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ABSTRACT. We consider the nonlinear Schrödinger equation with a short-range external potential, in a semi-classical scaling. We show that for fixed Planck constant, a complete scattering theory is available, showing that both the potential and the nonlinearity are asymptotically negligible for large time. Then, for data under the form of coherent state, we show that a scattering theory is also available for the approximate envelope of the propagated coherent state, which is given by a nonlinear equation. In the semi-classical limit, these two scattering operators can be compared in terms of classical scattering theory, thanks to a uniform in time error estimate. Finally, we infer a large time decoupling phenomenon in the case of finitely many initial coherent states.

1. INTRODUCTION

We consider the equation

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + |\psi^\varepsilon|^2\psi^\varepsilon, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^3, \tag{1.1}$$

and both semi-classical ($\varepsilon\to 0$) and large time ($t\to\pm\infty$) limits. Of course these limits must not be expected to commute, and one of the goals of this paper is to analyze this lack of commutation on specific asymptotic data, under the form of coherent states as described below. Even though our main result (Theorem 1.6) is proven specifically for the above case of a cubic three-dimensional equation, two important intermediate results (Theorems 1.4 and 1.5) are established in a more general setting. Unless specified otherwise, we shall from now on consider $\psi^\varepsilon : \mathbb{R}_t\times\mathbb{R}^d_x\to\mathbb{C}$, $d\geqslant 1$.

1.1. Propagation of initial coherent states.

In this subsection, we consider the initial value problem, as opposed to the scattering problem treated throughout this paper. More precisely, we assume here that the wave function is, at time $t=0$, given by the coherent state

$$\psi^\varepsilon(0,x) = \frac{1}{\varepsilon^{d/4}}a\left(\frac{x-q_0}{\sqrt{\varepsilon}}\right)e^{ip_0\cdot(x-q_0)/\varepsilon}, \tag{1.2}$$

where $q_0, p_0\in\mathbb{R}^d$ denote the initial position and velocity, respectively. The function a belongs to the Schwartz class, typically. In the case where a is a (complex) Gaussian, many explicit computations are available in the linear case (see [33]). Note that the L^2-norm of ψ^ε is independent of ε, $\|\psi^\varepsilon(t,\cdot)\|_{L^2(\mathbb{R}^d)} = \|a\|_{L^2(\mathbb{R}^d)}$.

Throughout this subsection, we assume that the external potential V is smooth and real-valued, $V\in C^\infty(\mathbb{R}^d;\mathbb{R})$, and at most quadratic, in the sense that

$$\partial^\alpha V\in L^\infty(\mathbb{R}^d), \quad \forall|\alpha|\geqslant 2.$$

This assumption will be strengthened when large time behavior is analyzed.

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1.1.1. *Linear case.* Resume (1.1) in the absence of nonlinear term:

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon, \quad x\in\mathbb{R}^d, \tag{1.3}$$

associated with the initial datum (1.2). To derive an approximate solution, and to describe the propagation of the initial wave packet, introduce the Hamiltonian flow

$$\dot q(t) = p(t), \quad \dot p(t) = -\nabla V\left(q(t)\right), \tag{1.4}$$

and prescribe the initial data $q(0)=q_0, p(0)=p_0$. Since the potential V is smooth and at most quadratic, the solution $(q(t),p(t))$ is smooth, defined for all time, and grows at most exponentially. The classical action is given by

$$S(t) = \int_0^t \left(\frac12|p(s)|^2 - V(q(s))\right) ds. \tag{1.5}$$

We observe that if we change the unknown function ψ^ε to u^ε by

$$\psi^\varepsilon(t,x) = \varepsilon^{-d/4}u^\varepsilon\left(t, \frac{x-q(t)}{\sqrt{\varepsilon}}\right) e^{i(S(t)+p(t)\cdot(x-q(t)))/\varepsilon}, \tag{1.6}$$

then, in terms of $u^\varepsilon = u^\varepsilon(t,y)$, the Cauchy problem (1.3)–(1.2) is equivalent to

$$i\partial_t u^\varepsilon + \frac12\Delta u^\varepsilon = V^\varepsilon(t,y)u^\varepsilon \quad ; \quad u^\varepsilon(0,y)=a(y), \tag{1.7}$$

where the external time-dependent potential V^ε is given by

$$V^\varepsilon(t,y) = \frac1\varepsilon\left(V(x(t)+\sqrt{\varepsilon}y) - V(x(t)) - \sqrt{\varepsilon}\left\langle \nabla V(x(t)), y\right\rangle\right). \tag{1.8}$$

This potential corresponds to the first term of a Taylor expansion of V about the point $q(t)$, and we naturally introduce $u=u(t,y)$ solution to

$$i\partial_t u + \frac12\Delta u = \frac12\left\langle Q(t)y,y\right\rangle u \quad ; \quad u(0,y)=a(y), \tag{1.9}$$

where

$$Q(t) := \nabla^2 V\left(q(t)\right), \quad \text{so that } \frac12\left\langle Q(t)y,y\right\rangle = \lim_{\varepsilon\to0} V^\varepsilon(t,y).$$

The obvious candidate to approximate the initial wave function ψ^ε is then:

$$\varphi^\varepsilon(t,x) = \varepsilon^{-d/4}u\left(t, \frac{x-q(t)}{\sqrt{\varepsilon}}\right) e^{i(S(t)+p(t)\cdot(x-q(t)))/\varepsilon}. \tag{1.10}$$

Indeed, it can be proven (see e.g. [2, 4, 17, 33, 35, 36]) that there exists $C>0$ independent of ε such that

$$\|\psi^\varepsilon(t,\cdot) - \varphi^\varepsilon(t,\cdot)\|_{L^2(\mathbb{R}^d)} \leqslant C\sqrt{\varepsilon}e^{Ct}.$$

Therefore, φ^ε is a good approximation of ψ^ε at least up to time of order $c\ln\frac1\varepsilon$ (Ehrenfest time).

1.1.2. *Nonlinear case.* When adding a nonlinear term to (1.3), one has to be cautious about the size of the solution, which rules the importance of the nonlinear term. To simplify the discussions, we restrict our analysis to the case of a gauge invariant, defocusing, power nonlinearity, $|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon$. We choose to measure the importance of nonlinear effects not directly through the size of the initial data, but through an ε-dependent coupling factor: we keep the initial datum (1.2) (with an L^2-norm independent of ε), and consider

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \varepsilon^\alpha|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon.$$

Since the nonlinearity is homogeneous, this approach is equivalent to considering $\alpha = 0$, up to multiplying the initial datum by $\varepsilon^{\alpha/(2\sigma)}$. We assume $\sigma > 0$, with $\sigma < 2/(d-2)$ if $d \geqslant 3$: for $a \in \Sigma$, defined by

$$\Sigma = \{f \in H^1(\mathbb{R}^d), \quad x \mapsto \langle x\rangle f(x) \in L^2(\mathbb{R}^d)\}, \quad \langle x\rangle = \left(1+|x|^2\right)^{1/2},$$

we have, for fixed $\varepsilon > 0$, $\psi^\varepsilon_{|t=0} \in \Sigma$, and the Cauchy problem is globally well-posed, $\psi^\varepsilon \in C(\mathbb{R}_t;\Sigma)$ (see e.g. [9]). It was established in [11] that the value

$$\alpha_c = 1 + \frac{d\sigma}{2}$$

is critical in terms of the effect of the nonlinearity in the semi-classical limit $\varepsilon \to 0$. If $\alpha > \alpha_c$, then $\varphi^\varepsilon_{\rm lin}$, given by (1.9)-(1.10), is still a good approximation of ψ^ε at least up to time of order $c\ln\frac{1}{\varepsilon}$. On the other hand, if $\alpha = \alpha_c$, nonlinear effects alter the behavior of ψ^ε at leading order, through its envelope only. Replacing (1.9) by

$$i\partial_t u + \frac{1}{2}\Delta u = \frac{1}{2}\langle Q(t)y, y\rangle u + |u|^{2\sigma}u, \tag{1.11}$$

and keeping the relation (1.10), φ^ε is now a good approximation of ψ^ε. In [11] though, the time of validity of the approximation is not always proven to be of order at least $c\ln\frac{1}{\varepsilon}$, sometimes shorter time scales (of the order $c\ln\ln\frac{1}{\varepsilon}$) have to be considered, most likely for technical reasons only. Some of these restrictions have been removed in [37], by considering decaying external potentials V.

1.2. **Linear scattering theory and coherent states.** We now consider the aspect of large time, and instead of prescribing ψ^ε at $t = 0$ (or more generally at some finite time), we impose its behavior at $t = -\infty$. In the linear case (1.3), there are several results addressing the question mentioned above, considering different forms of asymptotic states at $t = -\infty$. Before describing them, we recall important facts concerning quantum and classical scattering.

1.2.1. *Quantum scattering.* Throughout this paper, we assume that the external potential is short-range, and satisfies the following properties:

Assumption 1.1. *We suppose that V is smooth and real-valued, $V \in C^\infty(\mathbb{R}^d;\mathbb{R})$. In addition, it is short range in the following sense: there exists $\mu > 1$ such that*

$$|\partial^\alpha V(x)| \leqslant \frac{C_\alpha}{(1+|x|)^{\mu+|\alpha|}}, \quad \forall \alpha \in \mathbb{N}^d. \tag{1.12}$$

Our final result is established under the stronger condition $\mu > 2$ (a condition which is needed in several steps of the proof), but some results are established under the mere assumption $\mu > 1$. Essentially, the analysis of the approximate solution is valid for $\mu > 1$ (see Section 4), while the rest of the analysis requires $\mu > 2$.

Denote by

$$H_0^\varepsilon = -\frac{\varepsilon^2}{2}\Delta \quad \text{and} \quad H^\varepsilon = -\frac{\varepsilon^2}{2}\Delta + V(x)$$

the underlying Hamiltonians. For fixed $\varepsilon > 0$, the (linear) wave operators are given by

$$W_\pm^\varepsilon = \lim_{t\to\pm\infty} e^{i\frac{t}{\varepsilon}H^\varepsilon} e^{-i\frac{t}{\varepsilon}H_0^\varepsilon},$$

and the (quantum) scattering operator is defined by

$$S_{\rm lin}^\varepsilon = \left(W_+^\varepsilon\right)^* W_-^\varepsilon.$$

See for instance [20].

1.2.2. *Classical scattering.* Let V satisfying Assumption 1.1. For $(q^-,p^-) \in \mathbb{R}^d \times \mathbb{R}^d$, we consider the classical trajectories $(q(t),p(t))$ defined by (1.4), along with the prescribed asymptotic behavior as $t \to -\infty$:

$$\lim_{t\to-\infty} \left|q(t) - p^- t - q^-\right| = \lim_{t\to-\infty} |p(t) - p^-| = 0. \tag{1.13}$$

The existence and uniqueness of such a trajectory can be found in e.g. [20, 51], provided that $p^- \neq 0$. Moreover, there exists a closed set $\mathcal{N}_0$ of Lebesgue measure zero in $\mathbb{R}^{2d}$ such that for all $(q^-,p^-) \in \mathbb{R}^{2d} \setminus \mathcal{N}_0$, there exists $(q^+,p^+) \in \mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right)$ such that

$$\lim_{t\to+\infty} \left|q(t) - p^+ t - q^+\right| = \lim_{t\to+\infty} |p(t) - p^+| = 0.$$

The classical scattering operator is $S^{\rm cl} : (q^-,p^-) \mapsto (q^+,p^+)$. Choosing $(q^-,p^-) \in \mathbb{R}^{2d} \setminus \mathcal{N}_0$ implies that the following assumption is satisfied:

Assumption 1.2. *The asymptotic center in phase space, $(q^-,p^-) \in \mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right)$ is such that the classical scattering operator is well-defined,*

$$S^{\rm cl}(q^-,p^-) = (q^+,p^+), \quad p^+ \neq 0,$$

and the classical action has limits as $t \to \pm\infty$:

$$\lim_{t\to-\infty} \left|S(t) - t\frac{|p^-|^2}{2}\right| = \lim_{t\to+\infty} \left|S(t) - t\frac{|p^+|^2}{2} - S_+\right| = 0,$$

for some $S_+ \in \mathbb{R}$.

1.2.3. *Some previous results.* It seems that the first mathematical result involving both the semi-classical and large time limits appears in [27], where the classical field limit of non-relativistic many-boson theories is studied in space dimension $d \geqslant 3$.

In [56], the case of a short range potential (Assumption 1.1) is considered, with asymptotic states under the form of semi-classically concentrated functions,

$$e^{-i\frac{\varepsilon t}{2}\Delta}\psi^\varepsilon(t)_{|t=-\infty} = \frac{1}{\varepsilon^{d/2}}\widehat{f}\left(\frac{x-q^-}{\varepsilon}\right), \quad f \in L^2(\mathbb{R}^d),$$

where $\widehat{f}$ denotes the standard Fourier transform (whose definition is independent of ε). The main result from [56] shows that the semi-classical limit for $S_{\rm lin}^\varepsilon$ can be expressed in terms of the classical scattering operator, of the classical action, and of the Maslov index associated to each classical trajectory. We refer to [56] for a precise statement, and to [57] for the case of long range potentials, requiring modifications of the dynamics.

In [34, 35], coherent states are considered,

$$e^{-i\frac{\varepsilon t}{2}\Delta}\psi^\varepsilon(t)_{|t=-\infty} = \frac{1}{\varepsilon^{d/4}}u_-\left(\frac{x-q^-}{\sqrt{\varepsilon}}\right)e^{ip^-\cdot(x-q^-)/\varepsilon+iq^-\cdot p^-/(2\varepsilon)} =: \psi^\varepsilon_-(x). \tag{1.14}$$

More precisely, in [34, 35], the asymptotic state u_- is assumed to be a complex Gaussian function. Introduce the notation

$$\delta(t) = S(t) - \frac{q(t)\cdot p(t) - q^-\cdot p^-}{2}.$$

Then Assumption 1.2 implies that there exists $\delta^+ \in \mathbb{R}$ such that

$$\delta(t) \underset{t\to-\infty}{\longrightarrow} 0 \quad \text{and} \quad \delta(t) \underset{t\to+\infty}{\longrightarrow} \delta^+.$$

In [17, 35], we find the following general result (an asymptotic expansion in powers of $\sqrt{\varepsilon}$ is actually given, but we stick to the first term to ease the presentation):

Theorem 1.3. *Let Assumptions 1.1 and 1.2 be satisfied, and let*

$$u_-(y) = a_-\exp\left(\frac{i}{2}\langle\Gamma_- y, y\rangle\right),$$

where $a_- \in \mathbb{C}$ and Γ_- is a complex symmetric $d\times d$ matrix whose imaginary part is positive and non-degenerate. Consider ψ^ε solution to (1.3)*, with* (1.14)*. Then the following asymptotic expansion holds in $L^2(\mathbb{R}^d)$:*

$$S^\varepsilon_{\rm lin}\psi^\varepsilon_- = \frac{1}{\varepsilon^{d/4}}e^{i\delta^+/\varepsilon}e^{ip^+\cdot(x-q^+)/\varepsilon+iq^+\cdot p^+/(2\varepsilon)}\hat{R}(G_+)u_-\left(\frac{x-q^+}{\sqrt{\varepsilon}}\right)+\mathcal{O}(\sqrt{\varepsilon}),$$

where $\hat{R}(G_+)$ is the metaplectic transformation associated to $G_+ = \frac{\partial(q^+,p^+)}{\partial(q^-,p^-)}$.

