



HAL
open science

LOCAL SMOOTHING ESTIMATES FOR THE MASSLESS DIRAC-COULOMB EQUATION

Federico Cacciafesta, Eric Séré

► **To cite this version:**

Federico Cacciafesta, Eric Séré. LOCAL SMOOTHING ESTIMATES FOR THE MASSLESS DIRAC-COULOMB EQUATION. 2015. hal-01117768v1

HAL Id: hal-01117768

<https://hal.science/hal-01117768v1>

Preprint submitted on 3 Mar 2015 (v1), last revised 11 Dec 2015 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

LOCAL SMOOTHING ESTIMATES FOR THE MASSLESS DIRAC-COULOMB EQUATION

FEDERICO CACCIAFESTA AND ÉRIC SÉRÉ

ABSTRACT. We prove local smoothing estimates for the massless 3D Dirac equation with a Coulomb potential. Our strategy is inspired by [9] and relies on partial wave subspaces decomposition and spectral analysis of the Dirac-Coulomb operator.

1. INTRODUCTION AND GENERALITIES

The 3D massless Dirac equation with an electric Coulomb potential reads as

$$(1.1) \quad \begin{cases} iu_t + \mathcal{D}u + \frac{\nu}{|x|}u = 0, & u(t, x) : \mathbb{R}_t \times \mathbb{R}_x^3 \rightarrow \mathbb{C}^4 \\ u(0, x) = f(x) \end{cases}$$

where \mathcal{D} is the 3D massless Dirac operator, i.e.

$$\mathcal{D} = -i \sum_{k=1}^3 \alpha_k \partial_k = -i(\alpha \cdot \nabla)$$

and the 4×4 Dirac matrices can be written as

$$(1.2) \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3$$

in terms of the Pauli matrices

$$(1.3) \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Dirac equation represents one of the most important and investigated models in quantum mechanics, and it has been widely and successfully used in physics to describe relativistic particles having a spin-1/2 (see e.g. [22], [28]). From a mathematical viewpoint, there have been many results in the recent years on the stationary (massive) Dirac equation (we refer to the recent survey [17]). On the other hand, the state-of-the-art understanding of its dynamics is still far from being satisfactory.

The α_j matrices satisfy the following anticommutation relations

$$\alpha_i \alpha_k + \alpha_k \alpha_i = 2\delta_{ik} \mathbb{I}_4, \quad i, k = 1, 2, 3,$$

which imply that $\mathcal{D}^2 = -\Delta$ (we recall that the spectrum of \mathcal{D} is the whole line \mathbb{R}); in particular, this means that

$$(i\partial_t - \mathcal{D})(i\partial_t + \mathcal{D}) = (\Delta - \partial_{tt}^2)I_4,$$

Date: March 3, 2015.

which strictly links the dynamics of the *free* massless Dirac equation to a system of wave equations. Therefore, dynamical properties for the free Dirac equation can easily be derived from their wave counterparts: dispersive properties of the flow are thus mainly encoded in the celebrated family of *Strichartz estimates*, which are given by

$$(1.4) \quad \|e^{it\mathcal{D}} f\|_{L_t^p \dot{H}_x^{\frac{1}{q} - \frac{1}{p} - \frac{1}{2}}} \lesssim \|f\|_{L^2}$$

where the exponents (p, q) are wave admissible, i.e. satisfy

$$\frac{2}{p} + \frac{2}{q} = 1, \quad 2 < p \leq \infty, \quad 2 \leq q < \infty$$

and $e^{it\mathcal{D}}$ represents the propagator for the solutions to (3.2). We stress the fact that the estimate corresponding to the couple $(p, q) = (2, \infty)$, the so called *endpoint*, that is

$$(1.5) \quad \|e^{it\mathcal{D}} f\|_{L_t^2 L_x^\infty} \lesssim \|f\|_{\dot{H}^1}$$

is known to fail as the corresponding one for the 3D wave flow. In [23] the authors proved a refined version of this estimate involving angular spaces, and this allowed them to prove well-posedness for the celebrated cubic nonlinear Dirac equation assuming initial data in H^1 with slight additional regularity on the angular variable. The problem was afterwards completely solved in [5].

Another family of *a priori* estimates encoding informations about dispersion are the so called *local-smoothing type estimates*, for which different forms are available: to the best of our knowledge the most general one in this setting is the following

$$(1.6) \quad \|w_\sigma^{-1/2} e^{it\mathcal{D}} f\|_{L_t^2 L_x^2} \leq C \|f\|_{L^2}$$

where $w_\sigma(x) = |x|(1 + |\log|x||)^\sigma$, $\sigma > 1$. (Here and in the following we shall use the notation \dot{H}^s for the homogeneous Sobolev space with the norm $\|f\|_{\dot{H}^s} = \| |D|^s f \|_{L^2}$ where $|D| = (-\Delta)^{1/2}$, and $L_t^p L_x^q = L^p(\mathbb{R}_t; L^q(\mathbb{R}_x^n))$ for the mixed space-time Strichartz spaces). Both estimates (1.4) and (1.6) can be found in [14] (see also [16], [7]).

In recent years a lot of effort has been spent in order to investigate what happens to dispersive flows when potential perturbations come into play. In [14] the authors proved that both (1.4) and (1.6) hold for the flow $e^{it(D+V)}$ under the assumption

$$|V(x)| \leq \frac{\delta}{w_\sigma(x)}, \quad \sigma > 1$$

for some δ sufficiently small, which by the way is a sufficient condition to guarantee that the operator $\mathcal{D} + V$ is selfadjoint (see [28]). Similar results are available for magnetic potentials as well (see e.g. [8], [13]). In [11], [12] the authors generalized the angular endpoint estimates proved in [23] to small potential perturbations by introducing some new mixed Strichartz-smoothing estimates.

At the same time, some effort has been spent in order to find examples of potentials such that the corresponding flow does *not* disperse (in the sense we have discussed above). This problem has been tackled for the magnetic Dirac equation in [3], essentially showing that for vector potentials of the form $A(x) \cong |x|^{-\delta} Mx$ with $\delta \in (1, 2)$ most of the mass of the solution is localized around a non-dispersive function. Other previous results in this direction include [20] for the Schrödinger equation. These arguments suggest that, heuristically, the degree of homogeneity of the operator works as a threshold for

the validity of dispersive estimates: therefore, for the Dirac equation, the Coulomb potential is to be thought of as a *critical* case (we stress the fact that the restriction to the massless case seems unavoidable now). In the very last years the problem of understanding dispersive estimates for scaling invariant potentials has been approached for other dynamics: in [9] and [10] the authors have proved indeed that Morawetz and Strichartz estimates can be recovered for both the Schrödinger and wave equations with inverse square potentials, and in [18] the $L^1 \rightarrow L^\infty$ time decay estimate is proved in the electromagnetic case.

