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▶ **To cite this version:**

Rafe Mazzeo, Frank Pacard, Tatiana Zolotareva. HIGHER CODIMENSION ISOPERIMETRIC PROBLEMS. 2015. hal-01118388

HAL Id: hal-01118388
https://hal.science/hal-01118388

Preprint submitted on 22 Feb 2015

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HIGHER CODIMENSION ISOPERIMETRIC PROBLEMS

RAFE MAZZEO, FRANK PACARD, AND TATIANA ZOLOTAREVA

Abstract. We consider a variational problem for submanifolds $Q \subset M$ with nonempty boundary $\partial Q = K$. We propose the definition that the boundary K of any critical point Q have constant mean curvature, which seems to be a new perspective when $\dim Q < \dim M$. We then construct small nearly-spherical solutions of this higher codimension CMC problem; these concentrate near the critical points of a certain curvature function.

1. Introduction

Constant mean curvature (CMC) hypersurfaces are critical points of the area functional subject to a volume constraint. Examples include sufficiently smooth solutions to the isoperimetric problem. If K is an embedded submanifold in a Riemannian manifold (M^{m+1}, g), then its mean curvature vector H_K is the trace of its shape operator. When K is a hypersurface, then we say that K has CMC if this vector has constant length, and this is the only sensible definition in this case. However, when $\operatorname{codim} K > 1$, it is less obvious how to formulate the CMC condition, since there is more than one way one might regard the mean curvature vector as being constant. One definition that has perhaps received the most attention is to require that H_K be parallel. This is quite restrictive, and for that reason, not very satisfactory.

We propose here a different, and directly variational, definition. Building on ideas of Almgren [1], and extending one standard characterization of CMC hypersurfaces, we define constant mean curvature submanifolds to be boundaries of submanifolds which are critical for a certain energy functional. Roughly speaking, we say that K has constant mean curvature if $K = \partial Q$ where Q is minimal, K has CMC in Q, and H_K has no component orthogonal to Q.

The goal of this paper is to show that generic metrics on any compact manifold admit 'small' CMC submanifolds in this sense. The result proved here is a generalization of a well-known theorem by Ye [9], which constructs families of CMC hypersurfaces which are small perturbations of geodesic spheres centered at nondegenerate critical points of the scalar curvature

R.M. Supported by NSF-DMS-1105050.
F. Pacard and T. Zolotareva are partially supported by the ANR-2011-IS01-002 grant.
T. Zolotareva is partially supported by FMJH foundation.

function $\mathcal{R}$. The more recent paper [6] obtains such families of CMC hypersurfaces under general condition on the scalar curvature and in particular when it is constant; in that case, these hypersurfaces are centered near critical points of a different curvature invariant. These various results illustrate the sense in which the metric must be generic: some scalar function of the curvature must have nondegenerate critical points.

Let us now introduce the relevant curvature function. For any $(k+1)$-dimensional subspace $\Pi_p \subset T_pM$, define the partial scalar curvature

$$\mathcal{R}_{k+1}(\Pi_p) := -\sum_{i,j=1}^{k+1} \langle R(E_i, E_j)E_i, E_j\rangle.$$

where $E_1, \ldots, E_{k+1}$ is any orthonormal basis for Π_p. Note that $\mathcal{R}_{m+1}(T_pM)$ is the standard scalar curvature at p, while $\mathcal{R}_2(\Pi_p)$ is twice the sectional curvature of the 2-plane Π_p. The Grassman bundle $G_{k+1}(TM)$ is the fibre bundle over M with fibre at $p \in M$ the Grassmanian of all $(k+1)$-planes in T_pM. We regard $\mathcal{R}_{k+1}$ as a smooth function on $G_{k+1}(M)$.

Denote by $\mathcal{S}^k_\varepsilon(\Pi_p)$ and $\mathcal{B}^{k+1}_\varepsilon(\Pi_p)$ the images of the sphere and ball of radius ε in Π_p under the exponential map $\exp_p$, $p \in M$. We can now state our main result.

Theorem 1.1. *If Π_p is a nondegenerate critical point of $\mathcal{R}_{k+1}$, then for all ε sufficiently small, there exists a CMC submanifold $K_\varepsilon(\Pi_p)$ which is a normal graph over $\mathcal{S}^k_\varepsilon(\tilde{\Pi}_{\tilde{p}})$ by some section with $\mathcal{C}^{2,\alpha}$ norm bounded by $C\varepsilon^2$, and* $\operatorname{dist}(\tilde{\Pi}_{\tilde{p}}, \Pi_p) \leq c\,\varepsilon$.

Our construction of CMC submanifolds generalizes the method introduced in [6], and can also be carried out in certain cases when the partial scalar curvature has degenerate critical points, for example when (M, g) is Einstein has or constant partial scalar curvature.

Theorem 1.2. *There exists $\varepsilon_0 > 0$ and a smooth function*

$$\Psi : G_{k+1}(TM) \times (0, \varepsilon_0) \longrightarrow \mathbb{R},$$

defined in (9) *below, such that if $\varepsilon \in (0, \varepsilon_0)$ and Π_p is a critical point of $\Psi(\cdot, \varepsilon)$, then there exists an embedded k-dimensional submanifold $K_\varepsilon(\Pi_p)$ with constant mean curvature equal to k/ε. This submanifold is a normal graph over a geodesic sphere $\mathcal{S}^k_\varepsilon(\Pi_p)$ with respect to a vector field, the $\mathcal{C}^{2,\alpha}$ norm of which is bounded by $c\varepsilon^2$.*

The function Ψ is essentially just the associated energy functional restricted to a particular finite dimensional set of approximately CMC submanifolds.

The outline of this paper is as follows. We first give a more careful description of our proposed definition of constant mean curvature and its relationship to the associated energy functional. We introduce the linearization and second variation of this energy, then compute these operators in detail

for the round sphere $S^k \subset \mathbb{R}^{m+1}$. The construction of 'small' solutions of the CMC problem concentrating around critical points of the function Ψ proceeds in stages. We construct a family of approximate solutions, then solve the problem up to a finite dimensional defect. This defect depends on certain parameters in the approximate solution, and in the last step we employ a variational argument to choose the parameters appropriately to solve the exact problem. Certain long technical calculations are relegated to the appendices.

2. Preliminaries

In this section we begin by setting notation and recalling some standard formulæ. This is followed by the introduction of a variational notion of constant mean curvature for closed submanifolds of arbitrary codimension. We compute the first and second variations of the associated energy functional, and then explain what these look like for round spheres (of arbitrary codimension) in $\mathbb{R}^{m+1}$.

2.1. **The mean curvature vector.** Let (M^{m+1}, g) be a compact smooth Riemannian manifold, and consider smooth, closed k-dimensional submanifolds $K \subset M$ and $(k+1)$-dimensional submanifolds Q with nonempty boundary K, $1 \leq k \leq m$. We write ∇^Σ for the connection on any embedded submanifold Σ, and reserve ∇ for the full Levi-Civita connection on M.

The second fundamental form of Σ is the symmetric bilinear form on $T\Sigma$ taking values in the normal bundle $N\Sigma$ defined by

$$h(X, Y) := \nabla_X Y - \nabla^\Sigma_X Y = \pi_{N\Sigma} \nabla_X Y;$$

here $\pi_{N\Sigma}$ is the fibrewise orthogonal projection $T_\Sigma M \to N\Sigma$. The trace of h is a section of $N\Sigma$, and is called the mean curvature vector field

$$H_\Sigma := \operatorname{tr}^g h = \sum_{i=1}^{\dim \Sigma} h(E_i, E_i),$$

where $\{E_i\}$ is any orthonormal basis for $T_p\Sigma$. By definition, Σ is minimal provided $H_\Sigma \equiv 0$.

2.2. **Constant mean curvature in high codimension.** Let us now specialize to the case where $Q^{k+1} \subset M$ is a smooth, compact submanifold with boundary, with $\partial Q = K$. The normal bundle NK decomposes as an orthogonal direct sum

$$NK = NK^\perp \oplus NK^\parallel,$$

where $NK^\parallel = NK \cap TQ$ has rank 1 and $NK^\perp = N_K(NQ) = NK \cap NQ$ has rank $m - k$. We shall write n for the inward pointing unit normal to K in Q. Thus if $\Phi \in NK$, then $\Phi = [\Phi]^\perp + [\Phi]^\parallel = [\Phi]^\perp + \phi\, n$ for some scalar function ϕ.

Definition 2.1. *The closed submanifold $K \subset M$ is said to have constant mean curvature if $K = \partial Q$ where Q is minimal in M, K has constant mean curvature in Q and the Q-normal component $[H_K]^\perp \in NK^\perp$ vanishes.*

A key motivation is that this definition is variational, where the relevant energy is given by

$$\mathcal{E}_{h_0}(Q) := \mathrm{Vol}_k(\partial Q) - h_0 \,\mathrm{Vol}_{k+1}(Q). \tag{1}$$

Proposition 2.1. *The submanifold $K = \partial Q$ has constant mean curvature h_0 (in the sense of Definition 2.1) if and only if*

$$D\mathcal{E}_{h_0}|_Q = 0.$$

The meaning of the differential here is the usual one. Let Ξ be a smooth vector field on M and denote by ξ_t its associated flow. For t small, write $Q_t = \xi(t, Q)$ and $K_t := \partial Q_t = \xi(t, K)$. The requirement in the Proposition is then that for any smooth vector field Ξ,

$$\frac{d}{dt}\mathcal{E}_{h_0}(Q_t)\Big|_{t=0} = 0.$$

The proof is standard. The classical first variation formula (see Appendix) states that

$$\frac{d}{dt}\mathrm{Vol}(K_t)\Big|_{t=0} = -\int_K g(H_K, \Xi)\,\mathrm{dvol}_K,$$

and

$$\frac{d}{dt}\,\mathrm{Vol}(Q_t)\,\Big|_{t=0} = -\int_Q g(H_Q, \Xi)\,\mathrm{dvol}_Q - \int_K g(n, \Xi)\,\mathrm{dvol}_K\,.$$

It follow directly from these that

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}_{h_0}(Q_t) = 0,$$

for all vector fields Ξ if and only if $H_K = h_0\, n$ and $H_Q \equiv 0$, as claimed.

The definition above coincides with the standard meaning of CMC when K is a hypersurface in M which is the boundary of a region Q. In particular, if $K^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^{m+1}$ and K has CMC as a hypersurface in $\mathbb{R}^{k+1}$, then it has CMC in the sense of Definition 2.1. In particular, any round sphere $S^k \subset \mathbb{R}^{m+1}$ has CMC in this sense.

2.3. **The Jacobi operator.** Let us now study the differential of the mean curvature operator, which is known as the Jacobi operator. For this subsection, we revert to considering an arbitrary submanifold Σ, either closed or with boundary, and shall now recall the expression for this operator.

The Jacobi operator J_Σ is the differential of the mean curvature vector field with respect to perturbations of Σ. To describe this more carefully, set $B_\varepsilon(N\Sigma) = \{(q, v) \in T_\Sigma M : |v| < \varepsilon\}$ and consider the exponential map exp from an ε-neighborhood of the zero section in $T_\Sigma M$ into M. Since $\exp_*|_{\{v=0\}} = \mathrm{Id}$, If $\Phi \in \mathcal{C}^2(\Sigma; T_\Sigma M)$ has $||\Phi||_{\mathcal{C}^0}$ sufficiently small, then $\Sigma_\Phi := \{\exp_q(\Phi(q)) : q \in \Sigma\}$ is an embedded submanifold. We shall denote

the family of submanifolds $\Sigma_{s\Phi}$ by Σ_s, and their mean curvature vector fields by H_s. We also write $F_s : \Sigma \to \Sigma_s$ for the map $q \mapsto \exp_q(s\Phi(q))$. By definition,

$$J_\Sigma(\Phi) = \nabla_{\partial/\partial s} H_s\big|_{s=0}.$$

We shall be particularly interested in the case where Φ is a section of the normal bundle $N\Sigma$. When $\partial\Sigma \neq \emptyset$, we also require that $\Phi = 0$ on $\partial\Sigma$. The operator $\pi_{N\Sigma} \circ J_\Sigma$ will be denoted J^N_Σ. We recall in the appendix the proof of the standard formula

$$J^N_\Sigma = -\Delta^N_\Sigma + \mathrm{Ric}^N_\Sigma + \mathfrak{H}^{(2)}_\Sigma, \tag{2}$$

where Δ^N_Σ is the (positive definite) connection Laplacian on sections of $N\Sigma$,

$$\forall \Phi \in N\Sigma, \quad \Delta^N_\Sigma \Phi = \sum_{i=1}^{\dim(\Sigma)} \nabla^N_{E_i} \nabla^N_{E_i} \Phi - \nabla^N_{\nabla^N_{E_i} E_i} \Phi, \quad \nabla^N_X Y = \pi_{N_\Sigma} \circ \nabla_X Y$$

and the other two terms are the following symmetric endomorphisms of $N\Sigma$:

(i) The orthogonal projection $\mathrm{Ric}^N_\Sigma = \pi_{N_\Sigma} \circ \mathrm{Ric}_\Sigma$ of the partial Ricci curvature Ric_Σ, defined by

$$\begin{aligned} \langle \mathrm{Ric}_\Sigma X, Y\rangle &:= -\mathrm{tr}^{\,g} \langle R(\cdot, X)\cdot, Y\rangle \\ &= -\sum_{i=1}^{\dim \Sigma} \langle R(E_i, X)E_i, Y\rangle \quad \text{for all} \quad X, Y \in TM \end{aligned} \tag{3}$$

note that the curvature tensor appearing on the right is the one on all of M, and is not the curvature tensor for Σ;

(ii) the square of the shape operator, defined by

$$\mathfrak{H}^{(2)}_\Sigma(X) := \sum_{i,j=1}^{\dim \Sigma} \langle h(E_i, E_j), X\rangle h(E_i, E_j), \quad \text{for all} \quad X \in TM \tag{4}$$

In general, $J_\Sigma(\Phi) \neq J^N_\Sigma(\Phi)$ since $J_\Sigma(\Phi)$ has a nontrivial component $J^T_\Sigma(\Phi)$ which is parallel to Σ; as we show later, that part is canceled in our final formula so we do not need to make it explicit. Note, however, that $J^T_\Sigma(\Phi)$ vanishes when Σ is minimal. Indeed, writing the mean curvature vector field to $\Sigma_{s\Phi}$ in the form

$$H_s = \sum_\nu \langle H_s, N_\nu(s)\rangle N_\nu(s),$$

where $N_\nu(s)$, $\nu = \dim\Sigma + 1, \ldots, m+1$ is a local orthonormal frame for $N\Sigma_{s\Phi}$ we find

$$[J_\Sigma(\Phi)]^T = \sum_\nu \Big(\Big(\langle \nabla_{\partial/\partial s} H_s\big|_{s=0}, N_\nu(0)\rangle + \langle H_\Sigma, \nabla_{\partial/\partial s}\big|_{s=0} N_\nu(s)\rangle \Big) N_\nu(0)$$

$$\langle H_\Sigma, N_\nu(0)\rangle \, \nabla_{\partial/\partial s}\big|_{s=0} N_\nu \Big)^T = \sum_\nu \langle H_\Sigma, N_\nu\rangle \Big[\nabla_{\partial/\partial s} N_\nu(s)\big|_{s=0} \Big]^T,$$

and if $H_\Sigma = 0$, we have $J^T_\Sigma = 0$.

