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# Thermal blurring of a coherent Fermi gas

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Coherent gases of ultracold atoms confined in immaterial non-dissipative traps are unique examples of isolated macroscopic quantum systems. The value of their intrinsic coherence time is then a fundamental question. But it is also a practical issue for all the applications which exploit macroscopic coherence, such as interferometry [1], or quantum engineering where one generates non-trivial entangled states by coherent evolution [2–4]. Coherence time measurements are presently being performed in cold Bose gases [5–7]. Experiments on Fermi gases, that up to now focussed on traditional aspects of the  $N$ -body problem, such as thermodynamic properties [8, 9], are moving towards correlation and coherence measurements [10]. This turn will open a new research field, including the strong coupling regime : that of fermionic quantum optics [11]. However a theory predicting the coherence time of a pair-condensed Fermi gas was missing, except in the limiting case of zero temperature [12]. In this paper we present the first microscopic theory bridging this theoretical gap in a general way. Our results hold for other physical systems, such as mesoscopic Josephson Junctions, provided that the environment-induced decoherence is sufficiently reduced.

*Setting the stage:* For a Bose-condensed gas of bosons, the finite coherence time is due to the spreading of the condensate phase probability distribution. At zero temperature and in presence of interactions, a ballistic phase spreading is caused by atom number fluctuations in the sample. This effect has been measured by interfering two initially mutually-coherent condensates, whose particle number fluctuates due to partition noise [5, 7]. Contrarily to lasers, which are open quantum systems, and somehow unexpectedly [13], a ballistic spreading persists in Bose-Einstein condensates for a fixed atom number at non-zero temperature [14, 15]. Fluctuations of the energy, another conserved quantity, then play the same role as number fluctuations.

For an unpolarized pair-condensed Fermi gas, the study of coherence time presupposes a clear definition of the condensate phase and of the corresponding operator  $\hat{\theta}_0$  [12]. Furthermore, at non-zero temperature the speed of variation of the phase should include the contribution of two excitation branches: the fermionic pair-breaking one and the bosonic one exciting the pair motion. For the fermionic branch Anderson’s Random Phase Approximation (RPA) [16] is enough. For the bosonic branch however, we need the equivalent for fermions of the Bogoliubov method to construct quasiparticle creation  $\hat{b}_\alpha^\dagger$  and annihilation  $\hat{b}_\alpha$  operators, and to express  $d\hat{\theta}_0/dt$  in term of these operators. More than that, we need to include interactions among quasiparticles in the evolution of the  $\hat{b}_\alpha$ . This is a non-trivial consequence of the dependence of condensate wavefunction on the total number of particles  $N$  even for a spatially homogeneous system, and clearly goes beyond the RPA program.

*Correlation function decay:* Below the critical temperature, the time-correlation function of the pairing field  $\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r}')$  where  $\hat{\psi}_\sigma(\mathbf{r})$  is the fermionic field operator of the spin  $\sigma$  component, is dominated at long times by the condensate contribution:

$$g_1(t) = \langle \hat{a}_0^\dagger(t)\hat{a}_0(0) \rangle \quad (1)$$

where  $\hat{a}_0 = \int d^3r d^3r' \varphi_0(\mathbf{r}, \mathbf{r}') \hat{\psi}_\downarrow(\mathbf{r}) \hat{\psi}_\uparrow(\mathbf{r}')$  is the component of the pairing field on the condensate wavefunction [12]. At equilibrium the system is in a mixture of  $N$ -body eigenstates  $|\psi_\lambda\rangle$ , with weights  $\Pi_\lambda$ . We then study the correlation function  $g_1^\lambda(t)$  in the eigenstate  $|\psi_\lambda\rangle$  of energy  $E_\lambda$  and particle number  $N_\lambda$ . To exploit the weak relative fluctuations in the number of condensed pairs for a large system, we split  $\hat{a}_0$  into modulus and phase hermitian operators [12].

$$\hat{a}_0 = e^{i\hat{\theta}_0} \hat{N}_0^{1/2}, \quad (2)$$

and we approximate  $\hat{N}_0$  by its mean value  $\bar{N}_0$  in the equilibrium state, to obtain

$$g_1^\lambda(t) \simeq \bar{N}_0 e^{iE_\lambda t/\hbar} \langle \psi_\lambda | e^{-i(\hat{H} + \hat{W})t/\hbar} | \psi_\lambda \rangle \quad (3)$$

The operator  $\hat{W}$ , difference between  $\hat{H}$  transformed by  $e^{i\hat{\theta}_0}$  and  $\hat{H}$ ,

$$\hat{W} = e^{-i\hat{\theta}_0} \hat{H} e^{i\hat{\theta}_0} - \hat{H} = -i[\hat{\theta}_0, \hat{H}] - \frac{1}{2}[\hat{\theta}_0, [\hat{\theta}_0, \hat{H}]] + \dots \quad (4)$$

is approximatively  $N$  times smaller than  $\hat{H}$ . Indeed  $e^{i\hat{\theta}_0}$ , like  $\hat{a}_0$ , changes the total particle number by a quantity  $O(N^0)$ . While  $\hat{H}$  is an extensive observable,  $\hat{W}$  is intensive. In equation (3) formally appears the evolution operator of the Hamiltonian  $\hat{H}$  perturbed by  $\hat{W}$ , and restricted to the eigenstate  $|\psi_\lambda\rangle$  of  $\hat{H}$ . Up to a phase factor, the function  $g_1^\lambda/\bar{N}_0$  is then proportional to the probability amplitude that the system prepared in  $|\psi_\lambda\rangle$  is still in that state after a time  $t$ . A standard way to obtain a non-perturbative approximation of this amplitude is to use the Green function or the resolvent operator  $\hat{G}(z) = (z\hat{1} - (\hat{H} + \hat{W}))^{-1}$  of the perturbed Hamiltonian. Within the projectors method [17], we introduce an effective non hermitian Hamiltonian  $\hat{H}_{\text{eff}}(z)$  governing the evolution restricted to  $|\psi_\lambda\rangle$ , which gives

$\langle \psi_\lambda | \hat{G}(z) | \psi_\lambda \rangle = \left( z - \langle \psi_\lambda | \hat{H}_{\text{eff}}(z) | \psi_\lambda \rangle \right)^{-1}$ . Keeping in  $\hat{H}_{\text{eff}}(z)$  terms up to order two in  $\hat{W}$  and neglecting its  $z$  dependence  $\hat{H}_{\text{eff}}(z) \approx \hat{H}_{\text{eff}}(E_\lambda + i0^+)$  (pole approximation), we obtain

$$g_1^\lambda(t) \simeq \bar{N}_0 e^{-it \langle \psi_\lambda | \hat{W} | \psi_\lambda \rangle / \hbar} e^{-(i\delta_\lambda + \gamma_\lambda)t} \quad (5)$$