The aim of this paper is to try to adapt the techniques of [9] to the massless Dirac-Coulomb model, and to prove local smoothing estimates for solutions to (3.2). The strategy of proof of [9] is fairly straightforward: the basic idea is to use a spherical harmonics decomposition to reduce the equation to a radial one and then rely on the Hankel transform to diagonalize it. Several problems arise when trying to adapt this strategy to the Dirac setting. The first difficulty is given by the fact that the Dirac operator does not preserve radiality, and therefore one needs a more complex, but still classical, spherical harmonics decomposition related to the SU_2 group action (see [28]). Another issue is that we cannot use the Hankel transform, and we have to build up a relativistic analogue. To do this, we need to study the continuous spectrum of the Dirac-Coulomb model. The corresponding (approximated) eigenfunctions are well known (see for instance [22]), but their rich structure will raise some additional technical difficulties. We stress the fact that, to the best of our knowledge, this is the first result concerning dispersive properties of Dirac-Coulomb, an operator which plays a fundamental role in relativistic quantum chemistry. We hope that our result will be useful in the study of the dynamics of nonlinear models involving, for instance, several relativistic electrons in a molecule.

Before stating our main Theorem, we introduce some notations that will be used throughout the paper.

Notation We will indicate with L^p and H^s , respectively, the standard Lebesgue and Sobolev spaces. The Dirac operator with the Coulomb potential will be

$$\mathcal{D}_\nu = \mathcal{D} + \frac{\nu}{|x|};$$

we will need the fractional powers of this operator as well, that we will denote with $|\mathcal{D}_\nu|^\alpha$, that will be precisely defined the next section (see Proposition 2.3). With Ω^s we will denote the operator

$$(\Omega^s \phi)(x) = |x|^s \phi(x)$$

and with a little abuse of notation we will use the same symbol to indicate the operators which are pointwise equal for all times,

$$(\Omega^s \psi)(t, x) = |x|^s \psi(t, x).$$

Moreover, for any $j = 1/2 + n$ for $n \in \mathbb{N}$ we will denote with $\mathcal{H}_{\geq j}$ the direct sum

$$\mathcal{H}_{\geq j} = \bigoplus_{m_j, k_j} h_{j, m_j, k_j}$$

where the h spaces are defined in details in the next section.

We are now ready to state our main result.

Theorem 1.1. *Let u be solution of (3.2) and let $j = 1/2 + n$ for $n \in \mathbb{N}$. Then for any $1/2 < \alpha < \sqrt{(j + 1/2)^2 - \nu^2} + 1/2$ and any $f \in L^2((0, \infty)) \otimes \mathcal{H}_{\geq j}$ there exists a constant $C = C(\nu, \alpha, k)$ such that the following estimate holds*

$$(1.7) \quad \|\Omega^{-\alpha} |\mathcal{D}_\nu|^{1/2 - \alpha} u\|_{L_t^2 L_x^2} \leq C \|f\|_{L^2}.$$

Remark 1.1. The essential self-adjointness of the Dirac operator with a Coulomb potential is in fact guaranteed if $|\nu| \leq \frac{\sqrt{3}}{2}$. On the other hand, in the interval $\frac{\sqrt{3}}{2} < |\nu| < 1$ it is anyway possible to build a distinguished selfadjoint extension. We refer to [4] and references therein for a detailed discussion of the topic in a more general setting. We stress here the fact that, for ν approaching 1, the range of admissible exponents when $j = 1/2$ for our estimate (1.7) shrinks to zero.

Remark 1.2. The endpoint case $\alpha = 1/2$ in (1.7) would permit to recover with a standard argument the full set of Strichartz estimates for the perturbed flow (see e.g. [9]); even though we don't have a concrete counterexample, this estimate seems to fail. Forthcoming remark 3.1 gives more details on the issue.

The plan of the paper is the following: in section 2 we build the setup, reviewing the theory of partial wave subspaces, the spectrum of the Dirac-Coulomb operator and defining our analogue of the Hankel transform, while section 3 is devoted to the proof of Theorem (1.1). An appendix containing some generalities on special functions is included at the end.

Acknowledgments. The first named author is supported by the FIRB 2012 "Dispersive dynamics, Fourier analysis and variational methods". The second named author acknowledges support from the ANR "NoNAP" of the french ministry of research.

2. THE SETUP.

We devote this section to recall some classical facts about the radial Dirac operator and his spectrum, including a short discussion on how to explicitly calculate the eigenstates of the continuous spectrum. We refer to [22] and [28] for many more informations on the topic. Also, we construct an integral transform that will be crucial in our proof.

2.1. Partial wave decomposition and continuous spectrum. In the following proposition we collect some classical facts about the so called *partial wave decomposition*

Proposition 2.1. *There is a unitary isomorphism between the Hilbert spaces $L^2(\mathbb{R}^3)^4$ and $L^2((0, \infty)) \otimes h_{j, m_j, k_j}$ where the indices vary as*

$$j = \frac{1}{2}, \frac{3}{2}, \dots, \quad m_j = -j, -j + 1, \dots, +j, \quad k_j = \pm(j + 1/2).$$

where each 2-dimensional space h_{j, m_j, k_j} is spanned by the couple of orthogonal angular vectors Ξ_{m_j, k_j}^\pm where

$$\Xi_{m_j, \mp(j+1/2)}^+ = \begin{pmatrix} i\Omega_{j \mp 1/2}^{m_j} \\ 0 \end{pmatrix}, \quad \Xi_{m_j, \mp(j+1/2)}^- = \begin{pmatrix} 0 \\ \Omega_{j \pm 1/2}^{m_j} \end{pmatrix}$$

$$\Omega_{j-1/2}^{m_j} = \frac{1}{\sqrt{2j}} \begin{pmatrix} \sqrt{j+m_j} Y_{j-1/2}^{m_j-1/2} \\ \sqrt{j-m_j} Y_{j-1/2}^{m_j+1/2} \end{pmatrix}$$