2.4. **The second variation of $\mathcal{E}_{h_0}$.** We set

$$\mathcal{C}^{2,\alpha}_0(NQ) := \{V \in \mathcal{C}^{2,\alpha}(NQ) : V|_K = 0\}.$$

With this notation in mind, we have the:

Definition 2.2. *The minimal submanifold Q is nondegenerate if*

$$J_Q : \mathcal{C}^{2,\alpha}_0(NQ) \longrightarrow \mathcal{C}^{0,\alpha}(NQ),$$

is invertible.

Lemma 2.1. *If Q is nondegenerate, then there is a smooth mapping $\Phi \mapsto Q_\Phi$ from a neighbourhood of 0 in $\mathcal{C}^{2,\alpha}(NK)$ into the space of $(k+1)$-dimensional minimal submanifolds of M with $\mathcal{C}^{2,\alpha}$ boundary, such that Q_0 is the initial submanifold Q and $\partial Q_\Phi = K_\Phi$.*

Proof. Fix a continuous linear extension operator

$$\mathcal{C}^{2,\alpha}(NK) \ni \Phi \mapsto V_\Phi \in \mathcal{C}^{2,\alpha}(T_Q M).$$

Thus V_Φ is a vector field along Q which restricts to Φ on K. Without loss of generality, we can assume that $V_\Phi \in TQ$ if $[\Phi]^\perp = 0$ and $V_\Phi \in NQ$ when $[\Phi]^\parallel = 0$. Next, let W be a $\mathcal{C}^{2,\alpha}$ section of NQ which vanishes on K. If both $||\Phi||_{2,\alpha}$ and $||W||_{2,\alpha}$ are sufficiently small, then $\exp_Q(V_\Phi + W)$ is an embedded $\mathcal{C}^{2,\alpha}$ submanifold Q_U, $U = V_\Phi + W$, with boundary $K_\Phi := \partial Q_U$. Denoting its mean curvature vector by $H(\Phi, W)$, then

$$D_W H|_{(0,0)}(W) = J_Q W.$$

Since Q is minimal, $D_W H|_{(0,0)}(W)$ takes values in NQ, whereas $H(\Phi, W) \in NQ_U \subset T_{Q_U} M$, so we cannot directly apply the implicit function theorem. To remedy this, first let $\widetilde{H}(\Phi, W)$ be the parallel transport of $H(\Phi, W)$ along the geodesic $s \mapsto \exp_q(sU(q))$, from $s = 1$ to $s = 0$. Parallel transport preserves regularity (this reduces to the standard result on smooth dependence on initial conditions for the solutions of a family of ODE's), so $\widetilde{H}(\Phi, W)$ is a $\mathcal{C}^{0,\alpha}$ section of $T_Q M$. Now define

$$\widehat{H}(\Phi, W) := \pi_{NQ} \circ \widetilde{H}(\Phi, W),$$

where $\pi_{NQ} : T_Q M \to NQ$ is the orthogonal projection. Since $H(\Phi, W) \in N_{Q_U} M$ and since $||U||_{\mathcal{C}^1}$ is small, $\widetilde{H}(\Phi, W)$ lies in the nullspace of π at any $q \in Q$ if and only if it actually vanishes. Thus it is enough to look for solutions of $\widehat{H}(\Phi, W) = 0$. Notice that $D_W \widehat{H}|_{(0,0)} = J_Q$. We can now apply the implicit function theorem to conclude the existence of a $\mathcal{C}^{2,\alpha}$ map $\Phi \mapsto W(\Phi)$ such that $H(\Phi, W(\Phi)) = \widehat{H}(\Phi, W(\Phi)) \equiv 0$ for all small Φ. □

We henceforth denote by Q_Φ the minimal submanifold $\exp_Q(V_\Phi + W(\Phi))$. Observe that when $[\Phi]^\perp = 0$, the submanifold parametrized by $\exp_Q(V_\phi)$ is $\mathcal{O}(\|\Phi\|^2_{\mathcal{C}^{2,\alpha}})$ close to Q_Φ; this is easy to check when $\Phi := \phi\, n$ where ϕ is small. Therefore, in this 'tangential' case, we conclude that

$$U_\Phi = V_\Phi + \mathcal{O}(\|\Phi\|^2_{\mathcal{C}^{2,\alpha}}).$$

Next, when $[\Phi]^{\|} = 0$, we define Z_Φ as the solution of

$$J_Q Z_\Phi = 0, \qquad Z_\Phi|_K = \Phi,$$

and it is easy to check that the submanifold parametrized by $\exp_Q(Z_\phi)$ is also $\mathcal{O}(\|\Phi\|^2_{\mathcal{C}^{2,\alpha}})$ close to Q_Φ. We summarize all this in the

Lemma 2.2. *When $\|\Phi\|_{\mathcal{C}^{2,\alpha}}$ is small, we have the decomposition*

$$U_\Phi = V_{[\Phi]^{\|}} + Z_{[\Phi]^\perp} + \mathcal{O}(\|\Phi\|^2_{\mathcal{C}^{2,\alpha}}),$$

where $Z_{[\Phi]^\perp}$ is the solution of

$$J_Q Z_{[\Phi]^\perp} = 0, \qquad Z_{[\Phi]^\perp}\Big|_K = [\Phi]^\perp.$$

Now consider the energy $\mathcal{E}_{h_0}$ along a one-parameter family $s \mapsto Q_s := Q_{s\Phi}$ of minimal submanifolds with boundaries $K_s := \partial Q_s = K_{s\Phi}$. By the formulæ of the last subsection,

$$\frac{d}{ds}\mathcal{E}_{h_0}(Q_s) = -\int_{K_s} g(H_s - h_0\, n_s, \partial/\partial s)\, \mathrm{dvol}_{K_s},$$

where H_s is the mean curvature of K_s and n_s is the inward pointing unit normal to K_s in Q_s. Note that this first variation of energy is localized to the boundary; the interior terms vanish because of the minimality of the Q_s. Our task is to compute

$$\frac{d^2}{ds^2}\mathcal{E}_{h_0}(Q_s)\Big|_{s=0},$$

when Q is critical for $\mathcal{E}_{h_0}$.

Parametrize both K_s and Q_s by $y \mapsto F_s(y) := \exp_y(U_{s\Phi}(y))$ (with $y \in K$ or $y \in Q$, respectively). As before, choose a smooth local orthonormal frame E_α for TK, so that $(F_s)_* E_\alpha = E_\alpha(s)$ is a local (non-orthonormal) frame for $TK_{s\Phi}$. We then include $n(s)$, the unit inward normal to K_s in Q_s. Moreover, we extend $n(s)$ to a vector $\bar{n}(s) \in TQ_s$ so that it satisfies $\nabla^{Q_s}_{\bar{n}(s)}\bar{n}(s) = 0$. We supplement this to a complete local frame for $T_{Q_s}M$ (at least near points of K_s) by adding a local orthonormal frame $N_\mu(s) \in NQ_s$. Here we let the indices $\alpha, \beta, \ldots$ run from 1 to k while $\mu, \nu, \ldots$ run from $k+1$ to $m+1$.

Notation 2.1. *Set $\mathcal{H}(s) = H(K_s) - h_0\, H(Q_s)$, where $h_0 = H_K$. We also write*

$$L_Q = \nabla_{\partial/\partial s}\mathcal{H}_s\big|_{s=0}$$

Note that we can decompose $\mathcal{H}'(0)$ into $\mathcal{H}'(0)^{N_K} + \mathcal{H}'(0)^{T_K}$, its components perpendicular and parallel to K. Since $\mathcal{H}(s) \perp K_s$, we have that $\langle \mathcal{H}(s), E_\alpha(s)\rangle = 0$, so

$$\langle \mathcal{H}'(0), E_\alpha\rangle + \langle \mathcal{H}(0), E'_\alpha(0)\rangle = 0.$$

Since $\mathcal{H}(0) = 0$, we obtain $[L_Q]^{T_K} = 0$.

Next decompose $\Phi = [\Phi]^\perp + \phi\, n$ into parts perpendicular and parallel to Q (along K). Noting that we can choose the vector field V_Φ extending

Φ in Lemma 2.1 so that its component tangent to Q lies in the span of n, there is a similar decomposition $U_\Phi = [U_\Phi]^\perp + u_\phi\, \bar{n}(s)$ for the vector field U_Φ constructed in that Lemma, locally near K_Φ; note that $[U_\Phi]^\perp\big|_K = [\Phi]^\perp$ and $u_\phi|_K = \phi$.

To see that $E'_\alpha(0) = \nabla_{E_\alpha}\Phi$, choose a curve $c(t)$ in K with $c(0) = p$, $c'(0) = E_\alpha$ and define $G(t,s) = \exp_{c(t)}(s\Phi(c(t)))$; we then obtain that

$$\nabla_{\partial/\partial s} E_\alpha\big|_{s=0} = \nabla_{\partial/\partial s}\nabla_{\partial/\partial t}\big|_{s=t=0} G(t,s) = \nabla_{\partial/\partial t}\Phi(c(t))\big|_{t=0} = \nabla_{E_\alpha}\Phi,$$

as claimed. To compute $n'(0)$, observe that $(F_s)_*(n(0))$ is always tangent to Q_s and transverse, but not necessarily a unit normal, to K_s. We can adjust it, using the Gram-Schmidt process, to get that

$$n(s) = \Big((F_s)_*(n(0)) - \sum c_\alpha E_\alpha(s)\Big) \Big/ \Big|((F_s)_*(n(0)) - \sum c_\alpha E_\alpha(s)\Big|,$$

where

$$c_\alpha(s) = \langle E_\alpha(s), (F_s)_* n(0)\rangle / |E_\alpha(s)|^2.$$

Arguing as before, take a curve $d(t)$ in Q such that $d(0) = p$ and $d'(0) = n$ and define $\tilde{G}(t,s) = \exp_{d(t)}(U_{s\Phi}(d(t)))$. Note that $U_{s\Phi} = s(V_{[\Phi]^\parallel} + Z_{[\Phi]^\perp}) + \mathcal{O}(s^2\|\Phi\|^2_{\mathcal{C}^{2,\alpha}})$. We get

$$\nabla_{\partial/\partial s}(F_s)_* n(0)\big|_{s=0} = \nabla_{\partial/\partial s}\nabla_{\partial/\partial t}\tilde{G}(t,s)\Big|_{t=s=0} = \nabla_n(V_{[\Phi]^\parallel} + Z_{[\Phi]^\perp})$$

and since $c_\alpha(0) = 0$, we obtain

$$[n'(0)]^\perp = \Big[\nabla_n V_{[\Phi]^\parallel} + \nabla_n Z_{[\Phi]^\perp}\Big]\Big|^\perp_K = \Big[\nabla^\perp_n Z_{\Phi^\perp} + \phi\,\nabla^\perp_n \bar{n}\Big]\Big|_K.$$

Finally, the component $[n'(0)]^\parallel = 0$. Combining these calculations gives the

Proposition 2.2. *If Q is critical for $\mathcal{E}_{h_0}$, then*

$$L_Q\,\Phi = J_K^{N_K}\Phi - h_0\, D_Q\Phi,$$

where

$$D_Q\Phi = \Big[\nabla^\perp_n Z_\Phi + \phi\,\nabla^\perp_n \bar{n}\Big]\Big|_K$$

2.5. **The linearization at $K = S^k$.** We conclude this section by discussing the precise form of this linearization, and its nullspace, when

$$K = S^k \times \{0\} \subset Q = B^{k+1} \times \{0\} \subset \mathbb{R}^{m+1},$$

since this is our basic model later. It is easy to see that B^{k+1} is critical for $\mathcal{E}_k$.

