The elastic response for microlayered functionally graded media

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J. Jędrysiak[*1], **E. Wierzbicki**[**2], and **Cz. Woźniak**[***3]

[1] Department of Structural Mechanics, Łódź University of Technology, Al. Politechniki 6, 93-590 Łódź, Poland
[2,3] Institute of Mathematics and Computer Sciences, Częstochowa University of Technology, 42 200 Częstochowa, Poland

In this note microlayered composites having continuously varying macroscopic properties are considered. Such composites are referred to as the functionally graded laminates (FGL). The aim of this contribution is to derive a new averaged model describing the elastic response of the FGL, using the modified tolerance averaging technique, developed for periodic composites and structures by Woźniak and Wierzbicki (2000).

Following [3], materials, having properties continuously varying with position, are called *functionally graded materials* (*FGM*). Our considerations are related to composites with a laminated microstructure, which on the macro level have continuously varying properties, cf. Fig. 1. These composites will be called *functionally graded laminates* (*FGL*).

Different methods proposed to analyse functionally graded composites were shown in [3], but most of them neglect *the effect of the microstructure size*.

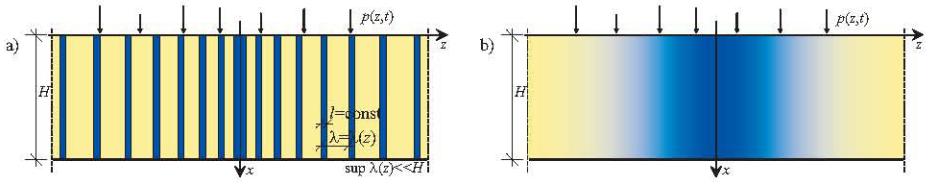

Fig. 1 An example of a functionally graded laminated layer: a) on the micro level, b) on the macro level.

The aim of this contribution is to derive two averaged models describing the elastic response of the FGL, employing the concepts introduced in the tolerance averaging technique for periodic composites and structures, [5]. Using these concepts and introducing some additional assumptions model equations with slowly-varying continuous coefficients will be derived. A similar approach was also proposed to FGL plates in [1], FGL shells in [4], FGL with microdefects in [2]. Moreover, an application of a new averaged model to vibrations of a FGL layer (cf. Fig. 1) will be presented.

Let us consider a plane problem of the layer in the orthogonal Cartesian coordinate system Ox_1x_2. Denote $x \equiv x_1$, $z \equiv x_2$. The layer has the thickness H. It is made of two linear-elastic materials distributed in laminas with varying thicknesses, which are perfectly bonded. The distribution of lamina thicknesses is extrapolated by a smooth *cell distribution function* $\lambda(z)$, which is assumed to be slowly varying. Moreover, every lamina consists of two sub-laminas, made of isotropic, homogeneous materials with mass densities - ρ^1, ρ^2 and tensors of elastic moduli - $\mathbf{C}^1$, $\mathbf{C}^2$. It is also assumed that *the volume fraction distribution* is described by functions $\varphi^1(z)$, $\varphi^2(z)$, such that $\varphi^1(z)+\varphi^2(z)=1$. In the modelling procedure of the modified tolerance averaging technique, some concepts and assumptions are employed, which were formulated for periodic composites in [5]. They are: a slowly varying function, a local periodic function, an oscillating function or *an averaging operation*, which is defined by $\langle f\rangle(z) \equiv [\lambda(z)]^{-1} \int_{-\lambda(z)/2}^{\lambda(z)/2} f(z+y)dy$ for any integrable function $f(z)$.

The main kinematic assumption is that the displacement field of the FGL $\mathbf{w}=\mathbf{w}(x,z,t)$, $x \in [0,H]$, can be expected in the form: $\mathbf{w}(x,z,t)=\mathbf{u}(x,z,t)+g(z)\mathbf{v}(x,z,t)$, where $\mathbf{u}$ are averaged parts of displacements $\mathbf{w}$, being slowly varying functions of z; g is the known shape fluctuation function, being a local periodic, oscillating function; $\mathbf{v}$ are new kinematic unknowns, being slowly varying functions of z. The product $g\mathbf{v}$ describes the fluctuation part of displacements $\mathbf{w}$. The averaged displacements $\mathbf{u}$ and the additional unknowns $\mathbf{v}$ will be called *macrodisplacements* and *fluctuation amplitudes*, respectively.

After some manipulations, employing the tolerance averaging technique, [5], we arrive at the model equations:

$$\begin{aligned} &\langle\rho\rangle\ddot{\mathbf{u}} - \partial_2\cdot(\langle\mathbf{C}\rangle:\nabla\mathbf{u}) - \partial_1\cdot(\langle\mathbf{C}\rangle:\nabla\mathbf{u}) - \partial_2\cdot(\mathbf{v}\cdot\langle\partial_2 g\cdot\mathbf{C}\rangle) - \partial_1\cdot(\langle\partial_2 g\cdot\mathbf{C}\rangle\cdot\mathbf{v}) = 0 \\ &\underline{\langle\rho(g)^2\rangle}\ddot{\mathbf{v}} + \langle\partial_2 g\cdot\mathbf{C}\cdot\partial_2 g\rangle\cdot\mathbf{v} - \underline{\langle\mathbf{C}(g)^2\rangle}:\partial_1\partial_1\mathbf{v} + \nabla\mathbf{u}:\langle\partial_2 g\cdot\mathbf{C}\rangle = 0 \end{aligned} \tag{1}$$

The derived differential equations describe the effect of *the microstructure size* (underlined terms) on the overall dynamic behaviour of the FGL. The model described by equations (1) will be called *the microstuctural model of the FGL*. Moreover, all coefficients of equations (1) are slowly varying functions of z. Boundary conditions have to be formulated for the macrodisplacements $\mathbf{u}$ on all boundaries and for the fluctuation amplitudes $\mathbf{v}$ only on the boundaries intersecting the laminas.