$$\Omega_{j+1/2}^{m_j} = \frac{1}{\sqrt{2j+2}} \begin{pmatrix} \sqrt{j+1-m_j} Y_{j+1/2}^{m_j-1/2} \\ -\sqrt{j+1+m_j} Y_{j+1/2}^{m_j+1/2} \end{pmatrix}$$

and Y_l^m are the usual spherical harmonics. The isomorphism is explicit by the expansion

$$(2.1) \quad \Phi(x) = \sum_{j,m_j,k_j} f_{m_j,k_j}(r) \Xi_{m_j,k_j}^+(\theta, \phi) + g_{m_j,k_j}(r) \Xi_{m_j,k_j}^-(\theta, \phi)$$

$$= \sum_{j,m_j,k_j} \begin{pmatrix} i f_{m_j,k_j}(r) \Omega_{j\mp 1/2}^{m_j}(\theta, \phi) \\ g_{m_j,k_j}(r) \Omega_{j\pm 1/2}^{m_j}(\theta, \phi) \end{pmatrix}$$

for any state $\Phi \in L^2(\mathbb{R}^3, \mathbb{C}^4)$.

Moreover, the Dirac-Coulomb operator leaves invariant the partial wave subspaces $C_0^\infty((0, \infty)) \otimes h_{j,m_j,k_j}$ and, with respect to the basis $\{\Xi_{m_j,k_j}^+, \Xi_{m_j,k_j}^-\}$ is represented by the radial matrix

$$(2.2) \quad d_{m_j,k_j} = \begin{pmatrix} \frac{\nu}{r} & -\frac{d}{dr} + \frac{k_j}{r} \\ \frac{d}{dr} + \frac{k_j}{r} & \frac{\nu}{r} \end{pmatrix}.$$

The Dirac-Coulomb operator $\mathcal{D} + \frac{\nu}{r}$ on $C_0^\infty(\mathbb{R}^3)^4$ is unitary equivalent to the direct sum of "partial wave Dirac operators" d_{m_j,k_j} ,

$$\mathcal{D} + \frac{\nu}{r} \cong \bigoplus_{j,m_j,k_j} d_{m_j,k_j},$$

and the restricted operators d_{m_j,k_j} are selfadjoint if and only if the operator $\mathcal{D} + \frac{\nu}{r}$ is selfadjoint.

Proof. See [28]. □

Remark 2.1. Notice that the radial Dirac operators as defined in (2.2) only depend on the quantum number k which in turns depends on j .

It is therefore natural to seek the wave function of the stationary states of (3.2) in the form

$$(2.3) \quad \Psi_\epsilon^{\nu,m_j,k_j}(r, \theta, \phi) = \begin{pmatrix} f_\epsilon^{\nu,m_j,k_j}(r) \Xi_{m_j,k_j}^+(\theta, \phi) \\ g_\epsilon^{\nu,m_j,k_j}(r) \Xi_{m_j,k_j}^-(\theta, \phi) \end{pmatrix}$$

For the rest of this section we fix the values of the parameters j , m_j , and we will omit the corresponding subscripts for the radial functions in order to keep notations light. Also, we will omit the dependance on j of the quantic number, simply writing k for k_j .

In view of (2.2) we have the following system for radial components of the continuous spectrum (we start from *positive values of ϵ*):

$$(2.4) \quad \begin{cases} f' + \frac{1+k}{r}f - (\epsilon - \frac{\nu}{r})g = 0 \\ g' + \frac{1-k}{r}g + (\epsilon - \frac{\nu}{r})f = 0. \end{cases}$$

We seek for the functions f and g in the form

$$f = \sqrt{\epsilon}e^{-\frac{1}{2}\rho}\rho^{\gamma-1}(Q_1 + Q_2),$$

$$g = -i\sqrt{\epsilon}e^{-\frac{1}{2}\rho}\rho^{\gamma-1}(Q_1 - Q_2),$$

with $\rho = -2i\epsilon r$. With this substitution and some algebraic manipulations we are led to the system

$$(2.5) \quad \rho Q_1'' + (2\gamma + 1 - \rho)Q_1' - (\gamma - i\nu)Q_1 = 0,$$

$$\rho Q_2'' + (2\gamma + 1 - \rho)Q_2' - (\gamma + 1 - i\nu)Q_2 = 0$$

which we recognize to be *confluent hypergeometric differential equations*, giving as solutions

$$(2.6) \quad f_\epsilon, g_\epsilon = C_{\gamma, \nu}(2\epsilon r)^{\gamma-1} \times \begin{Bmatrix} \text{Im} \\ \text{Re} \end{Bmatrix} \left\{ e^{i(\epsilon r + \xi)} {}_1F_1(\gamma - i\nu, 2\gamma + 1, -2\epsilon r) \right\}$$

where $\gamma = \sqrt{k^2 - \nu^2}$ where $e^{-2i\xi} = \frac{\gamma - i\nu}{k}$ is the phase shift and the constant

$$(2.7) \quad C_{\epsilon, \gamma, \nu} = \begin{Bmatrix} 1 \\ i \end{Bmatrix} \frac{4}{\sqrt{\pi}} e^{\frac{1}{2}\pi\nu} \frac{|\Gamma(\gamma + 1 + i\nu)|}{\Gamma(2\gamma + 1)},$$

where we mean that to f corresponds the choice $(+, \text{Im})$, while to g $(-, \text{Re})$. ${}_1F_1(a, b, x)$ here represents the *confluent hypergeometric function* (see the Appendix for definition and basic properties). Notice that with this choice the following normalization condition holds for (2.3)

$$(2.8) \quad \int_0^{+\infty} \overline{\Psi}_\epsilon(r) \Psi_{\epsilon'}(r) r^2 dr = \frac{\delta(\epsilon - \epsilon')}{\epsilon^2}$$

meaning with that that the function

$$K(r, r') = r^2 \int \epsilon^2 e^{-l\epsilon} \overline{\Psi}_\epsilon(r) \Psi_{\epsilon'}(r) d\epsilon$$

is a δ -approximation for $l \rightarrow 0$.

Remark 2.2. Due to our choice of normalization (2.8) we have that for every $\epsilon > 0$ $f_\epsilon(r) = f_r(\epsilon) = f(r\epsilon)$ and the same for g .

Remark 2.3. Note that the wave functions for repulsive fields are obtained by simply changing the sign of ν .