The unit inward normal to S^k in B^{k+1} is $n_{S^k}(\Theta) = -\Theta$. If $\Phi \in \mathcal{C}^{2,\alpha}(NS^k)$, then

$$\Phi = [\Phi]^\perp - \phi\,\Theta,$$

where the first term on the right is perpendicular to B^{k+1}. The operator $J^N_{S^k}$ acts on these two components separately, via $J^\perp_{S^k}$ and $J^\parallel_{S^k}$, respectively.

The first of these operators acts on sections of the trivial bundle of rank $m-k$. Obviously, $\mathrm{Ric}^N_{S^k} = 0$, cf. (3), and $(\mathfrak{H}^{(2)}_{S^k})^\perp = 0$ as well, so

$$J^\perp_{S^k} = \Delta_{S^k}$$

acting on $(m-k)$-tuples of functions. Its eigenvalues are $\ell(k+\ell-2)$. The operator $D_{B^{k+1}}$ also acts on sections of the trivial bundle $NB^{k+1}\big|_{S^k}$. In fact, since $J_{B^{k+1}} = \Delta_{B^{k+1}}$, this operator is simply the standard Dirichlet-to-Neumann operator for the Laplacian (acting on $\mathbb{R}^{m-k}$-valued functions). Its eigenfunctions are the restrictions to $r=1$ of the homogeneous harmonic polynomials $P(x)$, $x = r\Theta$, $\Theta \in S^k$. If P is homogeneous of order ℓ, then $P(x) = r^\ell P(\Theta)$, so $D_{B^{k+1}}P(\Theta) = -\ell P(\Theta)$ (recall we are using the inward-pointing normal). Combining these two operators, we see that $\Delta_{S^k} - kD_{B^{k+1}}$ has eigenvalues $-\ell(k+\ell-1) + k\ell = -\ell(\ell-1)$, hence

$$\left(J^\perp_{S^k} - kD_{B^{k+1}}\right)[\Phi]^\perp = 0 \Rightarrow [\Phi]^\perp \in \mathrm{span}\,\{(a_\mu + b_\mu x_\mu)E_\mu\},$$

where E_μ, $\mu = k+2, \ldots, m+1$ is an orthonormal basis for $NB^{k+1} = \mathbb{R}^{m-k}$.

The remaining part is

$$J^\parallel_{S^k} = \Delta_{S^k} + k,$$

since $\mathrm{Ric}_{S^k} = 0$ and $\mathfrak{H}^{(2)}_{S^k} = k\,\mathrm{Id}$. Thus

$$J^\parallel_{S^k}(\phi\,\Theta) = J^\parallel_{S^k}(\phi)\,\Theta = 0 \Rightarrow \phi \in \mathrm{span}\,\{x_1, \ldots, x_{k+1}\}.$$

We have now shown that the nullspace $\mathcal{K}$ of $L_{B^{k+1}}$ splits as $\mathcal{K}^\perp \oplus \mathcal{K}^\parallel$. The first of these summands is comprised by infinitesimal translations in $\mathbb{R}^{m-k}$ and infinitesimal rotations in the $\alpha\mu$ planes (now $\alpha \leq k+1$); the second summand corresponds to infinitesimal translations in $\mathbb{R}^{k+1}$.

3. Construction of constant mean curvature submanifolds

We now turn to the main task of this paper, which is to construct small constant mean curvature submanifolds concentrated near the critical points of $\mathcal{R}_{k+1}$. The first step is to define a family of approximate solutions, i.e., a family of pair $(Q_\varepsilon, K_\varepsilon)$ where Q_ε is minimal and has nearly CMC boundary. We then use a variational argument to perturb this to a minimal submanifold with exactly CMC boundary.

3.1. **Approximate solutions.** We adopt all the notation used earlier. Thus we fix $\Pi_p \in G_{k+1}(TM)$ and an orthonormal basis E_i, $1 \leq i \leq m+1$ of T_pM, where E_a, $1 \leq a \leq k+1$ span Π_p and E_μ, $\mu > k+1$, span $\Pi_p^\perp$. This induces a Riemann normal coordinate system $(x^1, \ldots, x^{m+1})$ near p, and it is standard that

$$g_{ij}(x) = g(\partial_{x^i}, \partial_{x^j}) = \delta_{ij} + \frac{1}{3}\sum_{k,\ell}(R_p)_{ikj\ell}\,x^k x^\ell + \mathcal{O}(|x|^3), \tag{5}$$

where δ is the Euclidean metric.

3.1.1. *Rescaling.* In terms of the map $F_\varepsilon : T_pM \to M$, $F_\varepsilon(y) = \exp_p(\varepsilon y)$, used earlier, define the metric

$$g_\varepsilon = \varepsilon^{-2} F_\varepsilon^* g$$

on T_pM, or equivalently, work in the rescaled coordinates $y^j = x^j/\varepsilon$.In either case,

$$g_\varepsilon = |dy|^2 + \varepsilon^2 h_\varepsilon(y, dy), \tag{6}$$

where h_ε is family of smooth symmetric two-tensors depending smoothly on $\varepsilon \in [0, \varepsilon_0]$. The mean curvature vectors H^g and H^{g_ε} with respect to g and g_ε satisfy

$$\varepsilon^2 \, H^g = (F_\varepsilon)_* \, H^{g_\varepsilon}, \quad \text{and} \qquad \|H^{g_\varepsilon}\|_{g_\varepsilon} = \varepsilon \, \|H^g\|_g.$$

Let $B^{k+1} = B^{k+1}(\Pi_p) \subset \Pi_p$ be the unit ball and $S^{k+1} = S^{k+1}(\Pi_p) = \partial B^{k+1}$, and denote their images under F_ε by $\mathcal{B}_\varepsilon$ and $\mathcal{S}_\varepsilon$. These have parametrizations

$$S^{k+1} \ni \Theta \longmapsto \exp_p^g(\varepsilon \, \Theta), \quad B^{k+1} \ni y \longmapsto \exp_p^g(\varepsilon \sum_{a=1}^{k+1} y^a E^a).$$

In the lemmas (3.1) and (3.2) below we give the expansion of the mean curvature of $\mathcal{B}_\varepsilon$ and $\mathcal{S}_\varepsilon$ in terms of ε. To this end we indroduce two supplementary curvature invariants which are restrictions of the Ricci curvature of the ambient manifold M:

$$\mathcal{R}ic_{k+1}(\Pi_p)(v_1, v_2) = -\sum_{i=1}^{k+1} R_p(E_i, v_1, E_i, v_2), \qquad v_1, v_2 \in \Pi_p$$

$$\mathcal{R}ic^\perp_{k+1}(\Pi_p)(v, N) = -\sum_{i=1}^{k+1} R_p(E_i, v, E_i, N), \qquad v \in \Pi_p, N \in \Pi_p^\perp.$$

Note that

$$\mathcal{R}ic^\perp_{k+1}(\Pi_p) = \left[\mathrm{Ric}^N_{\mathcal{B}_\varepsilon}\right]_p.$$

Lemma 3.1. *We have*

$$H^g(\mathcal{B}_\varepsilon)(y) = \sum_{\mu=k+1}^{m+1} \left(\frac{2\,\varepsilon}{3} \, \mathcal{R}ic^\perp_{k+1}(\Pi_p)(y, E_\mu) + \mathcal{O}(\varepsilon^2) \right) \mathcal{N}_\mu$$

where $\mathcal{N}_\mu$, $k+1 \leq \mu \leq m+1$ is an orthonormal basis of $N\mathcal{B}_\varepsilon$.

Remark 3.1. *Here and below, we write $\mathcal{O}(\varepsilon^k)$ for a function with $\mathcal{C}^{0,\alpha}$ norm bounded by $C\varepsilon^k$.*

Proof. Recall that

$$H^g(\mathcal{B}_\varepsilon) = \frac{1}{\varepsilon^2} \, (F_\varepsilon)_* \, H^{g_\varepsilon}(B^{k+1})$$

We denote $\mathcal{N}^\varepsilon_\mu$, $k+1 < \mu < m+1$ the orthonormal basis of the normal bundle of $B^{k+1} \in T_pM$ with respect to the metric g_ε obtained by applying

the Gram-Schmidt process to the vectors E_μ, $k+1 \leq \mu \leq m+1$. Remark that

$$g_\varepsilon(\mathcal{N}_\mu, E_\nu) = \delta_{\mu\nu} + \mathcal{O}(\varepsilon^2)$$

We denote $\mathcal{N}_\mu = \varepsilon \mathcal{N}_\mu^\varepsilon$ the orthonormal basis of the normal bundle to B^{k+1} with respect to the metric $(F_\varepsilon)^* g$. We identify $\mathcal{N}_\mu$ with $(F_\varepsilon)_* \mathcal{N}_\mu$; these last vector fields form an orthonormal basis of $N\mathcal{B}_\varepsilon$ with respect to the metric g. The Christoffel symbols corresponding to the metric g_ε are:

$$\begin{aligned}(\Gamma^{g_\varepsilon})_{ij}^\ell(y) &= \frac{1}{2} g_\varepsilon^{kq} \left(\partial_{y^j}(g_\varepsilon)_{iq} + \partial_{y^i}(g_\varepsilon)_{jq} - \partial_{y^q}(g_\varepsilon)_{ij}\right) \\ &= \delta^{q\ell} \frac{\varepsilon^2}{6} y^p \left(R_{ijqp} + R_{ipqj} + R_{jiqp} + R_{jpqi} - R_{iqjp} - R_{ipjq}\right) + \mathcal{O}(\varepsilon^3) \\ &= -\frac{\varepsilon^2}{3} \left(R_{ipj\ell} + R_{i\ell jp}\right) y^p + \mathcal{O}(\varepsilon^3)\end{aligned}$$

whence

$$g_\varepsilon(\nabla^{g_\varepsilon}_{\partial_{y^a}} \partial_{y^b}, \mathcal{N}_\mu^\varepsilon) = (\Gamma^{g_\varepsilon})_{ab}^\mu + \mathcal{O}(\varepsilon^4)$$

Taking the trace in the indices $a, b = 1, \ldots, k+1$ with respect to g_ε gives the result. □

Lemma 3.2. *We have*

$$H^g(\mathcal{S}_\varepsilon^k) = \left(\frac{k}{\varepsilon} - \frac{\varepsilon}{3} \mathcal{R}ic_{k+1}(\Pi_p)(\Theta, \Theta) + \mathcal{O}(\varepsilon^2)\right) n_{\mathcal{S}} + \sum_{\mu=k+1}^{m+1} \left(\frac{2\varepsilon}{3} \mathcal{R}ic^\perp_{k+1}(\Pi_p)(\Theta, E_\mu) + \mathcal{O}(\varepsilon^2)\right) \mathcal{N}_\mu$$

where $n_{\mathcal{S}}$ is a unit normal vector field to $\mathcal{S}_\varepsilon^k$ in $\mathcal{B}_\varepsilon^{k+1}$ with respect to the metric g.

Proof. The proof is similar to that of the previous lemma, but with several changes.

Let $u^1, \ldots, u^k \mapsto \Theta(u^1, \ldots, u^k)$ be a local parametrization of $S^k \subset \Pi_p$. The tangent bundle TS^k is spanned by the vector fields $\Theta_\alpha = \partial_{u^\alpha}\Theta$. As before, we have

$$H^g(\mathcal{S}_\varepsilon^k) = \frac{1}{\varepsilon^2} (F_\varepsilon)_* H_\varepsilon^g(S^k)$$

By the Gauss lemma,

$$g\left((F_\varepsilon)_*\Theta_\alpha, (F_\varepsilon)_*\Theta\right)(F_\varepsilon(\Theta)) = g_p(\Theta_\alpha, \Theta) = 0$$

and

$$g\left((F_\varepsilon)_* E_\mu, (F_\varepsilon)_*\Theta\right)(F_\varepsilon(\Theta)) = g_p(E_\mu, \Theta) = 0$$

this yields

$$g(\mathcal{N}_\mu, (F_\varepsilon)_*\Theta) = 0 \quad \text{and} \quad g_\varepsilon(\mathcal{N}_\mu^\varepsilon, \Theta) = 0$$

Finally we put $n_{\mathcal{S}} := -(F_\varepsilon)_*\Theta$. We have

$$\nabla^{g_\varepsilon}_{\partial_{u^\alpha}} \partial_{u^\beta} = \partial_{u^\alpha}\,\partial_{u^\beta}\Theta + (\Gamma^{g_\varepsilon})^\ell_{ij}(\Theta_\alpha)^i(\Theta_\beta)^j\, E_\ell$$

$\alpha, \beta = 1, \dots, k$, $i, j, \ell = 1, \dots, m+1$.

