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Thermal blurring of a coherent Fermi gas

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It is generally assumed that a condensate of paired fermions at equilibrium is characterized by a macroscopic wavefunction with a well-defined, immutable phase. In reality, all systems have a finite size and are prepared at non-zero temperature; the condensate has then a finite coherence time, even when the system is isolated in its evolution and the particle number N is fixed. The loss of phase memory is due to interactions of the condensate with the excited modes that constitute a dephasing environment. This fundamental effect, crucial for applications using the condensate of pairs’ macroscopic coherence, was scarcely studied. We link the coherence time to the condensate phase dynamics, and we show with a microscopic theory that the time derivative of the condensate phase operator $\hat{\theta}_0$ is proportional to a chemical potential operator that we construct including both the pair-breaking and pair-motion excitation branches. In a single realization of energy E , $\hat{\theta}_0$ evolves at long times as $-2\mu_{\text{mc}}(E)t/\hbar$ where $\mu_{\text{mc}}(E)$ is the microcanonical chemical potential; energy fluctuations from one realization to the other then lead to a ballistic spreading of the phase and to a Gaussian decay of the temporal coherence function with a characteristic time $\propto N^{1/2}$. In the absence of energy fluctuations, the coherence time scales as N due to the diffusive motion of $\hat{\theta}_0$. We propose a method to measure the coherence time with ultracold atoms, which we predict to be tens of milliseconds for the canonical ensemble unitary Fermi gas.

Fermi gas | Quantum fluids | Quantum coherence

Setting the stage

Coherent gases of ultracold atoms confined in immaterial non-dissipative traps are unique examples of isolated macroscopic quantum systems. The value of their intrinsic coherence time is therefore a fundamental question. But it is also a practical issue for all the applications which exploit macroscopic coherence, such as interferometry or quantum engineering where one generates non-trivial entangled states by coherent evolution [1, 2, 3]. Coherence time measurements are presently being performed in cold Bose gases [4, 5, 6]. Experiments on Fermi gases, which up to now focused on traditional aspects of the N -body problem, such as thermodynamic properties [7, 8], are moving towards correlation and coherence measurements [9]. This turn will open a new research field, including the strong coupling regime : that of fermionic quantum optics [10]. However, a theory predicting the coherence time of a pair-condensed Fermi gas was missing, except in the limiting case of zero temperature [11]. In this paper we present the first microscopic theory bridging this theoretical gap in a general way. Our approach holds for other physical systems, such as mesoscopic Josephson Junctions, provided that the environment-induced decoherence is sufficiently reduced.

For a Bose-condensed gas of bosons, the finite coherence time is due to the spreading of the condensate phase probability distribution. At zero temperature and in presence of interactions, a ballistic phase spreading is caused by atom number fluctuations in the sample. This effect has been measured by interfering two initially mutually-coherent condensates, whose particle number fluctuates due to partition noise [4, 6]. Contrarily to lasers, which are open quantum systems, and somehow unexpectedly, a ballistic spreading persists in Bose-Einstein condensates for a fixed atom number at non-

zero temperature [12, 13]. Fluctuations of the energy, another conserved quantity, then play the same role as number fluctuations.

For an unpolarized pair-condensed Fermi gas, the study of coherence time presupposes a clear definition of the condensate phase and of the corresponding operator $\hat{\theta}_0$ [11]. Furthermore, at non-zero temperature the speed of variation of the phase should include the contribution of two excitation branches: the fermionic pair-breaking one and the bosonic one exciting the pair motion. For the fermionic branch Anderson’s Random Phase Approximation (RPA) [14] is enough. For the bosonic branch however, we need the equivalent for fermions of the Bogoliubov method to construct quasiparticle creation \hat{b}_α^\dagger and annihilation \hat{b}_α operators and to express $d\hat{\theta}_0/dt$ in term of these operators. More than that, we need to include interactions among quasiparticles in the evolution of the \hat{b}_α . This is a non-trivial consequence of the dependence of condensate wavefunction on the total number of particles N even for a spatially homogeneous system, and clearly goes beyond the RPA program.

Correlation function decay

Below the critical temperature, the time-correlation function of the pairing field $\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r}')$ where $\hat{\psi}_\sigma(\mathbf{r})$ is the fermionic field operator of the spin σ component, is dominated at long times by the condensate contribution:

$$g_1(t) = \langle \hat{a}_0^\dagger(t)\hat{a}_0(0) \rangle \quad [1]$$

where $\hat{a}_0 = \int d^3r d^3r' \varphi_0(\mathbf{r}, \mathbf{r}')\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r}')$ is the component of the pairing field on the condensate wavefunction [11]. At equilibrium the system is in a mixture of N -body eigenstates $|\psi_\lambda\rangle$, with weights Π_λ . We therefore study the correlation function $g_1^\lambda(t)$ in the eigenstate $|\psi_\lambda\rangle$ of energy E_λ and particle number N_λ . To exploit the weak relative fluctuations in the number of condensed pairs for a large system, we split \hat{a}_0 into modulus and phase hermitian operators [11]

$$\hat{a}_0 = e^{i\hat{\theta}_0}\hat{N}_0^{1/2}, \quad [2]$$

and we approximate \hat{N}_0 by its mean value \bar{N}_0 in the equilibrium state to obtain

$$g_1^\lambda(t) \simeq \bar{N}_0 e^{iE_\lambda t/\hbar} \langle \psi_\lambda | e^{-i(\hat{H} + \hat{W})t/\hbar} | \psi_\lambda \rangle \quad [3]$$

The operator \hat{W} , difference between \hat{H} transformed by $e^{i\hat{\theta}_0}$ and \hat{H} ,

$$\hat{W} = e^{-i\hat{\theta}_0}\hat{H}e^{i\hat{\theta}_0} - \hat{H} = -i[\hat{\theta}_0, \hat{H}] - \frac{1}{2}[\hat{\theta}_0, [\hat{\theta}_0, \hat{H}]] + \dots \quad [4]$$

is approximately N times smaller than \hat{H} . Indeed $e^{i\hat{\theta}_0}$, like \hat{a}_0 , changes the total particle number by a quantity $O(N^0)$. While \hat{H} is an extensive observable, \hat{W} is intensive and the double commutator in (4) is of order $1/N$. In equation (3) formally appears the evolution operator of the Hamiltonian \hat{H} perturbed by \hat{W} , and restricted to the eigenstate $|\psi_\lambda\rangle$ of \hat{H} . Up to a phase factor, the function g_1^λ/\bar{N}_0 is then proportional to the probability amplitude that the system prepared in $|\psi_\lambda\rangle$ is still in that state after a time t .

A standard way to obtain a non-perturbative approximation of this amplitude is to use the Green function or the resolvent operator $\hat{G}(z) = (z\hat{1} - (\hat{H} + \hat{W}))^{-1}$ of the perturbed Hamiltonian. Within the projectors method (see §III.B.2 of [15]), we introduce an effective non hermitian Hamiltonian $\hat{H}_{\text{eff}}(z)$ governing the evolution restricted to $|\psi_\lambda\rangle$, *id est* $\langle\psi_\lambda|\hat{G}(z)|\psi_\lambda\rangle = (z - \langle\psi_\lambda|\hat{H}_{\text{eff}}(z)|\psi_\lambda\rangle)^{-1}$. Keeping in $\hat{H}_{\text{eff}}(z)$ terms up to order two in \hat{W} and neglecting its z dependence $\hat{H}_{\text{eff}}(z) \approx \hat{H}_{\text{eff}}(E_\lambda + i0^+)$ (pole approximation), we obtain¹

$$g_1^\lambda(t) \simeq \bar{N}_0 e^{-it\langle\psi_\lambda|\hat{W}|\psi_\lambda\rangle/\hbar} e^{-(i\delta_\lambda + \gamma_\lambda)t} \quad [5]$$

Introducing $\hat{Q}_\lambda = \hat{1} - |\psi_\lambda\rangle\langle\psi_\lambda|$, the projector orthogonal to $|\psi_\lambda\rangle$, and the notation $\langle\hat{A}\rangle_\lambda \equiv \langle\psi_\lambda|\hat{A}|\psi_\lambda\rangle$ one has²:

$$\hbar(\delta_\lambda - i\gamma_\lambda) = \langle\hat{W}\hat{Q}_\lambda \frac{\hat{Q}_\lambda}{(E_\lambda + i0^+)\hat{Q}_\lambda - \hat{Q}_\lambda\hat{H}\hat{Q}_\lambda} \hat{Q}_\lambda\hat{W}\rangle_\lambda \quad [6]$$

The leading term under the exponential in (5), $\langle\psi_\lambda|\hat{W}|\psi_\lambda\rangle$, is of order N^0 like \hat{W} . A key step in its interpretation is to remark that, according to the expansion in (4), in the Heisenberg picture

$$\hat{W}(t) = \hbar \frac{d\hat{\theta}_0}{dt} + O\left(\frac{1}{N}\right) \quad [7]$$

At this stage it may seem difficult to obtain a tractable explicit expression of $d\hat{\theta}_0/dt$ and to go beyond a purely formal result for the phase dynamics. Fortunately this is not the case and, as we will show in the next section, the coarse grained time average of $d\hat{\theta}_0/dt$ in a weakly excited gas is proportional to a chemical potential operator, which is in essence a thermodynamic quantity:

$$-\frac{\hbar}{2} \frac{d\hat{\theta}_0}{dt} = \mu_0(\hat{N}) + \sum_{s=F,B} \sum_{\alpha} \frac{d\epsilon_{s,\alpha}}{dN} \hat{n}_{s,\alpha} \quad [8]$$

The sum on the right hand side runs over both the gapped quasi-particles fermionic branch of excitation (in the homogeneous case α includes both an orbital and a spin index, $\alpha = \mathbf{k}, \sigma$) and the bosonic one which, in the thermodynamic limit and for an homogeneous system, has a phonic behaviour (α is then only orbital, $\alpha = \mathbf{q}$). By requiring that the gas is weakly excited we mean that the thermal depletion of the condensate of pairs must be small. This requires in particular that the average number of quasi-particles is a small fraction of the total particle number. The coarse grained time average is taken over a time long with respect to the inverse of the quasi-particle eigenfrequencies $\epsilon_{s,\alpha}/\hbar$ yet short with respect to the typical time-scale of variation of the occupation numbers $\hat{n}_{s,\alpha}$. Finally $\mu_0(N)$ is the zero temperature chemical potential of the gas with N particles, that is the derivative of the ground state energy with respect to N . We interpret the second term on the right hand side of (8) as a “chemical potential operator” in the sense that its quantum average is the adiabatic derivative of the quasi-particle gas energy $\sum_{s=F,B} \sum_{\alpha} \epsilon_{s,\alpha} \langle\hat{n}_{s,\alpha}\rangle$ with respect to N , that is at fixed quasi-particle populations $\langle\hat{n}_{s,\alpha}\rangle$. Equation (8) establishes the link between the phase derivative and the chemical potential at the level of quantum mechanical operators in a multimode microscopic theory. In that respect, it goes beyond the usual second Josephson relation for the phase of the superconducting order parameter (see §3.4 in [16]).

By taking the average of equation (8) in the stationary state $|\psi_\lambda\rangle$ and using the Eigenstate thermalization hypothesis [17]

to identify the quantum average in an eigenstate with the microcanonical average, we recognize the microcanonical chemical potential μ_{mc} at energy E_λ and particle number N_λ and obtain:

$$\langle\psi_\lambda|\hbar \frac{d\hat{\theta}_0}{dt}|\psi_\lambda\rangle = -2\mu_{\text{mc}}(E_\lambda, N_\lambda) \quad [9]$$

We omitted here the coarse grained time average as the quantum average is taken in an exact eigenstate of the system.