As a corollary, our main result yields another interpretation of the above statement. It turns out that a complete scattering theory is available for (1.9). As a particular case of Theorem 1.5 (which addresses the nonlinear case), given $u_- \in \Sigma$, there exist a unique $u \in C(\mathbb{R};\Sigma)$ solution to (1.9) and a unique $u_+ \in \Sigma$ such that

$$\|e^{-i\frac{t}{2}\Delta}u(t) - u_\pm\|_\Sigma \underset{t\to\pm\infty}{\longrightarrow} 0.$$

Then in the above theorem (where u_- is restricted to be a Gaussian), we have

$$u_+ = \hat{R}(G_+)u_-.$$

Finally, we mention in passing the paper [48], where similar issues and results are obtained for

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V\left(\frac{x}{\varepsilon}\right)\psi^\varepsilon + U(x)\psi^\varepsilon,$$

for V a short-range potential, and U is bounded as well as its derivatives. The special scaling in V implies that initially concentrated waves (at scaled ε) first undergo the effects of V, then exit a time layer of order ε, through which the main action of V corresponds to the above quantum scattering operator (but with $\varepsilon = 1$ due to the new scaling in the equation). Then, the action of V becomes negligible, and the propagation of the wave is dictated by the classical dynamics associated to U.

1.3. **Main results.** We now consider the nonlinear equation

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \varepsilon^\alpha|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon, \tag{1.15}$$

along with asymptotic data (1.14). We first prove that for fixed $\varepsilon > 0$, a scattering theory is available for (1.15): at this stage, the value of α is naturally irrelevant, as well as the form (1.14). To establish a large data scattering theory for (3.1), we assume that the attractive part of the potential,

$$(\partial_r V(x))_+ = \left(\frac{x}{|x|}\cdot\nabla V(x)\right)_+$$

is not too large, where $f_+ = \max(0, f)$ for any real number f.

Theorem 1.4. *Let $d \geqslant 3$, $\frac{2}{d} < \sigma < \frac{2}{d-2}$, and V satisfying Assumption 1.1 for some $\mu > 2$. There exists $M = M(\mu, d)$ such that if the attractive part of the potential $(\partial_r V)_+$ satisfies*

$$(\partial_r V(x))_+ \leqslant \frac{M}{(1+|x|)^{\mu+1}}, \quad \forall x \in \mathbb{R}^d,$$

one can define a scattering operator for (3.1) *in $H^1(\mathbb{R}^d)$: for all $\psi_-^\varepsilon \in H^1(\mathbb{R}^d)$, there exist a unique $\psi^\varepsilon \in C(\mathbb{R}; H^1(\mathbb{R}^d))$ solution to* (3.1) *and a unique $\psi_+^\varepsilon \in H^1(\mathbb{R}^d)$ such that*

$$\|\psi^\varepsilon(t) - e^{i\frac{\varepsilon t}{2}\Delta}\psi_\pm^\varepsilon\|_{H^1(\mathbb{R}^d)} \underset{t\to\pm\infty}{\longrightarrow} 0.$$

The (quantum) scattering operator is the map $S^\varepsilon : \psi_-^\varepsilon \mapsto \psi_+^\varepsilon$.

We emphasize the fact that several recent results address the same issue, under various assumptions on the external potential V: [58] treats the case where V is an inverse square (a framework which is ruled out in our contribution), while in [12], the potential is more general than merely inverse square. In [12], a magnetic field is also included, and the Laplacian is perturbed with variable coefficients. We make more comparisons with [12] in Section 3.

The second result of this paper concerns the scattering theory for the envelope equation:

Theorem 1.5. *Let $d \geqslant 1$, $\frac{2}{d} \leqslant \sigma < \frac{2}{(d-2)_+}$, and V satisfying Assumption 1.1 for some $\mu > 1$. One can define a scattering operator for* (1.11) *in Σ: for all $u_- \in \Sigma$, there exist a unique $u \in C(\mathbb{R}; \Sigma)$ solution to* (1.11) *and a unique $u_+ \in \Sigma$ such that*

$$\|e^{-i\frac{t}{2}\Delta}u(t) - u_\pm\|_\Sigma \underset{t\to\pm\infty}{\longrightarrow} 0.$$

As mentioned above, the proof includes the construction of a linear scattering operator, comparing the dynamics associated to (1.9) to the free dynamics $e^{i\frac{t}{2}\Delta}$. In the above formula, we have incorporated the information that $e^{i\frac{t}{2}\Delta}$ is unitary on $H^1(\mathbb{R}^d)$, but *not on* Σ (see e.g. [13]).

We can now state the nonlinear analogue to Theorem 1.3. Since Theorem 1.4 requires $d \geqslant 3$, we naturally have to make this assumption. On the other hand, we will need the approximate envelope u to be rather smooth, which requires a smooth nonlinearity, $\sigma \in \mathbb{N}$. Intersecting this property with the assumptions of Theorem 1.4 leaves only one case: $d = 3$ and $\sigma = 1$, that is (1.1), up to the scaling. We will see in Section 5 that considering $d = 3$ is also crucial, since the argument uses dispersive estimates which are known only in the three-dimensional case for V satisfying Assumption 1.1 with $\mu > 2$ (larger values for μ could be considered in higher dimensions, though). Introduce the notation

$$\Sigma^k = \{f \in H^k(\mathbb{R}^d), \quad x \mapsto |x|^k f(x) \in L^2(\mathbb{R}^d)\}.$$

Theorem 1.6. *Let Assumptions 1.1 and 1.2 be satisfied, with $\mu > 2$ and V as in Theorem 1.4. Consider ψ^ε solution to*

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \varepsilon^{5/2}|\psi^\varepsilon|^2\psi^\varepsilon, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^3,$$

and such that (1.14) *holds, with $u_- \in \Sigma^7$. Then the following asymptotic expansion holds in $L^2(\mathbb{R}^3)$:*

$$S^\varepsilon\psi^\varepsilon_- = \frac{1}{\varepsilon^{3/4}}e^{i\delta^+/\varepsilon}e^{ip^+\cdot(x-q^+)/\varepsilon+iq^+\cdot p^+/(2\varepsilon)}u_+\left(\frac{x-q^+}{\sqrt{\varepsilon}}\right)+\mathcal{O}(\sqrt{\varepsilon}), \tag{1.16}$$

where S^ε is given by Theorem 1.4 and u_+ stems from Theorem 1.5.

Remark 1.7. In the subcritical case, that is if we consider

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \varepsilon^{\alpha}|\psi^\varepsilon|^2\psi^\varepsilon, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^3,$$

along with (1.14), for some $\alpha > 5/2$, the argument of the proof shows that (1.16) remains true, but with u_+ given by the scattering operator associated to (1.9) (as opposed to (1.11)), that is, the same conclusion as in Theorem 1.3 when u_- is a Gaussian.

As a corollary of the proof of the above result, and of the analysis from [11], we infer:

Corollary 1.8 (Asymptotic decoupling)**.** *Let Assumption 1.1 be satisfied, with $\mu > 2$ and V as in Theorem 1.4. Consider ψ^ε solution to*

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \varepsilon^{5/2}|\psi^\varepsilon|^2\psi^\varepsilon, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^3,$$

with initial datum

$$\psi^\varepsilon(0,x) = \sum_{j=1}^N \frac{1}{\varepsilon^{3/4}}a_j\left(\frac{x-q_{0j}}{\sqrt{\varepsilon}}\right)e^{ip_{0j}\cdot(x-q_{0j})/\varepsilon} =: \psi_0^\varepsilon(x),$$

where $N \geqslant 2$, $q_{0j}, p_{0j} \in \mathbb{R}^3$, $p_{0j} \neq 0$ so that scattering is available as $t \to +\infty$ for $(q_j(t), p_j(t))$, in the sense of Assumption 1.2, and $a_j \in \mathcal{S}(\mathbb{R}^3)$. We suppose $(q_{0j}, p_{0j}) \neq (q_{0k}, p_{0k})$ for $j \neq k$. Then we have the uniform estimate:

$$\sup_{t\in\mathbb{R}}\left\|\psi^\varepsilon(t) - \sum_{j=1}^N \varphi_j^\varepsilon(t)\right\|_{L^2(\mathbb{R}^3)} \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

where φ_j^ε is the approximate solution with the j-th wave packet as an initial datum. As a consequence, the asymptotic expansion holds in $L^2(\mathbb{R}^3)$, as $\varepsilon \to 0$:

$$\left(W_\pm^\varepsilon\right)^{-1}\psi_0^\varepsilon = \sum_{j=1}^N \frac{1}{\varepsilon^{3/4}}e^{i\delta_j^\pm/\varepsilon}e^{ip_j^\pm\cdot(x-q_j^\pm)/\varepsilon+iq_j^\pm\cdot p_j^\pm/(2\varepsilon)}u_{j\pm}\left(\frac{x-q_j^\pm}{\sqrt{\varepsilon}}\right)+o(1),$$

where the inverse wave operators $\left(W_\pm^\varepsilon\right)^{-1}$ stem from Theorem 1.4, the $u_{j\pm}$'s are the asymptotic states emanating from a_j, and

$$\delta_j^\pm = \lim_{t\to\pm\infty}\left(S_j(t) - \frac{q_j(t)\cdot p_j(t) - q_{0j}\cdot p_{0j}}{2}\right) \in \mathbb{R}.$$

Remark 1.9. In the case $V = 0$, the approximation by wave packets is actually exact, since then $Q(t) \equiv 0$, hence $u^\varepsilon = u$. For one wave packet, Theorem 1.6 becomes empty, since it is merely a rescaling. On the other hand, for two initial wave packets, even in the case $V = 0$, Corollary 1.8 brings some information, reminiscent of profile decomposition. More precisely, define u^ε by (1.6), and choose (arbitrarily) to privilege the trajectory (q_1, p_1). The Cauchy problem is then equivalent to

$$\begin{cases} i\partial_t u^\varepsilon + \dfrac{1}{2}\Delta u^\varepsilon = |u^\varepsilon|^2 u^\varepsilon, \\ u^\varepsilon(0,y) = a_1(y) + a_2\left(y + \dfrac{q_{01}-q_{02}}{\sqrt{\varepsilon}}\right) e^{ip_{02}\cdot\delta q_0/\varepsilon - i\delta p_0\cdot y/\sqrt{\varepsilon}}, \end{cases}$$

where we have set $\delta p_0 = p_{01} - p_{02}$ and $\delta q_0 = q_{01} - q_{02}$. Note however that the initial datum is uniformly bounded in $L^2(\mathbb{R}^3)$, but in no $H^s(\mathbb{R}^3)$ for $s > 0$ (if $p_{01} \neq p_{02}$), while the equation is $\dot{H}^{1/2}$-critical, Therefore, even in the case $V = 0$, Corollary 1.8 does not seem to be a consequence of profile decompositions like in e.g. [21, 42, 45]. In view of (1.4), the approximation provided by Corollary 1.8 reads, in that case:

$$u^\varepsilon(t,y) = u_1(t,y) + u_2\left(t, y + \frac{t\delta p_0 + \delta q_0}{\sqrt{\varepsilon}}\right) e^{i\phi_2^\varepsilon(t,y)} + o(1) \quad \text{in } L^\infty(\mathbb{R}; L^2(\mathbb{R}^3)),$$

where the phase shift is given by

$$\begin{aligned} \phi_2^\varepsilon(t,y) &= \frac{1}{\varepsilon} p_{02}\cdot(t\delta p_0 + \delta q_0) - \frac{1}{\sqrt{\varepsilon}}\delta p_0\cdot y + \frac{t}{2\varepsilon}\left(|p_{02}|^2 - |p_{01}|^2\right) \\ &= \frac{1}{\varepsilon} p_{02}\cdot\delta q_0 - \frac{1}{\sqrt{\varepsilon}}\delta p_0\cdot y - \frac{t}{2\varepsilon}|\delta p_0|^2. \end{aligned}$$

Notation. We write $a^\varepsilon(t) \lesssim b^\varepsilon(t)$ whenever there exists C independent of $\varepsilon \in (0,1]$ and t such that $a^\varepsilon(t) \leqslant C b^\varepsilon(t)$.

2. Spectral properties and consequences

In this section, we derive some useful properties for the Hamiltonian

$$H = -\frac{1}{2}\Delta + V.$$

Since the dependence upon ε is not addressed in this section, we assume $\varepsilon = 1$.

First, it follows for instance from [46] that Assumption 1.1 implies that H has no singular spectrum. Based on Morawetz estimates, we show that H has no eigenvalue, provided that the attractive part of V is sufficiently small. Therefore, the spectrum of H is purely absolutely continuous. Finally, again if the attractive part of V is sufficiently small, zero is not a resonance of H, so Strichartz estimates are available for e^{-itH}.

2.1. Morawetz estimates and a first consequence.

In this section, we want to treat both linear and nonlinear equations, so we consider

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = V\psi + \lambda|\psi|^{2\sigma}\psi, \quad \lambda \in \mathbb{R}. \tag{2.1}$$

Morawetz estimate in the linear case $\lambda = 0$ will show the absence of eigenvalues. In the nonlinear case $\lambda > 0$, these estimates will be a crucial tool for prove scattering in the quantum case. The following lemma and its proof are essentially a rewriting of the presentation from [3].

Proposition 2.1 (Morawetz inequality)**.** *Let $d \geqslant 3$, and V satisfying Assumption 1.1 for some $\mu > 2$. There exists $M = M(\mu, d) > 0$ such that if the attractive part of the potential satisfies*

$$(\partial_r V(x))_+ \leqslant \frac{M}{(1+|x|)^{\mu+1}}, \quad \forall x \in \mathbb{R}^d,$$

then any solution $\psi \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^d))$ to (2.1) *satisfies*

$$\lambda \iint_{\mathbb{R}\times\mathbb{R}^d} \frac{|\psi(t,x)|^{2\sigma+2}}{|x|} dtdx + \iint_{\mathbb{R}\times\mathbb{R}^d} \frac{|\psi(t,x)|^2}{(1+|x|)^{\mu+1}} dtdx \lesssim \|\psi\|^2_{L^\infty(\mathbb{R};H^1)}. \tag{2.2}$$

In other words, the main obstruction to global dispersion for V comes from $(\partial_r V)_+$, which is the attractive contribution of V in classical trajectories, while $(\partial_r V)_-$ is the repulsive part, which does not ruin the dispersion associated to $-\Delta$ (it may reinforce it, see e.g. [8], but repulsive potentials do not necessarily improve the dispersion, see [32]).