Introducing  $\hat{Q}_\lambda = \hat{1} - |\psi_\lambda\rangle\langle\psi_\lambda|$ , the projector orthogonal to  $|\psi_\lambda\rangle$ , and the notation  $\langle \hat{A} \rangle_\lambda \equiv \langle \psi_\lambda | \hat{A} | \psi_\lambda \rangle$  one has:

$$\hbar(\delta_\lambda - i\gamma_\lambda) = \langle \hat{W} \hat{Q}_\lambda \frac{\hat{Q}_\lambda}{(E_\lambda + i0^+) \hat{Q}_\lambda - \hat{Q}_\lambda \hat{H} \hat{Q}_\lambda} \hat{Q}_\lambda \hat{W} \rangle_\lambda \quad (6)$$

The leading term under the exponential in (5),  $\langle \psi_\lambda | \hat{W} | \psi_\lambda \rangle$ , is of order  $N^0$  like  $\hat{W}$ . A first step in its interpretation is to remark that, according to the expansion (4),  $\hat{W} = \hbar d\hat{\theta}_0/dt + O(1/N)$ . We then use a central relation, proved in the next section, giving the coarse grained time average of  $d\hat{\theta}_0/dt$  in a weakly excited gas

$$-\frac{\hbar}{2} \frac{d\hat{\theta}_0}{dt} = \mu_0(\hat{N}) + \sum_{s=F,B} \sum_{\alpha} \frac{d\epsilon_{s,\alpha}}{dN} \hat{n}_{s,\alpha} \quad (7)$$

The sum on the right hand side runs over both the gapped quasi-particles fermionic branch of excitation (in the homogenous case  $\alpha$  includes both a spin and an orbital index,  $\alpha = \mathbf{k}, \sigma$ ) and the bosonic one which, in the thermodynamic limit and for an homogeneous system, has a phononic behaviour ( $\alpha$  is then only orbital,  $\alpha = \mathbf{q}$ ). The coarse grained time average is taken over a time long with respect to the inverse of the quasi-particle eigenfrequencies  $\epsilon_{s,\alpha}/\hbar$ , yet short with respect to the typical time-scale of variation of the occupation numbers  $\hat{n}_{s,\alpha}$ . Finally  $\mu_0(N)$  is the zero temperature chemical potential of the gas with  $N$  particles. We interpret  $d\hat{\theta}_0/dt$  as the adiabatic derivative of the energy that is at fixed quasi-particle populations. By taking the average of equation (7) in the stationary state  $|\psi_\lambda\rangle$  and using the Eigenstate thermalization hypothesis [18] to identify the quantum average in an eigenstate with the microcanonical average, we recognize the microcanonical chemical potential  $\mu_{\text{mc}}$  at energy  $E_\lambda$  and particle number  $N_\lambda$  and obtain:

$$\langle \psi_\lambda | \hbar \frac{d\hat{\theta}_0}{dt} | \psi_\lambda \rangle = -2\mu_{\text{mc}}(E_\lambda, N_\lambda) \quad (8)$$

The next-to-leading term under the exponential in (5), is of order  $1/N$ . In order to prove it, we express it in terms of the correlation function of  $d\hat{\theta}_0/dt$  in  $|\psi_\lambda\rangle$ :

$$\gamma_\lambda + i\delta_\lambda = \int_0^{+\infty} dt \left[ \left\langle \frac{d\hat{\theta}_0(t)}{dt} \frac{d\hat{\theta}_0(0)}{dt} \right\rangle_\lambda - \left\langle \frac{d\hat{\theta}_0}{dt} \right\rangle_\lambda^2 \right] \quad (9)$$

This is equivalent to (6) as can be checked by inserting a closure relation. The  $t = 0$  value of the integrand is

$\text{Var}_\lambda(d\hat{\theta}_0/dt) = O(1/N)$  (this comes from adding up the variances of independent quasi-particles numbers in the canonical ensemble and overestimates the microcanonical variance); the function then decays in a time  $\tau_c$  which is the typical collision time of quasi-particles and hence the correlation time of the  $\hat{n}_{s,\alpha}$ . Altogether we estimate  $|\gamma_\lambda + i\delta_\lambda| \approx \tau_c \text{Var}(d\hat{\theta}_0/dt) = O(1/N)$ . The energy shift  $\delta_\lambda$  is thus of the same order in  $N$  as the neglected term  $[\hat{\theta}_0, [\hat{\theta}_0, \hat{H}]]$  in  $\hat{W}$ , that is  $N$  times smaller than (8); we neglect it for a large system. In contrast, we keep  $\gamma_\lambda$ , the only term leading to an exponential decay of the correlation function  $g_1^\lambda$ . It is in fact the phase diffusion coefficient of a system prepared in the microcanonical ensemble corresponding to  $|\psi_\lambda\rangle$ ,  $\gamma_\lambda = D(E_\lambda, N_\lambda)$ . Finally:

$$g_1^\lambda(t) \simeq \bar{N}_0 e^{[2i\mu_{\text{mc}}(E_\lambda, N_\lambda)/\hbar - D(E_\lambda, N_\lambda)]t} \quad (10)$$

The final step is to take the statistical average over the probability distribution  $\Pi_\lambda$  of the states  $|\psi_\lambda\rangle$ . For large  $N$ , we assume that energy and atom number fluctuations around the mean values  $\bar{E}$  and  $\bar{N}$  are weak in relative value. This is the case if  $\Pi_\lambda$  describes a canonical or grand canonical ensemble. We assume Gaussian fluctuations and linearize  $\mu_{\text{mc}}$  around  $(\bar{E}, \bar{N})$ . To this order, we keep only the central value  $D(\bar{E}, \bar{N})$  of the next-to-leading term. Altogether this leads to the main result of this work :

$$g_1(t) \simeq \bar{N}_0 e^{2i\mu_{\text{mc}}(\bar{E}, \bar{N})t/\hbar} e^{-t^2/2t_{\text{br}}^2} e^{-D(\bar{E}, \bar{N})t} \quad (11)$$

At long times, the thermal blurring thus consists in a Gaussian decay of the correlation function  $g_1(t)$ , with a characteristic time

$$(2t_{\text{br}}/\hbar)^{-2} = \text{Var} \left( N \frac{\partial \mu_{\text{mc}}}{\partial N}(\bar{E}, \bar{N}) + E \frac{\partial \mu_{\text{mc}}}{\partial E}(\bar{E}, \bar{N}) \right) \quad (12)$$

which diverges  $N^{1/2}$  for normal fluctuations. The phase diffusion coefficient  $D$  leads to an exponential decay with a characteristic time diverging as  $N$ . As expected it is a subleading effect at long times, except if the system is prepared in the microcanonical ensemble.

*Microscopic derivation of the dynamical equation of the phase* : We give here the first (to our knowledge) microscopic derivation of equation (7), relating the evolution of the phase of a pair-condensed gas to what one could call the chemical potential operator.