The next-to-leading term under the exponential in (5), is of order $1/N$. In order to prove it, we express this term in terms of the correlation function of $d\hat{\theta}_0/dt$ in $|\psi_\lambda\rangle$: up to a contribution of order $1/N^2$,

$$\gamma_\lambda + i\delta_\lambda = \int_0^{+\infty} dt \left[\left\langle \frac{d\hat{\theta}_0(t)}{dt} \frac{d\hat{\theta}_0(0)}{dt} \right\rangle_\lambda - \left\langle \frac{d\hat{\theta}_0}{dt} \right\rangle_\lambda^2 \right] \quad [10]$$

This is equivalent to (6) as can be checked by inserting a closure relation. The $t = 0$ value of the integrand is $\text{Var}_\lambda(d\hat{\theta}_0/dt) = O(1/N)$ (this comes from adding up the variances of independent quasi-particles numbers in the canonical ensemble and it overestimates the microcanonical variance); the function then decays in a time τ_c which is the typical collision time of quasi-particles and hence the correlation time of the $\hat{n}_{s,\alpha}$. Altogether we estimate $|\gamma_\lambda + i\delta_\lambda| \approx \tau_c \text{Var}(d\hat{\theta}_0/dt) = O(1/N)$. The energy shift δ_λ is thus of the same order in N as the subleading term $[\hat{\theta}_0, [\hat{\theta}_0, \hat{H}]]$ in \hat{W} , *i.e.* N times smaller than (9); we neglect both terms for a large system. In contrast, we keep γ_λ , the only term which leads to an exponential decay of the correlation function g_1^λ . It is in fact the phase diffusion coefficient of a system prepared in the microcanonical ensemble corresponding to $|\psi_\lambda\rangle$, $\gamma_\lambda = D(E_\lambda, N_\lambda)$. If one can write kinetic equations for the quasi-particles numbers appearing in (8), one can calculate their time correlation functions and hence D from (10), as done for bosons in [18]. Finally:

$$g_1^\lambda(t) \simeq \bar{N}_0 e^{[2i\mu_{\text{mc}}(E_\lambda, N_\lambda)/\hbar - D(E_\lambda, N_\lambda)]t} \quad [11]$$

The final step is to take the statistical average over the probability distribution Π_λ of the states $|\psi_\lambda\rangle$. For large N , we assume that energy and atom number fluctuations around the mean values \bar{E} and \bar{N} are weak in relative value. This is the case if Π_λ describes a canonical or grand canonical ensemble. We assume Gaussian fluctuations and linearize μ_{mc} around (\bar{E}, \bar{N}) while, to this order, we keep only the central value $D(\bar{E}, \bar{N})$ of the next-to-leading term. We are led to the calculation of a Gaussian integral with a phase factor $\exp\{2i[(\partial\mu_{\text{mc}}(\bar{E}, \bar{N})/\partial E)(E - \bar{E}) + (\partial\mu_{\text{mc}}(\bar{E}, \bar{N})/\partial N)(N - \bar{N})]t/\hbar\}$. Altogether this leads to the main result of this work :

$$g_1(t) \simeq \bar{N}_0 e^{2i\mu_{\text{mc}}(\bar{E}, \bar{N})t/\hbar} e^{-t^2/2t_{\text{br}}^2} e^{-D(\bar{E}, \bar{N})t} \quad [12]$$

In presence of energy or atom number fluctuations, the thermal blurring at long times consists in a Gaussian decay of the correlation function $g_1(t)$, with a characteristic time

$$(2t_{\text{br}}/\hbar)^{-2} = \text{Var} \left(N \frac{\partial\mu_{\text{mc}}}{\partial N}(\bar{E}, \bar{N}) + E \frac{\partial\mu_{\text{mc}}}{\partial E}(\bar{E}, \bar{N}) \right) \quad [13]$$

¹The pole approximation implicitly assumes that $|\psi_\lambda\rangle$ is coupled to a broad energy continuum. As a consequence (5) holds only at times longer than the inverse frequency width of the continuum, *i.e.* longer than the quasi particle correlation time τ_c introduced below.

²One might object that in a finite size system the spectrum is discrete, the resolvent has no branch cut and there should be no imaginary part in equation (6). For a large system, however, the level spacing is so small that the dynamics imitates the exponential relaxation of an infinite system, see §C.I.3 of [15]. Given that the right hand side of equation (6) scales as $1/N$, the rigorous way to obtain the result is to take the thermodynamic limit after multiplication by N . To avoid the branch cut, one should first shift up the energy E_λ from the real axis by a finite quantity η that will go to zero in the end: $\hbar(\delta_\lambda - i\gamma_\lambda) \simeq \frac{1}{N} \lim_{\eta \rightarrow 0^+} \lim_{N \rightarrow +\infty} N \langle \hat{W} \hat{Q}_\lambda \frac{\hat{Q}_\lambda}{(E_\lambda + i\eta)\hat{Q}_\lambda - \hat{Q}_\lambda \hat{H} \hat{Q}_\lambda} \hat{Q}_\lambda \hat{W} \rangle_\lambda$.

which diverges as $N^{1/2}$ for normal fluctuations. The phase diffusion coefficient D leads to an exponential decay with a characteristic time diverging as N . As expected it is a sub-leading effect at long times, except if the system is prepared in the microcanonical ensemble in which case the intrinsic phase diffusion may be directly observed.

Microscopic derivation of the phase operator equation

We give here the first (to our knowledge) microscopic derivation of equation (8), relating the evolution of the phase operator of a pair-condensed gas to a “chemical potential operator”.

The contribution of the fermionic branch of excitations to $d\hat{\theta}_0/dt$ can be obtained from linearized equations of motion for small fluctuations of the pair operators $\hat{\psi}_\downarrow\hat{\psi}_\uparrow$, $\hat{\psi}_\uparrow^\dagger\hat{\psi}_\downarrow^\dagger$ and $\hat{\psi}_\sigma^\dagger\hat{\psi}_\sigma$ around the mean-field state in the Random Phase Approximation (RPA)[14]. Using equation (120) of reference [11] to extract the time average of $d\hat{\theta}_0/dt$, and rewriting equation (86) of the same reference in terms of the fermionic quasi-particle occupation numbers $\hat{n}_{F,\alpha}$, we get³

$$-\frac{\hbar}{2} \frac{d\hat{\theta}_0}{dt} \Big|_{\text{RPA}} = \mu(\bar{N}) + \frac{d\mu}{d\bar{N}}(\hat{N} - \bar{N}) + \sum_{\alpha=\mathbf{k},\sigma} \frac{d\epsilon_{F,\alpha}}{d\bar{N}} \hat{n}_{F,\alpha} \quad [14]$$

where $\epsilon_{F,\mathbf{k},\sigma}$ is the BCS excitation spectrum of an homogeneous system and \bar{N} the BCS average particle number in the grand canonical ensemble of chemical potential μ .

We encountered fundamental difficulties in deriving the phonon branch contribution to equation (8) within the RPA⁴. We therefore decided to treat the problem variationally with the most general time-dependent coherent state Ansatz⁵:

$$|\psi\rangle = \mathcal{N}(t) \exp \left(l^6 \sum_{\mathbf{r},\mathbf{r}'} \Gamma(\mathbf{r},\mathbf{r}';t) \hat{\psi}_\uparrow^\dagger(\mathbf{r}) \hat{\psi}_\downarrow^\dagger(\mathbf{r}') \right) |0\rangle, \quad [15]$$

Here \mathcal{N} ensures normalization and the $\Gamma(\mathbf{r},\mathbf{r}')$ form a set of independent variables. The space has been discretized on a cubic lattice of step l , which we take to zero in the end of the calculations. The field operators obey anticommutation relations of the kind: $\{\hat{\psi}_\sigma(\mathbf{r}), \hat{\psi}_{\sigma'}^\dagger(\mathbf{r}')\} = \delta_{\sigma,\sigma'} \delta_{\mathbf{r},\mathbf{r}'}/l^3$. Section §9.9b of reference [19] constructs from $\Gamma(\mathbf{r},\mathbf{r}')$, $\Gamma^*(\mathbf{r},\mathbf{r}')$ the set of canonically conjugate variables $\Phi(\mathbf{r},\mathbf{r}')$, $\Phi^*(\mathbf{r},\mathbf{r}')$.⁶ This field Φ should not be confused with the usual pairing field $\langle\hat{\psi}_\downarrow\hat{\psi}_\uparrow\rangle$.⁷ When the pairs in (15) are at rest, $\Gamma(\mathbf{r},\mathbf{r}',t)$ depends only on $\mathbf{r} - \mathbf{r}'$ and the Fourier transform of Φ reproduce the $V_{\mathbf{k}}$ amplitude of the $\mathbf{k} \uparrow, -\mathbf{k} \downarrow$ pair of BCS theory [11], while the Fourier transform of the pairing field is $-U_{\mathbf{k}}V_{\mathbf{k}} = -V_{\mathbf{k}}(1 - |V_{\mathbf{k}}|^2)^{1/2}$. For moving pairs we have no physical interpretation, but the squared norm of Φ is still half the mean number of particles in the state $|\psi\rangle$:

$$\frac{N}{2} = \|\Phi\|^2 \equiv l^6 \sum_{\mathbf{r},\mathbf{r}'} |\Phi(\mathbf{r},\mathbf{r}';t)|^2. \quad [16]$$

Its evolution is governed by the classical Hamiltonian:

$$\mathcal{H}(\Phi, \Phi^*) = \langle\psi|\hat{H}|\psi\rangle. \quad [17]$$

In the following we will need only the invariance of \mathcal{H} under a global phase change $\Phi(\mathbf{r},\mathbf{r}') \rightarrow e^{i\gamma}\Phi(\mathbf{r},\mathbf{r}')$, $\forall\gamma \in \mathbb{R}$ ($U(1)$ symmetry), consequence of the conservation of the particle number \hat{N} by evolution with \hat{H} . At zero temperature and for a fixed N the field $\Phi(\mathbf{r},\mathbf{r}')$ is frozen, up to a global phase factor, into the minimizer $\Phi_0(\mathbf{r},\mathbf{r}') = (N/2)^{1/2}\phi_0(\mathbf{r},\mathbf{r}')$ of \mathcal{H} . ϕ_0 is chosen real and normalized to one. It depends on N even in the spatially

homogeneous case and differs from the condensate wavefunction φ_0 in the same way that Φ differs from the pairing field $\langle\hat{\psi}_\downarrow\hat{\psi}_\uparrow\rangle$ (see note 7). At sufficiently low temperature one can expand \mathcal{H} in powers of the small deviations of Φ away from the circle $\gamma \mapsto e^{i\gamma}\Phi_0(\mathbf{r},\mathbf{r}')$, locus of the minima of \mathcal{H} for fixed N . To this end, we split the field into its components parallel and orthogonal to ϕ_0 :

$$\Phi(\mathbf{r},\mathbf{r}') = e^{i\theta} [n^{1/2}\phi_0(\mathbf{r},\mathbf{r}') + \Lambda(\mathbf{r},\mathbf{r}')] \quad [18]$$

The phase θ can reach arbitrarily large values while Λ is bounded. This framework allows us to develop a systematic perturbation theory in powers of the field Λ (see appendix A), the fermionic equivalent of the Bogoliubov $U(1)$ -symmetry conserving approach for bosons [20]. Provided that Λ stays small, the phase θ remains close to the condensate phase θ_0 as we shall see. We therefore write down the equations of motion of θ and of the fields Λ, Λ^* . At the end of the calculations we systematically eliminate the condensate variables with the relation $n = \|\Phi\|^2 - \|\Lambda\|^2$, consequence of (16), and we restrict ourselves to order 2 in Λ, Λ^* .

The main challenge of the calculation is the occurrence of a term linear in Λ, Λ^* in $d\theta/dt$, resulting from the fact that ϕ_0 depends on the number of pairs [11]. Without this term, one would simply expand the field Λ on the eigenmodes of its small linear oscillations obtained from a quadratization of the Hamiltonian \mathcal{H} at fixed N :

$$\begin{pmatrix} \Lambda(\mathbf{r},\mathbf{r}';t) \\ \Lambda^*(\mathbf{r},\mathbf{r}';t) \end{pmatrix} = \sum_{\alpha} b_{\alpha}(t) \begin{pmatrix} u_{\alpha}(\mathbf{r},\mathbf{r}') \\ v_{\alpha}(\mathbf{r},\mathbf{r}') \end{pmatrix} + b_{\alpha}^*(t) \begin{pmatrix} v_{\alpha}^*(\mathbf{r},\mathbf{r}') \\ u_{\alpha}^*(\mathbf{r},\mathbf{r}') \end{pmatrix} \quad [19]$$

where the sum runs over the eigenmodes of positive energy ϵ_{α} , normalized as $\|u_{\alpha}\|^2 - \|v_{\alpha}\|^2 = 1$. To this order, $b_{\alpha}(t) = b_{\alpha}(0)e^{-i\epsilon_{\alpha}t/\hbar}$. One would insert the expansion (19) into $d\theta/dt$ and take a coarse grained temporal average to get

³We use $\frac{\Delta_0}{\epsilon_{F,\mathbf{k},\sigma}} \hat{\zeta}_{\mathbf{k}} = \hat{n}_{F,\mathbf{k},\uparrow} + \hat{n}_{F,-\mathbf{k},\downarrow}$ where Δ_0 and $\hat{\zeta}_{\mathbf{k}}$ refer to notations of [11], and we use equation (74) of that reference to recognize $d\epsilon_{F,\mathbf{k},\sigma}/d\mu$. The trivial term $\mu(\bar{N})$ in the phase derivative (14) is absent in [11] due to the use of the grand canonical Hamiltonian.

