Elastic shallow shells with functionally graded structure

Jaroslaw Jedrysiak, Czeslaw Wozniak

Elastic shallow shells with functionally graded structure

Jarosław Jędrysiak[*,1], **Czesław Woźniak**[**,1]

[1] Technical University of Łódź, Chair of Structural Mechanics, al. Politechniki 6, 90-924 Łódź, Poland

Main objects of considerations are linear-elastic thin shallow shells with a tolerance-periodic structure along one direction. Moreover, their properties along the perpendicular direction can be described by continuous, slowly-varying functions. Hence, these objects can be treated as shells with a functionally graded structure. Here, the tolerance model of these shells is proposed.

1 Introduction

Our considerations are related to linear-elastic thin shallow shells having a tolerance-periodic structure along one direction, cf. Fig. 1a. It is assumed that functions describing their properties along the perpendicular direction are continuous and slowly-varying. Hence, shells of this kind can be treated as made of functionally graded materials, cf. the book [5], and will be called *the functionally graded shells* or *the FG-type shells*. Moreover, the size of the microstructure of these shells is assumed to be of an order the shell thickness h.

Dynamical problems of the FG-type shells can be desribed in the framework of various known methods shown in the book [5]. Unfortunately, *the effect of the microstructure size* is neglected in the most of them.

The aim of this note is twofold. Firstly – it is derived a new averaged model, describing the above effect on the elastic response of the FG-type shells, called *the tolerance model*. Secondly – it is proposed a simplified model, which neglects this effect and is called *the homogenization model*. Governing equations of these models have slowly-varying coefficients.

These equations are obtained by applying the concepts of *the tolerance averaging technique*, proposed for periodic composites and structures in [9] and extended on functionally graded media in [7]. Some applications of this approach to an analysis of various functionally graded structures are shown in a series of papers, e.g. [2], [8], [6], [3], [1], [4].

Let us consider a dynamical problem of the FG-type shell in the orthogonal Cartesian coordinate system $Ox_1x_2x_3$, cf. Fig. 1a; and denote $z=x_3$, $\mathbf{x}=(x_1,x_2)$. The projection of the shell on the plane $z=0$ is shown in Fig. 1b. It can be observed that $\mathbf{x}\in[-L_1/2,L_1/2]\times[-L_2/2,L_2/2]$. It is assumed that the shell has constant thickness h and is made of two linear-elastic materials. Their distribution along the x_1-axis is tolerance-periodic, but along the x_2-axis is slowly-varying. Hence, lengths of cells along the x_1-axis can be extrapolated by a smooth cell distribution function $\lambda(x_1)$, which is assumed to be slowly-varying and satisfy conditions $\lambda(x_1)<<L_1$, $\lambda(x_1)\in O(h)$. Hence, $\lambda(x_1)$ is called *the microstructure parameter*. Moreover, we assume that functions describing properties of the shell, i.e. mass density $\rho(x_1,x_2,z)$ and tensor of elastic moduli $C_{\alpha\beta\gamma\delta}(x_1,x_2,z)$ (under the plane stress assumption; Greek symbols run over 1, 2), are tolerance-periodic in x_1, slowly-varying in x_2 and even in z.

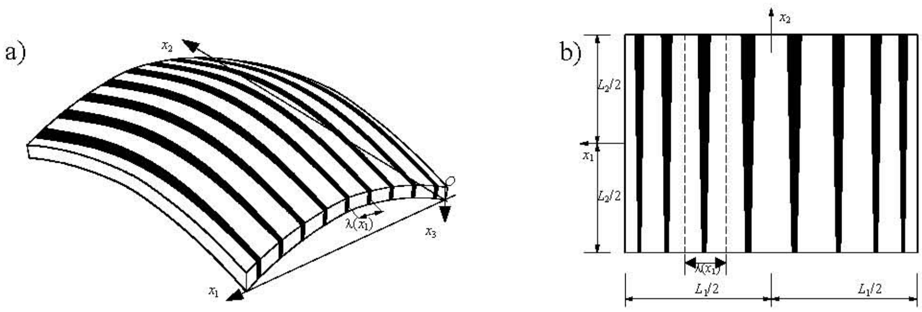

Fig. 1 a) A fragment of a thin shallow FG-type shell. b) The projection of the shell on the plane $z=0$.