The contribution of the fermionic branch of excitations to  $d\hat{\theta}_0/dt$  can be obtained from the linearized equations of motion for small fluctuations of the pair operators  $\hat{\psi}_\downarrow \hat{\psi}_\uparrow$ ,  $\hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow^\dagger$  and  $\hat{\psi}_\sigma^\dagger \hat{\psi}_\sigma$  around the mean-field state [16]. Using equation (120) of reference [12] to extract the time average of  $d\hat{\theta}_0/dt$ , and rewriting equation (86) of [12] in terms of the fermionic quasi-particle occupation numbers, we get

$$-\frac{\hbar}{2} \frac{d\hat{\theta}_0}{dt} \underset{\text{RPA}}{=} \mu(\bar{N}) + \frac{d\mu}{dN}(\hat{N} - \bar{N}) + \sum_{\alpha=\mathbf{k},\sigma} \frac{d\epsilon_{F,\alpha}}{dN} \hat{n}_{F,\alpha} \quad (13)$$

where  $\epsilon_{F,\mathbf{k},\sigma}$  is the BCS excitation spectrum of an homogeneous system and  $\bar{N}$  the BCS average particle number in the grand canonical ensemble of chemical potential  $\mu$

The result (13) does not include the contribution of the phonon branch because at the linear order of the RPA the phase of the condensate (formed of pairs at rest) is not coupled to moving pairs operators. One could expect to obtain such coupling by pushing the expansion to the quadratic order. Unfortunately the operators of the RPA, although linearly independent, are linked by nonlinear relations, as one can see by rearranging the pair operators using fermionic anticommutation relations. As a consequence, if one inserts the modal expansion obtained from the linearized equations of motion in a nonlinear expression such as  $d\hat{\theta}_0/dt$ , the coefficient of  $\hat{b}_{\mathbf{q}}^\dagger \hat{b}_{\mathbf{q}}$  is not uniquely determined. We then decided to treat the nonlinearity with a variational approach based on a time-dependent coherent state including moving pairs:

$$|\psi\rangle = \mathcal{N}(t) \exp \left( b^6 \sum_{\mathbf{r},\mathbf{r}'} \Gamma(\mathbf{r},\mathbf{r}';t) \hat{\psi}_{\uparrow}^\dagger(\mathbf{r}) \hat{\psi}_{\downarrow}^\dagger(\mathbf{r}') \right) |0\rangle, \quad (14)$$

Here  $\mathcal{N}$  ensures normalization and the  $\Gamma(\mathbf{r},\mathbf{r}')$  form a set of independent variables. The space has been discretized on a cubic lattice of step  $b$ , which we take to zero in the end of the calculations. The field operators obey anticommutation relations of the kind:  $\{\hat{\psi}_{\sigma}(\mathbf{r}), \hat{\psi}_{\sigma'}^\dagger(\mathbf{r}')\} = \delta_{\sigma,\sigma'} \delta_{\mathbf{r},\mathbf{r}'}/b^3$ . Reference [19] constructs from  $\Gamma(\mathbf{r},\mathbf{r}'), \Gamma^*(\mathbf{r},\mathbf{r}')$  the set of canonically conjugate variables  $\Phi(\mathbf{r},\mathbf{r}'), \Phi^*(\mathbf{r},\mathbf{r}')$ . This field  $\Phi$  has no simple relationship with the usual pairing field  $\langle \hat{\psi}_{\downarrow} \hat{\psi}_{\uparrow} \rangle$ . The square of its norm is exactly the mean number of pairs in  $|\psi\rangle$

$$\frac{N}{2} = \|\Phi\|^2 \equiv b^6 \sum_{\mathbf{r},\mathbf{r}'} |\Phi(\mathbf{r},\mathbf{r}';t)|^2. \quad (15)$$

Its evolution is governed by the classical Hamiltonian:

$$\mathcal{H}(\Phi, \Phi^*) = \langle \psi | \hat{H} | \psi \rangle. \quad (16)$$

In the following we will need only the invariance of  $\mathcal{H}$  under a global phase change  $\Phi(\mathbf{r},\mathbf{r}') \rightarrow e^{i\gamma} \Phi(\mathbf{r},\mathbf{r}')$ ,  $\forall \gamma \in \mathbb{R}$  ( $U(1)$  symmetry), consequence of the conservation of the particle number  $\hat{N}$  by evolution with  $\hat{H}$ . At zero temperature and for a fixed  $N$  the field  $\Phi(\mathbf{r},\mathbf{r}')$  is fixed, up to a global phase factor, into the minimizer  $\Phi_0(\mathbf{r},\mathbf{r}') = (N/2)^{1/2} \phi_0(\mathbf{r},\mathbf{r}')$  of  $\mathcal{H}$ .  $\phi_0$  is chosen real and normalized to one. It depends on  $N$  even in the spatially homogeneous case and differs from the condensate wavefunction  $\varphi_0$  in the same way that  $\Phi$  differs from the pairing field  $\langle \hat{\psi}_{\downarrow} \hat{\psi}_{\uparrow} \rangle$ . At sufficiently low temperature one can expand  $\mathcal{H}$  in powers of the small deviations of  $\Phi$  away from the circle  $\gamma \mapsto e^{i\gamma} \Phi_0(\mathbf{r},\mathbf{r}')$ , locus of the minima of  $\mathcal{H}$  for fixed  $N$ . We split the field into its components parallel and orthogonal to  $\phi_0$ :

$$\Phi(\mathbf{r},\mathbf{r}') = e^{i\theta} [n^{1/2} \phi_0(\mathbf{r},\mathbf{r}') + \Lambda(\mathbf{r},\mathbf{r}')] \quad (17)$$

The phase  $\theta$  can reach arbitrarily large values while  $\Lambda$  is bounded. This framework allows us to develop a systematic perturbation theory in powers of the field  $\Lambda$  (*cf.* Appendix A), fermionic equivalent of the Bogoliubov  $U(1)$ -symmetry conserving approach for bosons [20]. Provided that  $\Lambda$  stays small, the phase  $\theta$  remains close to the condensate phase  $\theta_0$  as we shall see. We then write down the equations of motion of  $\theta$  and of the fields  $\Lambda, \Lambda^*$ . At the end of the calculations we systematically eliminate the condensate variables with the relation  $n = \|\Phi\|^2 - \|\Lambda\|^2$ , consequence of (15), and we restrict ourselves to order 2 in  $\Lambda, \Lambda^*$ .

