Wye Junctions. *Water Distribution Systems Analysis* 2008, 1-10.
- Constans S., Brémond B., Morel P. 2003. Simulation and Control of Chlorine Levels in Water Distribution Networks. *J. Water Resour. Plan. Manage.*, 129(2), pp. 135-145.
- Deuerlein, J. W. 2008. Decomposition Model of a General Water Supply Network Graph. *Journal of Hydraulic Engineering*, 134(6), 822-832.
- Deuerlein, J. and Simpson A.R. 2012. A modified Global gradient algorithm based on topological decomposition. 14th Water Distribution Systems Analysis Conference. pp. 384-397.
- EPANET-BAM (2013). <http://www.sandia.gov/EPANET-BAM/download.htm>, accessed on June 5, 2013.
- Gancel, G. 2006. "Modélisation d'un problème inverse pour la qualité de l'eau dans les réseaux d'eau potable," Applied Mathematics thesis from the University of Bordeaux (PRES), defended on, 136 pages, Talence, France.
- Gill, W. N. and Sankarasubramanian, R. 1970. Exact analysis of unsteady convective diffusion. *Proceedings of the Royal Society of London Series A*, 316: 341-350.
- Ho, C.K., and S.S. Khalsa, 2008, EPANET-BAM: Water Quality Modeling with Incomplete Mixing in Pipe Junctions, in *Proceedings of the 2008 Water Distribution System Analysis Conference*, Kruger National Park, South Africa, August 17-20, 2008. (SAND2008-3065C), 11 pages.
- Jaumouillé, E., Piller, O., and Van Zyl, J. E. 2007. A hydraulic model for water distribution systems incorporating both inertia and leakage. *Water Management Challenges in Global Change (CCWI2007 and SUWM2007 Conference)*, Leicester, GBR, 3-5 September 2007, 8 pages.
- Lee, Y. 2004. Mass Dispersion in Intermittent Laminar Flow. PhD Dissertation, University of Cincinnati, Cincinnati, Ohio, 182 pages.
- Onizuka, k. 1986. System Dynamics Approach to Pipe Network Analysis. *Journal of Hydraulic Engineering*, 112(8), 728-747.
- Perelman, L., and Ostfeld, A. 2012. Water-Distribution Systems Simplifications through Clustering. *Journal of Water Resources Planning and Management*, 138(3), 218-229.
- Piller, O., and Propato, M. 2006. Slow transient pressure driven modeling in water distribution networks. 8th annual Water Distribution System Analysis Symposium, University of Cincinnati, Cincinnati, Ohio, USA, August 27-30 2006, printed by ASCE, 13.
- Porteau 2013, <http://porteur.irstea.fr/>, accessed on June 5, 2013.
- Rogalla B.U. and Wolters A. 1994a. Slow transients in closed conduit flow – Part I numerical methods. M.H. Chaudhry and L. W. Mays (eds.), *Computer Modeling of Free-Surface and Pressurized Flows*, © 1994 Kluwer Academic Publishers. Printed in the Netherlands. 613-642.
- Rogalla B.U. and Wolters A. 1994b. Slow transients in closed conduit flow – Part II validation and application of the numerical methods. M.H. Chaudhry and L. W. Mays (eds.), *Computer Modeling of Free-Surface and Pressurized Flows*, NATO ASI Series Volume 274, 1994 Kluwer Academic Publishers. Printed in the Netherlands. 643-671.
- Romero-Gomez, P., and Choi, C. 2011. Axial Dispersion Coefficients in Laminar Flows of Water-Distribution Systems. *Journal of Hydraulic Engineering*, 137(11), pp. 1500-1508.
- SecurEau 2013, <http://www.secureau.eu/>, accessed on February 21, 2013.
- Shimada, M. 1989. Graph-Theoretical Model for Slow Transient Analysis of Pipe Networks. *Journal of Hydraulic Engineering*, 115(N°9), 1165-1183.
- Sir3S 2013, <http://www.3sconsult.de/sir3s.html>, accessed on June 5, 2013.
- SMaRT-OnlineWDN 2013. <http://www.smart-onlinewdn.eu/>, accessed on June 05, 2013.
- Taylor, G. 1953. Dispersion of Soluble Matter in Solvent Flowing Slowly through a Tube. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 219(1137), 186-203.
- Taylor, G. 1954. The Dispersion of Matter in Turbulent Flow through a Pipe. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 223(1155), 446-468.