Proof. The proof follows standard arguments, based on virial identities with a suitable weight. We resume the main steps of the computations, and give more details on the choice of the weight in our context. For a real-valued function $h(x)$, we compute, for ψ solution to (3.1),

$$\frac{d}{dt}\int h(x)|\psi(t,x)|^2 dx = \operatorname{Im}\int \bar\psi(t,x)\nabla h(x)\cdot\nabla\psi(t,x)dx,$$

$$\begin{aligned}\frac{d}{dt}\operatorname{Im}\int \bar\psi(t,x)\nabla h(x)\cdot\nabla\psi(t,x)dx &= \int \nabla\bar\psi(t,x)\cdot\nabla^2 h(x)\nabla\psi(t,x)dx \\ &\quad -\frac14\int|\psi(t,x)|^2\Delta^2 h(x)dx - \int|\psi(t,x)|^2\nabla V\cdot\nabla h(x)dx \\ &\quad + \frac{\lambda\sigma}{\sigma+1}\int|\psi(t,x)|^{2\sigma+2}\Delta h(x)dx.\end{aligned} \tag{2.3}$$

In the case $V = 0$, the standard choice is $h(x) = |x|$, for which

$$\nabla h = \frac{x}{|x|}, \quad \nabla^2_{jk} h = \frac{1}{|x|}\left(\delta_{jk} - \frac{x_j x_k}{|x|^2}\right), \quad \Delta h \geqslant \frac{d-1}{h}, \quad \text{and } \Delta^2 h \leqslant 0 \text{ for } d \geqslant 3.$$

This readily yields Proposition 2.1 in the repulsive case $\partial_r V \leqslant 0$, since $\nabla h \in L^\infty$.

In the same spirit as in [3], we proceed by perturbation to construct a suitable weight when the attractive part of the potential is not too large. We seek a priori a radial weight, $h = h(|x|) \geqslant 0$, so we have

$$\begin{aligned}\Delta h &= h'' + \frac{d-1}{r}h', \\ \Delta^2 h &= h^{(4)} + 2\frac{d-1}{r}h^{(3)} + \frac{(d-1)(d-3)}{r^2}h'' - \frac{(d-1)(d-3)}{r^3}h', \\ \nabla^2_{jk} h &= \frac1r\left(\delta_{jk} - \frac{x_j x_k}{r^2}\right)h' + \frac{x_j x_k}{r^2}h''.\end{aligned}$$

We construct a function h such that $h', h'' \geqslant 0$, so the condition $\nabla^2 h \geqslant 0$ will remain. The goal is then to construct a radial function h such that the second line in (2.3) is non-negative, along with $\Delta h \geqslant \eta/|x|$ for some $\eta > 0$.

Case $d = 3$**.** In this case, the expression for $\Delta^2 h$ is simpler, and the above conditions read

$$\frac{1}{4}h^{(4)} + \frac{1}{r}h^{(3)} + \nabla V(x)\cdot\nabla h \leqslant 0,$$
$$h'' + \frac{2}{r}h' \geqslant \frac{\eta}{r}, \quad h', h'' \geqslant 0.$$

Since we do not suppose a priori that V is a radial potential, the first condition is not rigorous. We actually use the fact that for $h' \geqslant 0$, Assumption 1.1 implies

$$\nabla V(x)\cdot\nabla h \leqslant \left(\partial_r V(x)\right)_+ h'(r) \leqslant \frac{M}{(1+r)^{\mu+1}}h'(r).$$

To achieve our goal, it is therefore sufficient to require:

$$\frac{1}{4}h^{(4)} + \frac{1}{r}h^{(3)} + \frac{M}{(1+r)^{\mu+1}}h' \leqslant 0, \tag{2.4}$$

$$h'' + \frac{2}{r}h' \geqslant \frac{\eta}{r}, \quad h' \in L^\infty(\mathbb{R}_+),\ h', h'' \geqslant 0. \tag{2.5}$$

In view of (2.5), we seek

$$h'(r) = \eta + \int_0^r h''(\rho)d\rho.$$

Therefore, if $h'' \geqslant 0$ with $h'' \in L^1(\mathbb{R}_+)$, (2.5) will be automatically fulfilled. We now turn to (2.4). Since we want $h' \in L^\infty$, we may even replace h' by a constant in (2.4), and solve, for $C > 0$, the ODE

$$\frac{1}{4}h^{(4)} + \frac{1}{r}h^{(3)} + \frac{C}{(1+r)^{\mu+1}} = 0.$$

We readily have

$$h^{(3)}(r) = -\frac{4C}{r^4}\int_0^r \frac{\rho^4}{(1+\rho)^{\mu+1}}d\rho,$$

along with the properties $h^{(3)}(0) = 0$,

$$h^{(3)}(r) \underset{r\to\infty}{\sim} -\frac{k}{r^{\min(\mu,4)}}, \quad \text{for some } k > 0.$$

It is now natural to set

$$h''(r) = -\int_r^\infty h^{(3)}(\rho)d\rho,$$

so we have $h'' \in C([0,\infty);\mathbb{R}_+)$ and

$$h''(r) \underset{r\to\infty}{\sim} \frac{\kappa}{r^{\min(\mu-1,3)}}, \quad \text{for some } \kappa > 0.$$

This function is indeed in L^1 if and only if $\mu > 2$. We define h by $h(r) = \int_0^r h'(\rho)d\rho$,

$$h^{(3)}(r) = -\frac{K}{r^4}\int_0^r \frac{\rho^4}{(1+\rho)^{\mu+1}}d\rho, \tag{2.6}$$

for some $K > 0$, h'' and h' being given by the above relations: (2.5) is satisfied for any value of $K > 0$, and (2.4) boils down to an inequality of the form

$$-\frac{K}{4} + M\left(\eta + C(\mu)K\right) \leqslant 0, \tag{2.7}$$

where $C(\mu)$ is proportional to

$$\frac{1}{K}\|h'\|_{L^\infty} = \int_0^\infty \int_r^\infty \frac{1}{\rho^4}\int_0^\rho \frac{s^4}{(1+s)^{\mu+1}}dsd\rho dr.$$

We infer that (2.6) is satisfied for $K \gg \eta$, provided that $M < \frac{1}{4C(\mu)}$. Note then that by construction, we may also require

$$\frac{1}{4}\Delta^2 h + \nabla V \cdot \nabla h \leqslant \frac{-c_0}{(1+|x|)^{\mu+1}},$$

for $c_0 > 0$ morally very small.

Case $d \geqslant 4$. Resume the above reductions, pretending that the last two terms in $\Delta^2 h$ are not present: (2.6) just becomes

$$h^{(3)}(r) = -\frac{K}{r^{2d-2}} \int_0^r \frac{\rho^{2d-2}}{(1+\rho)^{\mu+1}} d\rho,$$

and we see that with h'' and h' defined like before, we have

$$rh'' - h' = -\eta - \int_0^r h'' + rh''.$$

Since this term is negative at $r = 0$ and has a non-positive derivative, we have $rh'' - h' \leqslant 0$, so finally $\Delta^2 h \leqslant 0$. □

We infer that H has no eigenvalue. Indeed, if there were an L^2 solution $\psi = \psi(x)$ to $H\psi = E\psi$, $E \in \mathbb{R}$, then $\psi \in H^2(\mathbb{R}^d)$, and $\psi(x)e^{-iEt}$ would be an H^1 solution to (2.1) for $\lambda = 0$. This is contradiction with the global integrability in time from (2.2), so $\sigma_{\rm pp}(H) = \emptyset$.

2.2. **Strichartz estimates.** In [3, Proposition 3.1], it is proved that zero is not a resonance of H, but with a definition of resonance which is not quite the definition in [52], which contains a result that we want to use. So we shall resume the argument.

By definition (as in [52]), zero is a resonance of H, if there is a distributional solution $\psi \notin L^2$, such that $\langle x\rangle^{-s}\psi \in L^2(\mathbb{R}^d)$ for all $s > \frac{1}{2}$, to $H\psi = 0$.

Corollary 2.2. *Under the assumptions of Proposition 2.1, zero is not a resonance of H.*

Proof. Suppose that zero is a resonance of H. Then by definition, we obtain a stationary distributional solution of (2.1) (case $\lambda = 0$), $\psi = \psi(x)$, and we may assume that it is real-valued. Since $\Delta\psi = 2V\psi$, Assumption 1.1 implies

$$\langle x\rangle^{\mu-s}\,\Delta\psi \in L^2(\mathbb{R}^d), \quad \forall s > \frac{1}{2}.$$

This implies that $\nabla\psi \in L^2$, by taking for instance $s = 1$ in

$$\int |\nabla\psi|^2 = -\int \langle x\rangle^{-s}\,\psi\,\langle x\rangle^{s}\,\Delta\psi.$$

By definition, for all test function φ,

$$\frac{1}{2}\int_{\mathbb{R}^d} \nabla\varphi(x)\cdot\nabla\psi(x)dx + \int_{\mathbb{R}^d} V(x)\varphi(x)\psi(x)dx = 0. \tag{2.8}$$

Let h be the weight constructed in the proof of Proposition 2.1, and consider

$$\varphi = \psi\Delta h + 2\nabla\psi\cdot\nabla h.$$

Since $\nabla h \in L^\infty$, $\nabla^2 h(x) = \mathcal{O}(\langle x\rangle^{-1})$, and $\nabla^3 h(x) = \mathcal{O}(\langle x\rangle^{-2})$, we see that $\varphi \in H^1$, and that this choice is allowed in (2.8). Integration by parts then yields (2.3) (where the left hand side is now zero):

$$0 = \int \nabla\psi\cdot\nabla^2 h\nabla\psi - \frac{1}{4}\int \psi^2\Delta^2 h - \int \psi^2\nabla V\cdot\nabla h.$$

By construction of h, this implies

$$\int_{\mathbb{R}^d} \frac{\psi(x)^2}{(1+|x|)^{\mu+1}}dx \leqslant 0,$$

hence $\psi \equiv 0$. □

Therefore, [52, Theorem 1.4] implies non-endpoint global in time Strichartz estimates. In the case $d = 3$, we know from [31] that (in view of the above spectral properties)

$$\|e^{-itH}\|_{L^1\to L^\infty} \leqslant C|t|^{-d/2}, \quad \forall t \neq 0,$$

a property which is stronger than Strichartz estimates, and yields the endpoint Strichartz estimate missing in [52], from [41]. On the other hand, this dispersive estimate does not seem to be known under Assumption 1.1 with $\mu > 2$ when $d \geqslant 4$: stronger assumptions are always present so far (see e.g. [7, 22]). However, endpoint Strichartz estimates for $d \geqslant 4$ are a consequence of [19, Theorem 1.1], under the assumptions of Proposition 2.1.

Proposition 2.3. *Let $d \geqslant 3$. Under the assumptions of Proposition 2.1, for all (q, r) such that*

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right), \quad 2 < q \leqslant \infty, \tag{2.9}$$

there exists $C = C(q, d)$ such that

$$\|e^{-itH}f\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d))} \leqslant C\|f\|_{L^2(\mathbb{R}^d)}, \quad \forall f \in L^2(\mathbb{R}^d).$$

It is classical that this homogeneous Strichartz estimate, a duality argument and Christ-Kiselev's Theorem imply the inhomogeneous counterpart. For two admissible pairs (q_1, r_1) and (q_2, r_2) (that is, satisfying (2.9)), there exists C_{q_1,q_2} independent of the time interval I such that if we denote by

$$R(F)(t,x) = \int_{I\cap\{s\leqslant t\}} e^{-i(t-s)H}F(s,x)ds,$$

we have

$$\|R(F)\|_{L^{q_1}(I;L^{r_1}(\mathbb{R}^d))} \leqslant C_{q_1,q_2}\|F\|_{L^{q_2'}(I;L^{r_2'}(\mathbb{R}^d))}, \quad \forall F \in L^{q_2'}(I; L^{r_2'}(\mathbb{R}^d)).$$

Note that the assumption $\mu > 2$ seems essentially sharp in order to have global in time Strichartz estimates. The result remains true for $\mu = 2$ ([5, 6]), but in [32], the authors prove that for repulsive potentials which are homogeneous of degree smaller than 2, global Strichartz estimates fail to exist.

3. Quantum scattering

In this section, we prove Theorem 1.4. Since the dependence upon ε is not measured in Theorem 1.4, we shall consider the case $\varepsilon = 1$, corresponding to

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = V\psi + |\psi|^{2\sigma}\psi. \tag{3.1}$$

We split the proof of Theorem 1.4 into two steps. First, we solve the Cauchy problem with data prescribed at $t = -\infty$, that is, we show the existence of wave operators. Then, given an initial datum at $t = 0$, we show that the (global) solution to (3.1) behaves asymptotically like a free solution, which corresponds to asymptotic completeness.