Remark 2.4. It is not difficult to show, using a charge conjugation argument or simply working on radial equations (2.4), that the corresponding formulas for the continuous spectrum for *negative* ϵ are understood by the relation

$$(2.9) \quad \Psi_{-\epsilon}^{\nu, m_j k_j}(r, \theta, \phi) = \begin{pmatrix} g_{\epsilon}^{-\nu, m_j, -k_j}(r) \Xi_{m_j, k_j}^+(\theta, \phi) \\ f_{\epsilon}^{-\nu, m_j, -k_j}(r) \Xi_{m_j, k_j}^-(\theta, \phi) \end{pmatrix}.$$

This allows us to use (2.6)-(2.7) to easily write down the the functions Ψ_{ϵ} for *any* value of $\epsilon \in \mathbb{R}$.

2.2. The integral transform. We now introduce the crucial integral transform, that consists in a projection on the continuous spectrum of the Dirac-Coulomb operator.

Definition 2.2. Let $\Phi_k \in L^2((0, \infty), dr) \otimes h_{j, m, k}$ for some fixed k and let $\varphi_k(r) = (\varphi_k^1(r), \varphi_k^2(r)) \in C_c^\infty(0, \infty)$ be the vector of its radial coordinates in decomposition (2.1). We define the following integral transform

$$(2.10) \quad \mathcal{H}_k \varphi_k(\epsilon) = \begin{pmatrix} \mathcal{H}_k^+ \varphi_k(\epsilon) \\ \mathcal{H}_k^- \varphi_k(\epsilon) \end{pmatrix} = \begin{pmatrix} \int_0^{+\infty} \Psi_{\epsilon}^{\nu, k}(r) \varphi_k(r) r^2 dr \\ \mathcal{C} \left(\int_0^{+\infty} \Psi_{-\epsilon}^{\nu, k}(r) \varphi_k(r) r^2 dr \right) \end{pmatrix} \\ = \int_0^{+\infty} H_k(\epsilon r) \cdot \varphi_k(r) r^2 dr$$

where we have introduced the matrix (see Remark 2.4)

$$(2.11) \quad H_k = \begin{pmatrix} f^{k, \nu}(\epsilon r) & g^{k, \nu}(\epsilon r) \\ g^{-k, -\nu}(\epsilon r) & f^{-k, -\nu}(\epsilon r) \end{pmatrix}$$

We collect in the following proposition some important properties of the operator \mathcal{H}_k .

Proposition 2.3. *For any fixed k and any $\Phi_k \in L^2((0, \infty), dr) \otimes h_{j, m, k}$ the following properties hold:*

- (1) \mathcal{H}_k is an L^2 -isometry.
- (2) $\mathcal{H}_k d_k = \Omega \mathcal{H}_k$.
- (3) The inverse transforms of \mathcal{H}_k is given by

$$(2.12) \quad \mathcal{H}_k^{-1} \varphi_k(r) = \int_0^{+\infty} H_k^*(\epsilon r) \cdot \varphi_k(\epsilon) \epsilon^2 d\epsilon$$

where $H_k^* = \begin{pmatrix} f^{k, \nu}(\epsilon r) & g^{-k, -\nu}(\epsilon r) \\ g^{k, \nu}(\epsilon r) & f^{-k, -\nu}(\epsilon r) \end{pmatrix}$ with the same notations as in Definition 2.2.

- (4) For every $\sigma \in \mathbb{R}$ we can define the fractional operators

$$(2.13) \quad A_k^\sigma \varphi_k(r) = \mathcal{H}_k \Omega^\sigma \mathcal{H}_k^{-1} \varphi_k(r) = \int_0^{+\infty} S_k^\sigma(r, s) \cdot \varphi_k(s) s^2 ds.$$

where the integral kernel $S_k(r, s)$ is a 2×2 matrix given by

$$(2.14) \quad S_k(r, s) = \int_0^{+\infty} H_k(\epsilon r) \cdot H_k^*(\epsilon s) \epsilon^{2+\alpha} d\epsilon$$

Remark 2.5. Property (1) allows, by standard arguments, to extend the definition of the operator \mathcal{H}_k to functions in L^2 .

Remark 2.6. When summing on k , Property (2.13) defines in a standard way fractional powers of the operator $|\mathcal{D}_\nu|$, which are used in the statement of Theorem (1.1).

Proof. Property (1) and (2) come from the definition of \mathcal{H}_k , once noticed that

$$\mathcal{H}_k^\pm(d_k\varphi_k) = \langle \Psi_{\pm\epsilon}^{\nu,k}, d_k\varphi_k \rangle = \langle d_k \Psi_{\pm\epsilon}^{\nu,k}, \varphi_k \rangle = \epsilon \langle \Psi_{\pm\epsilon}^{\nu,k}, \varphi_k \rangle$$

and from normalization relation (2.8). Property (3) is a consequence of (1) (H_k^* is the adjoint of \mathcal{H}_k).

To prove property (4) we use the definition of A_k to write

$$\begin{aligned} A_k^\sigma \varphi_k(r) &= \mathcal{H}_k \Omega^\sigma \mathcal{H}_k^{-1} \varphi_k(r) \\ &= \int_0^{+\infty} H_{k,\nu}(\epsilon r) \epsilon^{2+\sigma} \left(\int_0^{+\infty} H_{k,\nu}^*(\epsilon s) \varphi_k(s) s^2 ds \right) d\epsilon. \end{aligned}$$

Exchanging the order of the integrals yields (4). \square

3. PROOF OF THEOREM 1.1

We here prove our main result. Following [9], we rely on partial wave decomposition and on Proposition 2.3 to explicitly write the solutions on the single subspaces; then, a detailed and careful analysis of the kernel $S_k(r, s)$ of the operator A_k defined in (2.13) will be performed to sum back.