The vector field $\partial_{u^\alpha}\,\partial_{u^\beta}\Theta$ is tangent to $B^{k+1}(\Theta)$, so

$$g_\varepsilon\left(\nabla^{g_\varepsilon}_{\partial_{u^\alpha}}\partial_{u^\beta}, \mathcal{N}^\varepsilon_\mu\right) = (\Gamma^{g_\varepsilon})^\mu_{ab}\,(\Theta_\alpha)^a\,(\Theta_\beta)^b + \mathcal{O}(\varepsilon^3).$$

Taking trace in the indices α, β with respect to the metric induced on S^k from g_ε we get

$$g_\varepsilon(H^{g_\varepsilon}(S^k), \mathcal{N}_\mu) = \frac{2\,\varepsilon^2}{3}\,\mathcal{R}ic^\perp_{k+1}(\Pi_p)(\Theta, E_\mu) + \mathcal{O}(\varepsilon^3).$$

In order to find $[H^{g_\varepsilon}(S^k(\Pi_p))]^{||}$, recall the standard fact that if $\Sigma \subset M$ is an oriented hypersurface with unit inward pointing normal N_Σ, and if Σ_z is the family of hypersurfaces defined by

$$\Sigma \times \mathbb{R}(q, z) \mapsto \exp_q(zN_\Sigma(q)) \in \Sigma_z,$$

with induced metric g_z, then

$$|H_\Sigma| = -\frac{d}{dz}\,\log\sqrt{\det g_z}.$$

In our case, considering $S^k(\Pi_p) \subset B^{k+1}(\Pi_p)$ with metric g_ε, let $g_{\varepsilon z}$ be the induced metrics on the Euclidean sphere of radius $1 - z$. Then

$$\det g_{\varepsilon z} = (1-z)^{2k}\det g^S\left(1 - \frac{\varepsilon^2(1-z)^2}{3}\mathcal{R}ic_{S^k}(\Pi_p)(\vec{\Theta}, \vec{\Theta}) + \mathcal{O}(\varepsilon^3)\right),$$

where g^S is the standard spherical metric on $S^k(\Pi_p)$. From this we deduce that

$$g_\varepsilon\left(H^{g_\varepsilon}(S^k), -\Theta\right) = \frac{k}{\varepsilon} - \frac{\varepsilon}{3}\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta, \Theta) + \mathcal{O}(\varepsilon^2).$$

this completes the proof. □

Proposition 3.1. *Fix $\Pi_p \in G_{k+1}(TM)$. Then for $\varepsilon > 0$ small enough, there exists a minimal submanifold $Q_\varepsilon(\Pi_p)$ which is a small perturbation of $\mathcal{B}^{k+1}_\varepsilon(\Pi_p)$, whose boundary $K_\varepsilon(\Pi_p) = \partial Q_\varepsilon(\Pi_p)$ is a normal graph over $S^k_\varepsilon(\Pi_p)$ and whose mean curvature vector field satisfies*

$$H^g(K_\varepsilon(\Pi_p)) - \frac{k}{\varepsilon}\,n_K = g_p(\vec{a}, \Theta)\,n_K + \sum_{\mu=k+1}^{m+1}(g_p(\vec{c}_\mu, \Theta) + d_\mu)\,N_\mu \tag{7}$$

for some constant vectors $\vec{a} = \vec{a}(\varepsilon, \Pi_p)$, $\vec{c}_\mu = \vec{c}_\mu(\varepsilon, \Pi_p) \in \Pi_p$ and constants $d_\mu = d_\mu(\varepsilon, \Pi_p) \in \mathbb{R}$. Here n_K is a normal vector field to $K_\varepsilon(\Pi_p)$ in $Q_\varepsilon(\Pi_p)$ and N_μ form an orthonormal basis of $[NK_\varepsilon(\Pi_p)]^\perp$.

Proof. Take a vector field $\Phi \in \mathcal{C}^{2,\alpha}(T_pM)$ defined along the unit sphere $S^k(\Pi_p)$, such that

$$\Phi(\Theta) = -\phi(\Theta)\,\Theta + \sum_{\mu=k+1}^{m+1} \Phi^\mu(\Theta)\,E_\mu,$$

and write

$$S^k_\Phi = \left\{\Theta + \Phi(\Theta),\ \Theta \in S^k\right\}.$$

Then there exists a submanifold $B^{k+1}_{\varepsilon,\Phi}$ such that $\partial B^{k+1}_{\varepsilon,\Phi} = S^k_\Phi$ and which is minimal with respect to g_ε. The proof of this fact is almost the same as the proof of the Lemma (2.1); the only difference is that we use a "perturbed" metric and the starting submanifold is no longer minimal. Let V_Φ be a linear extension of Φ in B^{k+1} and take

$$W \in \mathcal{C}^{2,\alpha}(T_pM), \quad W = \sum_{\mu=k+1}^{m+1} W^\mu\,E_\mu, \quad W|_{S^k} = 0.$$

We let $H(\varepsilon,\Phi,W)$ denote the mean curvature with respect to the metric g_ε of the submanifold $\{U(y) = V_\Phi(y)+W(y),\ y \in B^{k+1}\}$. Note that $H(0,0,0) = 0$ and

$$D_3H|_{(0,0,0)} = J_{B^{k+1}} = \Delta_{B^{k+1}}.$$

We can then apply the implicit function theorem to $\hat{H}(\varepsilon,\Phi,W) = \pi \circ H(\varepsilon,\Phi,W)$, where π is the orthogonal projection onto the vertical subspace of T_pM, which is spanned by E_μ, $k+1 \geq \mu \leq m+1$. Then for ε and $\|\Phi\|_{\mathcal{C}^{2,\alpha}}$ small enough, there exists a mapping $(\varepsilon,\Phi) \mapsto W(\varepsilon,\Phi)$ such that

$$\hat{H}(\varepsilon,\Phi,W(\varepsilon,\Phi)) = 0 \quad \text{and} \quad H(\varepsilon,\Phi,W(\varepsilon,\Phi)) = 0.$$

Moreover,

$$U_{\varepsilon,\Phi} = V_\Phi + W(\varepsilon,\Phi) = V_\phi + Z_\Phi + W_\varepsilon + \mathcal{O}(\|\varepsilon^3\|) + \mathcal{O}(\varepsilon^2\|\,\Phi\|) + \mathcal{O}(\|\Phi^2\|)$$

where $V_\phi(y) = -\phi(y/\|y\|)\,y$, the vector field Z_Φ is the harmonic extansion of Φ in B^{k+1} and W_ε satisfies

$$\Delta_{B^{k+1}}\,W^\mu_\varepsilon = \frac{2\,\varepsilon^2}{3}\,\mathcal{R}ic^\perp_{k+1}(\Pi_p)(y,E_\mu), \quad W_\varepsilon = 0 \quad \text{on} \quad S^k$$

Remark 3.2. *A simple calculation shows that*

$$W_\varepsilon(y) = -\frac{\varepsilon^2}{3}\,\frac{1}{k+3}\,(1-|y|^2)\sum_{\mu=k+1}^{m+1} \mathcal{R}ic^\perp_{k+1}(\Pi_p)(y,E_\mu)\,E_\mu.$$

For the second step, we calculate the mean curvature of S^k_Φ with respect to the metric g_ε. First note that the vector fields

$$\tau_\alpha = (1-\phi)\,\Theta_\alpha - \partial_{u_\alpha}\phi\,\Theta + \sum_{\mu=k+1}^{m+1} \partial_{u_\alpha}\Phi\,E_\mu$$

locally frame TS^k_Φ, while

$$\Theta_\Phi = \Theta + \frac{1}{1-\phi}\nabla_{S^k}\phi, \quad \text{and} \quad (E_\mu)_\Phi = E_\mu - \frac{1}{1-\phi}\nabla_{S^k}\Phi^\mu$$

are a local basis for the normal bundle of S^k_Φ with respect to the Euclidean metric. Applying the Gram-Schmidt process with respect to the metric g_ε to these local frames yields the unit normal to S^k_Φ in $B^{k+1}_{\varepsilon,\Phi}$, which we denote n^ε_Φ, and the orthonormal frame $(\mathcal{N}_\Phi)^\varepsilon_\mu$ for the normal bundle of $B^{k+1}_{\varepsilon,\Phi}$ along S^k_Φ with respect to g_ε. These calculations show that

$$\langle n^\varepsilon_\Phi, -\Theta_\Phi/|\Theta_\Phi|_{g_{eucl}}\rangle_{g_\varepsilon} = 1 + \mathcal{O}(\varepsilon^2)$$

$$\langle (\mathcal{N}_\mu)_\Phi, (E_\mu)^\varepsilon_\Phi/|(E_\mu)_\Phi|_{g_{eucl}}\rangle_{g_\varepsilon} = 1 + \mathcal{O}(\varepsilon^2),$$

and $n^\varepsilon_0 = -\Theta$ and $(\mathcal{N}_\mu)^\varepsilon_0 = \mathcal{N}^\varepsilon_\mu$. We can then write

$$H^{g_\varepsilon}(S^k_\Phi) - k\, n_\Phi = \left(g_\varepsilon\left(H^{g_\varepsilon}(S^k_\Phi), n^\varepsilon_\Phi\right) - k\right) n^\varepsilon_\Phi + \sum_{\mu=k+1}^{m+1} g_\varepsilon\left(H^{g_\varepsilon}(S^k_\Phi), (\mathcal{N}_\Phi)^\varepsilon_\mu\right)(\mathcal{N}_\Phi)^\varepsilon_\mu.$$

Notation 3.1. *We let $L_{\Pi_p}(\Phi)$ denote any second order linear differential operator acting on Φ. The coefficients of $L_{\Pi_p}(\Phi)$ may depend on $\Pi_p \in G_{k+1}(TM)$ and $\varepsilon \in (0,1)$, but for all $j \in \mathbb{N}$ there exists a constant $C_j > 0$ independent of Π_p and ε such that*

$$\|L_{\Pi_p}(\Phi)\|_{\mathcal{C}^{j,\alpha}(S^k)} \leq C_j\, \|\Phi\|_{\mathcal{C}^{j+2,\alpha}(NS^k)}.$$

Similarly, for $\ell \in \mathbb{N}$, $Q^\ell_{\Pi_p}(\Phi)$ denotes some nonlinear operator in Φ, depending also on Π_p and ε, such that $Q^\ell_{\Pi_p}(0) = 0$ and which has the following properties. The coefficients of the Taylor expansion of $Q^\ell_{\Pi_p}(\Phi)$ in powers of the components of Φ and its derivatives satisfy that for any $j \geq 0$, there exists a constant $C_j > 0$, independent of $\Pi_p \in G_{k+1}(TM)$ and $\varepsilon \in (0,1)$,

$$\|Q^\ell_{\Pi_p}(\Phi_1) - Q^\ell_{\Pi_p}(\Phi_2)\|_{\mathcal{C}^{j,\alpha}(S^k)} \leq c\left(\|\Phi_1\|_{\mathcal{C}^{j+2,\alpha}(NS^k)} + \|\Phi_2\|_{\mathcal{C}^{j+k,\alpha}(NS^k)}\right)^{\ell-1}\|\Phi_1 - \Phi_2\|_{\mathcal{C}^{j+k,\alpha}(NS^k)}$$

provided $\|\Phi_i\|_{\mathcal{C}^1(NS^k)} \leq 1$, $i = 1, 2$.

Using that the Christoffel symbols of the metric g_ε are of order $\mathcal{O}(\varepsilon^2)$, we obtain

$$\begin{aligned} g_\varepsilon\left(H^{g_\varepsilon}(S^k_\Phi), n^\varepsilon_\Phi\right) - k &= -\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) + J^{\|}_{S^k}\phi \\ &\quad + \mathcal{O}(\varepsilon^3) + \varepsilon^2\, L_{\Pi_p}(\Phi) + Q^2_{\Pi_p}(\Phi), \\ g_\varepsilon\left(H^{g_\varepsilon}(S^k_\Phi), (\mathcal{N}_\Phi)^\varepsilon_\mu\right) &= -\mathcal{R}ic^{\perp}_{k+1}(\Pi_p)(\Theta, E_\mu) + \left(J^{\perp}_{S^k} - D_{B^{k+1}}\right)\,\Phi^\mu \\ &\quad + \mathcal{O}(\varepsilon^3) + \varepsilon^2\, L_{\Pi_p}(\Phi) + Q^2_{\Pi_p}(\Phi). \end{aligned}$$

As before, we let $\mathcal{K}^{\|}$ and $\mathcal{K}^{\perp}$ be the null-spaces of the operators

$$J^{\|}_{S^k} = \Delta_{S^k} + k \quad \text{and} \quad L^{\perp}_{B^{k+1}} = \Delta_{S^k} - D_{B^{k+1}}$$

and write $\mathcal{P}^{\|}$ and $\mathcal{P}^{\perp}$ for the orthogonal complements of $\mathcal{K}^{\|}$ and $\mathcal{K}^{\perp}$ in $L^2(S^k)$. Define

$$\mathfrak{E}_{\varepsilon,\Pi_p} := T_pM \times (T_pM \oplus \mathbb{R})^{m-k} \times \mathcal{P}^{\|} \times (\mathcal{P}^{\perp})^{m-k} \tag{8}$$

There exists an operator

$$\mathcal{G}_{\varepsilon,\Pi_p} : (\mathcal{C}^{0,\alpha}(S^k))^{m-k} \longrightarrow \mathfrak{E}_{\varepsilon,\Pi_p}$$

such that

$$\begin{aligned} &\mathcal{G}_{\varepsilon,\Pi_p}(f_0, f_1, \ldots, f_{m-k}) \\ &\qquad = \left(\vec{a}(\varepsilon,\Pi_p,f), \vec{c}_\mu(\varepsilon,\Pi_p,f), d_\mu(\varepsilon,\Pi_p,f), \phi(\varepsilon,\Pi_p,f), \Phi^{\perp}(\varepsilon,\Pi_p,f)\right) \end{aligned}$$

is the solution to

$$\begin{cases} J^{\|}_{S^k}\,\phi = g_p(\vec{a},\Theta) + f_0 \\ L^{\perp}_{B^{k+1}}\,\Phi^\mu = g_p(\vec{c}_\mu,\Theta) + d_\mu + f_{\mu-k}. \end{cases}$$

Applying a standard fixed point theorem for contraction mappings, we find that there exist constants $c \in \mathbb{R}$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \in (0,\varepsilon_0)$ and $\Pi_p \in G_{k+1}(TM)$ there is a unique

$$\left(\vec{a}(\varepsilon,\Pi_p), \vec{c}_\mu(\varepsilon,\Pi_p), d_\mu(\varepsilon,\Pi_p), \phi(\varepsilon,\Pi_p), \Phi^{\perp}_{\varepsilon,\Pi_p}\right) \in \mathfrak{E}_{\varepsilon,\Pi_p}.$$

(the indices are suppressed for simplicity) which belongs to a closed ball of radius $c\,\varepsilon^2$ in $\mathfrak{E}_{\varepsilon,\Pi_p}$ and such that

$$H^{g_\varepsilon}(S^k_\Phi) = -k\, n^\varepsilon_\Phi + g_p(\vec{a},\Theta)\, n^\varepsilon_\Phi + \sum_{\mu=k+1}^{m+1} \left(g_p(\vec{c}_\mu,\Theta) + d_\mu\right)\,(\mathcal{N}_\Phi)^\varepsilon_\mu.$$

Putting

$$n_K = (F_\varepsilon)_*\, n^\varepsilon_\Phi \quad \text{and} \quad N_\mu = (F_\varepsilon)_*\,(\mathcal{N}_\mu)^\varepsilon_\Phi$$

and taking $K_\varepsilon(\Pi_p) := F_\varepsilon(S^k_{\Phi(\varepsilon,\Pi_p)})$ and $Q_\varepsilon(\Pi_p) := F_\varepsilon(B^{k+1}_{\varepsilon,\Phi(\varepsilon,\Pi_p)})$ finishes the proof.