⁴The RPA result (14), restricted to the linear order in the pair operators, does not include the contribution of the phonon branch. One might hope to obtain this contribution by pushing the RPA calculation to the quadratic order in the pair operators as follows: First, one computes $d\theta_0/dt$ up to the quadratic order. Second, one collects all the RPA pair operators inducing a center of mass momentum change $\hbar\mathbf{q}$, that is $\hat{a}_{-\mathbf{k}-\mathbf{q}\downarrow}\hat{a}_{\mathbf{k}\uparrow}$, $\hat{a}_{\mathbf{k}+\mathbf{q}\uparrow}\hat{a}_{-\mathbf{k}\downarrow}$, $\hat{a}_{\mathbf{k}+\mathbf{q}\uparrow}\hat{a}_{\mathbf{k}\uparrow}$ and $\hat{a}_{-\mathbf{k}\downarrow}\hat{a}_{-\mathbf{k}\downarrow}$, where $\hat{a}_{\mathbf{k}\uparrow}$ annihilates a fermion of wave vector \mathbf{k} and spin \uparrow , and one writes the matrices $\mathcal{L}_{\mathbf{q}}$ of their closed linear RPA equations of motion. Each RPA matrix $\mathcal{L}_{\mathbf{q}}$ has two collective modes, with opposite energies linear in \mathbf{q} for small \mathbf{q} . The RPA operators are then expanded over the eigenmodes of $\mathcal{L}_{\mathbf{q}}$, with amplitudes $\hat{b}_{-\mathbf{q}}$ and $\hat{b}_{\mathbf{q}}$ on the collective modes that annihilate and create a collective excitation of momentum $\mp\hbar\mathbf{q}$. Third, one inserts this modal expansion in the RPA operators appearing in $d\theta_0/dt$ and obtains a quadratic expression in the modal amplitudes, hence terms in $\hat{b}_{\mathbf{q}}^\dagger\hat{b}_{\mathbf{q}}$ as in (8). Unfortunately, this RPA approach is not reliable and must be abandoned because the RPA operators, although linearly independent, are *not* quadratically independent, as one can see by rearranging the pair operators using fermionic anticommutation relations, e.g.

$$(\hat{a}_{\mathbf{k}+\mathbf{q}\uparrow}^\dagger\hat{a}_{\mathbf{k}\uparrow})(\hat{a}_{-\mathbf{k}-\mathbf{q}\downarrow}^\dagger\hat{a}_{-\mathbf{k}\downarrow}) = (\hat{a}_{\mathbf{k}+\mathbf{q}\uparrow}^\dagger\hat{a}_{-\mathbf{k}-\mathbf{q}\downarrow}^\dagger)(\hat{a}_{-\mathbf{k}\downarrow}\hat{a}_{\mathbf{k}\uparrow})$$

This shows that (i) there is no unique way of expressing $d\hat{\theta}_0/dt$ as a quadratic function of the RPA operators, (ii) the modal amplitudes are not quadratically independent, (iii) the coefficient of $\hat{b}_{\mathbf{q}}^\dagger\hat{b}_{\mathbf{q}}$ is not uniquely determined by this RPA approach.

⁵We use here for simplicity an Ansatz in which the number of particles has quantum fluctuations (not to be confused with the thermal fluctuations of the grand canonical ensemble). The use of an Ansatz with a fixed number of particles, possible although more difficult [16], would change the coefficients in the energy functional by a relative correction $O(N^{-1/2})$ and would not change the spectrum in the thermodynamic limit.

⁶If $\underline{\Gamma}$ and $\underline{\Phi}$ are the matrices of elements $l^3\Gamma(\mathbf{r},\mathbf{r}')$ and $l^3\Phi(\mathbf{r},\mathbf{r}')$ respectively, then $\underline{\Phi} = -\underline{\Gamma}(1 + \underline{\Gamma}^\dagger\underline{\Gamma})^{-1/2}$. Their variational equations of motion follow from the usual Lagrangian $L = i\hbar[\langle\psi|(d/dt)|\psi\rangle - \text{c.c.}]/2 - \langle\psi|H|\psi\rangle$. Φ is cleverly defined such that $L = i\hbar l^6 \sum_{\mathbf{r},\mathbf{r}'} [\Phi^*(\mathbf{r},\mathbf{r}')\partial_t\Phi(\mathbf{r},\mathbf{r}') - \text{c.c.}]/2 - \mathcal{H}(\Phi, \Phi^*)$, leading to $\partial_t\Phi = (i\hbar l^6)^{-1}\partial_{\Phi^*}\mathcal{H}$. This shows that the conjugate variable of Φ in the Hamiltonian formalism is $i\hbar l^6\Phi^*$ for the usual Poisson brackets, that is Φ^* for the Poisson brackets $\{\Phi, \Phi^*\} = (i\hbar l^6)^{-1}$, knowing that $\partial_t\Phi = \{\Phi, \mathcal{H}\}$.

⁷With the same matrix notation as in note 6 one has $\langle\hat{\psi}_\downarrow\hat{\psi}_\uparrow\rangle = -\underline{\Phi}(1 - \underline{\Phi}^\dagger\underline{\Phi})^{-1/2}$.

rid of the oscillating terms. $\overline{d\theta/dt}$ would then contain the expected linear combination of the numbers of bosonic quasi-particles $n_{B,\alpha} = |b_\alpha|^2$. In reality, the problem is more subtle: due to the interaction among the quasi-particles, $\overline{b_\alpha^t}$ does not vanish and is of order two in Λ and Λ^* . The contribution of the linear term in $d\theta/dt$ is then comparable to that of the quadratic ones. It is calculated in the appendix, using in particular the bounded nature of the field Λ (consequence of the $U(1)$ -symmetry preserving nature of expansion (18)) and the Hellmann-Feynman theorem. One finds

$$-\frac{\hbar}{2} \frac{d\overline{\theta}}{dt} = \mu_0(N) + \sum_\alpha \frac{d\epsilon_\alpha}{dN} |b_\alpha|^2 + O(\|\Lambda\|^3) \quad [20]$$

We now briefly discuss the form of the energy spectrum ϵ_α for a spatially homogeneous system, in the continuous limit $l \rightarrow 0$ for a s -wave contact interaction with a fixed scattering length between opposite spin fermions. For each value of the total wave vector \mathbf{q} , there exists (i) at most one discrete value $\epsilon_{B,\mathbf{q}}$, (ii) a continuum parametrized by two wave vectors $(\mathbf{k}_1, \uparrow; \mathbf{k}_2, \downarrow) \mapsto \epsilon_{F,\mathbf{k}_1,\uparrow} + \epsilon_{F,\mathbf{k}_2,\downarrow}$ of constant sum $(\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{q})$, where $\epsilon_{F,\mathbf{k},\sigma}$ is the BCS dispersion relation. The branch $\epsilon_{B,\mathbf{q}}$ coincides with that of reference [21], as we have checked. It has a phononic start and corresponds to the bosonic elementary excitations of the Fermi gas, whose contribution to the phase dynamics was missing. The continuum corresponds to the excitation of two fermionic quasi-particles. Indeed, since the Hamiltonian \hat{H} contains an even number of factors $\hat{\psi}$ and $\hat{\psi}^\dagger$, each annihilating or creating one quasi-particle, fermionic quasi-particles can only be created by pairs from the ground state. The corresponding biexcitations are not physically independent⁸, and are identical to what has been included via the RPA treatment.

Two more remarks are needed to obtain (8). (i) The fields $\langle \hat{\psi}_\downarrow \hat{\psi}_\uparrow \rangle$ and Φ differ and so do the phases $\hat{\theta}_0$ and θ . Their coarse grained temporal averages, however, only differ by a term of order $\|\Lambda\|^2$, which, bounded hence negligible in the long time limit, does not contribute to the phase blurring of the condensate of pairs⁹. (ii) The phase θ of our variational approach is a classical variable, whereas $\hat{\theta}_0$ in (8) is a quantum operator. This gap can be bridged by using the quantization procedure exposed in Chapter 11 of reference [19] where the b_α of the bosonic branch are in the end replaced by bosonic operators¹⁰ \hat{b}_α , $[\hat{b}_\alpha, \hat{b}_\alpha^\dagger] = 1$. We argue that equation (8), linking $d\hat{\theta}_0/dt$ to the chemical potential operator, and the resulting equation (12) should hold beyond the validity range of the microscopic variational derivation presented above, and should apply even to the strongly interacting regime, provided that the temperature is low enough for the quasi-particles lifetime to be much longer than the inverse of their eigenfrequency. Indeed, in the limiting case where one can neglect the fermionic excitation branch and drop the non-phononic part of the bosonic branch, Eq. (8) can be derived from the irrotational version of the quantum hydrodynamic theory of Landau and Khalatnikov [22, 23] (see appendix B).

Towards an experimental observation

Let us briefly explain how an experimental evidence of the thermal blurring of a condensate of pairs could be obtained. The key idea is to bosonize the atomic Cooper pairs into deeply bound weakly interacting dimers during the preparation and the measurement stage. This can be done in an adiabatic reversible way [24] by tuning the scattering length to a small and positive value thanks to a magnetic Feshbach resonance. It allows one to (i) produce a sample of dimers with weak

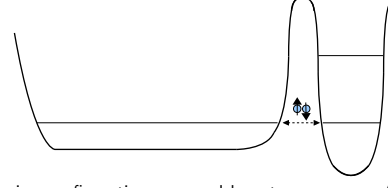


Fig. 1. Trapping configuration proposed here to measure $g_1(t)$ via Ramsey interferometry: the condensed paired Fermi gas is confined in the main trap (with a flat bottom on the figure); one transfers a small number of atoms (in the form of dimers) in the (very narrow) secondary trap via a resonant tunneling effect, which can be tuned by a barrier of adjustable height; in this way, one creates a phase reference, which is made to interfere with the condensate after an evolution time t .

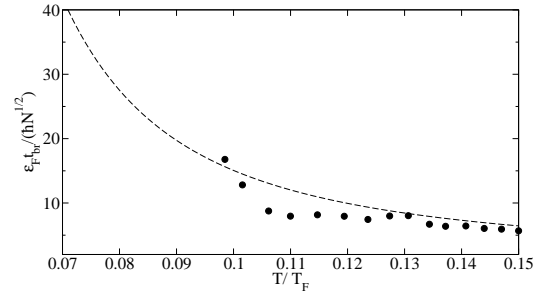


Fig. 2. Thermal blurring time of a coherent Fermi gas in the unitary limit in the canonical ensemble, as a function of temperature T in units of the Fermi temperature $T_F = \epsilon_F/k_B$. Discs: from the equation of state measured in reference [8]. Dashed line: expression (21) deduced from an approximated equation of state (see text).

number fluctuations from a melted Mott phase of an experimental realization of the Bose Hubbard model [25], (ii) control tunneling between the main trap (containing the N particles) and a very narrow secondary trap by adjusting the height of a potential barrier [2] (Fig.1), (iii) detect by fluorescence a single dimer [26] in this secondary trap. For the measurement of the $g_1(t)$ function, we adapt [27] the interferometric Ramsey method of two Rabi pulses at a time interval t to the case of paired fermions. The bosonized pairs are prepared initially in the main trap. A first pulse of angle ϵ transfers on average less than one dimer to the secondary trap; in this way, the thermal blurring is not masked by partition noise. Then the system evolves during a time t with interactions set to the value at which phase dynamics is to be studied. Last, the gas is rebosonized and a second pulse of angle ϵ closes the interferometer before the number n_{sec} of dimers in the secondary trap is measured. The average of n_{sec} over the realizations is an oscillating function of the time t , of angular frequency $2/\hbar$ times the difference of the two trapping zones chemical potentials, and of contrast equal to $|g_1(t)/g_1(0)|$.

⁸Exciting $\alpha = (\mathbf{k}_1, \uparrow; \mathbf{k}_2, \downarrow)$ and $\alpha' = (\mathbf{k}'_1, \uparrow; \mathbf{k}'_2, \downarrow)$ amounts to exciting $\alpha'' = (\mathbf{k}'_1, \uparrow; \mathbf{k}_2, \downarrow)$ and $\alpha''' = (\mathbf{k}_1, \uparrow; \mathbf{k}'_2, \downarrow)$.

⁹Expressing in $\langle \hat{a}_0 \rangle$ the pairing field in terms of Φ , one realizes that, for small Λ , $\langle \hat{a}_0 \rangle = e^{i\theta} \sqrt{N_0} (1 + O(\|\Lambda\|))$ and, since $\overline{\Lambda^t} = O(\|\Lambda\|^2)$, one has $\overline{\theta^t} = \overline{\theta} + O(\|\Lambda\|^2)$.