3.1. The strategy. We start with decomposition (2.1) and work on a single spherical space which will be identified by the triple $\{j, m_j, k_j\}$ with $j = \frac{1}{2}, \frac{3}{2}, \dots$, $m_j = -j, -j+1, \dots, +j$, $k_j = \pm(j+1/2)$ as in Proposition 2.1. Since, as already pointed out, the action of the Dirac-Coulomb operator on each of this subspace only depends on the quantic number k , we will omit the dependence on j and m_j and work on spinors Φ_k for $k \in \mathbb{Z} \setminus \{0\}$. We denote with $L_k f$ the solution to the initial value problem

$$(3.1) \quad \begin{cases} iu_t + d_k u = 0, \\ u(0, x) = f(x) \end{cases}$$

with d_k as in (2.2). We apply the operator \mathcal{H}_k , which is an L^2 -isometry to the LHS sides of estimate (1.7) and use Proposition 2.3 to obtain

$$\|\mathcal{H}_k \Omega^{-\alpha} |d_k|^{1/2-\alpha} L_k f\|_{L_t^2 L_x^2} = \|A_k^{-\alpha} \Omega^{1/2-\alpha} \mathcal{H}_k L_k f\|_{L_t^2 L_x^2} \leq C \|\mathcal{H}_k f\|_{L^2}$$

where we have used (2.13). Now, $\mathcal{H}_k L_k f$ solves (see (2) of Proposition 2.1)

$$(3.2) \quad \begin{cases} i\partial_t \mathcal{H}_k L_k f + \Omega \mathcal{H}_k L_k f = 0, \\ \mathcal{H}_k L_k f(0, \xi) = \mathcal{H}_k f(\xi), \end{cases}$$

so that the solution to this problem is given by

$$\mathcal{H}_k L_k f(t, \xi) = e^{it\xi} \mathcal{H}_k f(\xi).$$

Fourier transforming in time (since $\mathcal{F}_{t \rightarrow \tau}$ is an L^2 -isometry which commutes with A_k^σ) then gives

$$(\mathcal{F}_t \mathcal{H}_k L_k f)(\tau, \xi) = (\mathcal{H}_k f)(\rho) \delta(\tau + \xi).$$

We can then write

$$\begin{aligned} (A_k^{-\alpha} \Omega^{1/2-\alpha} \mathcal{F}_t \mathcal{H}_k L_k f)(\tau, \xi) &= \int_0^{+\infty} S_k^{-\alpha}(\xi, s) \delta(\tau + s) \mathcal{H}_k f(s) s^{2+1/2-\alpha} ds \\ &= -S_k^{-\alpha}(\xi, \tau) \mathcal{H}_k f(\tau) \tau^{5/2+\alpha}. \end{aligned}$$

In view of proving (1.7) we now need to calculate the L^2 norm in time and space of the quantity above which then gives

$$(3.3) \quad \int_0^{+\infty} \int_0^{+\infty} ((\mathcal{H}_k f)^*(\tau) S_k^{-\alpha}(\rho, \tau)^T) \cdot (S_k^{-\alpha}(\rho, \tau) (\mathcal{H}_k f)(\tau)) \tau^{5-2\alpha} \rho^2 d\rho.$$

Since $S_k^p(\rho, \tau)^T = S_k^p(\tau, \rho)$, the integral in $d\rho$ yields $S_k^{-2\alpha}(\tau, \tau)$ and we are therefore left with

$$(3.4) \quad \int_0^{+\infty} (\mathcal{H}_k f)^*(\tau) S_k^{-2\alpha}(\tau, \tau) (\mathcal{H}_k f)(\tau) \tau^{5-2\alpha} d\tau.$$

To conclude the proof of the estimate on the k -th space we need to bound (3.4) with the L^2 -norm of $\mathcal{H}_k f$ (which we recall to be the L^2 norm of f). The crucial and concluding step therefore relies in proving that

$$\begin{aligned} \int_0^{+\infty} (\mathcal{H}_k f)^*(\tau) S_k^{-2\alpha}(\tau, \tau) (\mathcal{H}_k f)(\tau) \tau^{5-2\alpha} d\tau &\leq \int_0^{+\infty} \text{Tr}(S_k^{-2\alpha}(\tau, \tau)) |\mathcal{H}_k f|^2(\tau) \tau^{5-2\alpha} d\tau \\ (3.5) \quad &\leq C_k \|\mathcal{H}_k f\|^2(\tau) \tau^2 d\tau \\ &= C_k \|f\|_{L^2}^2. \end{aligned}$$

with C_k a bounded constant of k , and therefore of γ . Expanding f in partial waves, and using the triangle inequality together with the orthogonality of spherical harmonics concludes the proof.

3.2. The main integral. The crucial step is thus to calculate

$$\begin{aligned} \text{Tr}(S^{-2\alpha}(\tau, \tau)) &= \text{Re} \left[\int_0^{+\infty} (f^{k,\nu}(\epsilon\tau) f^{k,\nu}(\epsilon\tau) + g^{k,\nu}(\epsilon\tau) g^{k,\nu}(\epsilon\tau) \epsilon^{2-2\alpha} d\epsilon) \right] \\ (3.6) \quad &+ \text{Re} \left[\int_0^{+\infty} (f^{-k,-\nu}(\epsilon\tau) f^{-k,-\nu}(\epsilon\tau) + g^{-k,-\nu}(\epsilon\tau) g^{-k,-\nu}(\epsilon\tau) \epsilon^{2-2\alpha} d\epsilon) \right] \\ &= \text{Re} [I_1(\tau) + I_2(\tau)]. \end{aligned}$$

More than giving an explicit solution of such integrals, which seems to be possible but significantly complicated, we are more interested in proving that

$$(3.7) \quad \text{Tr}(S^{-2\alpha}(\tau, \tau)) \cong \tilde{C}_k \tau^{-3+2\alpha}$$

where the constant \tilde{C}_k is bounded with respect to the parameter k . Recalling the structure of the eigenstates (2.6), we notice that integral (3.6) takes the form

$$\text{Tr}(S^{-2\alpha}(\tau, \tau)) = -\text{Re} \int Z_{k,\nu} \cdot Z_{k,\nu} \epsilon^{2-2\alpha} - \text{Re} \int Z_{-k,-\nu} \cdot Z_{-k,-\nu} \epsilon^{2-2\alpha}$$

where

$$Z_{k,\nu} = \frac{4}{\sqrt{\pi}} e^{\frac{1}{2}\pi\nu} \frac{|\Gamma(\gamma + 1 + i\nu)|}{\Gamma(2\gamma + 1)} (2\epsilon r)^{\gamma-1} e^{i(\epsilon r + \xi)} {}_1F_1(\gamma - i\nu, 2\gamma + 1, -2\epsilon r).$$

Therefore, integral I_1 reduces to the form

(3.8)

$$I_1(\tau) = C_{k,\nu} \tau^{2\gamma-2} \int_0^{+\infty} \epsilon^{2\gamma-2\alpha} e^{2i\epsilon\tau} {}_1F_1(\gamma - i\nu, 2\gamma + 1, -2i\epsilon\tau) {}_1F_1(\gamma - i\nu, 2\gamma + 1, -2i\epsilon\tau) d\epsilon$$

with

$$C_{k,\nu} = \frac{2^{2\gamma+2}}{\pi} e^{\pi\nu} \frac{|\Gamma(\gamma + 1 + i\nu)|^2}{\Gamma(2\gamma + 1)^2} e^{2i\xi}.$$

Notice that the only difference between I_1 and I_2 is encoded in the sign of ν since the quantic number k only appears squared into γ , and is virtually unessential.