Remark 3.3. *Notice that*

$$\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) \in \mathcal{P}^{\|} \quad \textit{and} \quad \mathcal{R}ic^{\perp}_{k+1}(\Pi_p)(\Theta, E_\mu) \in \mathcal{K}^{\perp}$$

Moreover, it was remarked in [6] *that*

$$\begin{aligned}\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) &= \sum_{a=1}^{k+1} \mathcal{R}ic_{k+1}(\Pi_p)_{aa}\,(\Theta^i)^2 + \sum_{a\neq b=1}^{k+1} \mathcal{R}ic_{k+1}(\Pi_p)_{ab}\,\Theta^a\,\Theta^b \\ &= \tfrac{1}{k+1}\mathcal{R}_{k+1}(\Pi_p) + \breve{\mathcal{R}}ic_{k+1}(\Pi_p)(\Theta,\Theta)\end{aligned}$$

where $\breve{\mathcal{R}}ic_{k+1}(\Pi_p)(\Theta,\Theta)$ *belongs to the eigenspace of* Δ_{S^k} *associated to the eigenvalue* $2(k+1)$*. Using this, one can easily verify that*

$$\phi_{\varepsilon,\Pi_p}(\Theta) = -\frac{\varepsilon^2}{3}\left(\frac{2}{k(k+2)}\,\mathcal{R}_{k+1}(\Pi_p) - \frac{1}{k+2}\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\right)\Theta + \mathcal{O}(\varepsilon^3),$$

$$[\Phi]^{\perp}_{\varepsilon,\Pi_p} = \mathcal{O}(\varepsilon^3).$$

□

3.2. **The variational argument.** We now employ a variational argument to prove that one can choose $\Pi_p \in G_k(M)$ in such a way that the submanifold $K_\varepsilon(\Pi_p)$ obtained in the previous Proposition has constant mean curvature.

To state our result, we introduce the following restrictions of the Riemann tensor of M:

$$\begin{aligned}&R_{k+1}(\Pi_p)(v_1,v_2,v_3,v_4) = g_p(R_p(v_1,v_2)v_3,v_4), \qquad && v_1,v_2,v_3,v_4 \in \Pi_p,\\ &R^{\perp}_{k+1}(\Pi_p)(v_1,v_2,v_3,N) = g_p(R_p(v_1,v_2)v_3,N), \qquad && v_1,v_2,v_3 \in \Pi_p, \quad N \in \Pi_p^{\perp},\end{aligned}$$

Finally, introduce the function $\mathbf{r}$ on $G_{k+1}(TM)$:

$$\begin{aligned}\mathbf{r}(\Pi_p) &= \tfrac{1}{36(k+5)}\Big(8\,\|\mathcal{R}ic_{k+1}(\Pi_p)\|^2 - 18\,\Delta^g_{k+1}\mathcal{R}_{k+1}(\Pi_p) - 3\,\|R_{k+1}(\Pi_p)\|^2\\ &\quad +5\,\mathcal{R}^2_{k+1}(\Pi_p) + 8\,\|\mathcal{R}ic^{\perp}_{k+1}(\Pi_p)\|^2 + 12\,\|R^{\perp}_{k+1}(\Pi_p)\|^2\Big)\\ &\quad +\tfrac{1}{9(k+2)}\Big(\tfrac{k+6}{k}\,\mathcal{R}^2_{k+1}(\Pi_p) - 2\,\|\mathcal{R}ic_{k+1}(\Pi_p)\|^2\Big)\\ &\quad -\tfrac{4k}{3(k+3)(k+5)}\,\|\mathcal{R}ic^{\perp}_{k+1}(\Pi_p)\|^2\end{aligned}$$

where $\Delta^g_{k+1}T(\Pi_p) = \sum_{i=1}^{k+1} \nabla^2_{E_i}T(p)$, for any tensor T on M.

Now consider the energy $\mathcal{E}_\varepsilon$ restricted to this finite dimensional space of submanifolds,

$$\mathcal{E}_\varepsilon(\Pi_p) := \mathrm{Vol}_k(K_\varepsilon(\Pi_p)) - \frac{k}{\varepsilon}\,\mathrm{Vol}_{k+1}(Q_\varepsilon(\Pi_p)),$$

which is a function on $G_{k+1}(TM)$. Tracing through the construction of $K_\varepsilon(\Pi_p)$ one obtains the relationship of this function to the curvature functions defined above.

Lemma 3.3. *There is an expansion*

$$\frac{(k+1)\,\mathcal{E}_\varepsilon(\Pi_p)}{\varepsilon^k\,\mathrm{Vol}(S^k)} = \left(1 - \frac{\varepsilon^2}{2(k+3)}\,\mathcal{R}_{k+1}(\Pi_p) + \frac{\varepsilon^4}{2(k+3)}\,\mathbf{r}(\Pi_p) + \mathcal{O}(\varepsilon^5)\right)$$

Proof. The proof is a technical calculation, contained in the Appendix. □

The main result of this section is the following proposition

Proposition 3.2. *If Π_p is a critical point of $\mathcal{E}_\varepsilon$, then $K_\varepsilon(\Pi_p)$ has constant mean curvature.*

Remark 3.4. *Theorems* (1.1) *and* (1.2) *are Corollaries of Proposition* (3.2)*. Indeed, if we define*

$$\Psi(\varepsilon,\Pi_p) = 2\,\varepsilon^{-2}\,(k+3)\left(1 - (k+1)\frac{\mathcal{E}_\varepsilon(\Pi_p)}{\varepsilon^k \mathrm{Vol}(S^k)}\right). \tag{9}$$

then for any $j \geq 0$, there exists a constant C_j which is independent of ε such that

$$\|\Psi(\cdot,\varepsilon) - \mathcal{R}_{k+1} + \varepsilon^2 \mathbf{r}(\Pi_p)\|_{\mathcal{C}^j(G_{k+1}(TM))} \leq C_j\,\varepsilon^3;$$

Proof of the Proposition. Let Π_p be a critical point of $\mathcal{E}_\varepsilon$. We show that the parameters $\vec{a}$, $\vec{c}$ and d must then necessarily vanish. We do this by considering the various types of perturbations of Π_p.

First consider the perturbations in $G_{k+1}(M)$ which correspond to parallel translations of Π_p. In other words, we suppose that the family of planes $\Pi_{\exp_p(t\xi)}$ in $G_{k+1}(M)$ are parallel translates of Π_p along the geodesic $\exp_p(t\xi)$.

The submanifold $K_\varepsilon(\Pi_{\exp_p(t\vec{\xi})})$ is a normal graph over $K_\varepsilon(\Pi)$ by a vector field $\Psi_{\varepsilon,\Pi_p,\xi,t}$ which depends smoothly on t. This defines a vector field on $K_\varepsilon(\Pi_p)$ by

$$Z_{\varepsilon,\Pi_p,\xi} = \partial_t \Psi_{\varepsilon,\Pi_p,\xi,t}\big|_{t=0}.$$

The first variation of the volume formula yields

$$\begin{aligned} 0 &= D\mathcal{E}_\varepsilon|_{\Pi_p}(\xi) \\ &= \textstyle\int_{K_\varepsilon(\Pi_p)} \left(g(H(K_\varepsilon(\Pi_p)), Z_{\varepsilon,\Pi_p,\xi}) - \frac{k}{\varepsilon}\,g(n, Z_{\varepsilon,\Pi_p,\xi})\right) dvol_{K_\varepsilon(\Pi_p)} \\ &\quad -\frac{k}{\varepsilon}\int_{Q_\varepsilon(\Pi_p)} g(H(Q_\varepsilon(\Pi_p)), Z_{\varepsilon,\Pi_p,\xi})\; dvol_{Q_\varepsilon(\Pi_p)}, \end{aligned} \tag{10}$$

and then the construction of $Q_\varepsilon(\Pi_p)$ and $K_\varepsilon(\Pi_p)$ gives that

$$\int \Big(g_p(\vec{a},\Theta) g(n, Z_{\varepsilon,\Pi_p,\xi}) + \sum_{\mu=k+1}^{m+1} (g_p(\vec{c}_\mu,\Theta) + d_\mu)\, g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu)\Big)\; dvol_{K_\varepsilon(\Pi_p)} = 0.$$

Let Ξ be the vector field obtained by parallel transport of ξ along geodesics issuing from p, and suppose that c is a constant independent of ε and ξ. Then

$$\|Z_{\varepsilon,\Pi_p,\xi} - \Xi\|_g \leq c\,\varepsilon^2\,\|\xi\|.$$

By construction of $K_\varepsilon(\Pi_p)$, we have

$$\|n + (F_\varepsilon)_*\Theta\|_g \leq c\,\varepsilon^2, \quad \text{and} \quad \|N_\mu - (F)_*E_\mu\|_g \leq c\,\varepsilon^2.$$

Now take $\xi \in \Pi_p \subset TM_p$, so that

$$g(n, Z_{\varepsilon,\Pi_p,\xi}) = g\left(-(F_\varepsilon)_*\Theta + (n + (F)_*\Theta)\,,\ \Xi + \left(Z_{\varepsilon,\Pi_p,\xi} - \Xi\right)\right),$$

and

$$g(N_\mu, Z_{\varepsilon,\Pi_p,\xi}) = g\left((F_\varepsilon)_*E_\mu + \left(N_\mu - (F_\varepsilon)_*\vec{E}_\mu,\ \Xi + \left(Z_{\varepsilon,\Pi_p,\xi} - \Xi\right)\right)\right).$$

Using the expansion of g near p, we conclude that

$$\left|g(n, Z_{\varepsilon,\Pi_p,\xi}) + g_p(\xi,\Theta)\right| \leq c\,\varepsilon^2\|\xi\|, \quad \text{and} \quad \left|g(N_\mu, Z_{\varepsilon,\Pi_p,\xi})\right| \leq c\,\varepsilon^2\|\xi\|,$$

hence

$$\begin{aligned}
\int_{K_\varepsilon(\Pi_p)} & g_p(\vec{a},\Theta)g_p(\xi,\Theta) \\
&\leq \Big| \int_{K_\varepsilon(\Pi_p)} g_p(\vec{a},\Theta)g_p(\xi,\Theta) + \int_{K_\varepsilon(\Pi_p)} g_p(\vec{a},\Theta)\, g(Z_{\varepsilon,\Pi_p,\xi}, n) \\
&\qquad + \sum_{\mu=k+1}^{m+1} \int_{K_\varepsilon(\Pi_p)} \left(g_p(\vec{c}_\mu,\Theta) + d_\mu\right) g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu)\Big| \\
&\leq c\,\varepsilon^2\,\|\xi\|\Big(\int_{K_\varepsilon(\Pi_p)} |g_p(\vec{a},\Theta)| + \sum_{\mu=k+1}^{m+1} \int_{K_\varepsilon(\Pi_p)} |g_p(\vec{c}_\mu,\Theta) + d_\mu| \Big)
\end{aligned}$$

Now let $\xi = \vec{a}$, so that

$$\begin{aligned}
\int_{K_\varepsilon(\Pi_p)} & |g_p(\vec{a},\Theta)|^2 \\
&\leq c\,\varepsilon^2\|\vec{a}\| \left(\int_{K_\varepsilon(\Pi_p)} |g_p(\vec{a},\Theta)| + \sum_{\mu=k+1}^{m+1} \int_{K_\varepsilon(\Pi_p)} |g_p(\vec{c}_\mu,\Theta) + d_\mu| \right)
\end{aligned}$$

In Euclidean space there is an equality

$$\mathrm{Vol}_k(S^k)\|v\|^2 = (k+1)\int_{S^k} \langle v,\Theta\rangle^2, \quad \text{for all} \ \ v \in \mathbb{R}^k.$$

By the expansion of the induced metric, we obtain for ε small enough

$$\frac{1}{2}\,\mathrm{Vol}_k(S^k)\,\varepsilon^k\,\|v\|^2 \leq (k+1)\int_{K_\varepsilon(\Pi_p)} |g_p(v,\Theta)|^2.$$