The main challenge of the calculation is the occurrence of a term linear in  $\Lambda, \Lambda^*$  in  $d\theta/dt$ , resulting from the fact that  $\phi_0$  depends on the number of pairs [12]. Without this term, one would simply expand the field  $\Lambda$  on the eigenmodes of its small linear oscillations, obtained from a quadratization of the Hamiltonian  $\mathcal{H}$  at fixed  $N$ :

$$\begin{pmatrix} \Lambda(\mathbf{r},\mathbf{r}';t) \\ \Lambda^*(\mathbf{r},\mathbf{r}';t) \end{pmatrix} = \sum_{\alpha} b_{\alpha}(t) \begin{pmatrix} u_{\alpha}(\mathbf{r},\mathbf{r}') \\ v_{\alpha}(\mathbf{r},\mathbf{r}') \end{pmatrix} + b_{\alpha}^*(t) \begin{pmatrix} v_{\alpha}^*(\mathbf{r},\mathbf{r}') \\ u_{\alpha}^*(\mathbf{r},\mathbf{r}') \end{pmatrix} \quad (18)$$

where the sum runs over the eigenmodes of positive energy  $\epsilon_{\alpha}$ , normalized as  $\|u_{\alpha}\|^2 - \|v_{\alpha}\|^2 = 1$ . At this order,  $b_{\alpha}(t) = b_{\alpha}(0)e^{-i\epsilon_{\alpha}t/\hbar}$ . One would insert the expansion (18) into  $d\theta/dt$  and take a coarse grained temporal average to get rid of the oscillating terms.  $\overline{d\theta/dt}^t$  would then contain the expected linear combination of the numbers of bosonic quasi-particles  $n_{B,\alpha} = |b_{\alpha}|^2$ . In reality, the problem is more subtle: due to the interaction among the quasi-particles,  $\overline{b_{\alpha}^t}$  does not vanish and is of order two in  $\Lambda$  and  $\Lambda^*$ . The contribution of the linear term in  $d\theta/dt$  is then comparable to the quadratic terms. It is calculated in the Appendix A, exploiting in particular the bounded nature of the field  $\Lambda$  (consequence of the  $U(1)$ -symmetry preserving nature of expansion (17)) and the Hellmann-Feynman theorem. One finds

$$-\frac{\hbar}{2} \frac{d\overline{\theta}}{dt} = \mu_0(N) + \sum_{\alpha} \frac{d\epsilon_{\alpha}}{dN} |b_{\alpha}|^2 + O(\|\Lambda\|^3) \quad (19)$$

We now briefly discuss the form of the energy spectrum  $\epsilon_{\alpha}$  for a spatially homogeneous system, in the continuous limit  $b \rightarrow 0$  for a  $s$ -wave contact interaction with a fixed scattering length between opposite spin fermions. For each value of the total wave vector  $\mathbf{q}$ , there exists (i) at most one discrete value  $\epsilon_{B,\mathbf{q}}$ , (ii) a continuum parametrized by two wave vectors  $(\mathbf{k}_1, \uparrow; \mathbf{k}_2, \downarrow) \mapsto \epsilon_{F,\mathbf{k}_1,\uparrow} + \epsilon_{F,\mathbf{k}_2,\downarrow}$  of constant sum ( $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{q}$ ), where  $\epsilon_{F,\mathbf{k},\sigma}$  is the BCS dispersion relation. The branch  $\epsilon_{B,\mathbf{q}}$  coincides with that of reference [21], as we have checked. It has a phononic start and corresponds to the bosonic elementary excitations of the Fermi gas, whose contribution to the phase dynamics was missing. It must be included in (19). The continuum corresponds to the excitation of two fermionic quasi-particles. Indeed, since the Hamiltonian  $\hat{H}$  contains an even number of factors  $\hat{\psi}$  and  $\hat{\psi}^\dagger$ , each annihilating or creating one quasi-particle,

fermionic quasi-particles can only be created by pairs from the ground state. The corresponding biexcitations are not physically independent and are redundant with the RPA contribution to  $d\hat{\theta}_0/dt$ . They must not be included in (19).

Two more remarks are needed to obtain (7). (i) The fields  $\langle\hat{\psi}_\downarrow\hat{\psi}_\uparrow\rangle$  and  $\Phi$  differ and so do the phases  $\hat{\theta}_0$  and  $\theta$ . Their coarse grained temporal averages, however, only differ by a term of order  $\|\Lambda\|^2$ , which, bounded hence negligible in the long time limit, does not contribute to the phase blurring of the condensate of pairs. (ii) The phase  $\theta$  of our variational approach is a classical variable, whereas  $\hat{\theta}_0$  in (7) is a quantum operator. This gap can be bridged by using the quantization procedure exposed in Chapter 11 of reference [19], where the  $b_\alpha$  are in the end replaced by bosonic operators  $\hat{b}_\alpha$ ,  $[\hat{b}_\alpha, \hat{b}_\alpha^\dagger] = 1$ . We argue that equation (7), linking  $d\hat{\theta}_0/dt$  to the chemical potential operator, and the resulting equation (11) should hold beyond the validity range of the microscopic variational derivation presented above, and should apply even to the strongly interacting regime, provided that the temperature is low enough for the quasi-particles lifetime to be much longer than the inverse of their eigenfrequency. Indeed, in the limiting case where one can neglect the fermionic excitation branch and drop the non-phononic part of the bosonic branch, equation (7) can be derived from the irrotational version of the quantum hydrodynamic theory of Landau and Khalatnikov [22] (see Appendix B).

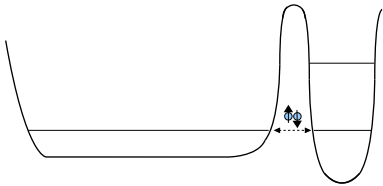


FIG. 1: Trapping configuration proposed here to measure  $g_1(t)$  via Ramsey interferometry: the condensed paired Fermi gas is confined in the main trap (with a flat bottom on the figure); one transfers a small number of atoms (in the form of dimers) in the (very narrow) secondary trap via a resonant tunneling effect, which can be tuned by a barrier of adjustable height; in this way, one creates a phase reference, which is made to interfere with the condensate after an evolution time  $t$ .

Let us briefly explain how an experimental evidence of the thermal blurring of a condensate of pairs could be obtained. The key idea is to bosonize the atomic Cooper pairs into deeply bound weakly interacting dimers during the preparation and the measurement stage. This can be done in an adiabatic reversible way [23] by tuning the scattering length to a small and positive value thanks to a magnetic Feshbach resonance. It allows one to (i) produce a sample of dimers with weak number fluctuations from a melted Mott phase of an experimental realization of the Bose Hubbard model [24], (ii) control tunneling between the main trap (containing the  $N$  particles) and

