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► **To cite this version:**

Cz. Wozniak, B. Michalak, M. Wozniak. Modelling of composites with a space-varying microstructure. PAMM, 2006, 6, pp.417-418. hal-01118090

HAL Id: hal-01118090

<https://hal.science/hal-01118090>

Submitted on 18 Feb 2015

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Modelling of composites with a space-varying microstructure

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The object of analysis is a microheterogeneous rigid heat conductor with apparent properties smoothly varying along the preferred direction. The aim of this contribution is to formulate two macroscopic models of this conductor.

Composite structural elements in most cases have the uniform microstructure. It means that on the macroscopic level they can be treated as homogeneous. In deterministic descriptions microstructure mentioned above is periodic. Hence, effective properties of composites with the uniform microstructure are constant in space and related macroscopic models are represented by PDEs with constant coefficients.

However, there exist situations in which composite microstructure is smoothly varying in space. These situations are typical for functionally graded materials (cf. Suresh and Mortensen, 1998) where effective properties are graded in a certain direction. The corresponding macroscopic models are described by PDEs with smooth and slowly varying coefficients.

The main aim of this contribution is to formulate a deterministic macroscopic model of the heat transfer in two-phase rigid conductors which is free of aforementioned drawbacks. For the sake of simplicity we restrict considerations to the plane problems. Setting $\xi \equiv \xi^1, \eta \equiv \xi^2, \partial_1 = \partial/\partial\xi, \partial_2 = \partial/\partial\eta, \nabla \equiv (\partial_1, \partial_2)$ and $\bar{\partial} \equiv (0, \partial_2), \partial \equiv (\partial_1, 0)$. Let assume that the composites have λ -periodic structure along the ξ^1 -coordinate and smoothly graded effective properties along the ξ^2 -coordinate.

It follows that heat conduction tensor \mathbf{K} and specific heat c satisfy conditions

$$\mathbf{K}(\xi, \eta) = \mathbf{K}(\xi + \lambda, \eta), \quad c(\xi, \eta) = c(\xi + \lambda, \eta) \quad (1)$$

in the whole domain of the definition in formulae (1). The generalized period λ of inhomogeneity is assumed to be sufficiently small when compared to the measure of the domain of argument ξ . The example of the above space-varying microstructure is shown in Fig.1.

Fig. 1 A vchfigure environment with a vchcaption. Figure and caption are leftindented.

The modeling will be investigated in the framework of the Fourier heat conduction equation in the form

$$c\dot{\theta} - \nabla \cdot (\mathbf{K} \cdot \nabla \theta) = 0 \quad (2)$$

where $\theta = \theta(\xi, \eta, t)$ is temperature at (ξ, η) and time t . Functions $\mathbf{K}(\cdot), c(\cdot)$ satisfy conditions (1) in the known orthogonal coordinates ξ, η . Let $\mathbf{e}_\alpha(\xi, \eta), \alpha = 1, 2$, stand for the orthogonal vector basis assigned to an arbitrary (ξ, η) . We also state that the composite has a curvilinear space-varying microstructure along ξ -direction. It means that increments of vector fields $\mathbf{e}_\alpha(\cdot, \eta)$ in every $(\xi, \xi + \lambda) \subset I$ can be neglected.

Let $F = F(\xi, \eta)$ for every η, t be an arbitrary integrable function of $\xi \subset I$. Setting

$$\langle F \rangle = \frac{1}{\lambda} \int_{-\lambda/2}^{\lambda/2} F(\xi + \zeta, \eta, t) d\zeta$$

we shall refer $\langle F \rangle$ to as the averaging of F . Moreover, let $g(\cdot, \eta)$ for every η be the known saw-like function which is continuous, λ -periodic in I , linear in every subinterval of I belonging to the fixed phase of the composite and satisfying conditions $\langle g \rangle = 0$.

Following the tolerance averaging technique detailed in Woźniak, Wierzbicki, 2000, for periodic composites, we shall formulate two heuristic assumptions.