¹⁰More precisely, these operators are bosonic only for a weak density of excitations. For a spatially homogeneous system and in a $U(1)$ symmetry breaking formalism ($\theta = 0$), we obtain from Eq.(11.81c) of [19] extended to the paired case ($\hat{a}_p, \hat{a}_h^\dagger$ replaced by the BCS fermionic quasi-particle operators $\hat{b}_{\mathbf{k}\sigma}$) and restricted to weakly excited bosonic images (\mathbb{B}^\dagger negligible): $\hat{b}_\mathbf{q} = \sum_{\mathbf{k}} X_{\mathbf{k}}^{\mathbf{q}} \hat{b}_{-\mathbf{k}\downarrow} \hat{b}_{\mathbf{k}+\mathbf{q}\uparrow} + Y_{\mathbf{k}}^{\mathbf{q}} \hat{b}_{\mathbf{k}-\mathbf{q}\uparrow} \hat{b}_{-\mathbf{k}\downarrow}^\dagger$. The real coefficients $X^{\mathbf{q}}$ and $Y^{\mathbf{q}}$ are linear combinations of the corresponding $u_{\mathbf{q}}$ and $v_{\mathbf{q}}$ in Eq.(19), and inherit the normalization condition $\sum_{\mathbf{k}} (X_{\mathbf{k}}^{\mathbf{q}})^2 - (Y_{\mathbf{k}}^{\mathbf{q}})^2 = 1$. Then in a state (15) with $\theta = 0$, $\hat{\delta}_\mathbf{q} \equiv [\hat{b}_\mathbf{q}, \hat{b}_\mathbf{q}^\dagger] - \hat{1}$ has a mean value and a variance $O(\|\Lambda\|^2/N)$ since $\hat{b}_{\mathbf{k}\sigma} |\psi\rangle = O(\|\Lambda\|/N^{1/2})$ if $\|\Lambda\|/N^{1/2} \rightarrow 0$.

¹¹The good agreement with the experimental data has to be taken cautiously. If one treats the two branches to all order in $k_B T$, one gets an upward shift of $t_{\text{br}} \epsilon_F / (\hbar N^{1/2})$ more or less constant and equal to 5 over the temperature range of figure 2.

Explicit results for the unitary gas

We first estimate the blurring time for a unitary Fermi gas prepared in the canonical ensemble, that is with energy fluctuations of variance $\text{Var } E = k_B T^2 \partial_T \bar{E}$. From the equation of state of the unpolarized unitary gas measured in reference [8], and for a spatially homogeneous system (in a flat bottom potential [28]) we find the thermal blurring times t_{br} plotted as discs in figure 2. For example, at a temperature $T = 0.12 T_F \simeq 0.7 T_c$, we find $t_{\text{br}} \approx 7N^{1/2} \hbar / \varepsilon_F$ corresponding to 20 milliseconds for a typical Fermi temperature $T_F = \varepsilon_F / k_B = 1 \mu\text{K}$ and a typical atom number $N = 10^5$. As in reference [29], one can also estimate the equation of state of the unitary gas from simple dispersion relations for the elementary excitations. For the bosonic branch one takes [29] $\varepsilon_{B,\mathbf{q}} = \hbar c q$ with c the $T = 0$ sound velocity, $mc^2 = \frac{2}{3} \xi \varepsilon_F$ and ξ the Bertsch parameter. For the fermionic branch, one takes [30] $\varepsilon_{F,\mathbf{k},\sigma} = \Delta + (\frac{\hbar^2 k^2}{2m} - \varepsilon_0) / (2f_0)$, where Δ is the gap, and ε_0 and f_0 give the location of the minimum and the curvature of the dispersion relation. Keeping each branch contribution to its leading order at low temperature [29] and using the experimental values [8, 31] $\xi = 0.376$, $\Delta = 0.44 \varepsilon_F$, $\varepsilon_0 = 0.85 \varepsilon_F$ and the theoretical value [30] $f_0 = 0.846 \varepsilon_F$, we find (see appendix C)

$$\frac{\hbar^2 N}{(t_{\text{br}} \varepsilon_F)^2} \simeq \left(\frac{\Theta}{0.46} \right)^5 \frac{(1+2r)^2}{(1+r)} \quad [21]$$

with $\Theta = T/T_F$ and $r \simeq \left(\frac{0.316}{\Theta} \right)^{9/2} e^{-0.44/\Theta}$ the relative weight of the two excitation branches. This formula¹¹, plotted as a dashed line in figure 2, is an exact equivalent to t_{br} for $\Theta \rightarrow 0$.

At finite times¹² $t = O(N^0) \ll t_{\text{br}}$, the contribution of D to $g_1(t)$ in equation (12) is *a priori* comparable to that of t_{br}^{-2} since both scale as $1/N$ in the canonical ensemble. D can be calculated at very low temperatures keeping only the phononic part of the spectrum. As the bosonic branch [21] is convex at low q for the unitary gas, phonons interact through Landau-Beliaev processes as in the weakly interacting Bose gas and one can adapt the treatment of [18] based on kinetic equations to obtain (see appendix D)

$$\frac{\hbar N D}{\varepsilon_F} \underset{\Theta \rightarrow 0}{\sim} C \Theta^4 \quad \text{with} \quad C \simeq 0.4 \quad [22]$$

With this we reach a complete picture of the thermal blurring of the unitary Fermi gas at low temperature.

Conclusion

We have presented the first microscopic theory of the thermal blurring of the phase of a condensate of pairs of fermions (12), revealing a ballistic blurring and a subleading phase diffusion. The blurring time depends on the variance of the total energy of the gas, and on the derivative of the microcanonical chemical potential with respect to the energy. This relies on the fact that the time derivative of the condensate phase is given by the chemical potential operator of the gas, see equation (8). We have derived this central relation in a fully microscopic way, including both the bosonic and the fermionic branches of excitation. Last, we have proposed a realistic experimental protocol to measure this blurring time, that we estimated to be tens of milliseconds for a coherent gas prepared in the unitary limit in the canonical ensemble.

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Appendix A: More on the variational calculation

Here we derive equation (20) of the time average of θ within the microscopic model based on the Ansatz (15), coherent state of moving pairs. In a first stage one should perform the expansion of the Hamiltonian \mathcal{H} treating the real quantity n and the complex field Λ as *independent* variables, that is, not fixing the value of $\|\Phi\|$. To include interactions among the quasi-particles, one must go to third order in Λ and Λ^* :

$$\mathcal{H}(\Phi, \Phi^*) = \mathcal{T}_0[n, \phi_0(N)] + \sum_{j=1}^3 \mathcal{T}_j[n, \phi_0(N)](\Lambda, \Lambda^*) + O(\|\Lambda\|^4) \quad [23]$$

where the tensor \mathcal{T}_j is of rank j so that $\mathcal{T}_j(\Lambda, \Lambda^*)$ is exactly of order j in Λ and Λ^* . It may be expressed in terms of the differential of order j of \mathcal{H} taken at $(\Phi, \Phi^*) = (n^{1/2} \phi_0, n^{1/2} \phi_0)$ and restricted to the subspace orthogonal to $(\phi_0, 0)$ and $(0, \phi_0)$. It does not depend on the phase θ due to $U(1)$ symmetry. For a fixed total number of particles, the energy does not vary to first order around the minimizer so that $\mathcal{T}_1[N/2, \phi_0(N)] = 0$, which is the famous gap equation when the system is spatially homogeneous. Furthermore, one can check that $\partial_n \mathcal{T}_0[N/2, \phi_0(N)] = 2\mu_0(N)$ where $\mu_0(N) = dE_0(N)/dN$ is the gas chemical potential at zero temperature, $E_0(N) = \mathcal{T}_0[N/2, \phi_0(N)]$ being the ground state energy.

The phase and the modulus square of the component of the field Φ on the mode ϕ_0 are canonically conjugate variables, so that $-\hbar d\theta/dt = \partial_n \mathcal{H}(\Phi, \Phi^*)$. Once this derivative is taken in (23) for fixed Λ and Λ^* , one can fix the norm of Φ to the value $(N/2)^{1/2}$ (that is the total particle number is fixed to N), and eliminate n through the identity $n = \|\Phi\|^2 - \|\Lambda\|^2$; the field Λ then remains the only dynamical variable of the problem. The resulting expression is useful up to order 2 in Λ, Λ^* :

$$-\hbar \frac{d\theta}{dt} = \partial_n \mathcal{T}_0[N/2, \phi_0(N)] - \|\Lambda\|^2 \partial_n^2 \mathcal{T}_0[N/2, \phi_0(N)] + \sum_{j=1}^2 \partial_n \mathcal{T}_j[N/2, \phi_0(N)](\Lambda, \Lambda^*) + O(\|\Lambda\|^3) \quad [24]$$

The Hamiltonian that determines the evolution of Λ at fixed particle number is obtained by replacing n with $N/2 - \|\Lambda\|^2$ in (23) and by expanding the resulting expression up to order three in Λ, Λ^* :

$$\mathcal{H}_N(\Lambda, \Lambda^*) = E_0(N) + \tilde{\mathcal{T}}_2[N](\Lambda, \Lambda^*) + \mathcal{T}_3[N/2, \phi_0(N)](\Lambda, \Lambda^*) - \|\Lambda\|^2 \partial_n \mathcal{T}_1[N/2, \phi_0(N)](\Lambda, \Lambda^*) + O(\|\Lambda\|^4) \quad [25]$$

with the quadratic form $\tilde{\mathcal{T}}_2[N](\Lambda, \Lambda^*)$ obtained by subtracting $2\mu_0(N)\|\Lambda\|^2$ from $\mathcal{T}_2[N/2, \phi_0(N)](\Lambda, \Lambda^*)$. To compute the coarse grained time average of $\partial_n \mathcal{T}_1[N/2, \phi_0(N)](\Lambda, \Lambda^*)$, we write the temporal derivative of the imaginary part of the component of the field Λ on the function $(N/2)^{1/2} d\phi_0/dN$,

$$Y = \frac{l^6}{2i} \sum_{\mathbf{r}, \mathbf{r}'} \left(\frac{N}{2} \right)^{1/2} \frac{d\phi_0(\mathbf{r}, \mathbf{r}')}{dN} (\Lambda(\mathbf{r}, \mathbf{r}') - \Lambda^*(\mathbf{r}, \mathbf{r}')) \quad [26]$$

Since $i\hbar \partial_t \Lambda = l^{-6} \partial_{\Lambda^*} \mathcal{H}_N(\Lambda, \Lambda^*)$, one gets

$$-2\hbar \frac{dY}{dt} = \mathcal{D} \cdot \mathcal{H}_N(\Lambda, \Lambda^*) = \mathcal{D} \cdot \tilde{\mathcal{T}}_2[N](\Lambda, \Lambda^*) + \mathcal{D} \cdot \mathcal{H}_N^{\text{cub}}(\Lambda, \Lambda^*) + O(\|\Lambda\|^3) \quad [27]$$

¹²Still we must have $t > \tau_c$ (see note 1). For the unitary gas $\tau_c \underset{\Theta \rightarrow 0}{\simeq} (0.149/\Theta)^5$, as shown in Appendix C.

where $\mathcal{H}_N^{\text{cub}}$ is the component of \mathcal{H}_N of order three in Λ, Λ^* . We have introduced the differential operator

$$\mathcal{D} = \sum_{\mathbf{r}, \mathbf{r}'} \left(\frac{N}{2} \right)^{1/2} \frac{d\phi_0(\mathbf{r}, \mathbf{r}')}{dN} (\partial_{\Lambda(\mathbf{r}, \mathbf{r}')} + \partial_{\Lambda^*(\mathbf{r}, \mathbf{r}')}) \quad [28]$$

We shall now take advantage of two identities that exactly hold for all Λ orthogonal to ϕ_0 :

$$2\mathcal{D} \cdot \tilde{\mathcal{T}}_2[N](\Lambda, \Lambda^*) = -\partial_n \mathcal{T}_1(\Lambda, \Lambda^*) \quad [29]$$

$$2\mathcal{D} \cdot \mathcal{H}_N^{\text{cub}}(\Lambda, \Lambda^*) = 2 \frac{d}{dN} \tilde{\mathcal{T}}_2(\Lambda, \Lambda^*) - \partial_n \mathcal{T}_2(\Lambda, \Lambda^*) + \|\Lambda\|^2 \partial_n^2 \mathcal{T}_0 \quad [30]$$

where all the tensors are taken at $[n = N/2, \phi_0(N)]$ or simply at N for $\tilde{\mathcal{T}}_2$. To prove these relations, one formally considers a field Φ' with $N + \delta N$ particles and determines in two different ways the quadratic expansion of $\mathcal{H}_{N+\delta N}(\Lambda', \Lambda'^*)$ in powers of Λ' and Λ'^* , where Λ' is as in (18) (written for $N + \delta N$ particles) the component of Φ' orthogonal to $\phi_0(N + \delta N)$. First, one simply replaces N with $N + \delta N$ and Λ with Λ' in (25), and then expands to first order in δN . The tensor $d\tilde{\mathcal{T}}_2/dN[N]$ naturally appears from this expansion. Second, one applies to $\mathcal{H}(\Phi', \Phi'^*)$ the expansion (23) around $\Phi_0(N)$ and takes into account the fact that, to first order in δN , the component of Φ' orthogonal to $\phi_0(N)$ contains, in addition to Λ'_\perp coming from Λ' , a contribution coming from $d\phi_0/dN$:

$$\Lambda = \delta N \left(\frac{N}{2} \right)^{1/2} \frac{d\phi_0}{dN} + \Lambda'_\perp + O(\delta N^2, \delta N \|\Lambda'\|, \|\Lambda'\|^2) \quad [31]$$

This infinitesimal shift proportional to δN along the direction of $d\phi_0/dN$ is responsible for the occurrence of the operator \mathcal{D} . Equations (29) and (30) are finally obtained by identification of the two resulting expressions of $\mathcal{H}_{N+\delta N}(\Lambda', \Lambda'^*)$ respectively to first and second order in Λ' and Λ'^* .