For each of these two steps, we first show that the nonlinearity is negligible for large time, and then recall that the potential is negligible for large time (linear scattering). This means that for any $\tilde{\psi}_- \in H^1(\mathbb{R}^d)$, there exists a unique $\psi \in C(\mathbb{R}; H^1(\mathbb{R}^d))$ solution to (3.1) such that

$$\|\psi(t) - e^{-itH}\tilde{\psi}_-\|_{H^1(\mathbb{R}^d)} \underset{t\to-\infty}{\longrightarrow} 0,$$

and for any $\varphi \in H^1(\mathbb{R}^d)$, there exist a unique $\psi \in C(\mathbb{R}; H^1(\mathbb{R}^d))$ solution to (3.1) and a unique $\tilde{\psi}_+ \in H^1(\mathbb{R}^d)$ such that

$$\|\psi(t) - e^{-itH}\tilde{\psi}_+\|_{H^1(\mathbb{R}^d)} \underset{t\to+\infty}{\longrightarrow} 0.$$

Then, we recall that the potential V is negligible for large time. We will adopt the following notations for the propagators,

$$U(t) = e^{i\frac{t}{2}\Delta}, \quad U_V(t) = e^{-itH}.$$

In order to construct wave operators which show that the nonlinearity can be neglected for large time, we shall work with an H^1 regularity, on the Duhamel's formula associated to (3.1) in terms of U_V, with a prescribed asymptotic behavior as $t \to -\infty$:

$$\psi(t) = U_V(t)\tilde{\psi}_- - i\int_{-\infty}^{t} U_V(t-s)\left(|\psi|^{2\sigma}\psi(s)\right)ds. \tag{3.2}$$

Applying the gradient to this formulation brings up the problem of non-commutativity with U_V. The worst term is actually the linear one, $U_V(t)\tilde{\psi}_-$, since

$$\nabla\left(U_V(t)\tilde{\psi}_-\right) = U_V(t)\nabla\tilde{\psi}_- - i\int_0^t U_V(t-s)\left((U_V(s)\tilde{\psi}_-)\nabla V\right)ds.$$

Since the construction of wave operators relies on the use of Strichartz estimates, it would be necessary to have an estimate of

$$\left\|\nabla\left(U_V(t)\tilde{\psi}_-\right)\right\|_{L^qL^r}$$

in terms of ψ_-, for admissible pairs (q,r). Proposition 2.3 yields

$$\left\|\nabla\left(U_V(t)\tilde{\psi}_-\right)\right\|_{L^qL^r} \lesssim \|\nabla\tilde{\psi}_-\|_{L^2} + \|(U_V(t)\tilde{\psi}_-)\nabla V\|_{L^{\tilde{q}'}L^{\tilde{r}'}},$$

for any admissible pair $(\tilde{q},\tilde{r})$. In the last factor, time is present only in the term $U_V(t)\tilde{\psi}_-$, so to be able to use Strichartz estimates again, we need to consider $\tilde{q} = 2$, in which case $\tilde{r} = 2^* := \frac{2d}{d-2}$:

$$\|(U_V(t)\tilde{\psi}_-)\nabla V\|_{L^2L^{2^{*\prime}}} \leqslant \|U_V(t)\tilde{\psi}_-\|_{L^2L^{2^*}}\|\nabla V\|_{L^{d/2}},$$

where Assumption 1.1 implies $\nabla V \in L^{d/2}(\mathbb{R}^d)$ as soon as $\mu > 1$. Using the endpoint Strichartz estimate from Proposition 2.3, we have

$$\|U_V(t)\tilde{\psi}_-\|_{L^2L^{2^*}} \lesssim \|\tilde{\psi}_-\|_{L^2},$$

and we have:

Lemma 3.1. *Let $d \geqslant 3$. Under the assumptions of Proposition 2.1, for all admissible pair (q,r),*

$$\|e^{-itH}f\|_{L^q(\mathbb{R};W^{1,r}(\mathbb{R}^d))} \lesssim \|f\|_{H^1(\mathbb{R}^d)}.$$

We shall rather use a vector-field, for we believe this approach may be interesting in other contexts.

3.1. **Vector-field.** We introduce a vector-field which naturally commutes with U_V, and is comparable with the gradient.

From Assumption 1.1, V is bounded, so there exists $c_0 \geqslant 0$ such that $V + c_0 \geqslant 0$. We shall consider the operator

$$A = \sqrt{H + c_0} = \sqrt{-\frac{1}{2}\Delta + V + c_0}.$$

Lemma 3.2. *Let $d \geqslant 3$, and V satisfying Assumption 1.1 with $V + c_0 \geqslant 0$. For every $1 < r < \infty$, there exists C_r, K_r such that for all $f \in W^{1,r}(\mathbb{R}^d)$,*

$$\|Af\|_{L^r} \leqslant C_r \left(\|f\|_{L^r} + \|\nabla f\|_{L^r}\right) \leqslant K_r \left(\|f\|_{L^r} + \|Af\|_{L^r}\right). \tag{3.3}$$

Proof. The first inequality is very close to [19, Theorem 1.2], and the proof can readily be adapted. On the other hand, the second inequality would require the restriction $4/3 < r < 4$ if we followed the same approach, based on Stein's interpolation theorem (a similar approach for followed in e.g. [43]). We actually take advantage of the smoothness of the potential V to rather apply Calderón–Zygmund result on the action of pseudo-differential operators.

We readily check that the two functions

$$a(x,\xi) = \sqrt{\frac{\frac{|\xi|^2}{2} + V(x) + c_0}{1 + |\xi|^2}}, \quad b(x,\xi) = \sqrt{\frac{|\xi|^2}{\frac{|\xi|^2}{2} + V(x) + c_0 + 1}},$$

are symbols of order zero, in the sense that they satisfy

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| + |\partial_x^\alpha \partial_\xi^\beta b(x,\xi)| \leqslant C_{\alpha,\beta} \left\langle \xi \right\rangle^{-|\beta|},$$

for all $\alpha, \beta \in \mathbb{N}^d$. This implies that the pseudo-differential operators of symbol a and b, respectively, are bounded on $L^r(\mathbb{R}^d)$, for all $1 < r < \infty$; see e.g. [53, Theorem 5.2]. In the case of a, this yields the first inequality in (3.3), and in the case of b, this yields the second inequality. □

3.2. **Wave operators.** With the tools presented in the previous section, we can prove the following result by adapting the standard proof of the case $V = 0$, as established in [29].

Proposition 3.3. *Let $d \geqslant 3$, $\frac{2}{d} \leqslant \sigma < \frac{2}{d-2}$, and V satisfying Assumption 1.1 for some $\mu > 2$. For all $\tilde{\psi}_- \in H^1(\mathbb{R}^d)$, there exists a unique*

$$\psi \in C((-\infty, 0]; H^1(\mathbb{R}^d)) \cap L^{\frac{4\sigma+4}{d\sigma}}((-\infty, 0); L^{2\sigma+2}(\mathbb{R}^d))$$

solution to (3.1) *such that*

$$\|\psi(t) - e^{-itH}\tilde{\psi}_-\|_{H^1(\mathbb{R}^d)} \underset{t\to-\infty}{\longrightarrow} 0.$$

Proof. The main part of the proof is to prove that (3.2) has a fixed point. Let

$$q = \frac{4\sigma + 4}{d\sigma}.$$

The pair $(q, 2\sigma+2)$ is admissible, in the sense that it satisfies (2.9). With the notation $L^\beta_T Y = L^\beta(]-\infty,-T]; Y)$, we introduce:

$$\begin{aligned} X_T := \Big\{ \psi \in C(]-\infty,-T]; H^1)\ ;\ & \|\psi\|_{L^q_T L^{2\sigma+2}} \leqslant K\|\tilde\psi_-\|_{L^2}, \\ \|\nabla\psi\|_{L^q_T L^{,2\sigma+2}} \leqslant K\|\tilde\psi_-\|_{H^1}, \quad & \|\psi\|_{L^\infty_T L^2} \leqslant 2\|\tilde\psi_-\|_{L^2}\,, \\ \|\nabla\psi\|_{L^\infty_T L^2} \leqslant K\|\tilde\psi_-\|_{H^1}, \quad & \|\psi\|_{L^q_T L^{2\sigma+2}} \leqslant 2\left\|U_V(\cdot)\tilde\psi_-\right\|_{L^q_T L^{2\sigma+2}} \Big\}, \end{aligned}$$

where K will be chosen sufficiently large in terms of the constants present in Strichartz estimates presented in Proposition 2.3. Set $r = s = 2\sigma+2$: we have

$$\frac{1}{r'} = \frac{1}{r} + \frac{2\sigma}{s}, \quad \frac{1}{q'} = \frac{1}{q} + \frac{2\sigma}{k},$$

where $q \leqslant k < \infty$ since $2/d \leqslant \sigma < 2/(d-2)$. Denote by $\Phi(\psi)$ the right hand side of (3.2). For $\psi \in X_T$, Strichartz estimates and Hölder inequality yield, for all admissible pairs (q_1, r_1):

$$\begin{aligned} \|\Phi(\psi)\|_{L^{q_1}_T L^{r_1}} &\leqslant C_{q_1}\|\tilde\psi_-\|_{L^2} + C\left\||\psi|^{2\sigma}\psi\right\|_{L^{q'}_T L^{r'}} \\ &\leqslant C_{q_1}\|\tilde\psi_-\|_{L^2} + C\|\psi\|^{2\sigma}_{L^k_T L^s}\|\psi\|_{L^q_T L^r} \\ &\leqslant C_{q_1}\|\tilde\psi_-\|_{L^2} + C\|\psi\|^{2\sigma\theta}_{L^q_T L^r}\|\psi\|^{2\sigma(1-\theta)}_{L^\infty_T L^r}\|\psi\|_{L^q_T L^r}, \end{aligned}$$

for some $0 < \theta \leqslant 1$, where we have used the property $r = s = 2\sigma+2$. Sobolev embedding and the definition of X_T then imply:

$$\|\Phi(\psi)\|_{L^{q_1}_T L^{r_1}} \leqslant C_{q_1}\|\tilde\psi_-\|_{L^2} + C\left\|U_V(\cdot)\tilde\psi_-\right\|^{2\sigma\theta}_{L^q_T L^r}\|\psi\|^{2\sigma(1-\theta)}_{L^\infty_T H^1}\|\psi\|_{L^q_T L^r}.$$

We now apply the operator A. Since A commutes with H, we have

$$\|A\Phi(\psi)\|_{L^{q_1}_T L^{r_1}} \lesssim \|A\tilde\psi_-\|_{L^2} + \left\|A\left(|\psi|^{2\sigma}\psi\right)\right\|_{L^{q'}_T L^{r'}}.$$

In view of Lemma 3.2, we have successively,

$$\begin{aligned} \|A\tilde\psi_-\|_{L^2} &\lesssim \|\tilde\psi_-\|_{H^1}, \\ \left\|A\left(|\psi|^{2\sigma}\psi\right)\right\|_{L^{q'}_T L^{r'}} &\lesssim \left\||\psi|^{2\sigma}\psi\right\|_{L^{q'}_T L^{r'}} + \left\|\nabla\left(|\psi|^{2\sigma}\psi\right)\right\|_{L^{q'}_T L^{r'}} \\ &\lesssim \|\psi\|^{2\sigma}_{L^k_T L^s}\left(\|\psi\|_{L^q_T L^r} + \|\nabla\psi\|_{L^q_T L^r}\right) \\ &\lesssim \|\psi\|^{2\sigma}_{L^k_T L^s}\left(\|\psi\|_{L^q_T L^r} + \|A\psi\|_{L^q_T L^r}\right). \end{aligned}$$

We infer along the same lines as above,

$$\|\nabla\Phi(\psi)\|_{L^{q_1}_T L^{r_1}} \lesssim \|\tilde\psi_-\|_{H^1} + \left\|U_V(\cdot)\tilde\psi_-\right\|^{2\sigma\theta}_{L^q_T L^r}\|\psi\|^{2\sigma(1-\theta)}_{L^\infty_T H^1}\left(\|\psi\|_{L^q_T L^r} + \|A\psi\|_{L^q_T L^r}\right).$$

We have also

$$\|\Phi(\psi)\|_{L^q_T L^r} \leqslant \left\|U_V(\cdot)\tilde\psi_-\right\|_{L^q_T L^r} + C\left\|U_V(\cdot)\tilde\psi_-\right\|^{2\sigma\theta}_{L^q_T L^r}\|\psi\|^{2\sigma(1-\theta)}_{L^\infty_T H^1}\|\psi\|_{L^q_T L^r}.$$

From Strichartz estimates, $U_V(\cdot)\tilde\psi_- \in L^q(\mathbb{R}; L^r)$, so

$$\left\|U_V(\cdot)\tilde\psi_-\right\|_{L^q_T L^r} \to 0 \quad \text{as } T \to +\infty.$$

Since $\theta > 0$, we infer that Φ sends X_T to itself, for T sufficiently large.

We have also, for $\psi_2, \psi_1 \in X_T$:

$$\begin{aligned}\|\Phi(\psi_2) - \Phi(\psi_1)\|_{L^q_T L^r} &\lesssim \max_{j=1,2} \|\psi_j\|^{2\sigma}_{L^k_T L^s} \|\psi_2 - \psi_1\|_{L^q_T L^r} \\ &\lesssim \left\|U_V(\cdot)\tilde\psi_-\right\|^{2\sigma\theta}_{L^q_T L^r} \|\tilde\psi_-\|^{2\sigma(1-\theta)}_{H^1} \|\psi_2 - \psi_1\|_{L^q_T L^r}\,.\end{aligned}$$

Up to choosing T larger, Φ is a contraction on X_T, equipped with the distance

$$d(\psi_2, \psi_1) = \|\psi_2 - \psi_1\|_{L^q_T L^r} + \|\psi_2 - \psi_1\|_{L^\infty_T L^2}\,,$$

which makes it a Banach space (see [13]). Therefore, Φ has a unique fixed point in X_T, solution to (3.2). It follows from (3.3) that this solution has indeed an H^1 regularity with

$$\|\psi(t) - e^{-itH}\tilde\psi_-\|_{H^1(\mathbb{R}^d)} \underset{t\to-\infty}{\longrightarrow} 0.$$

In view of the global well-posedness results for the Cauchy problem associated to (3.1) (see e.g. [13]), the proposition follows. □

3.3. **Asymptotic completeness.** There are mainly three approaches to prove asymptotic completeness for nonlinear Schrödinger equations (without potential). The initial approach ([28]) consists in working with a Σ regularity. This makes it possible to use the operator $x+it\nabla$, which enjoys several nice properties, and to which an important evolution law (the pseudo-conformal conservation law) is associated; see Section 4 for more details. This law provides important a priori estimates, from which asymptotic completeness follows very easily the the case $\sigma \geqslant 2/d$, and less easily for some range of σ below $2/d$; see e.g. [13].

The second historical approach relaxes the localization assumption, and allows to work in $H^1(\mathbb{R}^d)$, provided that $\sigma > 2/d$. It is based on Morawetz inequalities: asymptotic completeness is then established in [44, 29] for the case $d \geqslant 3$, and in [47] for the low dimension cases $d = 1, 2$, by introducing more intricate Morawetz estimates. Note that the case $d \leqslant 2$ is already left out in our case, since we have assumed $d \geqslant 3$ to prove Proposition 3.3.

The most recent approach to prove asymptotic completeness in H^1 relies on the introduction of interaction Morawetz estimates in [16], an approach which has been revisited since, in particular in [49] and [30]. See also [55] for a very nice alternative approach of the use of interaction Morawetz estimates. In the presence of an external potential, this approach was used in [12], by working with Morrey-Campanato type norms.

An analogue for the pseudo-conformal evolution law is available (see e.g. [13]), but it seems that in the presence of V satisfying Assumption 1.1, it cannot be exploited to get satisfactory estimates. We shall rather consider Morawetz estimates as in [29], and thus give an alternative proof of the corresponding result from [12]: note that for $\lambda = 1$, the first part of (2.2) provides exactly the same a priori estimate as in [29].