2 Tolerance averaging approach

In the modelling procedure of the modified tolerance averaging technique, some concepts and assumptions are applied, which are formulated in [9] and [7]. They are: a slowly-varying function, a tolerance-periodic function, an oscillating function or *an averaging operation*, which is defined by $<f>(x_1)\equiv\lambda^{-1}\int_{-\lambda(x_1)/2}^{\lambda(x_1)/2} f(y_1)dy_1$ for any integrable function $f(x_1)$. The

* Corresponding author: e-mail jarek@p.lodz.pl, Phone: +00 48 42 631 35 64, Fax: +00 48 42 631 35 64
** e-mail czeslaw.wozniak@p.lodz.pl, Phone: +00 48 42 631 35 64, Fax: +00 48 42 631 35 64

main kinematic modelling assumption, called *the micro-macro decomposition*, is that the displacements of the shell $w_i(\mathbf{x},z,t)$, i=1,2,3, are expected in the form: $w_3(\mathbf{x},z,t)=v_3(\mathbf{x},t)$, $w_\alpha(\mathbf{x},z,t)=[-\partial_\alpha v_3(\mathbf{x},t)+\varphi(x_1)u_\alpha(\mathbf{x},t)]z+v_\alpha(\mathbf{x},t)+\varphi(x_1)r_\alpha(\mathbf{x},t)$, where $\varphi(\cdot)$ is the known fluctuation shape function (being a tolerance-periodic, oscillating function); $v_i(\cdot,x_2,z,t)$ (i=1,2,3) and $u_\alpha(\cdot,x_2,z,t)$, $r_\alpha(\cdot,x_2,z,t)$ (α=1,2) are averaged displacements of the shell midsurface and additional kinematic unknowns, respectively, being slowly-varying functions in x_1. The averaged displacements v_i are called *macrodisplacements* and the new kinematic unknowns u_α, r_α are called *fluctuation amplitudes*.

3 Governing equations

Denote by μ, ϑ and $B_{\alpha\beta\gamma\delta}$, $D_{\alpha\beta\gamma\delta}$, averaged over the shell thickness h – the mass density, the rotational inertia and the shell stiffnesses, repectively; and by f^i the external loadings; by $b_{\alpha\beta}$ the metric tensor. After some manipulations, applying the tolerance averaging procedure, cf. [9], [7], the model equations take the form:

$$\begin{aligned}
&\partial_{\alpha\beta}(<B^{\alpha\beta\gamma\delta}>\partial_{\gamma\delta}v_3-<B^{\alpha\beta1\gamma}\partial_1\varphi>u_\gamma)-<D^{\alpha\beta\gamma\delta}b_{\gamma\delta}>\partial_\alpha v_\beta-<D^{\gamma\delta1\alpha}\partial_1\varphi b_{\gamma\delta}>r_\alpha+<D^{\alpha\beta\gamma\delta}b_{\alpha\beta}b_{\gamma\delta}>v_3+<\mu>\ddot{v}_3=<f^3>,\\
&<B^{\alpha\gamma1\delta}\partial_1\varphi>\partial_{\gamma\delta}v_3-<B^{1\alpha1\gamma}\partial_1\varphi\partial_1\varphi>u_\gamma+\underline{\underline{\partial_2(<B^{2\alpha2\gamma}\varphi\varphi>\partial_2u_\gamma)}}=0,\\
&\partial_\alpha(<D^{\alpha\beta\gamma\delta}>\partial_\gamma v_\delta+<D^{\alpha\beta1\gamma}\partial_1\varphi>r_\gamma-<D^{\alpha\beta\gamma\delta}b_{\gamma\delta}>v_3)-<\mu>\ddot{v}_\beta=-<f^\beta>,\\
&<D^{\alpha\gamma1\delta}\partial_1\varphi>\partial_\delta v_\gamma+<D^{1\alpha1\beta}\partial_1\varphi\partial_1\varphi>r_\beta-\underline{\underline{\partial_2(<D^{2\alpha2\beta}\varphi\varphi>\partial_2r_\beta)}}-<D^{\gamma\delta1\alpha}\partial_1\varphi b_{\gamma\delta}>v_3+\underline{<\mu\varphi\varphi>\ddot{r}_\alpha}=0.
\end{aligned}\tag{1}$$

The above equations represent *the tolerance model of thin shallow functionally graded shells* under consideration. Equations (1) describe the effect of the microstructure size on the overall behaviour of these shells by the underlined terms. All coefficients of these equations are slowly-varying functions.