It remains to combine equations (24), (27), (29), and (30) after a coarse grained temporal average to obtain

$$-\frac{\hbar}{2} \frac{d\overline{\theta}^t}{dt} = \mu_0(N) + \overline{\frac{d\tilde{\mathcal{T}}_2}{dN}[N](\Lambda, \Lambda^*)} + O(\|\Lambda\|^3) \quad [32]$$

where we used the crucial property that $\overline{d\Lambda/dt^t}$ vanishes (and so does $\overline{dY/dt^t}$), since the range of variation of the field Λ is bounded. The quadratic form $\tilde{\mathcal{T}}_2[N]$ is represented by the matrix $\eta\mathcal{L}[N]$ with $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, using a block notation and the scalar product $\langle \cdot, \cdot \rangle$ generating the norm $\|\cdot\|$:

$$\tilde{\mathcal{T}}_2[N](\Lambda, \Lambda^*) = \frac{1}{2} \left\langle (\Lambda, \Lambda^*), \eta\mathcal{L}[N] \begin{pmatrix} \Lambda \\ \Lambda^* \end{pmatrix} \right\rangle \quad [33]$$

Then one inserts the modal decomposition (19) in the derivative with respect to N of the equation (33); we recall that ϵ_α and (u_α, v_α) , $-\epsilon_\alpha$ and (v_α^*, u_α^*) are the eigenvalues and the eigenvectors of \mathcal{L} . The coarse grained temporal average $\overline{\quad}^t$ removes the crossed terms, and the Hellmann-Feynman theorem ensures that $\langle (u_\alpha^*, -v_\alpha^*), \frac{d\mathcal{L}[N]}{dN} \begin{pmatrix} u_\alpha \\ v_\alpha \end{pmatrix} \rangle = d\epsilon_\alpha/dN$, since $(u_\alpha, -v_\alpha)$ is the dual vector of (u_α, v_α) due to $(\eta\mathcal{L})^\dagger = \eta\mathcal{L}$, and finally leads to equation (20).

Appendix B: Irrotational quantum hydrodynamics

In this supplementary section we give an additional derivation of equation (8) based on quantum hydrodynamics. Although this derivation is not microscopic contrarily to the one presented in the paper, and although it neglects the internal

fermionic degrees of freedom treating the pairs of fermions at large spatial scales as a bosonic field, it has the advantage of relying only on the equation of state and thus of being valid in all interaction regimes.

To calculate the viscosity of superfluid helium at low temperature, Landau and Khalatnikov have developed in 1949 the theory of quantum hydrodynamics [22]. It allows one to determine, to leading order in T , the effect of a non-zero temperature on the quantum fluid, at least on the observables that only involve low energy scales and large length scales. Remarkably, the only specific ingredient is the zero-temperature equation of state of the fluid, which is here the energy per unit volume $e_0(\rho)$ of the ground state of the spatially homogeneous system of density ρ .

To obtain the time derivative of the phase operator of the condensate of fermion pairs, we refine the theory in two ways: by regularizing ultraviolet divergences and by specializing to the irrotational case.

First, we solve the issue of the Landau-Khalatnikov Hamiltonian ground state energy, that diverges due to the zero-point motion of the system eigenmodes. We discretize the space in a cubic lattice of spacing l , a sub-multiple of the size L of the quantization volume, which is much smaller than the typical wavelength $2\pi/q_{\text{th}}$ of the thermal excitations of the fluid but much larger than the mean interparticle distance $\rho^{-1/3}$,

$$\rho^{-1/3} \ll l \ll q_{\text{th}}^{-1} \quad [34]$$

both conditions being compatible at sufficiently low temperature. This is in the spirit of the validity range of hydrodynamics, which relies on a spatial coarse graining, and it provides a natural cut-off for the wave vectors \mathbf{q} by restricting them to the first Brillouin zone¹³ $\mathcal{D} = [-\pi/l, \pi/l]^3$. In the Hamiltonian one must then replace the differential operators such as the gradient, the divergence and the Laplacian, by their discrete versions, as we shall implicitly do below, and introduce the bare energy density $e_{0,0}(\rho)$, which depends on the lattice spacing l . Following the ideas of renormalization, the zero-point energy of the modes, that formally diverges when $l \rightarrow 0$, adds up to $e_{0,0}(\rho)$ to exactly reconstruct the *effective* or *true* energy density $e_0(\rho)$, that does not depend on l and is what is measured experimentally.

Second, we specialize the theory to the case of an irrotational velocity field operator $\hat{\mathbf{v}}(\mathbf{r}, t)$ that can then be written as the gradient of the phase field operator $\hat{\phi}(\mathbf{r}, t)$, itself canonically conjugate to the density field operator $\hat{\rho}(\mathbf{r}, t)$:

$$\hat{\mathbf{v}}(\mathbf{r}, t) = \frac{\hbar}{m} \text{grad } \hat{\phi}(\mathbf{r}, t) \quad \text{with} \quad [\hat{\rho}(\mathbf{r}, t), \hat{\phi}(\mathbf{r}', t)] = i \frac{\delta_{\mathbf{r}, \mathbf{r}'}}{l^3} \quad [35]$$

This amounts to neglecting the transverse component of the field $\hat{\mathbf{v}}(\mathbf{r}, t)$, as done in reference [22] to determine the phonon-phonon interaction and go forward in the computation of viscosity. In the particular scale invariant case of the unitary Fermi gas, this was justified within the effective field theory in reference [23]. We note *en passant* that the density of fermionic quasi-particles is exponentially small in $1/T$ and is directly omitted by hydrodynamic theory.

¹³We also require that $mc_l/\hbar > 1$, where c is the $T = 0$ sound velocity, so that the elementary excitations of the system remain phononic over the whole domain \mathcal{D} . This condition is weaker than the already assumed one, $l\rho^{1/3} \gg 1$, in the strongly interacting regime or in the BCS limit, since c is then of the order of the Fermi velocity. It must be added explicitly in the so-called BEC limit, where the fermion pairs can be considered as bosons and form a weakly interacting condensate.

The steps that follow are rather usual. One starts from the equations of motion of the fields in Heisenberg picture, that is the quantum continuity equation and the quantum Euler equation for the potential (whose gradient gives the quantum Euler equation for the velocity):

$$\partial_t \hat{\rho} + \text{div} \left[\frac{1}{2} \{ \hat{\rho}, \hat{\mathbf{v}} \} \right] = 0 \quad [36]$$

$$\hbar \partial_t \hat{\phi} = -\frac{1}{2} m \hat{\mathbf{v}}^2 - \mu_{0,0}(\hat{\rho}) \quad [37]$$

where $\{ \hat{A}, \hat{B} \} = \hat{A} \hat{B} + \hat{B} \hat{A}$ is the anticommutator of two operators and

$$\mu_{0,0}(\rho) = \frac{d}{d\rho} e_{0,0}(\rho) = e'_{0,0}(\rho) \quad [38]$$

is the bare ground state chemical potential at density ρ . These equations originate from the Hamiltonian

$$\hat{H}_{\text{hydro}} = l^3 \sum_{\mathbf{r}} \left[\frac{1}{2} m \hat{\mathbf{v}} \cdot \hat{\rho} \hat{\mathbf{v}} + e_{0,0}(\hat{\rho}) \right] \quad [39]$$

The quantum spatial density and phase fluctuations are weak provided that $\rho^{1/3} l$ is large enough; the thermal ones are weak if in addition $q_{\text{th}} l$ is small enough¹⁴. Under these conditions one can linearize as in [22] the equations of motion around the spatially uniform solution:

$$\hat{\rho}(\mathbf{r}, t) = \hat{\rho}_0 + \delta \hat{\rho}(\mathbf{r}, t) \quad [40]$$

$$\hat{\phi}(\mathbf{r}, t) = \hat{\phi}_0(t) + \delta \hat{\phi}(\mathbf{r}, t) \quad [41]$$

The operator $\hat{\rho}_0$ reduces to \hat{N}/L^3 , where \hat{N} is the operator giving the total number of particles, and is a constant of motion. The operator $\hat{\phi}_0$ is the phase operator of the condensate; one has here

$$\hat{\phi}_0 = \hat{\theta}_0/2 \quad [42]$$

since the phase operator $\hat{\theta}_0$ in equation (2) takes the pairs as the building block, whereas equations (36,37) are built on the fermionic particles. The spatial fluctuations $\delta \hat{\rho}$ and $\delta \hat{\phi}$, of vanishing (discrete) integral over the whole space, can be expanded on the plane waves of non-zero wave vector \mathbf{q} , and commute with $\hat{\rho}_0$. One solves the linearized equations for $\delta \hat{\rho}$ and $\delta \hat{\phi}$ and one can use the usual expansion on eigenmodes:

$$\delta \hat{\rho}(\mathbf{r}, t) = \frac{\hat{\rho}_0^{1/2}}{L^{3/2}} \sum_{\mathbf{q} \in \frac{2\pi}{L} \mathbb{Z}^{3*} \cap \mathcal{D}} \left(\frac{\hbar q}{2m \hat{c}_{0,0}} \right)^{1/2} (\hat{B}_{\mathbf{q}} + \hat{B}_{-\mathbf{q}}^\dagger) e^{i\mathbf{q} \cdot \mathbf{r}} \quad [43]$$

$$\delta \hat{\phi}(\mathbf{r}, t) = \frac{-i}{\hat{\rho}_0^{1/2} L^{3/2}} \sum_{\mathbf{q} \in \frac{2\pi}{L} \mathbb{Z}^{3*} \cap \mathcal{D}} \left(\frac{m \hat{c}_{0,0}}{2\hbar q} \right)^{1/2} (\hat{B}_{\mathbf{q}} - \hat{B}_{-\mathbf{q}}^\dagger) e^{i\mathbf{q} \cdot \mathbf{r}} \quad [44]$$

where the creation operator $\hat{B}_{\mathbf{q}}^\dagger$ and the annihilation operator $\hat{B}_{\mathbf{q}}$ of a phonon with wave vector \mathbf{q} and energy $\hbar q \hat{c}_{0,0}$ obey bosonic commutation relations $[\hat{B}_{\mathbf{q}}, \hat{B}_{\mathbf{q}'}^\dagger] = \delta_{\mathbf{q}, \mathbf{q}'}$ and where we introduced the zero-temperature bare sound velocity operator

$$\hat{c}_{0,0} \equiv \left(\frac{\hat{\rho}_0 \mu'_{0,0}(\hat{\rho}_0)}{m} \right)^{1/2} \quad [45]$$

It remains to expand the right-hand side of (37) up to second order in $\delta \hat{\rho}$ and $\delta \hat{\theta}$, to extract the zero wave vector Fourier component, to perform a coarse grained temporal average to get rid of the oscillating crossed terms $\hat{B}_{\mathbf{q}} \hat{B}_{-\mathbf{q}}$ and $\hat{B}_{-\mathbf{q}}^\dagger \hat{B}_{\mathbf{q}}^\dagger$, and to use the identity

$$\frac{d}{d\rho} [\rho \mu'_{0,0}(\rho)]^{1/2} = \frac{\mu'_{0,0}(\rho) + \rho \mu''_{0,0}(\rho)}{2[\rho \mu'_{0,0}(\rho)]^{1/2}} \quad [46]$$

to obtain

$$\hbar \frac{d}{dt} \hat{\phi}_0 = -\mu_{0,0}(\hat{\rho}_0) - \sum_{\mathbf{q} \in \frac{2\pi}{L} \mathbb{Z}^{3*} \cap \mathcal{D}} \left(\hbar q \frac{d}{dN} \hat{c}_{0,0} \right) \left(\hat{B}_{\mathbf{q}}^\dagger \hat{B}_{\mathbf{q}} + \frac{1}{2} \right) \quad [47]$$

At this order of the expansion, one can collect in (47) the zero-point contribution of the modes (the term 1/2 in between parentheses) and the bare chemical potential $\mu_{0,0}(\hat{\rho}_0)$ to form the true chemical potential $\mu_0(\hat{\rho}_0)$ of the fluid at zero temperature, and one can identify $\hat{c}_{0,0}$ in the prefactor of the phonon number operator $\hat{B}_{\mathbf{q}}^\dagger \hat{B}_{\mathbf{q}}$ with the true sound velocity at zero temperature, $\hat{c}_0 \equiv [\hat{\rho}_0 \mu'_0(\hat{\rho}_0)/m]^{1/2}$. One then obtains the (low temperature) phononic limit of relation (8), without any constraint on the interaction strength.