Proposition 3.4. *Let $d \geqslant 3$, $\frac{2}{d} < \sigma < \frac{2}{d-2}$, and V satisfying Assumption 1.1 for some $\mu > 2$. There exists $M = M(\mu, d)$ such that if the attractive part of the potential satisfies*

$$(\partial_r V(x))_+ \leqslant \frac{M}{(1+|x|)^{\mu+1}}, \quad \forall x \in \mathbb{R}^d,$$

then for all $\varphi \in H^1(\mathbb{R}^d)$, there exist a unique $\psi \in C(\mathbb{R}; H^1(\mathbb{R}^d))$ solution to (3.1) *with $\psi_{|t=0} = \varphi$, and a unique $\tilde\psi_+ \in H^1(\mathbb{R}^d)$ such that*

$$\|\psi(t) - e^{-itH}\tilde\psi_+\|_{H^1(\mathbb{R}^d)} \underset{t\to+\infty}{\longrightarrow} 0.$$

In addition, $\psi, \nabla\psi \in L^q(\mathbb{R}_+, L^r(\mathbb{R}^d))$ for all admissible pairs (q, r).

Proof. The proof follows that argument presented in [29] (and resumed in [26]), so we shall only described the main steps and the modifications needed in the present context. The key property in the proof consists in showing that there exists $2 < r < \frac{2d}{d-2}$ such that

$$\|\psi(t)\|_{L^r} \underset{t\to+\infty}{\longrightarrow} 0. \tag{3.4}$$

Since $\psi \in L^\infty(\mathbb{R}; H^1)$ (see e.g. [13]), we infer that the above property is true for *all* $2 < r < \frac{2d}{d-2}$. This aspect is the only one that requires some adaptation in our case. Indeed, once this property is at hand, the end of the proof relies on Strichartz estimates applied to Duhamel's formula. In our framework, since we first want to get rid of the nonlinearity only (and not the potential V yet), we consider

$$\psi(t) = U_V(t)\varphi - i\int_0^t U_V(t-s)\left(|\psi|^{2\sigma}\psi(s)\right)ds,$$

and thanks to Proposition 2.3, it is possible to follow exactly the same lines as in [29] (see also [54]) in order to infer Proposition 3.4.

Therefore, the only delicate point is to show that (3.4) holds for some $2 < r < \frac{2d}{d-2}$. This corresponds to Corollary 5.1 in [29] (Lemme 12.6 in [26]). The main technical remark is that once Morawetz estimate is available (the one given in Proposition 2.1, whose final conclusion does not depend on V), one uses dispersive properties of the group $U(t)$. As mentioned above, we do not want to use dispersive properties of $U_V(t)$, since they are known only in the case $d = 3$ (on the other hand, this means that the result is straightforward in the case $d = 3$, from [29] and [31]). So instead, we consider Duhamel's formula for (3.1) in terms of $U(t)$, which reads

$$\psi(t) = U(t)\varphi - i\int_0^t U(t-s)\left(|\psi|^{2\sigma}\psi(s)\right)ds - i\int_0^t U(t-s)\left(V\psi(s)\right)ds. \tag{3.5}$$

The new term compared to [29] is of course the last term in (3.5), and so the nonlinearity is now

$$f(\psi) = |\psi|^{2\sigma}\psi + V\psi.$$

Following the argument from [29] (or [26]), it suffices to prove the following two properties:

1. There exist $r_1 > 2^* = \frac{2d}{d-2}$ and $\alpha > 0$ such that

$$\left\|\int_{t_0}^{t-\ell} U(t-s)\left(V\psi(s)\right)ds\right\|_{L^{r_1}(\mathbb{R}^d)} \leqslant C\ell^{-\alpha}\|\psi\|_{L^\infty(\mathbb{R};H^1)}, \tag{3.6}$$

Consider a Lebesgue index r_1 slightly larger than 2^*,

$$\frac{1}{r_1} = \frac{1}{2^*} - \eta, \quad 0 < \eta \ll 1.$$

Let $\ell > 0$, and consider

$$I_1(t) = \left\|\int_{t_0}^{t-\ell} U(t-s)\left(V\psi(s)\right)ds\right\|_{L^{r_1}(\mathbb{R}^d)}.$$

Standard dispersive estimates for U yield

$$I_1(t) \lesssim \int_{t_0}^{t-\ell} (t-s)^{-\delta_1}\|V\psi(s)\|_{L^{r_1'}}ds,$$

where δ_1 is given by

$$\delta_1 = d\left(\frac{1}{2} - \frac{1}{r_1}\right) = 1 + \eta d.$$

Now we apply Hölder inequality in space, in view of the identity

$$\frac{1}{r_1'} = \frac{1}{2} + \frac{1}{d} - \eta = \underbrace{\frac{1}{2} - \frac{1}{d} + \eta}_{1/k} + \underbrace{\frac{2}{d} - 2\eta}_{1/q}.$$

For $\eta > 0$ sufficiently small, $V \in L^q(\mathbb{R}^d)$ since $\mu > 2$, and so

$$\|V\psi(s)\|_{L^{r_1'}} \leqslant \|V\|_{L^q}\|\psi(s)\|_{L^k} \lesssim \|\psi\|_{L^\infty(\mathbb{R};H^1)},$$

where we have used Sobolev embedding, since $2 < k < 2^*$. We infer

$$\begin{aligned} I_1(t) &\lesssim \int_{t_0}^{t-\ell} (t-s)^{-\delta_1} ds \|\psi\|_{L^\infty(\mathbb{R};H^1)} \lesssim \int_\ell^\infty s^{-\delta_1} ds \|\psi\|_{L^\infty(\mathbb{R};H^1)} \\ &\lesssim \ell^{1-\delta_1}\|\psi\|_{L^\infty(\mathbb{R};H^1)} = \ell^{-\eta d}\|\psi\|_{L^\infty(\mathbb{R};H^1)}. \end{aligned}$$

2. Now for fixed $\ell > 0$, let

$$I_2(t) = \left\| \int_{t-\ell}^t U(t-s)\left(V\psi(s)\right) ds \right\|_{L^{2\sigma+2}(\mathbb{R}^d)}.$$

We show that for any $\ell > 0$, $I_2(t) \to 0$ as $t \to \infty$. Dispersive estimates for $U(t)$ yield

$$I_2(t) \lesssim \int_{t-\ell}^t (t-s)^{-\delta} \|V\psi(s)\|_{L^{\frac{2\sigma+2}{2\sigma+1}}} ds, \quad \delta = d\left(\frac{1}{2} - \frac{1}{2\sigma+2}\right) = \frac{d\sigma}{2\sigma+2} < 1.$$

For (a small) α to be fixed later, Hölder inequality yields

$$\|V\psi(s)\|_{L^{\frac{2\sigma+2}{2\sigma+1}}} = \left\| |x|^\alpha V \frac{\psi(s)}{|x|^\alpha} \right\|_{L^{\frac{2\sigma+2}{2\sigma+1}}} \leqslant \||x|^\alpha V\|_{L^{\frac{\sigma+1}{\sigma}}} \left\| \frac{\psi(s)}{|x|^\alpha} \right\|_{L^{2\sigma+2}}.$$

Note that for $0 < \alpha \ll 1$, $\||x|^\alpha V\|_{L^{\frac{\sigma+1}{\sigma}}}$ is finite, since $\frac{\sigma+1}{\sigma} > \frac{d}{2}$ and $\mu > 2$. For $0 < \theta < 1$, write

$$\begin{aligned} \left\| \frac{\psi(s)}{|x|^\alpha} \right\|_{L^{2\sigma+2}} = \left\| \frac{|\psi(s)|^\theta}{|x|^\alpha} |\psi(s)|^{1-\theta} \right\|_{L^{2\sigma+2}} &\leqslant \left\| \frac{\psi(s)}{|x|^{\alpha/\theta}} \right\|_{L^{2\sigma+2}}^\theta \|\psi(s)\|_{L^{2\sigma+2}}^{1-\theta} \\ &\lesssim \left\| \frac{\psi(s)}{|x|^{\alpha/\theta}} \right\|_{L^{2\sigma+2}}^\theta \|\psi\|_{L^\infty(\mathbb{R};H^1)}^{1-\theta}. \end{aligned}$$

To use Morawetz estimate, we impose $\alpha/\theta = 1/(2\sigma+2)$, so that we have

$$\left\| \frac{\psi(s)}{|x|^\alpha} \right\|_{L^{2\sigma+2}} \lesssim \left(\int_{\mathbb{R}^d} \frac{|\psi(s,x)|^{2\sigma+2}}{|x|} dx \right)^{\theta/(2\sigma+2)} \|\psi\|_{L^\infty(\mathbb{R};H^1)}^{1-\theta}.$$

We conclude by applying Hölder inequality in time: since $\delta < 1$, the map $s \mapsto (t-s)^{-\delta}$ belongs to $L^q_{\rm loc}$ for $1 \leqslant q \leqslant 1+\gamma$ and $\gamma > 0$ sufficiently small. Let $q = 1+\gamma$ with $0 < \gamma \ll 1$ so that $s \mapsto (t-s)^{-\delta} \in L^q_{\rm loc}$: we have $q' < \infty$, and we can choose $0 < \theta \ll 1$ (or equivalently $0 < \eta \ll 1$) so that

$$\theta q' = 2\sigma + 2.$$

We end up with

$$I_2(t) \lesssim \ell^\beta \left(\iint_{[t-\ell,t]\times\mathbb{R}^d} \frac{|\psi(s,x)|^{2\sigma+2}}{|x|} ds dx \right)^{1/(2\sigma+2)q'},$$

for some $\beta > 0$. The last factor goes to zero as $t \to \infty$ from Proposition 2.1. □

3.4. **Scattering.** Under Assumption 1.1, a linear scattering theory is available, provided that $\mu > 1$; see e.g. [20, Section 4.6]. This means that the following strong limits exist in $L^2(\mathbb{R}^d)$,

$$\lim_{t\to-\infty} U_V(-t)U(t), \quad \text{and} \quad \lim_{t\to+\infty} U(-t)U_V(t),$$

where the second limit usually requires to project on the continuous spectrum. Recall that this projection is the identity in our framework.

Lemma 3.5. *Let $d \geqslant 3$, V satisfying Assumption 1.1 with $p > 1$. Then the strong limit*

$$\lim_{t\to-\infty} U_V(-t)U(t)$$

exists in $H^1(\mathbb{R}^d)$.

Proof. Following Cook's method ([51, Theorem XI.4]), it suffices to prove that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$t \mapsto \|U_V(-t)VU(t)\varphi\|_{H^1} \in L^1((-\infty,-1]).$$

For the L^2 norm, we have

$$\|U_V(-t)VU(t)\varphi\|_{L^2} = \|VU(t)\varphi\|_{L^2}\,.$$

Assumption 1.1 implies that $V \in L^q(\mathbb{R}^d)$ for all $q > d/\mu$. For $\mu > 1$, let q be given by

$$\frac{1}{q} = \frac{1}{d} + \eta, \text{ with } \eta > 0 \text{ and } q > \frac{d}{\mu}.$$

We apply Hölder inequality with the identity

$$\frac{1}{2} = \frac{1}{q} + \underbrace{\frac{1}{2} - \frac{1}{d} - \eta}_{1/r}\,.$$

Using dispersive estimates for $U(t)$, we have

$$\|VU(t)\varphi\|_{L^2} \lesssim \|U(t)\varphi\|_{L^r} \lesssim |t|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)}\|\varphi\|_{L^{r'}} = |t|^{-1-d\eta}\|\varphi\|_{L^{r'}},$$

hence the existence of the strong limit in L^2.

For the H^1 limit, recall that from Lemma 3.2,

$$\|\nabla U_V(-t)VU(t)\varphi\|_{L^2} \lesssim \|AU_V(-t)VU(t)\varphi\|_{L^2}$$

Since A commutes with U_V which is unitary on L^2, the right hand side is equal to

$$\|AVU(t)\varphi\|_{L^2} \lesssim \|VU(t)\varphi\|_{H^1},$$

where we have used Lemma 3.2 again. Now

$$\|VU(t)\varphi\|_{H^1} \leqslant \|VU(t)\varphi\|_{L^2} + \|\nabla V \times U(t)\varphi\|_{L^2} + \|VU(t)\nabla\varphi\|_{L^2},$$

and each term is integrable, like for the L^2 limit, from Assumption 1.1. □

In the case $d = 3$, the dispersive estimates established by Goldberg [31] make it possible to prove asymptotic completeness in H^1 by Cook's method as well: for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$t \mapsto \|U(-t)VU_V(t)\varphi\|_{H^1} \in L^1(\mathbb{R}),$$

a property which can be proven by the same computations as above, up to changing the order of the arguments. To complete the proof of Theorem 1.4, it therefore remains to prove that for $d \geqslant 4$, $\psi_+ \in H^1(\mathbb{R}^d)$ and

$$\|\psi(t) - U(t)\psi_+\|_{H^1(\mathbb{R}^d)} \underset{t\to\infty}{\longrightarrow} 0. \tag{3.7}$$

It follows from the above results that

$$\psi(t) = U(t)\psi_+ + i\int_t^{+\infty} U(t-s)\left(|\psi|^{2\sigma}\psi(s)\right)ds + i\int_t^{+\infty} U(t-s)\left(V(\psi(s)\right)ds,$$

and that $\psi, \nabla\psi \in L^q(\mathbb{R}; L^r(\mathbb{R}^d))$ for all admissible pairs (q,r). Since we have

$$\psi_+ = U(-t)\psi(t) - i\int_t^{+\infty} U(-s)\left(|\psi|^{2\sigma}\psi(s)\right)ds - i\int_t^{+\infty} U(-s)\left(V(\psi(s)\right)ds,$$

the previous estimates show that $\psi_+ \in H^1(\mathbb{R}^d)$, along with (3.7).

4. Scattering for the asymptotic envelope

In this section, we prove Theorem 1.5. The general argument is similar to the quantum case: we first prove that the nonlinear term can be neglected to large time, and then rely on previous results to neglect the potential. Recall that in view of Assumption 1.1, the time dependent harmonic potential $\frac{1}{2}\langle Q(t)y, y\rangle$ satisfies

$$\left\| \frac{d^\alpha}{dt^\alpha} Q(t)\right\| \lesssim \langle t\rangle^{-\mu-2-\alpha}, \quad \alpha \in \mathbb{N}, \tag{4.1}$$

where $\|\cdot\|$ denotes any matricial norm. We denote by

$$H_Q = -\frac{1}{2}\Delta + \frac{1}{2}\langle Q(t)y, y\rangle$$

the time-dependent Hamiltonian present in (1.11). Like in the quantum case, we show that the nonlinearity is negligible for large time by working on Duhamel's formula associated to (1.11) in terms of H_Q. Since H_Q depends on time, we recall that the propagator $U_Q(t,s)$ is the operator which maps u_0 to $u_{\rm lin}(t)$, where $u_{\rm lin}$ solves

$$i\partial_t u_{\rm lin} + \frac{1}{2}\Delta u_{\rm lin} = \frac{1}{2}\langle Q(t)y, y\rangle u_{\rm lin}; \quad u_{\rm lin}(s,y) = u_0(y).$$

It is a unitary dynamics, in the sense that $U_Q(s,s) = 1$, and $U_Q(t,\tau)U_Q(\tau,s) = U_Q(t,s)$; see e.g. [20]. Then to prove the existence of wave operators, we consider the integral formulation

$$u(t) = U_Q(t,0)\tilde{u}_- - i\int_{-\infty}^t U_Q(t,s)\left(|u|^{2\sigma}u(s)\right)ds. \tag{4.2}$$

A convenient tool is given by Strichartz estimates associated to U_Q. Local in time Strichartz estimates follow from general results given in [25], where local dispersive estimates are proven for more general potential. To address large time, we take advantage of the fact that the potential is exactly quadratic with respect to the space variable, so an explicit formula is available for U_Q, entering the general family of Mehler's formulas (see e.g. [23, 39]).