Also, because $\mathrm{Vol}_k(K_\varepsilon(\Pi_p)) = \mathcal{O}(\varepsilon^k)$, we deduce

$$\|\vec{a}\| \le c\,\varepsilon^2\Big(\|\vec{a}\| + \sum_{\mu=k+1}^{m+1}(\|\vec{c}_\mu\| + |d_\mu|)\Big). \tag{11}$$

Now move p in the direction of a vector $\xi \in \Pi_p^\perp$ to get

$$\big|g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu) - g_p(\xi, E_\mu)\big| \le c\,\varepsilon^2\|\xi\|, \quad \text{and} \quad |g(n, Z_{\varepsilon,\Pi_p,\xi})| \le c\,\varepsilon^2\|\xi\|.$$

Thus we can write

$$\begin{aligned}
\sum_{\mu=1}^{m-k}\int_{K_\varepsilon(\Pi_p)} & (g_p(\vec{c}_\mu,\Theta) + d_\mu)\; g_p(\xi, E_\mu) \\
&\le \Big|\sum_{\mu=k+1}^{m+1}\int_{K_\varepsilon(\Pi_p)} (g_p(\vec{c}_\mu,\Theta) + d_\mu)\, g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu) \\
&\quad - \sum_{\mu=k+1}^{m+1}\int_{K_\varepsilon(\Pi_p)} (g_p(\vec{c}_\mu,\Theta) + d_\mu)\, g_p(\xi, E_\mu) \\
&\quad + \int_{K_\varepsilon(\Pi_p)} g_p(\vec{a},\Theta) g(Z_{\varepsilon,\Pi_p,\xi}, n)\Big| \\
&\le c\,\varepsilon^2\|\xi\| \int_{K_\varepsilon(\Pi_p)} \Big(|g_p(\vec{a},\Theta)| + \sum_{\mu=k+1}^{m+1} |g_p(\vec{c}_\mu,\Theta) + d_\mu|\Big)
\end{aligned}$$

Taking $\xi = d_\nu\, E_\nu$ gives

$$\begin{aligned}
\int_{K_\varepsilon(\Pi_p)} d_\nu\; g_p(\vec{c}_\nu,\Theta) + {d_\nu}^2 \;&\le c\,\varepsilon^2|d_\nu|\Big(\textstyle\int_{K_\varepsilon(\Pi_p)} |g_p(\vec{a},\Theta)| \\
&\quad + \sum_{\mu=k+1}^{m+1}\int_{K_\varepsilon(\Pi_p)} |g_p(\vec{c}_\mu,\Theta) + d_\mu|\Big)
\end{aligned} \tag{12}$$

Next consider a perturbation of Π_p by a one-parameter family of rotations of Π_p in T_pM generated by an $(m+1)\times(m+1)$ skew matrix A. Then

$$D\mathcal{E}_\varepsilon|_{\Pi_p}(A) = \frac{d}{dt}\Big|_{t=0} \mathcal{E}_\varepsilon((I + tA + O(t^2))\Pi_p) = \frac{d}{dt}\Big|_{t=0} \mathcal{E}(A_t(K_\varepsilon(\Pi_p))),$$

where, in geodesic normal coordinates

$$A_t(x) = x + tAx + \mathcal{O}(t^2).$$

The coordinates of the vector field associated to this flow are

$$Z_{\varepsilon,\Pi_p,\xi}(x) = \frac{d}{dt}\Big|_{t=0} A_t(x) = Ax.$$

Considering only matrixes $A \in \mathfrak{o}(m)$ such that $A : \Pi_p \to \Pi_p^\perp$, we obtain

$$\big|g(Z_{\varepsilon,\Pi_p,\xi}, n)\big| \le c\,\varepsilon^2\|A\Theta\|, \quad \text{and} \quad \big|g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu) - g_p(A\Theta, E_\mu)\big| \le c\,\varepsilon^2\|A\Theta\|.$$

This gives, then,

$$\begin{aligned}\sum_{\mu=k+1}^{m+1} & \int_{K_\varepsilon(\Pi_p)} (g_p(\vec{c}_\mu,\Theta)+d_\mu)\, g_p(A\Theta,E_\mu) \\ & \leq \Big| \sum_{\mu=k+1}^{m+1} \int_{K_\varepsilon(\Pi_p)} (g_p(\vec{c}_\mu,\Theta)+d_\mu)\, g(Z_{\varepsilon,\Pi_p,\xi},N_\mu) \\ & - \sum_{\mu=k+1}^{m+1} \int_{K_\varepsilon(\Pi_p)} (g_p(\vec{c}_\mu,\Theta)+d_\mu)\, g_p(A\Theta,E_\mu) \\ & + \int_{K_\varepsilon(\Pi_p)} g_p(\vec{a},\Theta) g(Z_{\varepsilon,\Pi_p,\xi},n)\Big| \\ & \leq c\,\varepsilon^2 \int_{K_\varepsilon(\Pi_p)} \left(\|A\Theta\|\ |g_p(\vec{a},\Theta)| + \sum_{\mu=k+1}^{m+1} \|A\Theta\|\ |g_p(\vec{c}_\mu,\Theta)+d_\mu| \right).\end{aligned}$$

Let C_ν be the $(m-k)\times(k+1)$ matrix with column ν equal to the vector $\vec{c}_\nu \in \mathbb{R}^{k+1}$, and all other columns equal to 0. Then if

$$A = \begin{pmatrix} 0 & -C_\nu^T \\ C_\nu & 0 \end{pmatrix},$$

we get

(13)
$$\begin{aligned}\int_{K_\varepsilon(\Pi_p)} g_p(\vec{c}_\nu,\Theta)^2 & + g_p(\vec{c}_\nu,\Theta) d_\nu \leq C\varepsilon^2 \Big(\textstyle\int_{K_\varepsilon(\Pi_p)} |g_p(\vec{c}_\nu,\Theta)|\ |g_p(\vec{a},\Theta)| \\ & + \sum_{\mu=k+1}^{m+1} \int_{K_\varepsilon(\Pi_p)} |g_p(\vec{c}_\nu,\Theta)|\, |g_p(\vec{c}_\mu,\Theta)+d_\mu| \Big)\end{aligned}$$

Adding (12) and (13) now gives

$$\begin{aligned}\int_{K_\varepsilon(\Pi_p)} |d_\nu + g_p(\vec{c}_\nu,\Theta)|^2 \leq c\,\varepsilon^2 \Big(& \int_{K_\varepsilon(\Pi_p)} (|d_\nu| + |g_p(\vec{c}_\nu,\Theta)|)\, |g_p(\vec{a},\Theta)| \\ & + \sum_{\mu=k+1}^{m+1} (|d_\nu| + |g_p(\vec{c}_\nu,\Theta)|)\, |g_p(\vec{c}_\mu,\Theta)+d_\mu| \Big)\end{aligned}$$

In Euclidean space, if $v \in \mathbb{R}^{k+1}$ and $\alpha \in \mathbb{R}$ are arbitrary, then

$$\int_{S^k} |\alpha + \langle v,\Theta\rangle|^2 = \left(\alpha^2 + \frac{1}{k+1}\|v\|^2\right) \mathrm{Vol}_k(S^k).$$

Using, once again, the decomposition of the induced metric on $K_\varepsilon(\Pi_p)$, we see that when ε is small enough,

$$\frac{1}{2(k+1)} \varepsilon^k \,\mathrm{Vol}_k(S^k)\left(\alpha^2 + \|v\|^2\right) \leq \int_{K_\varepsilon(\Pi_p)} |\alpha + g_p(v,\Theta)|^2 . \tag{14}$$

which give

$$\begin{aligned}&\|\vec{c}_\nu\|^2+|d_\nu|^2\\&\leq c\,\frac{1}{\varepsilon^{k-2}}\,(\|\vec{c}_\nu\|+|d_\nu|)\,\Big(\int_{K_\varepsilon(\Pi_p)}|g_p(\vec{a},\Theta)|+\sum_{\mu=1}^{m-k}\int_{K_\varepsilon(\Pi_p)}|g_p(\vec{c}_\mu,\Theta)+d_\mu|\Big)\end{aligned}$$

Since $\mathrm{Vol}_k(K_\varepsilon(\Pi_p))=\mathcal{O}(\varepsilon^k)$, we get

$$\|\vec{c}_\nu\|+|d_\nu|\leq c\,\varepsilon^2\,(\|\vec{a}\|+\sum_{\mu=1}^{m-k}(\|\vec{c}_\mu\|+|d_\mu|)) \tag{15}$$

Adding (11) and (15) gives

$$\left(\|\vec{a}\|+\sum_{\mu=1}^{m-k}(\|\vec{c}_\mu\|+|d_\mu|)\right)\leq c\,\varepsilon^2\left(\|\vec{a}\|+\sum_{\mu=1}^{m-k}(\|\vec{c}_\mu\|+|d_\mu|)\right),$$

which implies finally that $\|\vec{a}\|=0$, $\|\vec{c}_\mu\|=0$ and $|d_\mu|=0$, $k+1\leq\mu$.

We conclude that if Π_p is a critical point of the functional $\mathcal{E}_\varepsilon$, then the manifold $K_\varepsilon(\Pi_p)$ is a constant mean curvature submanifold of M. □

4. Appendix 1

Mean curvature of submanifolds: Let $\Sigma^k\subset M^{m+1}$ be an embedded submanifold. Let $x^1,\dots,x^k$ be local coordinates on Σ and

$$E_\alpha=\partial_{x_\alpha}$$

the corresponding coordinate vector fields. Suppose that $E_{k+1},\dots,E_{m+1}$ is a local frame for $N\Sigma$. This gives local coordinates transverse to Σ by

$$p\in\Sigma\longmapsto\exp_p(\sum_{j=k+1}^{m+1}x^j\,E_j)$$

We make the convention that Greek indices run from 1 to$k$, while Latin indices run from $k+1$ to $m+1$. The induced metric on Σ has coefficients $\bar{g}_{\alpha\beta}$, while

$$\bar{h}^i_{\alpha\beta}:=\Gamma^i_{\alpha\beta}=g(\nabla_{E_\alpha}E_\beta,E_i)$$

are the coefficients of the shape operator. We also record the Christoffel symbols

$$\Gamma^j_{\alpha i}=g(\nabla_{E_\alpha}E_i,E_j)$$

The following result is standard, cf. [5] for a proof.

Lemma 4.1. *If* $X=\sum_{j=k+1}^{m+1}x^j\,E_j$, *then*

$$
\begin{aligned}
g_{\alpha\beta} &= \bar{g}_{\alpha\beta} - 2\,\bar{g}(\bar{h}_{\alpha\beta}, X) + g(R(E_\alpha, X)E_\beta, X) + g(\nabla_{E_\alpha} X, \nabla_{E_\beta} X) + \mathcal{O}(|x|^3) \\
&= \bar{g}_{\alpha\beta} - 2\,\bar{h}^i_{\alpha\beta}\, x^i + \Big(g(R(E_\alpha, E_i)E_\beta, E_j) + g^{\gamma\gamma'} \bar{h}^i_{\alpha\gamma}\, \bar{h}^j_{\gamma'\beta} + \Gamma^i_{\alpha\ell}\, \Gamma^j_{\ell\beta} \Big)\, x^i\, x^j + \mathcal{O}(|x|^3) \\
g_{\alpha j} &= -\Gamma^i_{\alpha j}\, x^i + \mathcal{O}(|x|^2) \\
g_{ij} &= \delta_{ij} + \tfrac{1}{3}\, g(R(E_i, E_\ell)E_j, E_{\ell'})\, x^\ell\, x^{\ell'} + \mathcal{O}(|x|^3)
\end{aligned}
$$

Let Φ be a smooth section of $N\Sigma$ and consider the normal graph $\Sigma_\Phi = \{\exp_p(\Phi(p)) : p \in \Sigma\}$. Now let us use the previous lemma to expand the metric and volume form on Σ_Φ. To state this result properly, introduce ∇^N, the induced connection on $N\Sigma$,

$$\nabla^N \Phi = \pi_{N\Sigma} \circ \nabla \Phi$$

Using the definitions of §2, we find that

Lemma 4.2.

$$
\begin{aligned}
\mathrm{Vol_k}(\Sigma_\Phi) &= \mathrm{Vol}_k(\Sigma) - \int_\Sigma g(H(\Sigma), \Phi)\, \mathrm{dvol}_\Sigma \\
&+ \frac{1}{2} \int_\Sigma \left(|\nabla^N \Phi|^2_g - g((\mathrm{Ric}_\Sigma + \mathfrak{H}^2_\Sigma)\, \Phi, \Phi) \right)\, \mathrm{dvol}_\Sigma \\
&+ \frac{1}{2} \int_\Sigma (g(H(\Sigma), \Phi))^2\, \mathrm{dvol}_\Sigma + ...
\end{aligned}
$$

Proof. First of all we expand the induced metric on Σ_Φ. Using the result of the previous Lemma, we find

$$
\begin{aligned}
(\bar{g}_\Phi)_{\alpha\beta} &= \bar{g}_{\alpha\beta} - 2\, g(\bar{h}_{\alpha\beta}, \Phi) + g(R(E_\alpha, \Phi)\, E_\beta, \Phi) + g(\nabla_{E_\alpha} \Phi, \nabla_{E_\beta} \Phi) + \dots \\
&= \bar{g}_{\alpha\beta} - 2\, g(\bar{h}_{\alpha\beta}, \Phi) + g(R(E_\alpha, \Phi)\, E_\beta, \Phi) \\
&+ \bar{g}^{\gamma\gamma'} g(\bar{h}_{\alpha\gamma}, \Phi)\, g(\bar{h}_{\gamma\beta}, \Phi) + g(\nabla^N_{E_\alpha} \Phi, \nabla^N_{E_\beta} \Phi) + \dots
\end{aligned}
$$