Appendix C: Thermodynamics of non interacting quasi particles, application to the unitary gas

In this supplementary section we explain how to obtain the approximated expression (21) giving the blurring time of the unpolarized unitary Fermi gas, prepared in thermal equilibrium in the canonical ensemble. The gas is here spatially homogeneous in a cubic quantization volume $V = L^3$ with periodic boundary conditions. The approach we shall use is a variant of the one used in reference [29] and it requires that the temperature is sufficiently low.

Effective Hamiltonian at low energy: At low energy we expect that the partition function and the thermodynamic quantities of the gas can be deduced with good accuracy from an effective low energy Hamiltonian

$$H_{\text{eff}} = E_0(N, V) + \sum_{\gamma} \epsilon_{\gamma}(N, V) \hat{b}_{\gamma}^\dagger \hat{b}_{\gamma} \quad [48]$$

where N is the total number of particles in the gas, $E_0(N, V)$ is its ground state energy in the quantization volume V , and the excited energy levels are those of an ideal gas of quasi-particles whose creation and annihilation operators are \hat{b}_{γ}^\dagger and \hat{b}_{γ} and whose dispersion relation $\gamma \mapsto \epsilon_{\gamma}(N, V)$ depends on N and V , but not on the temperature.

We do not need at this stage to specify the ensemble of variation of the index γ and the quantum statistics (bosonic or fermionic) of the operators \hat{b}_{γ} , nor the dispersion relation. In practice the index γ will represent a composite index (B, \mathbf{q}) or (F, \mathbf{k}, σ) , \mathbf{q} or \mathbf{k} being the quasi-particle wave vector, $\sigma = \pm 1/2$ is a spin index, and $s \in \{B, F\}$ labels the excitation branch which is bosonic (with a phononic start at $q \rightarrow 0$) or fermionic (displaying an energy gap and varying quadratically with k around its minimum).

Going back to the canonical ensemble: The expression (13) involves the microcanonical chemical potential $\mu_{\text{mc}}(N, V, E)$ and its derivative with respect to the total energy E . In the limit of a large system however, the microcanonical and canonical chemical potentials can be identified provided the two ensembles have the same mean energy, that is the canonical mean energy $E_{\text{can}}(N, V, T)$ for a well chosen temperature T coincides with the energy E of the microcanonical

¹⁴ One first checks that the density fluctuations on a given lattice site are small in relative value, using (43). At $T = 0$, $\langle \delta \hat{\rho}^2 \rangle / \rho^2 \approx \frac{\hbar}{mcl} \frac{1}{\rho l^3} \ll 1$, using (34) and the note 13. At $T > 0$,

there is the additional thermal contribution $\approx \frac{\hbar}{mcl} \frac{(k_{\text{th}} l)^4}{\rho l^3}$ which is $\ll 1$ for the same reasons. Second, one checks that the phase fluctuations between two neighboring lattice sites are small in absolute value. To this end, one notes from (44) that $l^2 \langle (\mathbf{grad} \delta \hat{\theta})^2 \rangle = (mcl/\hbar)^2 \langle \delta \hat{\rho}^2 \rangle / \rho^2$. To conclude, it remains to use $mc/(\hbar \rho^{1/3}) = O(1)$, a property that holds in the whole BEC-BCS crossover, as well as the previous estimates of $\langle \delta \hat{\rho}^2 \rangle / \rho^2$.

ensemble [12]:

$$\mu_{mc}(N, V, E_{\text{can}}(N, V, T)) \simeq \mu_{\text{can}}(N, V, T) \quad [49]$$

By taking the derivative with respect to T , one then obtains

$$\partial_E \mu_{mc}(N, V, E_{\text{can}}(N, V, T)) \simeq \frac{\partial_T \mu_{\text{can}}(N, V, T)}{\partial_T E_{\text{can}}(N, V, T)} \quad [50]$$

The free energy and its derivatives: The partition function of the effective Hamiltonian in the canonical ensemble at temperature T is simply given by

$$Z = e^{-\beta E_0} \prod_{\gamma} \sum_{\{n_{\gamma}\}} e^{-\beta n_{\gamma} \epsilon_{\gamma}} \quad [51]$$

where $\beta = 1/(k_B T)$ and where the occupation numbers n_{γ} vary in the ensemble $\{0, 1\}$ for the Fermi statistics and in that of the natural integers \mathbb{N} for the Bose statistics. The free energy $F = -k_B T \ln Z$ can then be written as

$$F = E_0 - k_B T \sum_{\gamma} \ln \left(\sum_{\{n_{\gamma}\}} e^{-\beta n_{\gamma} \epsilon_{\gamma}} \right) \quad [52]$$

By taking the derivative of F with respect to the temperature or the number of particles, one obtains the mean energy, the energy variance and the chemical potential of the gas in the canonical ensemble:

$$E_{\text{can}} = \partial_{\beta}(\beta F) = E_0 + \sum_{\gamma} \epsilon_{\gamma} \bar{n}_{\gamma} \quad [53]$$

$$\text{Var}_{\text{can}} E = k_B T^2 \partial_T E_{\text{can}} \quad [54]$$

$$\mu_{\text{can}} = \partial_N F = \mu_0 + \sum_{\gamma} (\partial_N \epsilon_{\gamma}) \bar{n}_{\gamma} \quad [55]$$

where

$$\mu_0 \equiv \partial_N E_0 \quad [56]$$

is the chemical potential of the gas at zero temperature, that is in the absence of quasi-particles, and where

$$\bar{n}_{\gamma} \equiv \frac{\sum_{\{n_{\gamma}\}} n_{\gamma} e^{-\beta n_{\gamma} \epsilon_{\gamma}}}{\sum_{\{n_{\gamma}\}} e^{-\beta n_{\gamma} \epsilon_{\gamma}}} \quad [57]$$

is the mean number of quasi-particles in the mode γ following the Bose or Fermi-Dirac distribution. We have now at hand all the useful expressions to determine the blurring time t_{br} in equation (13) when the Fermi gas is prepared in the canonical ensemble.

Dispersion relations: The idea is reduce the dispersion relations of the two excitation branches, that are not known exactly, to their essential feature at low temperature. The *bosonic branch* is reduced to its phononic start at $q \rightarrow 0$, *i.e.* we take a linear dispersion relation corresponding to the sound velocity c of the zero temperature gas,

$$\epsilon_{B,\mathbf{q}} \simeq \hbar c q \quad [58]$$

the cutoff at large q being ensured by the Bose law $\bar{n}_{B,\mathbf{q}} \simeq [\exp(\beta \hbar c q) - 1]^{-1}$. In the thermodynamic limit, the mean phonon energy and its derivative with respect to T are then given by

$$E_{B,\text{can}} = \frac{\pi^2}{30} V \frac{(k_B T)^4}{(\hbar c)^3} \quad [59]$$

$$\partial_T E_{B,\text{can}} = \frac{2\pi^2}{15} V \frac{k_B (k_B T)^3}{(\hbar c)^3} \quad [60]$$

The sound velocity and the contribution $\mu_{B,\text{can}}$ of the phonons to the gas chemical potential can be obtained very simply at the unitary limit. Indeed, due to scaling invariance, the zero temperature equation of state reduces to $\mu_0 = \xi \varepsilon_F$, where the Bertsch parameter ξ , a number, has been precisely measured [8], and

$$\varepsilon_F = \frac{\hbar^2}{2m} \left(3\pi^2 \frac{N}{V} \right)^{2/3} \quad [61]$$

is the Fermi energy of an ideal unpolarized gas of spin 1/2 fermions with the same total density $\rho = N/V$ as the unitary gas, m being the mass of a fermion. Then the exact hydrodynamic relation $mc^2 = \rho \frac{d}{d\rho} \mu_0$ simplifies into $mc^2 = \frac{2}{3} \mu_0$ and c varies as $\rho^{1/3}$, so that $\partial_N \epsilon_{B,\mathbf{q}} = \epsilon_{B,\mathbf{q}}/(3N)$ in equation (55) and

$$\mu_{B,\text{can}} = \frac{1}{3N} E_{B,\text{can}} \quad [62]$$

$$\partial_T \mu_{B,\text{can}} = \frac{1}{3N} \partial_T E_{B,\text{can}} \quad [63]$$

The *fermionic branch* at unitarity has a minimum, the gap Δ , for $k = k_0$, around which it varies quadratically [30], which should be a sufficient description at low temperature, as the dominant contribution to the free energy and to its derivatives should come from a neighbourhood of this minimum. We then take as in reference [30]

$$\epsilon_{F,\mathbf{k},\sigma} \simeq \epsilon_{\mathbf{k}} \equiv \Delta + \frac{\left(\frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 k_0^2}{2m} \right)^2}{2f_0} \quad [64]$$

where we recall that $\sigma = \pm 1/2$. The values of the parameters Δ and k_0 have been measured [31], while the curvature parameter f_0 has only been estimated theoretically [30].

According to equation (53), the mean energy corresponding to fermionic quasi-particles in the thermodynamic limit is

$$E_{F,\text{can}} = 2V \int_{\mathbb{R}^3} \frac{d^3 k}{(2\pi)^3} \frac{\epsilon_{\mathbf{k}}}{e^{\beta \epsilon_{\mathbf{k}}} + 1} \quad [65]$$

To leading order in temperature, one can neglect the 1 with respect to $\exp(\beta \epsilon_{\mathbf{k}}) \geq \exp(\Delta/k_B T) \gg 1$ in the denominator, making an exponentially small error when $T \rightarrow 0$; within the same order of approximation, after integrating over the direction of \mathbf{k} , one can extend the integration over the modulus k to the whole real axis, using (64) for all $k \in \mathbb{R}$. This gives

$$E_{F,\text{can}} \simeq \frac{V e^{-\beta \Delta}}{\pi^2} \int_{\mathbb{R}} dk k^2 \epsilon_{\mathbf{k}} e^{-\beta(\epsilon_{\mathbf{k}} - \Delta)} \quad [66]$$

where we made evident the activation law associated to the width Δ of the energy gap. When $T \rightarrow 0$, the Boltzmann factor $e^{-\beta(\epsilon_{\mathbf{k}} - \Delta)}$ gets more and more peaked around its maximum, reached in $k = k_0$ and of value 1, with a width in k that goes to zero as $T^{1/2}$. One then makes an error that is polynomial in T (more precisely, it is a linear combination of $k_B T/\Delta$ and $m^2 k_B T f_0 / (\hbar^2 k_0^2)^2$) when one replaces the prefactor by its leading approximation, $k^2 \epsilon_{\mathbf{k}} \simeq k_0^2 \Delta$, and when one quadratizes the dispersion relation in the exponential:

$$\epsilon_{\mathbf{k}} - \Delta \simeq \frac{1}{2f_0} \left[\frac{\hbar^2 k_0 (k - k_0)}{m} \right]^2 \quad [67]$$

It is then simple to perform the resulting Gaussian integral, and to calculate the derivative of the result with respect to

T to leading order, that is here by differentiating only the activation factor $\exp(-\beta\Delta)$:

$$E_{F,\text{can}} \simeq \frac{V}{\pi^2} \frac{mk_0}{\hbar^2} (2\pi f_0 k_B T)^{1/2} \Delta e^{-\beta\Delta} \quad [68]$$

$$\partial_T E_{F,\text{can}} \simeq \frac{\Delta}{k_B T^2} E_{F,\text{can}} \quad [69]$$

Again for the spectrum (64), the fermionic quasi-particles contribution to the chemical potential in the thermodynamic limit is, according to equation (55),

$$\mu_{F,\text{can}} = 2V \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{\partial_N \epsilon_{\mathbf{k}}}{e^{\beta\epsilon_{\mathbf{k}}} + 1} \quad [70]$$

As we have just seen, for $T \rightarrow 0$ we are led to approximate $\partial_N \epsilon_{\mathbf{k}}$ with $\partial_N \Delta$ in the prefactor of the thermal occupation number. Considering the scale invariance of the unitary gas, which imposes $\Delta \propto \rho^{2/3}$, we immediately obtain to leading order in T :

$$\mu_{F,\text{can}} \simeq \frac{2}{3N} E_{F,\text{can}} \quad [71]$$

$$\partial_T \mu_{F,\text{can}} \simeq \frac{2}{3N} \partial_T E_{F,\text{can}} \quad [72]$$