4.1. **Mehler's formula.** Consider, for $t_0 \ll -1$,

$$i\partial_t u + \frac{1}{2}\Delta u = \frac{1}{2}\langle Q(t)y, y\rangle u \quad ; \quad u(t_0, y) = u_0(y).$$

We seek a solution of the form

$$u(t,y) = \frac{1}{h(t)}\int_{\mathbb{R}^d} e^{\frac{i}{2}(\langle M_1(t)y,y\rangle + \langle M_2(t)z,z\rangle + 2\langle P(t)y,z\rangle)} u_0(z)dz, \tag{4.3}$$

with symmetric matrices $M_1, M_2, P \in \mathcal{S}_d(\mathbb{R})$. Experience shows that no linear term is needed in this formula, since the potential is exactly quadratic (see e.g. [18]).

We compute:

$$\begin{aligned} i\partial_t u = &-i\frac{\dot h}{h}u - \frac{1}{2}\left\langle \dot M_1(t)y, y\right\rangle u \\ &+ \frac{1}{h}\int e^{\frac{i}{2}(\dots)}\left(-\frac{1}{2}\left\langle \dot M_2(t)z, z\right\rangle - \left\langle \dot P(t)y, z\right\rangle\right) u_0(z)dz, \end{aligned}$$

$$\partial_j^2 u = \frac{1}{h}\int e^{\frac{i}{2}(\dots)}\left(-\left((M_1(t)y)_j + (P(t)z)_j\right)^2 - i\left(M_1\right)_{jj}\right) u_0(z)dz,$$

hence

$$\begin{aligned} &i\partial_t u + \frac{1}{2}\Delta u = -i\frac{\dot h}{h}u + \frac{i}{2}\operatorname{tr} M_1 - \frac{1}{2}\left\langle \dot M_1(t)y, y\right\rangle u \\ &+ \frac{1}{2h}\int e^{\frac{i}{2}(\langle M_1(t)y,y\rangle + \langle M_2(t)z,z\rangle + 2\langle P(t)y,z\rangle)} u_0(z)\times \\ &\times\left(-\left\langle \dot M_2(t)z, z\right\rangle - 2\left\langle \dot P(t)y, z\right\rangle - |M_1(t)y|^2 - |P(t)z|^2 - 2\langle M_1(t)y, P(t)z\rangle\right) dz. \end{aligned}$$

Identifying the quadratic forms (recall that the matrices M_j and P are symmetric), we find:

$$\begin{aligned} &\frac{\dot h}{h} = \frac{1}{2}\operatorname{tr} M_1, \\ &\dot M_1 + M_1^2 + Q = 0, \\ &\dot M_2 + P^2 = 0, \\ &\dot P + PM_1 = 0. \end{aligned}$$

Dispersion is given by

$$h(t) = h(t_1)\exp\left(\frac{1}{2}\int_{t_1}^t \operatorname{tr} M_1(s)ds\right),$$

where M_1 solves the matrix Riccati equation

$$\dot M_1 + M_1^2 + Q = 0; \quad M_1(t_0) = \frac{1}{t_0}\mathrm{I}_d. \tag{4.4}$$

Note that in general, solutions to Riccati equations develop singularities in finite time. What saves the day here is that (4.4) is not translation invariant, and can be considered, for $t \leqslant t_0 \ll -1$, as a perturbation of the Cauchy problem

$$\dot M + M^2 = 0; \quad M(t_0) = \frac{1}{t_0}\mathrm{I}_d,$$

whose solution is given by

$$M(t) = \frac{1}{t}\mathrm{I}_d.$$

Lemma 4.1. *Let Q be a symmetric matrix satisfying* (4.1) *for $\mu > 1$. There exists $t_0 < 0$ such that* (4.4) *has a unique solution $M_1 \in C((-\infty, t_0]; \mathcal{S}_d(\mathbb{R}))$. In addition, it satisfies*

$$M_1(t) = \frac{1}{t}\mathrm{I}_d + \mathcal{O}\left(\frac{1}{t^2}\right) \quad \text{as } t \to -\infty.$$

Proof. Seek a solution of the form $M_1(t) = \frac{1}{t}\mathrm{I}_d + R(t)$, where R is s symmetric matrix solution of

$$\dot{R} + \frac{2}{t}R + R^2 + Q = 0; \quad R(t_0) = 0.$$

Equivalently, the new unknown $\tilde{R} = t^2 R$ must satisfy

$$\dot{\tilde{R}} + \frac{1}{t^2}\tilde{R}^2 + t^2 Q = 0; \quad \tilde{R}(t_0) = 0. \tag{4.5}$$

Cauchy-Lipschitz Theorem yields a local solution: we show that it is defined on $(-\infty, t_0]$, along with the announced decay. Integrating between t_0 and t, we find

$$\tilde{R}(t) = -\int_{t_0}^{t} \frac{1}{s^2}\tilde{R}(s)^2 ds - \int_{t_0}^{t} s^2 Q(s) ds.$$

Note that $s \mapsto s^2 Q$ is integrable as $s \to -\infty$ from (4.1) (we assume $\mu > 1$). Setting

$$\rho(t) = \sup_{t \leqslant s \leqslant t_0} \|\tilde{R}(s)\|,$$

where $\|\cdot\|$ denotes any matricial norm, we have

$$\rho(t) \leqslant \frac{C}{t_0}\rho(t)^2 + \frac{C}{t_0^{\mu-1}},$$

for some constant C. Choosing $t_0 \ll -1$, global existence follows from the following bootstrap argument (see [1]): Let $f = f(t)$ be a nonnegative continuous function on $[0, T]$ such that, for every $t \in [0, T]$,

$$f(t) \leqslant \varepsilon_1 + \varepsilon_2 f(t)^\theta,$$

where $\varepsilon_1, \varepsilon_2 > 0$ and $\theta > 1$ are constants such that

$$\varepsilon_1 < \left(1 - \frac{1}{\theta}\right) \frac{1}{(\theta\varepsilon_2)^{1/(\theta-1)}}, \quad f(0) \leqslant \frac{1}{(\theta\varepsilon_2)^{1/(\theta-1)}}.$$

Then, for every $t \in [0, T]$, we have

$$f(t) \leqslant \frac{\theta}{\theta - 1}\,\varepsilon_1.$$

This shows that for $|t_0|$ sufficiently large, the matrix R (hence M_1) is defined on $(-\infty, t_0]$. Moreover, since $\tilde{R}$ is bounded, $R(t) = \mathcal{O}(t^{-2})$ as $t \to -\infty$, hence the result. □

We infer

$$h(t) \underset{t \to -\infty}{\sim} c|t|^{d/2},$$

which is the same dispersion as in the case without potential. Putting this result together with local dispersive estimates from [25], we have:

Lemma 4.2. *Let Q be a symmetric matrix satisfying* (4.1) *for $\mu > 1$. Then for all admissible pairs (q, r), there exists $C = C(q, d)$ such that for all $s \in \mathbb{R}$,*

$$\|U_Q(\cdot, s)f\|_{L^q(\mathbb{R}; L^r(\mathbb{R}^d))} \leqslant C\|f\|_{L^2(\mathbb{R}^d)}, \quad \forall f \in L^2(\mathbb{R}^d).$$

For two admissible pairs (q_1, r_1) and (q_2, r_2), there exists C_{q_1,q_2} such that for all time interval I, if we denote by

$$R(F)(t,y) = \int_{I\cap\{s\leqslant t\}} U_Q(t,s)F(s,y)ds,$$

we have

$$\|R(F)\|_{L^{q_1}(I;L^{r_1}(\mathbb{R}^d))} \leqslant C_{q_1,q_2}\|F\|_{L^{q_2'}(I;L^{r_2'}(\mathbb{R}^d))}, \quad \forall F \in L^{q_2'}(I;L^{r_2'}(\mathbb{R}^d)).$$

Remark 4.3. Since we have dispersive estimates, end-point Strichartz estimates ($q = 2$ when $d \geqslant 3$) are also available from [41].

4.2. **Wave operators.** In this section, we prove:

Proposition 4.4. *Let $d \geqslant 1$, $\frac{2}{d} \leqslant \sigma < \frac{2}{(d-2)_+}$, and V satisfying Assumption 1.1 for some $\mu > 1$. For all $\tilde{u}_- \in \Sigma$, there exists a unique $u \in C(\mathbb{R};\Sigma)$ solution to* (1.11) *such that*

$$\|U_Q(0,t)u(t) - \tilde{u}_-\|_\Sigma \underset{t\to-\infty}{\longrightarrow} 0.$$

Remark 4.5. The assumption $\sigma \geqslant \frac{2}{d}$ could easily be relaxed, following the classical argument (see e.g. [13]). We do not present the argument, since Theorem 1.4 is proven only for $\sigma > \frac{2}{d}$.

Proof. The proof follows closely the approach without potential ($Q = 0$). From this perspective, a key tool is the vector field

$$J(t) = y + it\nabla.$$

It satisfies three important properties:

- It commutes with the free Schrödinger dynamics,
$$\left[i\partial_t + \frac{1}{2}\Delta, J\right] = 0.$$
- It acts like a derivative on gauge invariant nonlinearities. If $F(z)$ is of the form $F(z) = G(|z|^2)z$, then
$$J(t)\left(F(u)\right) = \partial_z F(u)J(t)u - \partial_{\bar{z}}F(u)\overline{J(t)u}.$$
- It provides weighted Gagliardo-Nirenberg inequalities:
$$\|f\|_{L^r} \lesssim \frac{1}{|t|^{\delta(r)}}\|f\|_{L^2}^{1-\delta(r)}\|J(t)f\|_{L^2}^{\delta(r)}, \quad \delta(r) = d\left(\frac{1}{2} - \frac{1}{r}\right),$$
$$\text{with } \begin{cases} 2 \leqslant r \leqslant \infty \text{ if } d = 1, \\ 2 \leqslant r < \infty \text{ if } d = 2, \\ 2 \leqslant r \leqslant \dfrac{2d}{d-2} \text{ if } d \geqslant 3. \end{cases}$$

The last two properties stem from the factorization $J(t)f = ite^{i\frac{|y|^2}{2t}}\nabla\left(e^{-i\frac{|y|^2}{2t}}f\right)$. Note that the commutation property does not incorporate the quadratic potential:

$$[i\partial_t - H_Q, J] = itQ(t)y = itQ(t)J(t) + t^2Q(t)\nabla.$$

Now the important remark is that $t \mapsto t^2Q(t)$ is integrable, from (4.1) since $\mu > 1$.

To prove Proposition 4.4, we apply a fixed point argument to the Duhamel's formula (4.2). As in the case of the quantum scattering operator, we have to deal with the fact that the gradient does not commute with U_Q, leading to the problem described in Section 3.1. Above, we have sketched how to deal with the inhomogeneous term in (4.2), while in Section 3.1, we had underscored the difficulty related to the homogeneous term. We therefore start by showing that for any admissible pair (q_1, r_1), there exists K_{q_1} such that

$$\|\nabla U_Q(t,0)f\|_{L^{q_1}(\mathbb{R};L^{r_1})} + \|J(t)U_Q(t,0)f\|_{L^{q_1}(\mathbb{R};L^{r_1})} \leqslant K_{q_1}\|f\|_\Sigma. \tag{4.6}$$

To prove this, denote

$$v_0(t) = U_Q(t,0)f, \quad v_1(t) = \nabla U_Q(t,0)f, \quad v_2(t) = J(t)U_Q(t,0)f.$$

Since $yv_0 = v_2 - itv_1$, we have:

$$\begin{aligned} &i\partial_t v_1 = H_Q v_1 + Q(t)yv_0 = Hv_1 + Q(t)v_2 - itQ(t)v_1; \quad v_1(0,y) = \nabla f(y),\\ &i\partial_t v_2 = H_Q v_2 + itQ(t)v_2 + t^2 Q(t)v_1; \quad v_2(0,y) = yf(y). \end{aligned}$$

Lemma 4.2 yields

$$\begin{aligned} \|v_1\|_{L^{q_1}(\mathbb{R};L^{r_1})} + \|v_2\|_{L^{q_1}(\mathbb{R};L^{r_1})} \lesssim \|f\|_\Sigma &+ \int_{-\infty}^{\infty} \|\langle t\rangle Q(t)v_2(t)\|_{L^2}dt\\ &+ \int_{-\infty}^{\infty} \|\langle t\rangle^2 Q(t)v_1(t)\|_{L^2}dt, \end{aligned}$$

where we have chosen $(q_2, r_2) = (\infty, 2)$. The fact that U_Q is unitary on L^2 and (4.1) imply

$$\|\langle t\rangle Q(t)v_2(t)\|_{L^2} \lesssim \langle t\rangle^{-\mu-1}\|yf\|_{L^2}, \quad \|\langle t\rangle^2 Q(t)v_1(t)\|_{L^2} \lesssim \langle t\rangle^{-\mu}\|\nabla f\|_{L^2},$$

hence (4.6). We then apply a fixed point argument in

$$\begin{aligned} X(T) = \Big\{ &u \in L^\infty((-\infty,-T];H^1),\\ &\sum_{B\in\{\mathrm{Id},\nabla,J\}} \left(\|Bu\|_{L^\infty((-\infty,-T];L^2)} + \|Bu\|_{L^q((-\infty,-T];L^r)}\right) \leqslant \mathbf{K}\|\tilde{u}_-\|_\Sigma\Big\}, \end{aligned}$$

where the admissible pair (q,r) is given by

$$(q,r) = \left(\frac{4\sigma+4}{d\sigma}, 2\sigma+2\right),$$

and the constant $\mathbf{K}$ is related to the constants C_q from Strichartz inequalities (Lemma 4.2), and K_q from (4.6), whose value we do not try to optimize. The fixed point argument is applied to the Duhamel's formula (4.2): we denote by $\Phi(u)$ the left hand side, and let $u \in X(T)$. We have

$$\|\Phi(u)\|_{L^\infty((-\infty,-T];L^2)} \leqslant \|\tilde{u}_-\|_{L^2} + C\left\||u|^{2\sigma}u\right\|_{L_T^{q'}L^{r'}},$$

where L_T^a stands for $L^a((-\infty,-T])$. Hölder inequality yields

$$\left\||u|^{2\sigma}u\right\|_{L_T^{q'}L^{r'}} \leqslant \|u\|^{2\sigma}_{L_T^k L^r}\|u\|_{L_T^q L^r},$$

where k is given by

$$\frac{1}{q'} = \frac{1}{q} + \frac{2\sigma}{k}, \text{ that is } k = \frac{4\sigma(\sigma+1)}{2-(d-2)\sigma}.$$

Weighted Gagliardo-Nirenberg inequality and the definition of $X(T)$ yield

$$\|u(t)\|_{L^r} \lesssim \frac{1}{|t|^{\frac{d\sigma}{2\sigma+2}}}\|u_-\|_\Sigma.$$

We check that for $\sigma \geqslant \frac{2}{d}$,

$$k \times \frac{d\sigma}{2\sigma+2} = \frac{2d\sigma^2}{2-(d-2)\sigma} \geqslant 2,$$

and so

$$\|u\|^k_{L^k_T L^r} = \mathcal{O}\left(\frac{1}{T}\right) \text{ as } T \to \infty.$$

By using Strichartz estimates again,

$$\|\Phi(u)\|_{L^q_T L^r} \leqslant C_q\|\tilde{u}_-\|_{L^2} + C\left\||u|^{2\sigma}u\right\|_{L^{q'}_T L^{r'}},$$

which shows, like above, that if T is sufficiently large, $\|\Phi(u)\|_{L^q_T L^r} \leqslant 2C_q\|\tilde{u}_-\|_{L^2}$.