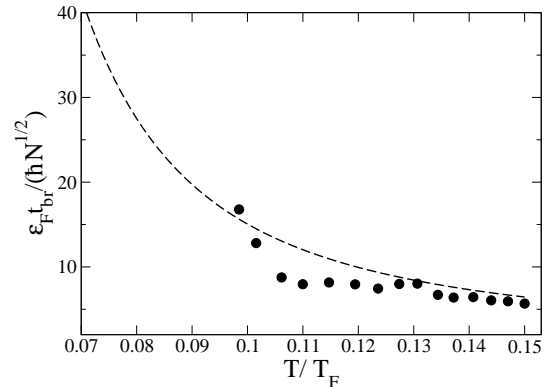


FIG. 2: Thermal blurring time of a coherent Fermi gas in the unitary limit in the canonical ensemble, as a function of temperature  $T$  in units of the Fermi temperature  $T_F = \epsilon_F/k_B$ . Discs: from the equation of state measured in reference [9]. Dashed line: expression (20) deduced from an approximated equation of state (see text).

a very narrow secondary trap by adjusting the height of a potential barrier [3] (Fig.1), (iii) detect by fluorescence a single dimer [25] in this secondary trap. For the measurement of the  $g_1(t)$  function, we adapt [26] the interferometric Ramsey method of two Rabi pulses at a time interval  $t$  to the case of paired fermions. The bosonized pairs are prepared initially in the main trap. A first pulse of angle  $\epsilon$  transfers on average less than one dimer to the secondary trap; in this way, the thermal blurring is not masked by partition noise. Then the system evolves during a time  $t$  with interactions set to the value at which phase dynamics is to be studied. Last, the gas is rebosonized and a second pulse of angle  $\epsilon$  closes the interferometer, and the number  $n_{\text{sec}}$  of dimers in the secondary trap is measured. The average of  $n_{\text{sec}}$  over the realizations is an oscillating function of time  $t$ , of angular frequency  $2/\hbar$  times the difference of the two trapping zones chemical potentials, with a contrast equal to  $|g_1(t)/g_1(0)|$ .

Finally, we estimate the blurring time for a unitary Fermi gas prepared in the canonical ensemble, that is with energy fluctuations of variance  $\text{Var } E = k_B T^2 \partial_T \bar{E}$ . From the equation of state of the unpolarized unitary gas measured in reference [9], and for a spatially homogeneous system (in a flat bottom potential [27]) we find the thermal blurring times  $t_{\text{br}}$  plotted as discs in figure 2. For example, at a temperature  $T = 0.12 T_F \simeq 0.7 T_c$ , we find  $t_{\text{br}} \approx 7N^{1/2}\hbar/\epsilon_F$  corresponding to 20 milliseconds for a typical Fermi temperature  $T_F = \epsilon_F/k_B = 1\mu\text{K}$  and a typical atom number  $N = 10^5$ . As in reference [28], one can also estimate the equation of state of the unitary gas from simple dispersion relations for the elementary excitations. For the bosonic branch one takes [28]  $\epsilon_{B,\mathbf{q}} = \hbar c q$  with  $c$  the  $T = 0$  sound velocity,  $mc^2 = \frac{2}{3}\xi\epsilon_F$  and  $\xi$  the Bertsch parameter. For the fermionic branch, one takes [29]  $\epsilon_{F,\mathbf{k},\sigma} = \Delta + (\frac{\hbar^2 k^2}{2m} - \epsilon_0)/(2\phi_0)$ , where  $\Delta$  is the gap, and  $\epsilon_0$  and  $\phi_0$  give the location of the minimum and the curvature of the dispersion relation. Keeping each branch

contribution to its leading order at low temperature, as in reference [28], and using the experimental values [9, 30]  $\xi = 0.376$ ,  $\Delta = 0.44\epsilon_F$ ,  $\epsilon_0 = 0.85\epsilon_F$  and the theoretical value [29]  $\phi_0 = 0.846\epsilon_F$ , we find

$$\frac{N\hbar^2}{(t_{\text{br}}\epsilon_F)^2} \simeq \left(\frac{\theta}{0.296}\right)^5 \frac{(1+2r)^2}{9(1+r)} \quad (20)$$

where  $\theta = T/T_F$  and  $r \simeq \left(\frac{0.316}{\theta}\right)^{9/2} e^{-0.44/\theta}$  gives the relative weight of the two excitation branches. This formula, plotted as a dashed line in figure 2, is an exact equivalent to  $t_{\text{br}}$  in the limit  $\theta \rightarrow 0$ . The good agreement with the experimental data has to be taken cautiously. If one treats the two branches to all order in  $k_B T$ , one gets an upward shift of  $t_{\text{br}}\epsilon_F/(\hbar N^{1/2})$  more or less constant and equal to 5 over the temperature range of figure 2.

*Conclusion* : We have presented the first microscopic theory of the thermal blurring of the phase of a condensate of pairs of fermions (11), revealing a ballistic blurring

and a subleading phase diffusion. The blurring time depends on the variance of the total energy of the gas, and on the derivative of the microcanonical chemical potential with respect to the energy. This relies crucially on the fact that the time derivative of the condensate phase is given by the chemical potential operator of the gas, see equation (7). We have derived this central relation in a fully microscopic way, including both the bosonic and the fermionic branches of excitation. Last, we have proposed a realistic experimental protocol to measure this blurring time, that we estimated to be tens of milliseconds for a coherent gas prepared in the unitary limit in the canonical ensemble.

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## Appendix A: More on the pair coherent state variational calculation

In a first stage, it is convenient to treat the real quantity  $n$  and the complex field  $\Lambda$  as independent variables, that is one does not fix the value of  $\|\Phi\|$ . To include interactions among the quasi-particles, one must go up to third order in  $\Lambda$  and  $\Lambda^*$ , so that

$$\mathcal{H}(\Phi, \Phi^*) = \mathcal{T}_0[n, \phi_0(N)] + \sum_{j=1}^3 \mathcal{T}_j[n, \phi_0(N)](\Lambda, \Lambda^*) + O(\|\Lambda\|^4) \quad (A1)$$

where the tensor  $\mathcal{T}_j$  is of rank  $j$  so that  $\mathcal{T}_j(\Lambda, \Lambda^*)$  is exactly of order  $j$  in  $\Lambda$  and  $\Lambda^*$ . It may be expressed in terms of the differential of order  $j$  of  $\mathcal{H}$  taken at  $(\Phi, \Phi^*) = (n^{1/2}\phi_0, n^{1/2}\phi_0)$  and restricted to the subspace orthogonal to  $(\phi_0, 0)$  and  $(0, \phi_0)$ . It does not depend on the phase  $\theta$  due to the  $U(1)$  symmetry. For a fixed total number of particles, the energy does not vary to first order around the minimizer so that  $\mathcal{T}_1[N/2, \phi_0(N)] = 0$ . This apparently not very inspiring equation is actually the famous gap equation when the system is spatially homogeneous. Furthermore, one can check that  $\partial_n \mathcal{T}_0[N/2, \phi_0(N)] = 2\mu_0(N)$  where  $\mu_0(N) = dE_0(N)/dN$  is the gas chemical potential at zero temperature,  $E_0(N) = \mathcal{T}_0[N/2, \phi_0(N)]$  being the ground state energy.