Now use the well known expansions

$$\det(I + A) = 1 + \mathrm{Tr}\, A + \frac{1}{2}\, \left((\mathrm{Tr} A)^2 - \mathrm{Tr}(A^2) \right) + ...$$

together with $\sqrt{1+x} = 1 + \frac{1}{2}\, x - \frac{1}{8}\, x^2 + ...$ to conclude that

$$
\begin{aligned}
\sqrt{\det \bar{g}_\Phi} = \big(1 - g(H(\Sigma), \Phi) + \tfrac{1}{2} \big(|\nabla^N \Phi|^2_g &- g((\mathrm{Ric}_\Sigma + (\mathfrak{H})^2_\Sigma)\, \Phi, \Phi) \\
&+ (g(H(\Sigma), \Phi))^2 \big) + ... \big)\, \sqrt{\det \bar{g}}
\end{aligned}
$$

This completes the proof. □

From this we obtain the first and second variations of the volume functional,

$$D_\Phi \mathrm{Vol_k}(\Sigma_\Phi)|_\Phi \Psi = - \int_\Sigma g(H(\Sigma_\Phi), \Psi)\, \mathrm{dvol}_{\Sigma_\Phi}, \tag{16}$$

and

$$\begin{aligned} D^2_\Phi \mathrm{Vol_k}(\Sigma_\Phi)|_{\Phi=0}(\Psi,\Psi) &= \int_\Sigma \left(|\nabla^N \Psi|^2 - g((\mathrm{Ric}_\Sigma + \mathfrak{H}^2_\Sigma)\,\Psi,\Psi)\right)\,\mathrm{dvol}_\Sigma \\ &+ \int_\Sigma (g(H(\Sigma),\Psi))^2\,\mathrm{dvol}_\Sigma. \end{aligned}$$

On the other hand, differentiating (16) once more gives

$$\begin{aligned} D^2_\Phi \mathrm{Vol_k}(\Sigma_\Phi)|_{\Phi=0}(\Psi,\Psi) &= -\int_\Sigma g(D_\Phi H(\Sigma_\Phi)|_{\Phi=0}\Psi,\Psi)\,\mathrm{dvol}_\Sigma \\ &+ \int_\Sigma (g(H(\Sigma),\Psi))^2\,\mathrm{dvol}_K. \end{aligned}$$

Comparing the two formulæ implies that the orthogonal projection of the Jacobi operator to $N\Sigma$ equals

$$J^N_\Sigma := D_\Phi H(\Sigma_\Phi)|_{\Phi=0} = \Delta^N_g + \mathrm{Ric}^N_\Sigma + \mathfrak{H}^2_\Sigma\,,$$

5. Appendix 2

Let $K_\varepsilon(\Pi_p)$ be the constant mean curvature submanifold constructed in Proposition (3.1) and denote by $F : T_pM \longrightarrow M$ the exponential mapping. Recall that

$$K_\varepsilon(\Pi_p) = F(S^k_{\varepsilon,\Phi}),$$

where $S^k_{\varepsilon,\Phi}$ is a submanifold of T_pM parametrized by $\left\{\varepsilon\,(1-\phi)\,\Theta + \varepsilon\,\Phi^\perp,\ \Theta \in S^k\right\}$. It follows from the proof of that proposition that

$$\phi(\Theta) = \frac{\varepsilon^2}{3}\left(\frac{2}{k(k+2)}\mathcal{R}_{k+1}(\Pi_p) - \frac{1}{k+2}\mathcal{R}ic(\Pi_p)(\Theta,\Theta)\right) + \mathcal{O}(\varepsilon^3),$$

$$\Phi^\perp = \mathcal{O}(\varepsilon^3).$$

There is also the minimal submanifold $Q_\varepsilon(\Pi_p) = F(B^{k+1}_{\varepsilon,\Phi})$,

where $B^{k+1}_{\varepsilon,\Phi} = \left\{\varepsilon\,y + \varepsilon\,U_\Phi(y),\ y \in B^{k+1}\right\}$ and

$$U_\Phi(y) = \phi\,(y/\|y\|) + W(y) + \mathcal{O}(p)(\varepsilon^3),$$

$$W(y) = \frac{1}{(k+3)}\sum_{i=1}^{k+1}\mathcal{R}ic^\perp(\Pi_p)_{i\mu}(|y|^2-1)\,y_i\,E_\mu.$$

We shall calculate the volume forms of $S^k_{\varepsilon,\Phi}$ and $B^{k+1}_{\varepsilon,\Phi}$ with respect to F^*g. To prepare for this, recall that near $x = 0$

$$\begin{aligned}(F^*g)_{ij} = \delta_{ij} &+ \frac{1}{3}\, g_p(R_p(x,E_i)x,E_j) + \frac{1}{6}\, g_p(\nabla_x R_p(x,E_i)x,E_j) \\ &+ \frac{1}{20}\, g_p(\nabla_x\nabla_x R_p(x,E_i)x,E_j) \\ &+ \sum_{\ell=1}^{m+1} \frac{2}{45}\, g_p(R_p(x,E_i)x,E_\ell)\, g_p(R_p(x,E_j)x,E_\ell) + \mathcal{O}_p(|x|^5)\end{aligned}$$

where R_p is the curvature tensor of M at the point p, cf. [7].

5.1. **Volume of the CMC sphere.** We first calculate the metric on $S^k_{\varepsilon,\Phi}$. In terms of the coordinate vector fields $\Theta_\alpha, \alpha = 1,\dots,k$ which are tangent to S^k, we can write the tangent vector fields to $S^k_{\varepsilon,\Phi}$ as

$$\tau_\alpha = \varepsilon\,(1-\phi(\Theta))\,\Theta_\alpha - \varepsilon\,\partial_\alpha\,\phi\,\Theta + \sum_{\mu=k+1}^{m+1} \varepsilon\,\partial_\alpha\Phi^\mu\,E_\mu, \quad \alpha = 1,\dots,k.$$

The metric coefficients then equal

$$\begin{aligned}g^K_{\alpha,\beta} &= \varepsilon^2(1-\phi)^2\, g^S_{\alpha,\beta} + \varepsilon^2\,\partial_\alpha\phi\,\partial_\beta\phi + \frac{\varepsilon^4}{3}\,(1-\phi)^4\, g_p(R_p(\Theta,\Theta_\alpha)\Theta,\Theta_\beta) \\ &+\frac{\varepsilon^5}{6}\, g_p(\nabla_\Theta R_p(\Theta,\Theta_\alpha)\Theta,\Theta_\beta) + \frac{\varepsilon^6}{20} g_p(\nabla_\Theta\nabla_\Theta R_p(\Theta,\Theta_\alpha)\Theta,\Theta_\beta) \\ &+\sum_{l=1}^{k+1} \frac{2\varepsilon^6}{45}\, g_p(R_p(\Theta,\Theta_\alpha)\Theta,E_l)\, g_p(R_p(\Theta,\Theta_\beta)\Theta,E_l) \\ &+\sum_{\mu=k+1}^{m+1} \frac{2\varepsilon^6}{45}\, g_p(R_p(\Theta,\Theta_\alpha)\Theta,E_\mu)\, g_p(R_p(\Theta,\Theta_\beta)\Theta,E_\mu) + \mathcal{O}(\varepsilon^7)\end{aligned}$$

Using

$$\sqrt{\det(I+A)} = 1 + \frac{1}{2}\mathrm{tr}A + \frac{1}{8}(\mathrm{tr}A)^2 - \frac{1}{4}\mathrm{tr}(A^2) + \mathcal{O}(|A|^3),$$

we get

$$\begin{aligned}\varepsilon^{-k}\,\frac{\sqrt{\det g^K}}{\sqrt{\det g^S}} &= 1 - k\phi + \frac{k(k-1)}{2}\phi^2 + \frac{1}{2}\,|\nabla_{S^k}\phi|^2 \\ &-\frac{\varepsilon^2}{6}(1-(k+2)\phi)\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) - \frac{\varepsilon^3}{12}\,\nabla_\Theta\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) \\ &-\frac{\varepsilon^4}{40}\nabla^2_\Theta\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) + \frac{\varepsilon^4}{72}\,(\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta))^2 \\ &-\frac{\varepsilon^4}{180}\sum_{i,j=1}^{k+1} g_p(R_p(\Theta,E_i)\Theta,E_j)^2 \\ &+\frac{\varepsilon^4}{45}\sum_{i=1}^{k+1}\sum_{\mu=k+1}^{m+1} g_p(R_p(\Theta,E_i)\Theta,E_\mu)^2 + \mathcal{O}_p(\varepsilon^5).\end{aligned}$$

5.2. **Volume of the minimal ball.** The tangent vectors to $B^{k+1}_{\varepsilon,\Phi}$ are

$$T_i(y) = \varepsilon\,(1-u(y))\,E_i + \varepsilon\,\partial_{y_i}u(y)\,y + \varepsilon\sum_{\mu=k+1}^{m+1}\partial_{y_i}W^\mu(y)\,E_\mu + \mathcal{O}_p(\varepsilon^4),$$

where $u(y)=\phi(y/|y|)$. The corresponding metric coefficients are

$$\begin{aligned}
\varepsilon^{-2}\,g^Q_{ij} &= (1-u)^2\,\delta_{ij} + (1-u)\left(\partial_{y_i}u\,y_j + \partial_{y_j}u\,y_i\right) + |y|^2\partial_{y_i}u\,\partial_{y_j}u + \sum_{\mu=k+1}^{m+1}\partial_{y_i}W^\mu\,\partial_{y_j}W^\mu\\
&+\frac{\varepsilon^2}{3}\,(1-u)^4\,g_p(R_p(y,E_i)y,E_j) + \frac{\varepsilon^2}{3}\sum_{\mu=k+1}^{m+1}\Big(W^\mu\,g_p(R_p(E_\mu,E_i)y,E_j)\\
&+W^\mu\,g_p(R_p(y,E_i)E_\mu,E_j) + \partial_{y_i}W^\mu g_p(R_p(y,E_\mu)y,E_j) + \partial_{y_j}W^\mu\,g_p(R_p(y,E_i)y,E_\mu)\Big)\\
&+\frac{\varepsilon^3}{6}\,g_p(\nabla_y R_p(y,E_i)y,E_j) + \frac{\varepsilon^4}{20}\,g_p(\nabla_y\nabla_y R_p(y,E_i)y,E_j)\\
&+\frac{2\varepsilon^4}{45}\sum_{l=1}^{k+1}g_p(R_p(y,E_i)y,E_l)\,g_p(R_p(y,E_i)y,E_l)\\
&+\frac{2\varepsilon^4}{45}\sum_{\mu=k+1}^{m+1}g_p(R_p(y,E_i)y,E_\mu)\,g_p(R_p(y,E_i)y,E_\mu) + \mathcal{O}(\varepsilon^5)
\end{aligned}$$

Putting $y=r\Theta$, $r\in(0,1)$ we calculate the volume element of $Q_\varepsilon(\Pi_p)$:

$$\begin{aligned}
\varepsilon^{-(k+1)}\sqrt{\det g^Q} &= 1-(k+1)\phi + \frac{k(k+1)}{2}\,\phi^2 + \sum_{\mu=k+1}^{m+1}\frac{1}{2}|\nabla_{S^k}W^\mu|^2\\
&-\frac{\varepsilon^2}{6}\,r^2\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) + \frac{\varepsilon^2}{6}\,r^2\,(k+3)\phi\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\\
&-\frac{\varepsilon^2}{3}\,r^2\sum_{i=1}^{k+1}\sum_{\mu=k+1}^{m+1}\Big(W^\mu\,g_p(R_p(\Theta,E_i,E_\mu,E_i) + \partial_{y_i}W^\mu\,g_p(R_p(\Theta,E_i)\Theta,E_\mu)\Big)\\
&\frac{\varepsilon^3}{12}\,r^3\,\nabla_\Theta\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) - \frac{\varepsilon^4}{40}\,r^4\,\nabla^2_\Theta\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\\
&+\frac{\varepsilon^4}{72}\,r^4\,(\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta))^2 - \frac{\varepsilon^4}{180}\,r^4\sum_{i,j=1}^{k+1}g_p(R_p(\Theta,E_i)\Theta,E_j)^2\\
&+\frac{\varepsilon^4}{45}\,r^4\sum_{i=1}^{k+1}\sum_{\mu=k+1}^{m+1}g_p(R_p(\Theta,E_i)\Theta,E_\mu)^2 + \mathcal{O}_p(\varepsilon^5).
\end{aligned}$$

5.3. **Expansion of the energy functional.** Collecting the results above gives that

$$
\begin{aligned}
&\varepsilon^{-k}\Big(\ \mathrm{Vol}(K_\varepsilon(\Pi_p)) - \frac{k}{\varepsilon}\,\mathrm{Vol}(Q_\varepsilon(\Pi_p)) \ \Big) \\
&\quad = \frac{1}{k+1}\,\mathrm{Vol}(S^k) - \frac{\varepsilon^2}{2}\frac{1}{k+3}\int_{S^k} \mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\,d\sigma + \int_{S^k}\frac{\varepsilon^2}{6}\,\mathcal{R}ic_{k+1}(\Pi_p)\,\phi\,d\sigma \\
&\quad +\varepsilon^4\,\frac{5}{k+5}\int_{S^k}\Big[-\frac{1}{40}\,\nabla^2_\Theta\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) \\
&\quad +\frac{1}{72}\,\big(\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\big)^2 - \frac{1}{180}\sum_{i,j=1}^{k+1} g_p(R_p(\Theta,E_i)\Theta,E_j)^2 \\
&\quad +\frac{1}{45}\sum_{i=1}^{k+1}\sum_{\mu=k+1}^{m+1} g_p(R_p(\Theta,E_i)\Theta,E_\mu)^2\Big]\,d\sigma \\
&\quad +\sum_{\mu=k+1}^{m+1}\frac{k}{2}\int_{B^{k+1}} W^\mu\,\Delta_{B^{k+1}}\,W^\mu\,dy \\
&\quad +\frac{\varepsilon^2}{3}\,k\sum_{i=1}^{k+1}\int_{B^{k+1}}\Big(W^\mu\,g_p(R_p(\Theta,E_i,E_\mu,E_i) + \partial_{y_i}W^\mu\,R_p(\Theta,E_i,\Theta,E_\mu)\Big) + \mathcal{O}(\varepsilon^5)
\end{aligned}
$$