Firstly, considerations are restricted to temperature field θ given by

$$\theta(\xi, \eta, t) = \vartheta(\xi, \eta, t) + g(\xi, \eta)\psi(\xi, \eta, t) \quad (3)$$

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where $\vartheta(\cdot, \eta, t)$, $\psi(\cdot, \eta, t)$ are slowly-varying in every $(\xi, \xi + \lambda) \subset I$

$$\vartheta(\cdot, \eta, t) \in SV_\varepsilon(\lambda), \psi(\cdot, \eta, t) \in SV_\varepsilon(\lambda) \quad (4)$$

Here ε represents a certain tolerance parameter, $0 < \varepsilon \ll 1$ cf. Zeeman, 1965, Fichera, 1992 (where the tolerance parameter was referred to as the upper bound for negligibles) and Woźniak, Wierzbicki, 2000.

Secondly, it is assumed that in calculations of averages $\langle \cdot \rangle$ involving slowly-varying functions terms $O(\varepsilon)$ can be neglected. Under this assumption $\langle \theta \rangle = \vartheta$. Hence formulae (4), (5) represent the decomposition of temperature into the averaged and the fluctuation parts, respectively. That is why ψ will be referred to as temperature fluctuation amplitude. Fields $\vartheta(\cdot)$ and $\psi(\cdot)$ are basic unknowns.

Let L stand for the left-hand side of equation (1) combined with formulae (3). The governing equations for ϑ, ψ can be derived from conditions $\langle L \rangle = 0$, $\langle gL \rangle = 0$. Let c', \mathbf{K}' and c'', \mathbf{K}'' be constants specifying material properties of both phases of the composite conductor. The volume fractions of these components will be denoted by $\nu' = \nu'(\xi, \eta)$, $\nu'' = \nu''(\xi, \eta)$.

Setting

$$[\mathbf{k}] \equiv 2\sqrt{3}(\mathbf{K}' - \mathbf{K}'') \cdot \mathbf{e}_1 \quad \{k\} \equiv 12 \mathbf{e}_1 \cdot (\nu' \mathbf{K}'' + \nu'' \mathbf{K}') \cdot \mathbf{e}_1 \quad \nu \equiv \sqrt{\nu' \nu''}$$

and applying the first and the second modeling assumption, after rather lengthy manipulations, we obtain finally

$$\begin{aligned} \langle c \rangle \dot{\vartheta} - \nabla \cdot (\langle \mathbf{K} \rangle \cdot \nabla \vartheta + [\mathbf{k}] \psi) &= 0 \\ \nu^2 \lambda^2 (\langle c \rangle \dot{\psi} - \bar{\partial} \cdot (\langle \mathbf{K} \rangle \cdot \bar{\partial} \psi)) + \{k\} \psi + [\mathbf{k}] \cdot \partial \vartheta &= 0 \end{aligned} \quad (5)$$

Model equations (5) together with formulae (3), (4) represent the macroscopic model of the rigid heat conductor under consideration. It has to be emphasised that equations (5), in contrast to equations (2), have smoothly varying functional coefficients and hence can be used as a tool for the analysis of specific problems.

The characteristic feature of the model is the presence of generalized microstructure period λ in equations (5). By the formal limit passage $\lambda \rightarrow 0$ we derive a certain approximate form of equations (5). After denotation

$$\mathbf{K}^o = \langle \mathbf{K} \rangle - \{k\}^{-1} [\mathbf{k}] \otimes [\mathbf{k}]$$

from equation (5) we get

$$\begin{aligned} \langle c \rangle \dot{\vartheta} - \nabla \cdot (\mathbf{K}^o \cdot \nabla \vartheta) &= 0 \\ \psi - \{k\}^{-1} [\mathbf{k}] \cdot \partial \vartheta &= 0 \end{aligned} \quad (6)$$

Equations (6) together with formulae (3), (4) represent the asymptotic model of the heat conductor under consideration.

The above models will be applied to the analysis of some specific problems which will be presented in the separate contribution.

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