The phase and the modulus square of the amplitude of the field  $\Phi$  on the mode  $\phi_0$  are canonically conjugate variables, so that  $-\hbar d\theta/dt = \partial_n \mathcal{H}(\Phi, \Phi^*)$ . Once this derivative is taken in (A1) for fixed  $\Lambda$  and  $\Lambda^*$ , one can fix the norm of  $\Phi$  to the value  $(N/2)^{1/2}$  (that is the total particle number is fixed to  $N$ ), and eliminate  $n$  through the identity  $n = \|\Phi\|^2 - \|\Lambda\|^2$ ; the field  $\Lambda$  then remains the only dynamical variable of the problem. The resulting expression is useful up to order 2 in  $\Lambda, \Lambda^*$ :

$$-\hbar \frac{d\theta}{dt} = \partial_n \mathcal{T}_0[N/2, \phi_0(N)] - \|\Lambda\|^2 \partial_n^2 \mathcal{T}_0[N/2, \phi_0(N)] + \sum_{j=1}^2 \partial_n \mathcal{T}_j[N/2, \phi_0(N)](\Lambda, \Lambda^*) + O(\|\Lambda\|^3) \quad (A2)$$

The Hamiltonian that determines the evolution of  $\Lambda$  at fixed particle number is obtained by replacing  $n$  with  $N/2 - \|\Lambda\|^2$  in (A1) and by expanding the resulting expression up to order three in  $\Lambda, \Lambda^*$ :

$$\mathcal{H}_N(\Lambda, \Lambda^*) = E_0(N) + \tilde{\mathcal{T}}_2[N](\Lambda, \Lambda^*) + \mathcal{T}_3[N/2, \phi_0(N)](\Lambda, \Lambda^*) - \|\Lambda\|^2 \partial_n \mathcal{T}_1[N/2, \phi_0(N)](\Lambda, \Lambda^*) + O(\|\Lambda\|^4) \quad (A3)$$

where the quadratic form  $\tilde{\mathcal{T}}_2[N](\Lambda, \Lambda^*)$  is obtained by subtracting  $2\mu_0(N)\|\Lambda\|^2$  from  $\mathcal{T}_2[N/2, \phi_0(N)](\Lambda, \Lambda^*)$ . The trick is then to directly write the temporal derivative of the imaginary part of the component of the field  $\Lambda$  on the function  $(N/2)^{1/2} d\phi_0/dN$ ,

$$Y = \frac{b^6}{2i} \sum_{\mathbf{r}, \mathbf{r}'} \left(\frac{N}{2}\right)^{1/2} \frac{d\phi_0(\mathbf{r}, \mathbf{r}')}{dN} (\Lambda(\mathbf{r}, \mathbf{r}') - \Lambda^*(\mathbf{r}, \mathbf{r}')) \quad (A4)$$

Since  $i\hbar\partial_t\Lambda = b^{-6}\partial_{\Lambda^*}\mathcal{H}_N(\Lambda, \Lambda^*)$ , one gets

$$-2\hbar\frac{dY}{dt} = D \cdot \mathcal{H}_N(\Lambda, \Lambda^*) = D \cdot \check{\mathcal{T}}_2[N](\Lambda, \Lambda^*) + D \cdot \mathcal{H}_N^{\text{cub}}(\Lambda, \Lambda^*) + O(\|\Lambda\|^3) \quad (\text{A5})$$

where  $\mathcal{H}_N^{\text{cub}}$  is the component of  $\mathcal{H}_N$  of order three in  $\Lambda, \Lambda^*$ . We have introduced the differential operator

$$D = \sum_{\mathbf{r}, \mathbf{r}'} \left(\frac{N}{2}\right)^{1/2} \frac{d\phi_0(\mathbf{r}, \mathbf{r}')}{dN} (\partial_{\Lambda(\mathbf{r}, \mathbf{r}')} + \partial_{\Lambda^*(\mathbf{r}, \mathbf{r}')}) \quad (\text{A6})$$

We shall now take advantage of two identities that exactly hold for all  $\Lambda$  orthogonal to  $\phi_0$ :

$$\partial_n \mathcal{T}_1(\Lambda, \Lambda^*) + 2D \cdot \check{\mathcal{T}}_2[N](\Lambda, \Lambda^*) = 0 \quad (\text{A7})$$

$$2D \cdot \mathcal{H}_N^{\text{cub}}(\Lambda, \Lambda^*) = 2\frac{d}{dN}\check{\mathcal{T}}_2(\Lambda, \Lambda^*) - \partial_n \mathcal{T}_2(\Lambda, \Lambda^*) + \|\Lambda\|^2 \partial_n^2 \mathcal{T}_0 \quad (\text{A8})$$

where all the tensors are taken at  $[n = N/2, \phi_0(N)]$  or simply at  $N$  for  $\check{\mathcal{T}}_2$ . To prove these relations, one formally considers a field  $\Phi'$  with  $N + \delta N$  particles and determines in two different ways the quadratic expansion of  $\mathcal{H}_{N+\delta N}(\Lambda', \Lambda'^*)$  in powers of  $\Lambda'$  and  $\Lambda'^*$ , where  $\Lambda'$  is as in (17) (written for  $N + \delta N$  particles) the component of  $\Phi'$  orthogonal to  $\phi_0(N + \delta N)$ . First, one simply replaces  $N$  with  $N + \delta N$  and  $\Lambda$  with  $\Lambda'$  in (A3), and then expands to first order in  $\delta N$ . The tensor  $d\check{\mathcal{T}}_2/dN[N]$  naturally appears from this expansion. Second, one applies to  $\mathcal{H}(\Phi', \Phi'^*)$  the expansion (A1) around  $\Phi_0(N)$  and takes into account the fact that, to first order in  $\delta N$ , the component of  $\Phi'$  orthogonal to  $\phi_0(N)$  contains, in addition to  $\Lambda'_\perp$  coming from  $\Lambda'$ , a contribution coming from  $d\phi_0/dN$ :

$$\Lambda(\mathbf{r}, \mathbf{r}') = \delta N \left(\frac{N}{2}\right)^{1/2} \frac{d\phi_0}{dN}(\mathbf{r}, \mathbf{r}') + \Lambda'_\perp(\mathbf{r}, \mathbf{r}') + O(\delta N^2, \delta N \|\Lambda'\|, \|\Lambda'\|^2) \quad (\text{A9})$$

This infinitesimal shift proportional to  $\delta N$  along the direction of  $d\phi_0/dN$  is responsible for the occurrence of the operator  $D$ . Equations (A7) and (A8) are finally obtained by identification of the two resulting expressions of  $\mathcal{H}_{N+\delta N}(\Lambda', \Lambda'^*)$  respectively to first and second order in  $\Lambda'$  and  $\Lambda'^*$ .