Integrals of these forms have already been object of study in literature (see [21], [19], [27], [6], [1]) since they are connected with some different physical problems, as for instance matrix multipole elements for the Dirac Coulomb model, and they present some difficulties since they are not a priori convergent at infinity as well as in the origin (see (4.3)). Singularity at the origin can be dealt with by a standard integration by parts argument (as the one used to define Γ function for negative real parts) and is discussed in [27] provided $2\gamma - 2\alpha + 1$ is not zero or a negative integer, while singularity at infinity is slightly more difficult to be handled. To make things work, we need first of all to define $I_j(r, s)$ off-diagonal and then use a limiting procedure. Also, the introduction of a term of the form $e^{-\delta\epsilon}$ with $\delta \in \mathbb{R}^+$ in the integral is needed in order to make it convergent; we will now denote with I_j^δ this new integral:

$$I_j^\delta(r, s)$$

$$= C_{k,\nu} \tau^{2\gamma-2} \int_0^{+\infty} \epsilon^{2\gamma-2\alpha} e^{\epsilon(i(r+s)-\delta)} {}_1F_1(\gamma - i\nu, 2\gamma + 1, -2i\epsilon r) {}_1F_1(\gamma - i\nu, 2\gamma + 1, -2i\epsilon s) d\epsilon.$$

This modified integral can be explicitly computed, and its solution is written in terms of a double Appel series, which unfortunately turns to be not convergent for the values we are interested in. To deal with this problem, it will be necessary to introduce an analytic continuation of such a series: we don't include the proofs of these results here as they can be found in corresponding references. Since, as already noticed, the discussion of I_1 and I_2 is completely analogous, we will limit ourselves to I_1 .

We begin with the following

Lemma 3.1. *Let k, k' such that $|k| + |k'| < |h|$ and d not a negative integer nor zero. Then*

$$(3.9) \quad \int_0^{+\infty} t^{d-1} e^{-ht} F(a_1, b_1, kt) F(a_2, b_2, k't) dt = h^{-d} \Gamma(d) F_2(d; a_1, a_2; b_1, b_2; \frac{k}{h}, \frac{k'}{h})$$

where F_2 is defined in (4.5).

Proof. See Lemma 1 in [25]. □

We should now point out that the ray of convergence of $F(\alpha; a_1, a_2; b_1; b_2; x, y)$ is $|x| + |y| < 1$, which provides a first obstacle to the application of this result to our case. To overcome this difficulty, we rely on the following analytic continuation of F_2 .

Lemma 3.2. *If $\operatorname{Re}(b_1 - a_1) > 0$, $\operatorname{Re}(a_1) > 0$*

$$(3.10) \quad \begin{aligned} & F_2(d; a_1, a_2, b_1, b_2; 1, 1) \\ &= e^{-i\pi(b-a-d)} \frac{\Gamma(b_1)\Gamma(d-b_1+1)}{\Gamma(a_1)\Gamma(d-a_1+1)} {}_3F_2(d, b_2 - a_2, d - b_1 + 1; b_2, d - a_1 + 1; 1) \\ &+ e^{i\pi a} \frac{\Gamma(b_1)\Gamma(d-b_1+1)}{\Gamma(b_1-a_1)\Gamma(d+a_1+1-b_1)} {}_3F_2(d, a_2, d+1-b_1; b_2, d+a+1-b_1; 1) \end{aligned}$$

Proof. See [6]. □

For the sake of generality we have stated this Lemma with general values of the parameters; we list here for convenience the necessary substitutions to treat our case:

$$(3.11) \quad \begin{aligned} d &= 2\gamma + 1 - 2\alpha; \\ a_1 = a_2 &= \gamma - i\nu; \\ b_1 = b_2 &= 2\gamma + 1. \end{aligned}$$

Since necessary conditions of Lemma 3.2 are fulfilled with these values, this result allows us to pass to the limit $I_j^\delta(r, s)$ for $\delta \rightarrow 0$, $s \rightarrow r =: \tau$, and thus leaves us with

$$\begin{aligned} & I_1(\tau) \\ &= C_{k,\nu} \tau^{2\gamma-2} (2i\tau)^{-(2\gamma+1-2\alpha)} \Gamma(2\gamma+1-2\alpha) F_2(2\gamma+1-2\alpha; \gamma-i\nu, \gamma-i\nu; 2\gamma+1, 2\gamma+1; 1, 1) \end{aligned}$$

with

$$C_{k,\nu} = \frac{2^{2\gamma+2}}{\pi} e^{\pi\nu} \frac{|\Gamma(\gamma+1+i\nu)|^2}{\Gamma(2\gamma+1)^2} e^{2i\xi}.$$

Using now representation (3.10) for F_2 and plugging in values (3.11), we are led to

$$(3.12) \quad I_1(\tau) = \tau^{2\alpha-3} C_{k,\nu} (2i)^{-(2\gamma+1-2\alpha)} \Gamma(2\gamma+1-2\alpha) (A + B)$$

where

$$\begin{aligned} A &= e^{i\pi(\gamma-2\alpha-i\nu)} \frac{\Gamma(2\gamma+1)\Gamma(1-2\alpha)}{\Gamma(\gamma-i\nu)\Gamma(\gamma+2-2\alpha+i\nu)} \\ &\times \sum_m \frac{(2\gamma+1-2\alpha)_m (\gamma+1+i\nu)_m (1-2\alpha)_m}{(2\gamma+1)_m (\gamma+2-2\alpha+i\nu)_m m!}, \\ B &= e^{i\pi(\gamma-i\nu)} \frac{\Gamma(2\gamma+1)\Gamma(1-2\alpha)}{\Gamma(\gamma+1+i\nu)\Gamma(\gamma+1-2\alpha-i\nu)} \\ &\times \sum_m \frac{(2\gamma+1-2\alpha)_m (\gamma-i\nu)_m (1-2\alpha)_m}{(2\gamma+1)_m (\gamma+1-2\alpha-i\nu)_m m!}. \end{aligned}$$