We now recall some identities. First,

$$
\int_{S^k}(\Theta^i)^2\,d\sigma = \frac{1}{k+1}\,\mathrm{Vol}(S^k),
$$

$$
\textstyle\int_{S^k}(\Theta^i)^4\,d\sigma = 3\displaystyle\int_{S^k}(\Theta^i\,\Theta^j)^2\,d\sigma = \frac{3}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)
$$

and second, if $a_{ijpq}\in\mathbb{R}$ $i,j,p,q = 1,\dots,k+1$, then

$$
\begin{aligned}
\sum_{p,q,l,n=1}^{k+1}\int_{S^k} a_{pqln}\,\Theta^p\,\Theta^q\,\Theta^l\,\Theta^n\,d\sigma \ &= \tfrac{3}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\sum_{i=1}^{k+1} a_{pppp} \\
&\quad +\tfrac{1}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\sum_{q\neq p=1}^{k+1}(a_{ppqq}+a_{pqpq}+a_{pqqp})\ . \\
&= \tfrac{1}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\sum_{p,q=1}^{k+1}(a_{ppqq}+a_{pqpq}+a_{pqqp})
\end{aligned}
$$

We now calculate each term:

$$
\begin{aligned}
\int_{S^k}\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\,d\sigma \ &= \sum_{i,j=1}^{k+1}\int_{S^k}\mathcal{R}ic_{k+1}(\Pi_p)(E_i,E_j)\,\Theta^k\,\Theta^l\,d\sigma \\
&= \sum_{i=1}^{k+1}\mathcal{R}ic_{k+1}(\Pi_p)(E_i,E_i)\,(\Theta^i)^2\,d\sigma \\
&= \frac{1}{k+1}\,\mathrm{Vol}(S^k)\,\mathcal{R}_{k+1}(\Pi_p);
\end{aligned}
$$

$$\begin{aligned}\int_{S^k}(\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta))^2 d\sigma &= \frac{1}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\left(2\sum_{i,j=1}^{k+1}(\mathcal{R}ic_{k+1}(\Pi_p)(E_i,E_j))^2\right)\\ &\quad+\sum_{i,j=1}^{k+1}\mathcal{R}ic_{k+1}(\Pi_p)(E_i,E_i)\,\mathcal{R}ic_{k+1}(\Pi_p)(E_j,E_j)\\ &= \frac{1}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\left(2\,\|\mathcal{R}ic_{k+1}(\Pi_p)\|^2+\mathcal{R}_{k+1}(\Pi_p)^2\right);\end{aligned}$$

$$\begin{aligned}&\sum_{\alpha,\beta=1}^{k}\int_{S^k} g_p(R_p(\Theta,\Theta_\alpha)\Theta,\Theta_\beta)^2\,d\sigma\\ &\quad=\sum_{i,j=1}^{k+1}\int_{S^k} g_p(R_p(\Theta,E_i)\Theta,E_j)^2\,d\sigma\\ &\quad=\frac{1}{(k+1)(k+3)}\mathrm{Vol}(S^k)\sum_{i,j,p,q=1}^{k+1}\left(R^2_{ipjq}+R_{ipjp}\,R_{iqjq}+R_{ipjq}\,R_{iqjp}\right)\\ &\quad=\frac{1}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\left(\|\mathcal{R}ic_{k+1}(\Pi_p)\|^2+\frac{3}{2}\,\|R_{k+1}(\Pi_p)\|^2\right);\end{aligned}$$

(we use here that $R^2_{ijpq}=(R_{ipjq}-R_{iqjp})^2=R^2_{ipjq}+R^2_{iqjp}-2\,R_{ipjq}\,R_{iqjp}$);

$$\begin{aligned}&\sum_{\alpha=1}^{k}\sum_{\mu=k+1}^{m+1}\int_{S^k} g_p(R_p(\Theta,\Theta_\alpha)\Theta,E_\mu)^2\,d\sigma\\ &\quad=\sum_{i=1}^{k+1}\sum_{\mu=k+1}^{m+1}\int_{S^k} g_p(R_p(\Theta,E_i)\Theta,E_\mu)^2\,d\sigma\\ &\quad=\frac{1}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\left(\left\|\mathcal{R}ic^{\perp}_{k+1}(\Pi_p)\right\|^2+\frac{3}{2}\left\|R^{\perp}_{k+1}(\Pi_p)\right\|^2\right);\end{aligned}$$

$$\begin{aligned}\int_{S^k}\nabla^2_\Theta\,\mathcal{R}ic_{k+1}(\Theta,\Theta)\,d\sigma &= \frac{1}{(k+1)(k+3)}\sum_{i,j=1}^{k+1}\Big(\nabla^2_{E_i}\,\mathcal{R}ic_{k+1}(\Pi_p)(E_j,E_j)\\ &\quad+2\,\nabla_{E_i}\nabla_{E_j}\,\mathcal{R}ic_{k+1}(E_i,E_j)\Big)\\ &= \frac{2}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\,\Delta^g_{k+1}\mathcal{R}_{k+1}(\Pi_p);\end{aligned}$$

$$\begin{aligned}-\sum_{\mu=k+1}^{m+1}\int_B^{k+1}|\nabla_{B^{k+1}}W^\mu|^2\,dy &= \sum_{\mu=k+1}^{m+1}\int_{B^{k+1}}W^\mu\,\Delta_{B^{k+1}}W^\mu\,dy\\ &= -\frac{2\,\varepsilon^4}{9}\,\frac{1}{k+3}\sum_{\mu=k+1}^{m+1}\int_{B^{k+1}}\sum_{i,j=1}^{k+1}R^2_{ijiu}\,(y^j)^2\,(1-|y|^2)\,dy\\ &= -\frac{2\,\varepsilon^4}{9}\,\frac{1}{(k+3)(k+1)}\,\mathrm{Vol}(S^k)\,\|\mathcal{R}ic^{\perp}_{k+1}\|^2\left(\tfrac{1}{k+3}-\tfrac{1}{k+5}\right)\\ &= -\frac{\varepsilon^4}{9}\,\frac{4}{(k+1)(k+3)^2(k+5)}\mathrm{Vol}(S^k)\,\|\mathcal{R}ic^{\perp}_{k+1}\|^2;\end{aligned}$$

$$\begin{aligned}\sum_{\mu=1}^{k+1}\int_{B^{k+1}} W^\mu \sum_{i,p=1}^{k+1} R_{ipi\mu}\, y^p\, dy &= -\frac{\varepsilon^2}{3}\,\frac{1}{(k+3)}\,\int_{B^{k+1}} R^2_{iji\mu}\,(y^j)^2(1-|y|^2)\,dy\\ &= -\frac{\varepsilon^2}{3}\,\frac{2}{(k+1)(k+3)^2(k+5)}\mathrm{Vol}(S^k)\,\|\mathcal{R}ic^\perp_{k+1}\|^2;\end{aligned}$$

and

$$\begin{aligned}&\sum_{\mu=k+1}^{m+1}\ \int_{B^{k+1}} \partial_{y^i} W^\mu R_{piq\mu}\, y^p\, y^q\, dy\\ &\quad= -\frac{\varepsilon^2}{3}\,\frac{1}{(k+3)}\sum_{\mu=k+1}^{m+1}\int_{B^{k+1}}\sum_{i,p,q=1}^{k+1}\Big(\mathcal{R}ic(\Pi_p)^\perp_{i\mu}\,R_{piq\mu}\,y^p\,y^q\,(1-|y|^2)\\ &\quad-2\sum_{j=1}^{k+1}\mathcal{R}ic(\Pi_p)^\perp_{j\mu}\,R_{piq\mu}\,y^j\,y^i\,y^p\,y^q\Big)\,dy\\ &\quad= \frac{\varepsilon^2}{3}\,\frac{2}{(k+1)(k+3)^2(k+5)}\mathrm{Vol}(S^k)\Big[-\|\mathcal{R}ic^\perp_{k+1}\|^2\\ &\quad+\sum_{p,q=1}^{k+1}\sum_{\mu=k+1}^{m+1}\big(\mathcal{R}ic(\Pi_p)^\perp_{p\mu}\,R_{qpq\mu}+\mathcal{R}ic(\Pi_p)^\perp_{q\mu}\,R_{ppq\mu}+\mathcal{R}ic(\Pi_p)^\perp_{q\mu}\,R_{qpp\mu}\big)\Big]\\ &\quad= -\frac{2\,\varepsilon^2}{3}\,\frac{1}{(k+1)(k+3)^2(k+5)}\,\mathrm{Vol}(S^k)\,\|\mathcal{R}ic^\perp_{k+1}\|^2.\end{aligned}$$

This gives finally

$$\begin{aligned}\frac{(k+1)\,\mathcal{E}(\Pi_p)}{\varepsilon^k\,\mathrm{Vol}(S^k)} &= 1-\frac{\varepsilon^2}{2}\frac{1}{k+3}\,\mathcal{R}_{k+1}(\Pi_p)\\ &+\frac{\varepsilon^4}{72}\,\frac{1}{(k+3)(k+5)}\Big(8\,\|\mathcal{R}ic_{k+1}(\Pi_p)\|^2-18\,\Delta^g_{k+1}\mathcal{R}_{k+1}(\Pi_p)-3\,\|R_{k+1}(\Pi_p)\|^2\\ &+5\,\mathcal{R}_{k+1}(\Pi_p)^2+8\,\|\mathcal{R}ic^\perp_{k+1}(\Pi_p)\|^2+12\,\|R^\perp_{k+1}(\Pi_p)\|^2\Big)\\ &+\frac{\varepsilon^4}{18}\Big(\frac{2}{k(k+2)}\mathcal{R}^2_{k+1}(\Pi_p)-\frac{1}{(k+2)(k+3)}\mathcal{R}^2_{k+1}(\Pi_p)\\ &-\frac{2}{(k+2)(k+3)}\|\mathcal{R}ic_{k+1}(\Pi_p)\|^2-\frac{12k}{(k+3)^2(k+5)}\|\mathcal{R}ic^\perp_{k+1}\|^2\Big)+\mathcal{O}(\varepsilon^5),\end{aligned}$$

or after simplification,

$$\begin{aligned}\frac{(k+1)\,\mathcal{E}(\Pi_p)}{\varepsilon^k\,\mathrm{Vol}(S^k)} &= 1-\frac{\varepsilon^2}{2}\frac{1}{k+3}\,\mathcal{R}_{k+1}(\Pi_p)\\ &+\frac{\varepsilon^4}{72}\,\frac{1}{(k+3)(k+5)}\Big(8\,\|\mathcal{R}ic_{k+1}(\Pi_p)\|^2-18\;\Delta^g_{k+1}\mathcal{R}_{k+1}(\Pi_p)-3\,\|R_{k+1}(\Pi_p)\|^2\\ &+5\,\mathcal{R}_{k+1}(\Pi_p)^2+8\,\|\mathcal{R}ic^{\perp}_{k+1}(\Pi_p)\|^2+12\,\|R^{\perp}_{k+1}(\Pi_p)\|\Big)\\ &+\frac{\varepsilon^4}{18}\,\frac{1}{(k+2)(k+3)}\Big(\tfrac{k+6}{k}\,\mathcal{R}^2_{k+1}(\Pi_p)-\|\mathcal{R}ic_{k+1}(\Pi_p)\|^2\\ &-\frac{12\,k(k+2)}{(k+3)(k+5)}\,\|\mathcal{R}ic^{\perp}_{k+1}(\Pi_p)\|^2\Big)+\mathcal{O}(\varepsilon^5)\\ &= 1-\frac{\varepsilon^2}{2}\frac{1}{k+3}\,\mathcal{R}_{k+1}(\Pi_p)+\frac{\varepsilon^4}{2(k+3)}\,\mathbf{r}(\Pi_p)+\mathcal{O}(\varepsilon^5).\end{aligned}$$

6. Problems

The results above produce a collection of k-dimensional spheres. It is reasonable to suspect that there are other compact k-dimensional embedded constant mean curvature submanifolds in $\mathbb{R}^n$? Find some other examples!

Is it possible to build noncompact k-dimensional (nonzero) constant mean curvature submanifolds which are not contained in a $(k+1)$-dimensional subspace ? For zero mean curvature submanifolds, the half plane, a half helicoid (observe that there are two ways to cut the helicoid in half along a straight line) which has boundary a straight line, and a fundamental piece of a Riemann surface, whose boundary are 2 parallel straight lines, are nontrivial examples. Are there any analogues in this setting?

It should follow by unique continuation that if $K=\partial Q=\partial Q'$ is a constant mean curvature submanifold, with $H_K\neq 0$, then $Q=Q'$. When Q is a hypersurface, so K has codimension 2, this is true by the Hopf boundary maximum principle.

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