At this stage most of the work has been done. It remains to combine equations (A2), (A5), (A7), and (A8) after a coarse grained temporal average (over a time scale much longer than the oscillation period of the modal amplitudes  $b_\alpha$  but much shorter than the evolution time of the quasi-particle numbers  $|b_\alpha|^2$ ), to obtain

$$-\frac{\hbar}{2} \overline{\frac{d\theta}{dt}}^t = \mu_0(N) + \overline{\frac{d\check{\mathcal{T}}_2}{dN}[N](\Lambda, \Lambda^*)}^t + O(\|\Lambda\|^3) \quad (\text{A10})$$

where we used the crucial property that  $\overline{d\Lambda/dt}^t$  vanishes (and so does  $\overline{dY/dt}^t$ ), since the range of variation of the field  $\Lambda$  is bounded. The quadratic form  $\check{\mathcal{T}}_2[N]$  is represented by the matrix  $\eta\mathcal{L}[N]$  with  $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , using a block notation and the scalar product  $\langle, \rangle$  generating the norm  $\| \cdot \|$ :

$$\check{\mathcal{T}}_2[N](\Lambda, \Lambda^*) = \frac{1}{2} \left\langle (\Lambda, \Lambda^*), \eta\mathcal{L}[N] \begin{pmatrix} \Lambda \\ \Lambda^* \end{pmatrix} \right\rangle \quad (\text{A11})$$

Then one inserts the modal decomposition (18) in the derivative with respect to  $N$  of the equation (A11); we recall that  $\epsilon_\alpha$  and  $(u_\alpha, v_\alpha)$ ,  $-\epsilon_\alpha$  and  $(v_\alpha^*, u_\alpha^*)$  are the eigenvalues and the eigenvectors of  $\mathcal{L}$ . The coarse grained temporal average  $\overline{\quad}^t$  gets rid of the crossed terms, and the Hellmann-Feynman theorem ensures that [33]  $\langle (u_\alpha^*, -v_\alpha^*), \frac{d\mathcal{L}[N]}{dN} \begin{pmatrix} u_\alpha \\ v_\alpha \end{pmatrix} \rangle = d\epsilon_\alpha/dN$  and finally leads to equation (19).

## Appendix B: Irrotational quantum hydrodynamics

To calculate the viscosity of superfluid helium at low temperature, Landau and Khalatnikov have developed in 1949 the theory of quantum hydrodynamics [22, 31]. It allows one to determine, to leading order in  $T$ , the effect of a non-zero temperature on the quantum fluid, at least on the observables that only involve low energy scales and large

length scales. Remarkably, the only specific ingredient is the zero-temperature equation of state of the fluid, which is here the energy per unit volume  $e_0(\rho)$  of the ground state of the spatially homogeneous system of density  $\rho$ .

To obtain the time derivative of the phase operator of the condensate of fermion pairs, we refine the theory in two ways: by regularizing ultraviolet divergences and by specializing to the irrotational case.

First, we solve the issue of the Landau-Khalatnikov Hamiltonian ground state energy, that diverges due to the zero-point motion of the system eigenmodes. We discretize the space in a cubic lattice of spacing  $b$ , a sub-multiple of the size  $L$  of the quantization volume, which is much smaller than the typical wavelength  $2\pi/q_{\text{th}}$  of the thermal excitations of the fluid but much larger than the mean interparticle distance  $\rho^{-1/3}$ ,

$$\rho^{-1/3} \ll b \ll q_{\text{th}}^{-1} \quad (\text{B1})$$

both conditions being compatible at sufficiently low temperature. This is in the spirit of the validity range of hydrodynamics, which relies on a spatial coarse graining, and it provides a natural cut-off for the wave vectors  $\mathbf{q}$  by restricting them to the first Brillouin zone [34]  $\mathcal{D} = [-\pi/b, \pi/b]^3$ . In the Hamiltonian one must then replace the differential operators such as the gradient, the divergence and the Laplacian, by their discrete versions, as we shall implicitly do below, and introduce the *bare* energy density  $e_{0,0}(\rho)$ , which depends on the lattice spacing  $b$ . Following the ideas of renormalization, the zero-point energy of the modes, that formally diverges when  $b \rightarrow 0$ , adds up to  $e_{0,0}(\rho)$  to exactly reconstruct the *effective* or *true* energy density  $e_0(\rho)$ , that does not depend on  $b$  and is what is measured experimentally.

Second, we specialize the theory to the case of an irrotational velocity field operator  $\hat{\mathbf{v}}(\mathbf{r}, t)$  that can then be written as the gradient of the phase field operator  $\hat{\phi}(\mathbf{r}, t)$ , itself canonically conjugate to the density field operator  $\hat{\rho}(\mathbf{r}, t)$ :

$$\hat{\mathbf{v}}(\mathbf{r}, t) = \frac{\hbar}{m} \text{grad} \hat{\phi}(\mathbf{r}, t) \quad \text{with} \quad [\hat{\rho}(\mathbf{r}, t), \hat{\phi}(\mathbf{r}', t)] = i \frac{\delta_{\mathbf{r}, \mathbf{r}'}}{b^3} \quad (\text{B2})$$

This amounts to neglecting the transverse component of the field  $\hat{\mathbf{v}}(\mathbf{r}, t)$ , as done in reference [22] to determine the phonon-phonon interaction and go forward in the computation of viscosity. In the particular scale invariant case of the unitary Fermi gas, this was justified within the effective field theory in reference [32]. We note *en passant* that the density of fermionic quasi-particles is exponentially small in  $1/T$  and is directly omitted by hydrodynamic theory.

The steps that follow are rather usual. One starts from the equations of motion of the fields in Heisenberg picture, that is the quantum continuity equation and the quantum Euler equation for the potential (whose gradient gives the quantum Euler equation for the velocity):

$$\partial_t \hat{\rho} + \text{div} \left[ \frac{1}{2} \{ \hat{\rho}, \hat{\mathbf{v}} \} \right] = 0 \quad (\text{B3})$$

$$\hbar \partial_t \hat{\phi} = -\frac{1}{2} m \hat{\mathbf{v}}^2 - \mu_{0,0}(\hat{\rho}) \quad (\text{B4})$$

where  $\{ \hat{A}, \hat{B} \} = \hat{A}\hat{B} + \hat{B}\hat{A}$  is the anticommutator of two operators and

$$\mu_{0,0}(\rho) = \frac{d}{d\rho} e_{0,0}(\rho) = e'_{0,0}(\rho) \quad (\text{B5})$$

is the bare ground state chemical potential at density  $\rho$ . The quantum spatial density and phase fluctuations are weak provided that  $\rho^{1/3}b$  is large enough; the thermal ones are weak if in addition  $q_{\text{th}}b$  is small enough [35]. Under these conditions one can linearize as in [22] the equations of motion around the spatially uniform solution:

$$\hat{\rho}(\mathbf{r}, t) = \hat{\rho}_0 + \delta\hat{\rho}(\mathbf{r}, t) \quad (\text{B6})$$

$$\hat{\phi}(\mathbf{r}, t) = \hat{\phi}_0(t) + \delta\hat{\phi}(\mathbf{r}, t) \quad (\text{B7})$$