It can be observed that neglecting the underlined terms in equations (1) we arrive at the following system of equations:

$$\begin{aligned}
&\partial_{\alpha\beta}(<B^{\alpha\beta\gamma\delta}>\partial_{\gamma\delta}v_3-<B^{\alpha\beta1\gamma}\partial_1\varphi>u_\gamma)-<D^{\alpha\beta\gamma\delta}b_{\gamma\delta}>\partial_\alpha v_\beta-<D^{\gamma\delta1\alpha}\partial_1\varphi b_{\gamma\delta}>r_\alpha+<D^{\alpha\beta\gamma\delta}b_{\alpha\beta}b_{\gamma\delta}>v_3+<\mu>\ddot{v}_3=<f^3>,\\
&<B^{\alpha\gamma1\delta}\partial_1\varphi>\partial_{\gamma\delta}v_3-<B^{1\alpha1\gamma}\partial_1\varphi\partial_1\varphi>u_\gamma=0,\\
&\partial_\alpha(<D^{\alpha\beta\gamma\delta}>\partial_\gamma v_\delta+<D^{\alpha\beta1\gamma}\partial_1\varphi>r_\gamma-<D^{\alpha\beta\gamma\delta}b_{\gamma\delta}>v_3)-<\mu>\ddot{v}_\beta=-<f^\beta>,\\
&<D^{\alpha\gamma1\delta}\partial_1\varphi>\partial_\delta v_\gamma+<D^{1\alpha1\beta}\partial_1\varphi\partial_1\varphi>r_\beta-<D^{\gamma\delta1\alpha}\partial_1\varphi b_{\gamma\delta}>v_3=0,
\end{aligned}\tag{2}$$

which neglect the effect of the microstructure size. The model described by equations (2) will be called *the homogenization model of thin shallow functionally graded shells* under consideration.

4 Final remarks

In this note a problem of mathematical modelling of thin linear-elastic shells with functionally graded structure (called functionally graded shells) is presented. As a tool of the analysis the tolerance averaging technique is applied, cf. [7], [9]. Summarizing our considerations we can formulate some new results and remarks.

It can be observed that the tolerance averaging technique makes it possible to pass from differential equations with highly oscillating, non-continuous coefficients to model equations having smooth, slowly-varying coefficients.

Moreover, the derived equations of the tolerance model describe the effect of the microstructure size on the overall behaviour of the functionally graded shells, in the contrary to the homogenization model. It can be marked that the obtained equations take into account this effect not only in non-stationary processes

More detailed analysis and examples of applications of the tolerance model of the functionally graded shells will be presented in forthcoming papers.

Acknowledgements This contribution is supported by the Ministry of Science and Higher Education of Poland under grant No. N N506 398535.

References

[1] J. Jędrysiak, and A. Radzikowska, J. Theor. Appl. Mech. **45**, 239-257 (2007).
[2] J. Jędrysiak, J. Rychlewska, and Cz. Woźniak, in: Shell Structures: Theory and Applications, edited by W. Pietraszkiewicz, Cz. Szymczak, (Balkema, London-Leiden, 2005), 119-123.
[3] J. Jędrysiak, E. Wierzbicki, and Cz. Woźniak, PAMM 6/1, 385-386 (2006).
[4] B. Michalak, Cz. Woźniak, and M. Woźniak, Arch. Appl. Mech. **77**, 823-834 (2007).
[5] S. Suresh, and A. Mortensen, Fundamentals of functionally graded materials (The University Press, Cambridge, 1998).
[6] J. Szymczyk, Cz. Woźniak, Arch. Mech. **58**, 445-458 (2006).
[7] Cz. Woźniak, B. Michalak, and J. Jędrysiak (eds.), Thermomechanics of microheterogeneous solids and structures. Tolerance averaging approach (Wydawnictwo Politechniki Łódzkiej, Łódź (Poland), 2008).
[8] Cz. Woźniak, J. Rychlewska, and E. Wierzbicki, in: Shell Structures: Theory and Applications, edited by W. Pietraszkiewicz, Cz. Szymczak, (Balkema, London-Leiden, 2005), 187-191.
[9] Cz. Woźniak, and E. Wierzbicki, Averaging techniques in thermomechanics of composite solids (Wydawnictwo Politechniki Częstochowskiej, Częstochowa (Poland), 2000).