We now apply ∇ and $J(t)$ to Φ, and get a closed system of estimates:

$$\begin{aligned}
\nabla\Phi(u) &= \nabla U_Q(t,0)\tilde{u}_- - i\int_{-\infty}^t U_Q(t,s)\nabla\left(|u|^{2\sigma}u(s)\right)ds \\
&\quad - i\int_{-\infty}^t U_Q(t,s)\left(Q(s)J(s)\Phi(u)\right)ds - \int_{-\infty}^t U_Q(t,s)\left(sQ(s)\nabla\Phi(u)\right)ds, \\
J(t)\Phi(u) &= J(t)U_Q(t,0)\tilde{u}_- - i\int_{-\infty}^t U_Q(t,s)J(s)\left(|u|^{2\sigma}u(s)\right)ds \\
&\quad + \int_{-\infty}^t U_Q(t,s)\left(sQ(s)J(s)\Phi(u)\right)ds - i\int_{-\infty}^t U_Q(t,s)\left(s^2Q(s)\nabla\Phi(u)\right)ds,
\end{aligned}$$

where we have used the same algebraic properties as in the proof of (4.6). Set

$$M(T) = \sum_{B\in\{\nabla,J\}}\left(\|B(t)\Phi(u)\|_{L^\infty_T L^2} + \|B(t)\Phi(u)\|_{L^q_T L^r}\right).$$

Lemma 4.2 and (4.6) yield

$$\begin{aligned}
M(T) &\lesssim \|\tilde{u}_-\|_\Sigma + \sum_{B\in\{\nabla,J\}}\left\||u|^{2\sigma}Bu\right\|_{L^{q'}_T L^{r'}} \\
&\quad + \|\langle t\rangle Q(t)J(t)\Phi(u)\|_{L^1_T L^2} + \|\langle t\rangle^2 Q(t)\nabla\Phi(u)\|_{L^1_T L^2},
\end{aligned}$$

where we have also used the fact that $J(t)$ acts like a derivative on gauge invariant nonlinearities. The same Hölder inequalities as above yield

$$\left\||u|^{2\sigma}Bu\right\|_{L^{q'}_T L^{r'}} \leqslant \|u\|^{2\sigma}_{L^k_T L^r}\|Bu\|_{L^q_T L^r} \lesssim \frac{1}{T^{2\sigma/k}}\|Bu\|_{L^q_T L^r}.$$

On the other hand, from (4.1),

$$\|\langle t\rangle Q(t)J(t)\Phi(u)\|_{L^1_T L^2} + \|\langle t\rangle^2 Q(t)\nabla\Phi(u)\|_{L^1_T L^2} \lesssim \frac{1}{T^{\mu-1}}M(T),$$

and so

$$M(T) \lesssim \|\tilde{u}_-\|_\Sigma + \frac{1}{T^{2\sigma/k}}\sum_{B\in\{\nabla,J\}}\|Bu\|_{L^q_T L^r} + \frac{1}{T^{\mu-1}}M(T).$$

By choosing T sufficiently large, we infer

$$M(T) \lesssim \|\tilde{u}_-\|_\Sigma + \frac{1}{T^{2\sigma/k}} \sum_{B\in\{\nabla,J\}} \|Bu\|_{L^q_T L^r},$$

and we conclude that Φ maps $X(T)$ to $X(T)$ for T sufficiently large. Up to choosing T even larger, Φ is a contraction on $X(T)$ with respect to the weaker norm $L^q_T L^r$, since for $u, v \in X(T)$, we have

$$\begin{aligned}\|\Phi(u)-\Phi(v)\|_{L^q_T L^r} &\lesssim \left\||u|^{2\sigma}u - |v|^{2\sigma}v\right\|_{L^{q'}_T L^{r'}} \lesssim \left(\|u\|^{2\sigma}_{L^k_T L^r} + \|v\|^{2\sigma}_{L^k_T L^r}\right)\|u-v\|_{L^q_T L^r}\\ &\lesssim \frac{1}{T^{2\sigma/k}}\|u-v\|_{L^q_T L^r},\end{aligned}$$

where we have used the previous estimate. Therefore, there exists $T > 0$ such that Φ has a unique fixed point in $X(T)$. This solution actually belongs to $C(\mathbb{R};\Sigma)$ from [10]. Unconditional uniqueness (in Σ, without referring to mixed space-time norms) stems from the approach in [54]. □

4.3. **Vector field.** It is possible to construct a vector field adapted to the presence of Q, even though it is not needed to prove Proposition 4.4. Such a vector field will be useful in Section 5, and since its construction is very much in the continuity of Section 4.1, we present it now. Set, for a scalar function f,

$$\mathcal{A}f = iW(t)e^{i\phi(t,y)}\nabla\left(e^{-i\phi(t,y)}f\right) = W(t)\left(f\nabla\phi + i\nabla f\right),$$

where W is a matrix and the phase ϕ solves the eikonal equation

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}\left\langle Q(t)y, y\right\rangle = 0.$$

Since the underlying Hamiltonian is quadratic, ϕ has the form

$$\phi(t,y) = \frac{1}{2}\left\langle K(t)y, y\right\rangle,$$

where $K(t)$ is a symmetric matrix. For $\mathcal{A}$ to commute with $i\partial_t - H_Q$, we come up with the conditions

$$\dot{K} + K^2 + Q = 0, \quad \dot{W} = W\nabla^2\phi = WK.$$

We see that we can take $K = M_1$ as in the proof of Lemma 4.1, and $\mathcal{A}$ will then satisfy the same three properties as J, up to the fact that the commutation property now includes the quadratic potential.

Since the construction of this vector field boils down to solving a matricial Riccati equation with initial data prescribed at large time (see (4.4)), we naturally construct two vector fields $\mathcal{A}_\pm$, associated to $t \to \pm\infty$. In view of Lemma 4.1, $\mathcal{A}_-$ is defined on $(-\infty, -T]$, while $\mathcal{A}_+$ is defined on $[T,\infty)$, for a common $T \gg 1$, with

$$\mathcal{A}_\pm = W_\pm(t)\left(\nabla\phi_\pm + i\nabla\right), \quad \phi_\pm(t,y) = \frac{1}{2}\left\langle K_\pm(t)y, y\right\rangle,$$

where $K_\pm$ and $W_\pm$ satisfy

$$\dot{K}_\pm + K_\pm^2 + Q = 0, \quad \dot{W}_\pm = W_\pm K_\pm,$$

so that Lemma 4.1 also yields

$$K_\pm(t) \sim \frac{1}{t}\mathrm{I}_d, \quad W_\pm(t) \sim t\mathrm{I}_d \quad \text{as } t \to \pm\infty. \tag{4.7}$$

We construct commuting vector fields for large time only, essentially because on finite time intervals, the absence of commutation is not a problem, so we can use ∇, y or J.

4.4. **Asymptotic completeness.** In this section we prove:

Proposition 4.6. *Let $d \geqslant 1$, $\frac{2}{d} \leqslant \sigma < \frac{2}{(d-2)_+}$, and V satisfying Assumption 1.1 for some $\mu > 1$. For all $u_0 \in \Sigma$, there exists a unique $\tilde{u}_+ \in \Sigma$ such that the solution $u \in C(\mathbb{R}; \Sigma)$ to* (1.11) *with $u_{|t=0} = u_0$ satisfies*

$$\sum_{\Gamma \in \{\mathrm{Id}, \nabla, J\}} \|\Gamma(t)u(t) - \Gamma(t)U_Q(t,0)\tilde{u}_+\|_{L^2} \underset{t\to+\infty}{\longrightarrow} 0.$$

Proof. In the case $Q = 0$, such a result is a rather direct consequence of the *pseudo-conformal conservation law*, established in [28]. Recalling that $J(t) = y + it\nabla$, this law reads

$$\frac{d}{dt}\left(\frac{1}{2}\|J(t)u\|_{L^2}^2 + \frac{t^2}{\sigma+1}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}\right) = \frac{t}{\sigma+1}(2 - d\sigma)\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}.$$

A way to derive this relation is to apply J to (1.11). The operator J commutes with the linear part ($Q = 0$), and the standard L^2 estimate, which consists in multiplying the outcome by $\overline{Ju}$, integrating in space, and taking the imaginary part, yields:

$$\frac{1}{2}\frac{d}{dt}\|J(t)u\|_{L^2}^2 = \mathrm{Im}\int \overline{Ju} J\left(|u|^{2\sigma}u\right).$$

Since we have $J = ite^{i\frac{|y|^2}{2t}}\nabla\left(\cdot e^{-i\frac{|y|^2}{2t}}\right)$,

$$J\left(|u|^{2\sigma}u\right) = (\sigma+1)|u|^{2\sigma}Ju + \sigma u^{\sigma+1}\bar{u}^{\sigma-1}\overline{Ju}.$$

The first term is real, and the rest of the computation consists in expanding the remaining term.

In the case where $Q \neq 0$, we resume the above approach: the new contribution is due to the fact that J does not commute with the external potential, so we find:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|J(t)u\|_{L^2}^2 &= \text{like before} + \mathrm{Re}\int tQ(t)xu\cdot\overline{Ju}\\ &= \text{like before} + t\,\mathrm{Re}\int_{\mathbb{R}^d}\langle Q(t)J(t)u, J(t)u\rangle + t^2\,\mathrm{Im}\int_{\mathbb{R}^d}\langle Q(t)\nabla u, Ju\rangle.\end{aligned}$$

On the other hand, we still have

$$\frac{d}{dt}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} = 2(\sigma+1)\int|u|^{2\sigma}\,\mathrm{Re}\,(\bar{u}\partial_t u) = 2(\sigma+1)\int|u|^{2\sigma}\,\mathrm{Re}\left(\bar{u}\times\frac{i}{2}\Delta u\right),$$

and so,

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|J(t)u\|_{L^2}^2 + \frac{t^2}{\sigma+1}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}\right) &= \frac{t}{\sigma+1}(2-d\sigma)\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}\\ &\quad + t\,\mathrm{Re}\int_{\mathbb{R}^d}\langle Q(t)J(t)u, J(t)u\rangle + t^2\,\mathrm{Im}\int_{\mathbb{R}^d}\langle Q(t)\nabla u, Ju\rangle.\end{aligned}$$

Thus for $t \geqslant 0$ and $\sigma \geqslant \frac{2}{d}$, (4.1) implies

$$\frac{d}{dt}\left(\frac{1}{2}\|J(t)u\|_{L^2}^2 + \frac{t^2}{\sigma+1}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}\right) \lesssim \langle t\rangle^{-\mu-1}\|J(t)u\|_{L^2}^2 + \langle t\rangle^{-\mu}\|\nabla u\|_{L^2}\|Ju\|_{L^2}.$$

Even though there is no conservation of the energy for (1.11) since the potential depends on time, we know from [37] that $u \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^d))$. As a matter of fact, the proof given

in [37, Section 4] concerns the case $\sigma = 1$ in $d = 2$ or 3, but the argument, based on energy estimates, remains valid for $d \geqslant 1$, $\sigma < \frac{2}{(d-2)_+}$, since we then know that $u \in C(\mathbb{R}; \Sigma)$. Since $\mu > 1$, we infer

$$Ju \in L^\infty(\mathbb{R}_+; L^2). \tag{4.8}$$

Writing Duhamel's formula for (1.11) with initial datum u_0, in terms of U_Q, we have

$$u(t) = U_Q(t,0)u_0 - i\int_0^t U_Q(t,s)\left(|u|^{2\sigma}u(s)\right)ds.$$

Resuming the computations presented in the proof of Proposition 4.4, (4.8) and (weighted) Gagliardo-Nirenberg inequalities make it possible to prove that

$$Bu \in L^{q_1}(\mathbb{R}_+; L^{r_1}),\ \forall (q_1,r_1) \text{ admissible},\ \forall B \in \{\mathrm{Id}, \nabla, J\}.$$

Duhamel's formula then yields, for $0 < t_1 < t_2$,

$$U_Q(0,t_2)u(t_2) - U_Q(0,t_1)u(t_1) = -i\int_{t_1}^{t_2} U_Q(0,s)\left(|u|^{2\sigma}u(s)\right)ds.$$

From Strichartz estimates,

$$\|U_Q(0,t_2)u(t_2) - U_Q(0,t_1)u(t_1)\|_{L^2} \lesssim \left\||u|^{2\sigma}u\right\|_{L^{q'}([t_1,t_2]:L^{r'})},$$

and the right hand side goes to zero as $t_1, t_2 \to +\infty$. Therefore, there exists (a unique) $\tilde{u}_+ \in L^2$ such that

$$\|U_Q(0,t)u(t) - \tilde{u}_+\|_{L^2} \underset{t\to+\infty}{\longrightarrow} 0,$$

and we have

$$u(t) = U_Q(t,0)\tilde{u}_+ + i\int_t^\infty U_Q(t,s)\left(|u|^{2\sigma}u(s)\right)ds.$$

Using the same estimates as in the proof of Proposition 4.4, we infer

$$\begin{aligned}\|\nabla u(t) - \nabla U_Q(t,0)\tilde{u}_+\|_{L^2} + \|J(t)u(t) - J(t)U_Q(t,0)\tilde{u}_+\|_{L^2}&\\ \lesssim \left\||u|^{2\sigma}\nabla u\right\|_{L^{q'}(t,\infty;L^{r'})} + \left\||u|^{2\sigma}Ju\right\|_{L^{q'}(t,\infty;L^{r'})}&\\ + \left\|\langle s\rangle^{-\mu-1}J(s)u\right\|_{L^1(t,\infty;L^2)} + \left\|\langle s\rangle^{-\mu}\nabla u\right\|_{L^1(t,\infty;L^2)}.&\end{aligned}$$

The right hand side goes to zero as $t \to \infty$, hence the proposition. □

Remark 4.7. As pointed out in the previous section, it would be possible to prove the existence of wave operators by using an adapted vector field $\mathcal{A}$. On the other hand, if $Q(t)$ is not proportional to the identity matrix, it seems that no (exploitable) analogue of the pseudo-conformal conservation law is available in terms of $\mathcal{A}$ rather than in terms of J.