Both the ${}_3F_2$ functions above turn to be convergent: we can indeed explicitly check convergence condition (4.2) to give in both cases $\alpha > 0$. Notice also that assumption $\alpha < \gamma + 1/2$ guarantees boundedness on the term $\Gamma(2\gamma+1-2\alpha)$. To conclude with, we

now need to take the real part of (3.12) and show that it is a bounded function of γ . After some simplifications, we have

$$\begin{aligned} \operatorname{Re}(I_1(\tau)) &= \frac{2^{1+2\alpha} e^{2\pi\nu}}{\pi} \frac{\Gamma(2\gamma+1-2\alpha)\Gamma(1-2\alpha)}{\Gamma(2\gamma+1)} \\ &\times \operatorname{Re} \left[e^{-i\frac{\pi}{2}(2\alpha+1)} \frac{\Gamma(\gamma+1+i\nu)(\gamma-i\nu)}{\Gamma(\gamma+2-2\alpha-i\nu)} \sum_m \frac{(2\gamma+1-2\alpha)_m (\gamma+1+i\nu)_m (1-2\alpha)_m}{(2\gamma+1)_m (\gamma+2-2\alpha+i\nu)_m m!} \right. \\ &\left. + e^{i\frac{\pi}{2}(2\alpha-1)} \frac{\Gamma(\gamma+1-i\nu)}{\Gamma(\gamma+1-2\alpha-i\nu)} \sum_m \frac{(2\gamma+1-2\alpha)_m (\gamma-i\nu)_m (1-2\alpha)_m}{(2\gamma+1)_m (\gamma+1-2\alpha-i\nu)_m m!} \right] \end{aligned}$$

where we have written $i^{-(2\gamma+1-2\alpha)} = e^{-i\frac{\pi}{2}(2\gamma+1-2\alpha)}$. Using now the basic property of the Pochhammer symbol $(x)_m = \Gamma(x+m)/\Gamma(x)$ many other terms simplify, leaving us with

$$\begin{aligned} \operatorname{Re}(I_1(\tau)) &= \frac{2^{1+2\alpha} e^{2\pi\nu}}{\pi} \\ &\times \operatorname{Re} \left[e^{-i\frac{\pi}{2}(2\alpha+1)} (\gamma-i\nu) \sum_m \frac{\Gamma(2\gamma+1-2\alpha+m)\Gamma(\gamma+1+m+i\nu)\Gamma(1-2\alpha+m)}{\Gamma(2\gamma+1+m)\Gamma(\gamma+2-2\alpha+m-i\nu)m!} \right. \\ &\left. + e^{i\frac{\pi}{2}(2\alpha-1)} (\gamma-i\nu) \sum_m \frac{\Gamma(2\gamma+1-2\alpha+m)\Gamma(\gamma+m-i\nu)\Gamma(1-2\alpha+m)}{\Gamma(2\gamma+1+m)\Gamma(\gamma+1-2\alpha+m+i\nu)m!} \right]. \end{aligned}$$

Stirling formula for Gamma function

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z \left(1 + O\left(\frac{1}{z}\right)\right)$$

shows that both the series above are asymptotic (in γ) to γ^{-1} , and therefore the proof is concluded.

Remark 3.1. It should be noticed that when $\alpha \rightarrow 1$ the term $\Gamma(1-2\alpha)$ produces a singularity of order 1, as it is seen by applying well known property $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$. Anyway, it can be checked that this singularity is balanced by the real part. Calculations are very easy in the unperturbed case (i.e. when $\nu = 0$): in this case indeed everything is real except for the exponential terms, which become respectively $e^{-i\frac{3}{2}\pi}$ and $e^{i\frac{1}{2}\pi}$ which thus have no real part. In the perturbed case things are a bit more complicated, but the problem has already been dealt with in literature by some limiting argument (see [6] and references therein). On the other hand, as α approaches 1/2 the singularity produced by the term $\Gamma(1-2\alpha)$ is not balanced anymore (exponential terms have a nonzero real part): this rules out the case $\alpha = 1/2$ and therefore the chance of recovering Strichartz estimates with standard arguments from our proof. The analysis of this problem will be the object of forthcoming works.

4. APPENDIX: SPECIAL FUNCTIONS, PROPERTIES AND INTEGRALS.

For reader's convenience we include a short appendix in which we recall definitions and basic properties of the special functions coming into play. We refer to [26] and [15] for further details and for a deeper insight of the argument.

Definition 4.1. Given a_1, \dots, a_p and b_1, \dots, b_q complex numbers, we define the *generalized hypergeometric function* as

$$(4.1) \quad {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{n!}$$

provided the sum of the series is finite, where we are using the *Pochhammer symbols*

$$(a)_0 = 1, \quad (a)_n = a(a+1) \dots (a+n-1), \quad n \geq 1.$$

The first thing to investigate is, of course, under which conditions these series converge. As a very general framework, we can summarize the situation as follows

- if $p \leq q$ the series is convergent for all values of z ;
- if $p = q + 1$ the series is convergent for $|z| < 1$, and for the special values $z = 1$ if

$$(4.2) \quad \operatorname{Re} \left(\sum_{n=1}^q b_n - \sum_{m=1}^p a_m \right) > 0$$

and $z = -1$ if

$$\operatorname{Re} \left(\sum_{n=1}^q b_n - \sum_{m=1}^p a_m \right) > -1;$$

- if $p > q + 1$ the series never converges except for $z = 0$.

In this work we are especially interested in the *confluent hypergeometric function* ${}_1F_1(a, c, z)$, which is thus the case $p = q = 1$. This series is therefore known to be convergent for every finite value of z (if c is not zero nor a negative integer). Moreover, if a is a negative integer or zero, ${}_1F_1$ reduces to a polynomial of degree $|a|$.