In terms of the ratio r : One obtains a relatively simple expression of the blurring time by introducing the dimensionless ratio r between the contributions of the bosonic and the fermionic branch to $\partial_T E_{\text{can}}$, that is to the denominator of the right hand side of equation (50):

$$r \equiv \frac{\partial_T E_{F,\text{can}}}{\partial_T E_{B,\text{can}}} \underset{T \rightarrow 0}{\sim} \frac{10(2\pi)^{1/2}}{3^{1/2}\pi^4} \frac{\epsilon_0^{1/2} f_0^{1/2} \mu_0^{3/2} \Delta^2}{(k_B T)^{9/2}} e^{-\beta\Delta} \quad [73]$$

where we introduced $\epsilon_0 = \hbar^2 k_0^2 / (2m)$ and we used equations (60,69). By replacing the parameters with their approximated values given just above equation (21), one obtains the expression of r given below this same equation. Thanks to the scale invariance of the unitary gas, the ratio between the bosonic and the fermionic contributions to $\partial_T \mu_{\text{can}}$ is simply related to r at low temperature, see equations (63,72):

$$\frac{\partial_T \mu_{F,\text{can}}}{\partial_T \mu_{B,\text{can}}} \underset{T \rightarrow 0}{\sim} \frac{\frac{2}{3N} \partial_T E_{F,\text{can}}}{\frac{1}{3N} \partial_T E_{B,\text{can}}} = 2r \quad [74]$$

so that one arrives at the approximation

$$\frac{N \partial_T \mu_{\text{can}}}{\partial_T E_{\text{can}}} \simeq \frac{1 + 2r}{3(1 + r)} \quad [75]$$

where μ_{can} is the full chemical potential of the gas and E_{can} its total mean energy when it is prepared in the canonical ensemble. One easily obtains the energy variance by using the general relation (54) and (61) to eliminate the volume V :

$$\frac{1}{N} \text{Var}_{\text{can}} E \simeq \frac{3^{3/2} \pi^4}{20} (1 + r) \frac{(k_B T)^5}{\epsilon_F^{3/2} \mu_0^{3/2}} \quad [76]$$

Last, one has to combine these relations with the identity $(2t_{\text{br}}/\hbar)^{-2} = \left(\frac{\partial_T \mu_{\text{can}}}{\partial_T E_{\text{can}}}\right)^2 \text{Var}_{\text{can}} E$ which follows from the equations (13,50), to finally obtain

$$\frac{N \hbar^2}{(\epsilon_F t_{\text{br}})^2} \simeq \frac{3^{3/2} \pi^4}{45} \frac{(1 + 2r)^2}{(1 + r)} \left(\frac{\epsilon_F}{\mu_0}\right)^{3/2} \left(\frac{k_B T}{\epsilon_F}\right)^5 \quad [77]$$

from which equation (21) which we wanted to justify, is obtained by simple numerical substitution.

Appendix D: phase diffusion coefficient of the unitary Fermi gas at low temperature

This supplementary section explains how to obtain the expression (22) of the diffusion coefficient $D(E, N)$ of the condensate phase in a spatially homogeneous unpolarized spin 1/2 unitary Fermi gas prepared in the microcanonical ensemble. We start from equation (10) and compute the right-hand-side by a straightforward adaptation of the kinetic equations approach of reference [18]. This adaptation consists mainly in a rescaling of the wave vectors, of the excitation spectrum and of the Beliaev-Landau scattering amplitudes.

Principle of the computation of D from kinetic equations: Let us first recall why $\gamma_\lambda = D(E_\lambda, N_\lambda)$ as it is expressed in (10) can be obtained from kinetic equations, as done in detail in reference [18]. Replacing the derivative of the phase by its coarse grained average (8) yields time integrals of the quasi-particles occupation numbers $\hat{n}_{s,\alpha}$ correlation functions. When the quasi-particles are weakly interacting, the state of the system at time zero can be seen as a microcanonical statistical mixture of Fock states $|(n_{s,\alpha})\rangle$ of quasi-particles with well-defined integer occupation numbers $n_{s,\alpha}$. All one needs is then to determine the correlation functions¹⁵

$$\begin{aligned} & \langle (n_{s,\alpha}) | \hat{n}_{s_2,\alpha_2}(t) \hat{n}_{s_1,\alpha_1}(0) | (n_{s,\alpha}) \rangle = \\ & \langle (n_{s,\alpha}) | \hat{n}_{s_2,\alpha_2}(t) | (n_{s,\alpha}) \rangle n_{s_1,\alpha_1}(0) \quad [78] \end{aligned}$$

hence the evolution of the mean quasi-particle number in each mode when the interactions among them are taken into account. This is exactly the purpose of kinetic equations. In a typical initial configuration, the occupation numbers $(n_{s,\alpha})$ are in practice very close to their microcanonical mean value $(\bar{n}_{s,\alpha})$, where $\bar{n}_{s,\alpha} = \langle \hat{n}_{s,\alpha} \rangle$, and $\langle \dots \rangle$ is the microcanonical average, and the kinetic equations can therefore be linearized around the set of $\bar{n}_{s,\alpha}$. After performing the average over the initial Fock states, we obtain the system

$$\frac{d\vec{X}(t)}{dt} = M\vec{X}(t) \quad [79]$$

where M is the matrix of the linearized kinetic equations restricted to the subspace $\delta E = 0$ of zero energy fluctuation (that is the subspace of vectors \vec{x} satisfying $\vec{\epsilon} \cdot \vec{x} = 0$, with $\vec{\epsilon}$ the vector of coordinates of the energies $(\epsilon_{s,\alpha})$) and the unknown vector $\vec{X}(t)$ contains the correlation functions

$$\begin{aligned} X_{s,\alpha}(t) & \equiv \langle (\hat{n}_{s,\alpha}(t) - \bar{n}_{s,\alpha}) \left[-\frac{d\hat{\theta}_0}{dt}(0) \right] \rangle \\ & = \sum_{s',\alpha'} A_{s',\alpha'} \langle (\hat{n}_{s,\alpha}(t) - \bar{n}_{s,\alpha}) (\hat{n}_{s',\alpha'}(0) - \bar{n}_{s',\alpha'}) \rangle \quad [80] \end{aligned}$$

Here \vec{A} is the vector whose coordinates are the coefficients of $\hat{n}_{s,\alpha}$ in $-\frac{d\hat{\theta}_0}{dt}$, that is, from equation (8)¹⁶:

$$A_{s,\alpha} = \frac{2}{\hbar} \frac{d}{dN} \epsilon_{s,\alpha} \quad [81]$$

¹⁵ At this level of approximation, $\delta_\lambda = 0$ in equation (10), since the correlation function in (78) is real.

¹⁶ Equation (8) has a factor 2 with respect to its equivalent (2) in [18], hence the factor 2 in the numerator of $A_{s,\alpha}$. Since $\vec{X}(0)$ is a linear function of \vec{A} , see (80), this leads in turn to a factor 4 in D .

The computation of the initial value $\vec{X}(0)$ is done in details in reference [18] and will be sketched later on. The phase diffusion coefficient D follows from $\vec{X}(t)$ by a time integration :

$$D = \int_0^{+\infty} dt \vec{A} \cdot \vec{X}(t) = -(P\vec{A}) \cdot M^{-1} \vec{X}(0) \quad [82]$$

The projector P , hermitian conjugate of the projector onto the zero energy fluctuation subspace¹⁷, subtracts the component parallel to the energy vector $\vec{\epsilon}$, *i.e.* $P\vec{\epsilon} = \vec{0}$. We act with it on \vec{A} in (82) when a multiplication by the matrix M^{-1} defined only in the sub-space $\delta E = 0$ occurs. Such a precaution is unnecessary for $\vec{X}(0)$ since by construction $\vec{\epsilon} \cdot \vec{X}(0) = 0$ in the microcanonical ensemble¹⁸.

Importance of the curvature of the excitation spectrum: We now consider a gas of fermions prepared in the microcanonical ensemble at an arbitrarily small yet non zero temperature T . We can then neglect the fermionic branch of excitations and limit ourselves to the region of arbitrarily small wave vectors q of the bosonic one. This is the validity range of Landau and Khalatnikov’s quantum hydrodynamics (see our supplementary information) which we shall use. A physically interesting point, already mentioned by Landau and Khalatnikov in their calculation of the viscosity of Helium III [22], is that taking a linear approximation for the excitation spectrum leads to wrong conclusions. Indeed, even to leading order in T the result does depend on the dimensionless curvature parameter γ defined as

$$\epsilon_{B,\mathbf{q}} \underset{\hbar q/(mc) \rightarrow 0}{=} \hbar c q + \frac{\gamma \hbar^3 q^3}{8m^2 c} + O(q^5) \quad [83]$$

where c is the zero temperature sound velocity and m the mass of a fermion. The role of γ is double: First, its sign distinguishes between two worlds in the formalism of the kinetic equations: when $\gamma > 0$ the bosonic quasi-particles interaction processes of the type $2 \rightarrow 1$ or $1 \rightarrow 2$ (two quasi-particles disappear and a new one is formed, or the other way around, see figure 3) dominate at low temperature whereas they are forbidden by energy conservation when $\gamma < 0$ and the $2 \rightarrow 2$ processes take over. Second, the absolute value of γ also matters as suggested by the result of reference [22] for the viscosity which depends on $|\gamma|$ even to leading order in T . We expect a similar behavior for D : with a bosonic branch linear in q ,

we are condemned to have $-\frac{\hbar}{2} \frac{d\hat{\rho}_0}{dt} - \mu_0(\hat{N})$ [in (8) and limited to the bosonic branch $s = \hat{B}$] proportional to the energy of the quasi-particles $\sum_{\mathbf{q}} \epsilon_{B,\mathbf{q}} \hat{n}_{B,\mathbf{q}}$, a constant of motion for the kinetic equations, hence not subject to diffusion. We thus expect the phase diffusion coefficient to vanish, $D \rightarrow 0$, in the limit of a spectrum with zero curvature¹⁹.

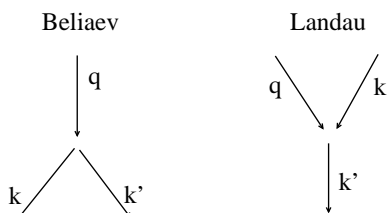


Fig. 3. Bosonic quasi-particles interaction processes $1 \rightarrow 2$ (Beliaev) and $2 \rightarrow 1$ (Landau) involving wave vectors \mathbf{q} , \mathbf{k} and \mathbf{k}' , with $\mathbf{k}' = \mathbf{q} - \mathbf{k}$ and $\mathbf{k}' = \mathbf{q} + \mathbf{k}$ respectively.

Kinetic equations for the unitary gas at low temperature: At unitarity, the numerical calculation of the bosonic excitation spectrum according to the equation of reference [21] yields a curvature parameter γ which, thanks to scale invariance, is a pure number independent of the density of the gas:

$$\gamma \simeq 0.1 \quad [84]$$

Since $\gamma > 0$ the bosonic branch is convex in the vicinity of $\mathbf{q} = \mathbf{0}$, and we can focus on $2 \rightarrow 1$ and $1 \rightarrow 2$ quasi-particle interaction processes. Qualitatively the situation is then similar to that of a weakly interacting Bose gas, and so should be the kinetic equations. Quantitatively however there are some differences: the dispersion relation $\mathbf{q} \mapsto \epsilon_{B,\mathbf{q}}$ is not the same, nor are the dimensionless Landau-Beliaev scattering amplitudes $\mathcal{A}_{\mathbf{k}_1, \mathbf{k}_2}^{\mathbf{q}}$ that appear in the Hamiltonian describing to lowest order the interactions between the quasi-particles:

$$H_{2 \leftrightarrow 1} = \frac{mc^2}{(\rho L^3)^{1/2}} \sum_{\mathbf{q}, \mathbf{k}_1, \mathbf{k}_2 \neq \mathbf{0}} \mathcal{A}_{\mathbf{k}_1, \mathbf{k}_2}^{\mathbf{q}} \delta_{\mathbf{k}_1 + \mathbf{k}_2, \mathbf{q}} \left[\hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2}^\dagger \hat{b}_{\mathbf{q}} + \text{h.c.} \right] \quad [85]$$

where ρ is the total density, L^3 the volume of the quantization box and the Kronecker δ ensures momentum conservation. Despite these differences however, the low temperature limit

$$\varepsilon \equiv \frac{k_B T}{mc^2} \ll 1 \quad [86]$$

is universal, meaning that it does not depend on the microscopic details of the physical system, since the wave vectors appearing in the kinetic equation have a small typical thermal value

$$q_{\text{therm}} \approx \frac{k_B T}{\hbar c} = \varepsilon \frac{mc}{\hbar} \ll \frac{mc}{\hbar} \quad [87]$$

Let us explain in two steps why and how this universal limit for D is obtained.

Limit of the matrix M when $\varepsilon \rightarrow 0$: First, one can artificially rewrite (83) in a way that mimics the Bogoliubov form (the true Bogoliubov spectrum is obtained setting $\gamma = 1$) and which is valid up to order 3 in $\hbar q/(mc)$:

$$\epsilon_{B,\mathbf{q}} \simeq \left[\frac{\hbar^2 q^2}{2m} \left(2mc^2 + \frac{\gamma \hbar^2 q^2}{2m} \right) \right]^{1/2} \quad [88]$$

At the thermodynamic limit, the kinetic equation of the time derivative of the mean number of quasi-particles of wave vector \mathbf{q} contains an integral over the wave vectors \mathbf{k}_1 and \mathbf{k}_2 involved in the corresponding $2 \leftrightarrow 1$ processes (see figure 3). Integrating over one of those wave vectors, say \mathbf{k}' , is straightforward thanks to the momentum conservation Dirac δ . Integrating over the angle between \mathbf{q} and the remaining wave vector \mathbf{k} is harder, but can be done analytically for the Bogoliubov spectrum thanks to the energy conservation Dirac δ [18]. This calculation, which yields k' as a function of k and q , can be recycled here by simply adapting it to the Bogoliubov-like spectrum (88)²⁰. Using the rotational invariance of the unknowns $X_{B,\mathbf{q}}(t)$ there remains a one dimensional integral over the wave vector k .