* Corresponding author: e-mail: jarek@p.lodz.pl, Phone: +48 42 6313564
** Second author: e-mail: ewarystwierzbicki@interia.pl, Phone: +48 42 6313565, Fax: +48 34 3250324
*** Third author: e-mail: czwozniak@interia.pl, Phone: +48 41 34224560, Fax: +48 34 3250324

It can be observed that neglecting the underlined terms in equations (1) the following equations are derived:

$$\langle\rho\rangle\ddot{\mathbf{u}} - \nabla\cdot(\langle\mathbf{C}\rangle:\nabla\mathbf{u}) = \nabla\cdot(\mathbf{v}\cdot\langle\partial_2 g\cdot\mathbf{C}\rangle), \qquad \langle\partial_2 g\cdot\mathbf{C}\cdot\partial_2 g\rangle\cdot\mathbf{v} = -\nabla\mathbf{u}:\langle\partial_2 g\cdot\mathbf{C}\rangle \tag{2}$$

The model described by equations (2) neglects the effect of the microstructure size on the overall dynamic behaviour of the FGL and will be called *the macrostructural model of the FGL.*

Let us consider vibrations of the layer shown in Fig. 1, with the thickness *H*, loaded on the upper boundary by *p*(*z*,*t*) and rested on an undeformable base. The layer has a length *L* along the *z*-axis. It is non-periodically reinforced along the *z*-axis.

The aim of the example is to determine a distribution function of materials in the FGL layer loaded on the upper boundary, under the assumption that the macrodeflections **u** are independent of z. The problem will be solved in the framework of *the macrostructural model*.

Neglecting the component u_2 of the macrodisplacements and setting on the boundaries z=$\pm L/2$ that $\partial_2 u_1$=0, we assume $u_1(x,z,t)$=$u(x,t)$. The function u will be called *the macrodeflection*. Assuming the loadings on the upper boundary $p(z,t)$=$Q_0 q(z)\exp(i\omega t)$ and the macrodeflection $u(z,t)$=$U(z)\exp(i\omega t)$, and denoting $\kappa\equiv$C$_{1111}$, $\delta\equiv\kappa^2/\kappa^1$, $\beta\equiv\rho^2/\rho^1$, $\eta(z)\equiv\lambda(z)/l$, $\chi\equiv\kappa^1[1-\delta+\delta\eta(z)]$, $\varsigma\equiv\rho^1[1-\delta+\delta\eta(z)]$, equations (2) can be written in the form describing *macrovibrations*

$$\partial_1\partial_1 U + \varsigma\chi^{-1}\omega^2 U = 0 \tag{3}$$

with the boundary conditions: $\partial_1 U(0)$=$-Qq(z)\eta(z)/\chi$, $U(H)$=0. Assuming δ=β and $q(z)$=$[1-\delta+\delta\eta(z)]/\eta(z)$, the macrodisplacement u_1 is really independent of z. Denoting $\xi\equiv x/L$, $\Omega^2\equiv\omega^2 H^2\varsigma/\chi$, $()'\equiv\partial_\xi$, $\phi(\xi)\equiv\kappa^1(HQ)^{-1}U(\xi H)$ equation (3) can be written in the non-dimensional form

$$\phi'' + \Omega^2\phi = 0 \tag{4}$$

with the boundary conditions: $\phi'(0)$=-1, $\phi(1)$=0. Solutions to equation (4) take the following form:

1) for the stationary problem (Ω^2=0) $\quad\phi(\xi) = 1-\xi$,

2) for the non-stationary problems ($\Omega^2 > 0$) $\quad\phi(\xi)$=$\Omega^{-1}[\tan\Omega\cos(\Omega\xi) - \sin(\Omega\xi)]$,

3) solutions do not exist (resonance macrovibrations) for $\quad\Omega^2$=$\pi^2(\frac{1}{2}+n)^2$.

Diagrams of the first and the second solutions are shown in Fig. 2.

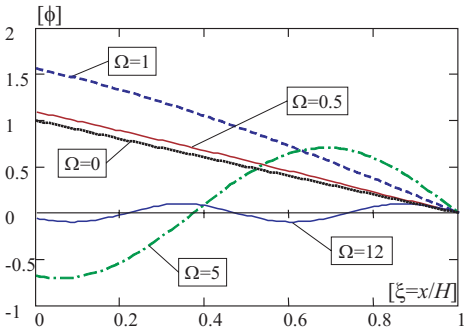

Fig. 2 Plots of solutions for the 1) and the 2) case.

Summarising, some remarks can be formulated:

· The new modelling approach proposed to describe the FGL, based on the concepts of the tolerance averaging technique formulated for periodic structures, leads to averaged models - the microstructural and the macrostructural of the FGL.

· The microstructural model makes it possible to investigate the effect of the microstructure size on the dynamic behaviour of the FGL, in the contrast to the macrostructural model.

· It has been shown that if the loading and the function of properties of the FGL layer are assumed in the same form the macrodeflection u is independent of z.

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