The operator  $\hat{\rho}_0$  reduces to  $\hat{N}/L^3$ , where  $\hat{N}$  is the operator giving the total number of particles, and is a constant of motion. The operator  $\hat{\phi}_0$  is the phase operator of the condensate; one has here

$$\hat{\phi}_0 = \hat{\theta}_0/2 \quad (\text{B8})$$

since the phase operator  $\hat{\theta}_0$  in equation (2) takes the pairs as the building block, whereas equations (B3,B4) are build on the fermionic particles. The spatial fluctuations  $\delta\hat{\rho}$  and  $\delta\hat{\phi}$ , of vanishing (discrete) integral over the whole space,



can be expanded on the plane waves of non-zero wave vector  $\mathbf{q}$ , and commute with  $\hat{\rho}_0$ . One solves the linearized equations for  $\delta\hat{\rho}$  and  $\delta\hat{\phi}$  and one can use the usual expansion on eigenmodes:

$$\delta\hat{\rho}(\mathbf{r}, t) = \frac{\hat{\rho}_0^{1/2}}{L^{3/2}} \sum_{\mathbf{q} \in \frac{2\pi}{L}\mathbb{Z}^{3*} \cap \mathcal{D}} \left( \frac{\hbar q}{2m\hat{c}_{0,0}} \right)^{1/2} (\hat{B}_{\mathbf{q}} + \hat{B}_{-\mathbf{q}}^\dagger) e^{i\mathbf{q}\cdot\mathbf{r}} \quad (\text{B9})$$

$$\delta\hat{\phi}(\mathbf{r}, t) = \frac{-i}{\hat{\rho}_0^{1/2} L^{3/2}} \sum_{\mathbf{q} \in \frac{2\pi}{L}\mathbb{Z}^{3*} \cap \mathcal{D}} \left( \frac{m\hat{c}_{0,0}}{2\hbar q} \right)^{1/2} (\hat{B}_{\mathbf{q}} - \hat{B}_{-\mathbf{q}}^\dagger) e^{i\mathbf{q}\cdot\mathbf{r}} \quad (\text{B10})$$

where the creation operator  $\hat{B}_{\mathbf{q}}^\dagger$  and the annihilation operator  $\hat{B}_{\mathbf{q}}$  of a phonon with wave vector  $\mathbf{q}$  and energy  $\hbar q \hat{c}_{0,0}$  obey bosonic commutation relations  $[\hat{B}_{\mathbf{q}}, \hat{B}_{\mathbf{q}'}^\dagger] = \delta_{\mathbf{q}, \mathbf{q}'}$  and where we introduced the zero-temperature bare sound velocity operator

$$\hat{c}_{0,0} \equiv \left( \frac{\hat{\rho}_0 \mu'_{0,0}(\hat{\rho}_0)}{m} \right)^{1/2} \quad (\text{B11})$$

It remains to expand the right-hand side of (B4) up to second order in  $\delta\hat{\rho}$  and  $\delta\hat{\phi}$ , to extract the zero wave vector Fourier component, to perform a coarse grained temporal average to get rid of the oscillating crossed terms  $\hat{B}_{\mathbf{q}}\hat{B}_{-\mathbf{q}}$  and  $\hat{B}_{-\mathbf{q}}^\dagger\hat{B}_{\mathbf{q}}^\dagger$ , and to use the identity

$$\frac{d}{d\rho} [\rho \mu'_{0,0}(\rho)]^{1/2} = \frac{\mu'_{0,0}(\rho) + \rho \mu''_{0,0}(\rho)}{2[\rho \mu'_{0,0}(\rho)]^{1/2}} \quad (\text{B12})$$

to obtain

$$\overline{\frac{d}{dt} \hat{\phi}_0}^t = -\mu_{0,0}(\hat{\rho}_0) - \sum_{\mathbf{q} \in \frac{2\pi}{L}\mathbb{Z}^{3*} \cap \mathcal{D}} \left( \hbar q \frac{d}{dN} \hat{c}_{0,0} \right) \left( \hat{B}_{\mathbf{q}}^\dagger \hat{B}_{\mathbf{q}} + \frac{1}{2} \right) \quad (\text{B13})$$

At this order of the expansion, one can collect in (B13) the zero-point contribution of the modes (the term  $1/2$  in between parentheses) and the bare chemical potential  $\mu_{0,0}(\hat{\rho}_0)$  to form the true chemical potential  $\mu_0(\hat{\rho}_0)$  of the fluid at zero temperature, and one can identify  $\hat{c}_{0,0}$  in the prefactor of the phonon number operator  $\hat{B}_{\mathbf{q}}^\dagger \hat{B}_{\mathbf{q}}$  with the true sound velocity at zero temperature,  $\hat{c}_0 \equiv [\hat{\rho}_0 \mu'_{0,0}(\hat{\rho}_0)/m]^{1/2}$ . One then obtains the (low temperature) phononic limit of relation (7), without any constraint on the interaction strength.

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- [33]  $(u_\alpha, -v_\alpha)$  is the dual vector of  $(u_\alpha, v_\alpha)$  since  $(\eta\mathcal{L})^\dagger = \eta\mathcal{L}$ .
- [34] We also require that  $mcb/\hbar > 1$ , where  $c$  is the  $T = 0$  sound velocity, so that the elementary excitations of the system remain phononic over the whole domain  $\mathcal{D}$ . This condition results from the already assumed one,  $b\rho^{1/3} \gg 1$ , in the strongly interacting regime or in the BCS limit, since  $c$  is then of the order of the Fermi velocity. It must be added explicitly in the so-called BEC limit, where the fermion pairs form a weakly interacting condensate.
- [35] One first checks that the density fluctuations on a given lattice site are small in relative value, using (B9). At  $T = 0$ ,  $\langle \delta\rho^2 \rangle / \rho^2 \approx \frac{\hbar}{mcb} \frac{1}{\rho b^3} \ll 1$ , using (B1) and the note [34]. At  $T > 0$ , there is the additional thermal contribution  $\approx \frac{\hbar}{mcb} \frac{(k_{\text{th}} b)^4}{\rho b^3}$  which is  $\ll 1$  for the same reasons. Second, one checks that the phase fluctuations between two neighboring lattice sites are small in absolute value. To this end, one notes from (B10) that  $b^2 \langle (\mathbf{grad} \delta\theta)^2 \rangle = (mcb/\hbar)^2 \langle \delta\rho^2 \rangle / \rho^2$ . To conclude, it remains to use  $mc/(\hbar\rho^{1/3}) = O(1)$ , a property that holds in the whole BEC-BCS crossover, as well as the previous estimates of  $\langle \delta\rho^2 \rangle / \rho^2$ .