¹⁷ Careful, these are not orthogonal projectors, see equation (41) of reference [18].

¹⁸ This is actually true at all times $\vec{\epsilon} \cdot \vec{X}(t) = 0, \forall t > 0$, since the energy is conserved by the kinetic equations.

¹⁹ Mathematically, this results from the fact that $P\vec{A} = 0$ in (82) since \vec{A} is then colinear to $\vec{\epsilon}$.

²⁰ This amounts to replacing $1 + \tilde{k}'^2$ by $1 + \gamma \tilde{k}'^2$ and $1 + \tilde{k}''^2$ by $1 + \gamma \tilde{k}''^2$ in the denominators of equations (A9) and (A10) of reference [18]. Equation (A10) and (A11) of the same reference are also changed to $\gamma \tilde{k}'^2 = \sqrt{1 + \gamma(\tilde{\epsilon}_q + \tilde{\epsilon}_k)^2} - 1$ and $\gamma \tilde{k}''^2 = \sqrt{1 + \gamma(\tilde{\epsilon}_k - \tilde{\epsilon}_q)^2} - 1$. We have introduced the rescalings $\tilde{k} = \hbar k / (\sqrt{2} mc)$ and $\tilde{\epsilon}_k = \epsilon_{B,\mathbf{k}} / (mc^2)$.

²¹ Our rescaling procedure mainly amounts to replacing ρg by mc^2 in [18] since in a weakly interacting Bose gas, $mc^2 = \rho g$ where g is the s -wave coupling constant..

Second, the limit $\varepsilon \rightarrow 0$ is taken in the kinetic equation after rescaling the wave vectors to their typical thermal value:

$$\tilde{q} \equiv \frac{\hbar qc}{\sqrt{2}k_B T} \quad [89]$$

and the time t to the inverse of the typical collision rate between quasi-particles:

$$\tilde{t} \equiv \varepsilon^5 \left(\frac{\sqrt{2}mc}{\hbar} \right)^3 \frac{mc^2}{2\pi^2 \hbar \rho} t \quad [90]$$

The dimensionless rescaled quantities \tilde{q} , \tilde{k} and \tilde{t} are kept constant when taking the limit. Definitions (89) and (90) are consistent with equations (D4,A5,A7) of reference [18], up to the inclusion of a trivial ε^5 factor in \tilde{t} . Similarly the definition (86) is consistent with equation (D1) of the same reference²¹. Doing so, we find that the dependence on the curvature parameter γ disappears from the coefficients of the kinetic equations, hence from M . To leading order in ε , only the sign γ matters (it allows resonant $2 \leftrightarrow 1$ processes) not its absolute value. In particular, as shown by expanding the quantum hydrodynamic Hamiltonian (39) up to order three in density (43) and phase (44) fluctuations, the scattering amplitudes for the unitary gas have an equivalent independent of γ :

$$A_{\mathbf{k},\mathbf{k}'}^{\mathbf{q}} \underset{\varepsilon \rightarrow 0}{\sim} \frac{3(1+\Lambda)}{2^{7/4}} \varepsilon^{3/2} (\tilde{q}\tilde{k}\tilde{k}')^{1/2} \quad [91]$$

$$A_{\mathbf{k},\mathbf{q}}^{\mathbf{k}'} \underset{\varepsilon \rightarrow 0}{\sim} \frac{3(1+\Lambda)}{2^{7/4}} \varepsilon^{3/2} (\tilde{q}\tilde{k}\tilde{k}')^{1/2}, \quad [92]$$

Momentum and energy conservation lead to $\tilde{k}' = \tilde{q} - \tilde{k}$ in the right-hand-side of (91) and to $\tilde{k}' = \tilde{q} + \tilde{k}$ in the right-hand-side of (92). In other words, at low energy the only resonant $2 \leftrightarrow 1$ scattering processes are those where the three wave vectors are almost colinear. We have introduced

$$\Lambda = \frac{\rho \mu_0''(\rho)}{3\mu_0'(\rho)} \quad [93]$$

where $\mu_0(\rho)$ is the zero temperature chemical potential, such that $mc^2 = \rho \mu_0'(\rho)$. Here $\Lambda = -1/9$ since the scale invariance of the unitary gas leads to $\mu_0(\rho) \propto \rho^{2/3}$. For a weakly interacting Bose gas, $\mu_0(\rho) \propto \rho$, hence $\Lambda = 0$ and equations (91,92) reproduce equations (D8) and (D9) of reference [18], as they should. Altogether this leads to the kinetic equations for the correlation functions (80) by introducing the notations $X_{B,\mathbf{q}}(t) \equiv X_{\tilde{q}}(\tilde{t})$ and $\bar{n}_{B,\mathbf{q}} \equiv \bar{n}_{\tilde{q}}$:

$$\begin{aligned} \frac{d}{d\tilde{t}} X_{\tilde{q}} &= -(\tilde{\Gamma}_{\tilde{q}}^{(B)} + \tilde{\Gamma}_{\tilde{q}}^{(L)}) X_{\tilde{q}} + \frac{9\pi}{4} (1+\Lambda)^2 \left[\int_0^{+\infty} d\tilde{k} \tilde{k}^2 (\tilde{q} + \tilde{k})^2 \right. \\ &\quad \times (\bar{n}_{\tilde{k}+\tilde{q}} - \bar{n}_{\tilde{q}}) X_{\tilde{k}} + \int_{\tilde{q}}^{+\infty} d\tilde{k} \tilde{k}^2 (\tilde{q} - \tilde{k})^2 (1 + \bar{n}_{\tilde{k}-\tilde{q}} + \bar{n}_{\tilde{q}}) X_{\tilde{k}} \\ &\quad \left. + \int_0^{\tilde{q}} d\tilde{k} \tilde{k}^2 (\tilde{q} - \tilde{k})^2 (\bar{n}_{\tilde{q}-\tilde{k}} - \bar{n}_{\tilde{q}}) X_{\tilde{k}} \right] \quad [94] \end{aligned}$$

where

$$\tilde{\Gamma}_{\tilde{q}}^{(B)} = \frac{9\pi}{8} (1+\Lambda)^2 \int_0^{\tilde{q}} d\tilde{k} \tilde{k}^2 (\tilde{q} - \tilde{k})^2 (1 + \bar{n}_{\tilde{k}} + \bar{n}_{\tilde{q}-\tilde{k}}) \quad [95]$$

$$\tilde{\Gamma}_{\tilde{q}}^{(L)} = \frac{9\pi}{4} (1+\Lambda)^2 \int_0^{+\infty} d\tilde{k} \tilde{k}^2 (\tilde{q} + \tilde{k})^2 (\bar{n}_{\tilde{k}} - \bar{n}_{\tilde{k}+\tilde{q}}), \quad [96]$$

that with we successfully compare to (D10,D11,D12) of reference [18]. Note that we have obtained the Beliaev-Landau

decay rates $\tilde{\Gamma}_{\tilde{q}}^{(B)}$ and $\tilde{\Gamma}_{\tilde{q}}^{(L)}$ of the quasi-particle of wave vector \mathbf{q} . They are dimensionless and rescaled according to (90).

Limit of $\vec{X}(0)$ and $P\vec{A}$ and behaviour of D when $\varepsilon \rightarrow 0$: To leading order in ε , the curvature parameter γ only appears in the initial value $\vec{X}(0)$ of the unknown vector and in the projection $P\vec{A}$ of the coefficient vector, see (82). These two quantities are in fact proportional to γ . The initial value $\vec{X}(0)$ is obtained by acting with the microcanonical quasi-particle numbers covariance matrix on the vector \vec{A} of equation (81). This covariance matrix appearing in (80) is derived by applying the projector onto the zero energy fluctuation subspace and its hermitian conjugate to the well-known canonical covariance matrix, see equations (21,41,47,48,49) of reference [18] for the derivation and equation (43) of the same reference setting $\eta = 0$ for the final result. The parameter γ appears in the coordinates of \vec{A} , and more indirectly in the covariance matrix through the occupation numbers $\bar{n}_{B,\mathbf{q}} = 1/[\exp(\epsilon_{B,\mathbf{q}}/k_B T) - 1]$, which themselves depend on the energies $\epsilon_{B,\mathbf{q}}$ and therefore on γ . The rather long ensuing calculation follows step-by-step the one conducted in [18]. The starting point are the low-temperature expansions for fixed \tilde{q}

$$\epsilon_{B,\mathbf{q}} \underset{\varepsilon \rightarrow 0}{=} \sqrt{2}k_B T \tilde{q} \left[1 + \frac{1}{4} \gamma \varepsilon^2 \tilde{q}^2 + O(\varepsilon^4) \right] \quad [97]$$

$$\hbar A_{B,\mathbf{q}} \underset{\varepsilon \rightarrow 0}{=} \frac{2\sqrt{2}}{3N} k_B T \tilde{q} \left[1 - \frac{1}{4} \gamma \varepsilon^2 \tilde{q}^2 + O(\varepsilon^4) \right] \quad [98]$$

where, in the second relation, we have used $Ndc/dN = c/3$ and $d\gamma/dN = 0$, a consequence of the scale invariance. The result for $\vec{X}(0)$ is $4\gamma/3$ times that of [18] for the same value of ε , provided we express them both in units of the quantity $mc^2/(N\hbar)$ corresponding to the considered system (weakly interacting Bose gas or unitary Fermi gas) and we consider them as functions of the corresponding \tilde{q} as in (A17) and (D14) of reference [18]. The same conclusion holds for the projection $P\vec{A}$ for which one can refer to equations (A16) and (D13) of reference [18]. Introducing the constant c_1 obtained by numerically solving the kinetic equations (94) for $\Lambda = 0$, see equations (50) and (D17) of [18],

$$c_1 \simeq 0.3036 \quad [99]$$

we finally obtain the phase diffusion coefficient of the unitary Fermi gas to leading order in temperature

$$\frac{\hbar ND}{mc^2} \underset{T \rightarrow 0}{\sim} \frac{(4\gamma/3)^2}{(1+\Lambda)^2} c_1 \left(\frac{k_B T}{mc^2} \right)^4 \quad [100]$$

Using the experimental numerical value of the parameter ξ of the equation of state $\mu_0(\rho) = \xi \varepsilon_F$ where $\varepsilon_F \equiv \hbar^2(3\pi^2\rho)^{2/3}/(2m)$, see the body of the article, leads to $mc^2/\varepsilon_F = 2\xi/3 = 0.251$ and to the expression of D given in (22).

In short : Let us now give a synthetic, if not convincing, explanation of the result (100), starting from the one obtained in [18], $\hbar ND/(mc^2) = c_1 (k_B T/mc^2)^4$, and keeping track of all the corrective factors allowing to pass from the weakly interacting Bose gas to the unitary Fermi gas. Knowing that D is a quadratic function of \vec{A} (see equations (80,82)): (i) the factor 2 in the coefficients of $\frac{d\theta_{\mathbf{q}}}{dt}$, see equation (81), leads to a factor 4 in D , (ii) the equation of state is changed from $\mu_0(\rho) \propto \rho$ to $\mu_0(\rho) \propto \rho^{2/3}$ hence $(N/c)dc/dN$ changes from $1/2$ to $1/3$, which leads to a factor $2/3$ in \vec{A} , hence a factor $4/9$ in D , (iii) $P\vec{A}$ and $\vec{X}(0)$, that are equal to zero for an excitation branch purely linear in q , are proportional at low

temperature to the curvature parameter γ , which is equal to 1 in [18], hence a factor γ^2 in D . All this explains well the factor $(4\gamma/3)^2$ in the numerator of equation (100). Last, the factor $(1 + \Lambda)^2$ in the denominator of (100) originates from the fact that the Beliaev-Landau amplitudes for the unitary gas are $(1 + \Lambda)$ times those of the weakly interacting Bose gas (for the same rescaled infinitesimal wave vectors), which increases the correlation time of the quasi-particle occupation numbers, and hence D , by a global factor $(1 + \Lambda)^{-2}$. Keeping equation (55)

of reference [18] in mind, this further leads to the expression of the finite correlation time τ_c of $\frac{d}{dt}\hat{\theta}_0$ induced by collisions among quasi-particles,

$$\tau_c \underset{T \rightarrow 0}{\sim} \frac{c_3}{(1 + \Lambda)^2} \frac{\hbar \rho}{mc^2 \varepsilon^5} \left(\frac{\hbar}{\sqrt{2}mc} \right)^3 \text{ with } c_3 \simeq 0.05472 \text{ [101]}$$

whose value for the unitary gas is given in the footnote 12 of our paper.

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