Confluent hypergeometric functions can be thought of as a limiting case of the celebrated *Gauss hypergeometric serie*, that namely is case $p = 2, q = 1$ in (4.1), via the relation

$${}_1F_1(a, c; z) = \lim_{b \rightarrow \infty} {}_2F_1(a, b; c; z/b).$$

It is not difficult to show that ${}_1F_1(a; c; z)$ satisfies the differential equation

$$zu'' + (c - z)u' - au = 0.$$

which in fact recovers (2.5). The asymptotic behaviour of ${}_1F_1$ can be derived by writing

$$(4.3) \quad {}_1F_1(a, b, z) \cong \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a} G(a, a-b+1, z) + \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} G(b-a, 1-a, z)$$

where G has the asymptotic series representation

$$G(a, b, z) = 1 + \frac{ab}{1!z} + \frac{a(a+1)b(b+1)}{2!z^2} + \dots$$

Among the properties that confluent hypergeometric functions satisfy we recall the following one

$$(4.4) \quad F(a, c, z) = e^z F(c-a, c, -z).$$

Among the possible generalizations of hypergeometric function a very useful one is given by the so called *Appell functions* (see [2]). Historically, the motivation for introducing series of this type is connected to investigating different kind of products of two

different Gauss functions. In particular, we are here interested in the following, which is sometimes called Appell function of the second type,

$$(4.5) \quad F_2(d; a_1, a_2; b_1; b_2; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(d)_{m+n} (a_1)_m (a_2)_n}{(b_1)_m (b_2)_n m! n!} x^m y^n.$$

It can be seen that convergence condition for F_2 is given by $|x| + |y| < 1$.

REFERENCES

- [1] H. Almstrom and P. Olsson. Analytic properties of certain radial matrix elements *J. Math. Phys.* 8 (10), 2013–2021 (1967).
- [2] P. Appell. Sur les fonctions hypergeomtriques de plusieurs variables. *Mémoir. Sci. Math. Paris* Gauthier-Villars (1925).
- [3] N. Arrizabalaga, L. Fanelli and A. García. On the lack of dispersion for a class of magnetic Dirac flows. *J. Evol. Equ.* 13 (1), 89–106 (2013).
- [4] N. Arrizabalaga, J. Duoandikoetxea and L. Vega. Self-adjoint extensions of Dirac operators with Coulomb type singularity. *J. Math. Phys.* 54, 041504 (2013).
- [5] I. Bejenaru and S. Herr. The cubic Dirac equation: Small initial data in $H^1(\mathbb{R}^3)$. to appear in *Comm. Math. Phys.*
- [6] L.C. Biedenharn, D.S. Onley, and J.T. Reynolds. Some Exact Radial Integrals for Dirac-Coulomb Functions, *J. Math. Phys.* 5, 411 (1964).
- [7] N. Boussaid. Stable directions for small nonlinear Dirac standing waves. *Comm. Math. Phys.* 268 (3), 757–817 (2006).
- [8] N. Boussaid, P. D’Ancona and L. Fanelli. Virial identity and weak dispersion for the magnetic Dirac equation. *J. Math. Pures et App.*, 95, 137–150, 2011.
- [9] N. Burq, F. Planchon, J.G. Stalker and A. Tahvildar-Zadeh Shadi. Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential. *J. Funct. Anal.* 203 (2), 519–549 (2003).
- [10] N. Burq, F. Planchon, J.G. Stalker and A. Tahvildar-Zadeh Shadi. Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay, *Indiana Univ. Math. J.* 53 (6), 1665–1680 (2004).
- [11] F. Cacciafesta. Global small solutions to the critical Dirac equation with potential. *Nonlinear Analysis* 74, 6060–6073 (2011).
- [12] F. Cacciafesta and P. D’Ancona. Endpoint estimates and global existence for the nonlinear Dirac equation with a potential. *J. Differential Equations* 254, 2233–2260 (2013).
- [13] F. Cacciafesta. Virial identity and dispersive estimates for the n-dimensional Dirac equation. *J. Math. Sci. Univ. Tokyo* 18, 1–23 (2011).
- [14] P. D’Ancona and L. Fanelli. Smoothing estimates for the Schrödinger equation with unbounded potentials. *J. Differential Equations* 246 (12), 4552–4567 (2009).
- [15] A. Erdelyi et al. Higher transcendental functions. McGraw-Hill Book Company, Inc., New York, 1953, Vol. 1.
- [16] M. Escobedo and L. Vega. A semilinear Dirac equation in $H^s(\mathbf{R}^3)$ for $s > 1$. *SIAM J. Math. Anal.* 28(2), 338–362 (1997).
- [17] M.J. Esteban, M. Lewin and É. Séré. Variational methods in relativistic quantum mechanics. *Bull. Amer. Math. Soc. (N.S.)* 45 (4), 535–593 (2008).
- [18] L. Fanelli, V. Felli, M. Fontelos and A. Primo. Time decay of scaling critical electromagnetic Schrödinger flows, *Comm. Math. Phys.* 324 (3), 1033–1067 (2013).
- [19] W.W. Gargaro and D.S. Onley. Real and Virtual Radiation in Electron-Nucleus Scattering, *Phys. Rev. C* 4 (4), 1032–1043 (1971).
- [20] M. Goldberg, L. Vega and N. Visciglia. Counterexamples of Strichartz inequalities for Schrödinger equations with repulsive potentials. *Int. Math. Res. Not.* Art. ID 13927 (2006).
- [21] L.D. Landau and L.M. Lifshitz. Quantum mechanics - Nonrelativistic theory.
- [22] L.D. Landau and L.M. Lifshitz. Quantum mechanics - Relativistic quantum theory.

- [23] S. Machihara, M. Nakamura, K. Nakanishi, and T. Ozawa. Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation. *J. Funct. Anal.*, 219(1), 1–20 (2005).
- [24] T. Ozawa and K.M. Rogers. Sharp Morawetz estimates. *J. Anal. Math.* 121, 163–175 (2013).
- [25] N. Saad and R. Hall. Integrals containing confluent hypergeometric functions with applications to perturbed singular potentials. *J. Phys. A* 36 (28), 7771–7788 (2003).
- [26] L.J. Slater. Generalized hypergeometric functions, Cambridge at the university press (1966).
- [27] K. Sun and L.E. Wright. Radial integrals with finite energy loss for Dirac Coulomb functions *J. Math. Phys.* 17, 2175–2181 (1976).
- [28] B. Thaller. The Dirac Equation, Springer-Verlag, Texts and Monographs in Physics (1992).

FEDERICO CACCIAFESTA: SAPIENZA — UNIVERSITÀ DI ROMA, DIPARTIMENTO DI MATEMATICA,
PIAZZALE A. MORO 2, I-00185 ROMA, ITALY

E-mail address: cacciafe@mat.uniroma1.it

URL: www.mat.uniroma1.it/people/~cacciafesta

ÉRIC SÉRÉ: CEREMADE (UMR CNRS 7534), UNIVERSITÉ PARIS-DAUPHINE, PLACE DE LATRE
DE TASSIGNY, F-75775 PARIS CEDEX 16

E-mail address: sere@ceremade.dauphine.fr

URL: www.ceremade.dauphine.fr/~sere