4.5. **Conclusion.** Like in the case of quantum scattering, we use a stronger version of the linear scattering theory:

Proposition 4.8. *Let $d \geqslant 1$, V satisfying Assumption 1.1 with $\mu > 1$. Then the strong limits*

$$\lim_{t\to\pm\infty} U_Q(0,t)U(t) \quad \textit{and} \quad \lim_{t\to\pm\infty} U(-t)U_Q(t,0) \quad \textit{and}$$

exist in Σ.

Proof. For the first limit (existence of wave operators), again in view of Cook's method, we prove that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$t \mapsto \|U_Q(0,t)\,\langle Q(t)y, y\rangle\, U(t)\varphi\|_\Sigma \in L^1(\mathbb{R}).$$

For the L^2 norm, we have, in view of (4.1),

$$\|U_Q(0,t)\,\langle Q(t)y, y\rangle\, U(t)\varphi\|_{L^2} \lesssim \langle t\rangle^{-\mu-2} \sum_{j=1}^d \|y_j^2 U(t)\varphi\|_{L^2}.$$

Write

$$y_j^2 = (y_j + it\partial_j)^2 + t^2\partial_j^2 - 2ity_j\partial_j = (y_j + it\partial_j)^2 - t^2\partial_j^2 - 2it(y_j + it\partial_j)\partial_j,$$

to take advantage of the commutation

$$(y_j + it\partial_j)U(t) = U(t)y_j,$$

and infer

$$\|U_Q(0,t)\,\langle Q(t)y, y\rangle\, U(t)\varphi\|_{L^2} \lesssim \langle t\rangle^{-\mu-2} \left(\||y|^2\varphi\|_{L^2} + t^2\|\Delta\varphi\|_{L^2}\right) \lesssim \langle t\rangle^{-\mu}.$$

The right hand side is integrable since $\mu > 1$, so the strong limits

$$\lim_{t\to\pm\infty} U_Q(0,t)U(t)$$

exist in L^2. To infer that these strong limits actually exist in Σ, we simply invoke (4.6) in the case $(q_1, r_1) = (\infty, 2)$, so the above computation are easily adapted.

For asymptotic completeness, we can adopt the same strategy. Indeed, it suffices to prove that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$t \mapsto \|U(-t)\,\langle Q(t)y, y\rangle\, U_Q(t,0)\varphi\|_\Sigma \in L^1(\mathbb{R}).$$

For the L^2 norm, we have

$$\begin{aligned}\|U(-t)\,\langle Q(t)y, y\rangle\, U_Q(t,0)\varphi\|_{L^2} &= \|\langle Q(t)y, y\rangle\, U_Q(t,0)\varphi\|_{L^2} \\ &\lesssim \langle t\rangle^{-\mu-2} \sum_{j=1}^d \left\|y_j^2 U_Q(t,0)\varphi\right\|_{L^2}.\end{aligned}$$

We first proceed like above, and write

$$y_j^2 = (y_j + it\partial_j)^2 - t^2\partial_j^2 - 2it(y_j + it\partial_j)\partial_j.$$

The operator J does not commute with U_Q, but this lack of commutation is harmless for our present goal, from (4.6). By considering the system satisfied by

$$(y_j + it\partial_j)^2 U_Q(t,0)\varphi, \partial_j^2 U_Q(t,0)\varphi, \partial_j(y_j + it\partial_j)U_Q(t,0)\varphi,$$

we obtain

$$\begin{aligned}&\sum_{j=1}^d \left(\|(y_j + it\partial_j)^2 U_Q(t,0)\varphi\|_{L^2} + \|\partial_j^2 U_Q(t,0)\varphi\|_{L^2} + \|\partial_j(y_j + it\partial_j)U_Q(t,0)\varphi\|_{L^2}\right) \\ &\qquad \leqslant C\|\varphi\|_{\Sigma^2},\end{aligned}$$

where Σ^k is the space of H^k functions with k momenta in L^2, and C does not depend on time. Finally, we also have a similar estimate by considering one more derivative or momentum. The key remark in the computation is that the external potential $\langle Q(t)y, y\rangle$ is exactly quadratic in space, and so differentiating it three times with any space variables yields zero. □

5. Proof of Theorem 1.6

The main result of this section is:

Theorem 5.1. *Let $d = 3$, $\sigma = 1$, V as in Theorem 1.4, and $u_- \in \Sigma^7$. Suppose that Assumption 1.2 is satisfied. Let ψ^ε be given by Theorem 1.4, u be given by Theorem 1.5, φ^ε defined by* (1.10)*. We have the uniform error estimate:*

$$\sup_{t\in\mathbb{R}} \|\psi^\varepsilon(t) - \varphi^\varepsilon(t)\|_{L^2(\mathbb{R}^3)} = \mathcal{O}\left(\sqrt{\varepsilon}\right).$$

Theorem 1.6 is a direct consequence of the above result, whose proof is the core of Section 5. From now on, we assume $d = 3$ and $\sigma = 1$.

5.1. Extra properties for the approximate solution.

Further regularity and localization properties on u will be needed.

Proposition 5.2. *Let $\sigma = 1$, $1 \leqslant d \leqslant 3$, $k \geqslant 2$ and V satisfying Assumption 1.1 for some $\mu > 1$. If $u_- \in \Sigma^k$, then the solution $u \in C(\mathbb{R};\Sigma)$ provided by Theorem 1.5 satisfies $u \in C(\mathbb{R};\Sigma^k)$. The momenta of u satisfy*

$$\||y|^\ell u(t,y)\|_{L^2(\mathbb{R}^d)} \leqslant C_\ell \langle t\rangle^\ell, \quad 0 \leqslant \ell \leqslant k,$$

where C_ℓ is independent of $t \in \mathbb{R}$.

Proof. We know from the proof of Theorem 1.5 that since $u_- \in \Sigma$,

$$u, \nabla u, Ju \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^d)).$$

The natural approach is then to proceed by induction on k, to prove that

$$\nabla^k u, J^k u \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^d)).$$

We have, as we have seen in the proof of Proposition 4.4,

$$\begin{aligned} i\partial_t \nabla u &= H_Q \nabla u + Q(t) y u + \nabla\left(|u|^2 u\right) \\ &\quad + H_Q \nabla u + Q(t) J(t) u - itQ(t)\nabla u + \nabla\left(|u|^2 u\right), \\ i\partial_t J u &= H_Q J u + itQ(t) y u + J\left(|u|^2 u\right) \\ &= H_Q J u + itQ(t) J(t) u + t^2 Q(t)\nabla u + J\left(|u|^2 u\right). \end{aligned}$$

Applying the operators ∇ and J again, we find

$$\begin{aligned} i\partial_t \nabla^2 u &= H_Q \nabla^2 u + 2Q(t) y \nabla u + Q(t) u + \nabla^2\left(|u|^2 u\right) \\ &\quad + H_Q \nabla u + 2Q(t) J(t) \nabla u - 2itQ(t)\nabla^2 u + Q(t) u + \nabla^2\left(|u|^2 u\right), \\ i\partial_t J^2 u &= H_Q J^2 u - 2t^2 Q(t) y \nabla u - t^2 Q(t) u + J^2\left(|u|^2 u\right) \\ &= H_Q J^2 u - 2t^2 Q(t) J \nabla u + 2it^3 Q(t) J^2 u + itQ(t) u + J^2\left(|u|^2 u\right). \end{aligned}$$

In view of (4.1), we see that $t \mapsto t^3 Q(t)$ need not be integrable (unless we make stronger and stronger assumptions of μ, as k increases), so the commutator seems to be fatal to this approach. To overcome this issue, we use the vector field mentioned in Section 4.3. For bounded time $t \in [-T, T]$, the above mentioned lack of commutation is not a problem, and we can use the operator J, which is defined for all time. We note that either of the operators $\mathcal{A}_\pm$ or J satisfies more generally the pointwise identity

$$B\left(u_1 \overline{u}_2 u_3\right) = (Bu_1)\,\overline{u}_2 u_3 + u_1\left(\overline{Bu_2}\right) u_3 + u_1 \overline{u}_2\,(Bu_3),$$

for all differentiable functions u_1, u_2, u_3.

Now we have all the tools to proceed by induction, and mimic the proof from [9, Appendix]. The main idea is that the proof is similar to the propagation of higher regularity for energy-subcritical problems, with the difference that large time is handled thanks to vector fields. We leave out the details, which are not difficult but rather cumbersome: considering

$$B(t) = \begin{cases} \mathcal{A}_-(t) \text{ for } t \leqslant -T, \\ \quad J(t) \text{ for } t \in [-T, T], \\ \mathcal{A}_+(t) \text{ for } t \geqslant T, \end{cases}$$

we can then prove that

$$\nabla^k u, B^k u \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^d)).$$

Back to the definition of $\mathcal{A}_\pm$,

$$\mathcal{A}_\pm(t) = W_\pm(t) K_\pm(t) y + i W_\pm(t) \nabla,$$

(4.7) then yields the result. □

5.2. **Strichartz estimates.** Introduce the following notations, taking the dependence upon ε into account:

$$H^\varepsilon = -\frac{\varepsilon^2}{2}\Delta + V(x), \quad U_V^\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H^\varepsilon}.$$

Since we now work only in space dimension $d = 3$, we can use the result from [31]. Resuming the proof from [31] (a mere scaling argument is not sufficient), we have, along with the preliminary analysis from Section 2, the global dispersive estimate

$$\|U_V^\varepsilon(t)\|_{L^1(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{(\varepsilon|t|)^{3/2}}, \quad t \neq 0. \tag{5.1}$$

For $|t| \leqslant \delta$, $\delta > 0$ independent of ε, the above relation stems initially from [25]. As a consequence, we can measure the dependence upon ε in Strichartz estimates. We recall the definition of admissible pairs related to Sobolev regularity.

Definition 5.3. *Let $d = 3$ and $s \in \mathbb{R}$. A pair (q, r) is called $\dot{H}^s$-admissible if*

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2} - s.$$

For $t_0 \in \mathbb{R} \cup \{-\infty\}$, we denote by

$$R_{t_0}^\varepsilon(F)(t) = \int_{t_0}^t U_V^\varepsilon(t-s) F(s) ds$$

the retarded term related to Duhamel's formula. Since the dispersive estimate (5.1) is the same as the one for $e^{i\varepsilon t\Delta}$, we get the same scaled Strichartz estimates as for this operator, which can in turn be obtained by scaling arguments from the case $\varepsilon = 1$.

Lemma 5.4 (Scaled L^2-Strichartz estimates)**.** *Let $t_0 \in \mathbb{R} \cup \{-\infty\}$, and let (q_1, r_1) and (q_2, r_2) be L^2-admissible pairs, $2 \leqslant r_j \leqslant 6$. We have*

$$\varepsilon^{\frac{1}{q_1}} \|U_V^\varepsilon(\cdot) f\|_{L^{q_1}(\mathbb{R}; L^{r_1}(\mathbb{R}^3))} \lesssim \|f\|_{L^2(\mathbb{R}^3)},$$

$$\varepsilon^{\frac{1}{q_1}+\frac{1}{q_2}} \|R_{t_0}^\varepsilon(F)\|_{L^{q_1}(I; L^{r_1}(\mathbb{R}^3))} \leqslant C_{q_1,q_2} \|F\|_{L^{q_2'}(I; L^{r_2'}(\mathbb{R}^3))},$$

where C_{q_1,q_2} is independent of ε, t_0, and of I such that $t_0 \in \bar{I}$.

We will also use Strichartz estimates for non-admissible pairs, as established in [40] (see also [15, 24]).

Lemma 5.5 (Scaled inhomogeneous Strichartz estimates)**.** *Let $t_0 \in \mathbb{R} \cup \{-\infty\}$, and let (q_1, r_1) be an $\dot{H}^{1/2}$-admissible pair, and (q_2, r_2) be an $\dot{H}^{-1/2}$-admissible pair, with*

$$3 \leqslant r_1, r_2 < 6.$$

We have

$$\varepsilon^{\frac{1}{q_1}+\frac{1}{q_2}} \|R^\varepsilon_{t_0}(F)\|_{L^{q_1}(I;L^{r_1}(\mathbb{R}^3))} \leqslant C_{q_1,q_2} \|F\|_{L^{q_2'}(I;L^{r_2'}(\mathbb{R}^3))},$$

where C_{q_1,q_2} is independent of ε, t_0, and of I such that $t_0 \in \bar{I}$.

5.3. **Preparing the proof.** Subtracting the equations satisfied by ψ^ε and φ^ε, respectively, we obtain as in [11]: $w^\varepsilon = \psi^\varepsilon - \varphi^\varepsilon$ satisfies

$$i\varepsilon\partial_t w^\varepsilon + \frac{\varepsilon^2}{2}\Delta w^\varepsilon = V w^\varepsilon - \mathcal{L}^\varepsilon + \varepsilon^{5/2}\left(|\psi^\varepsilon|^2\psi^\varepsilon - |\varphi^\varepsilon|^2\varphi^\varepsilon\right), \tag{5.2}$$

along with the initial condition

$$e^{-i\frac{\varepsilon t}{2}\Delta} w^\varepsilon_{|t=-\infty} = 0,$$

where the source term is given by

$$\mathcal{L}^\varepsilon(t,x) = \left(V(x) - V\left(q(t)\right) - \sqrt{\varepsilon}\left\langle \nabla V\left(q(t)\right), y\right\rangle - \frac{\varepsilon}{2}\left\langle Q(t)y, y\right\rangle\right)\Big|_{y=\frac{x-q(t)}{\sqrt{\varepsilon}}} \varphi^\varepsilon(t,x).$$

Duhamel's formula for w^ε reads

$$\begin{aligned} w^\varepsilon(t) = &-i\varepsilon^{3/2}\int_{-\infty}^t U_V^\varepsilon(t-s)\left(|\psi^\varepsilon|^2\psi^\varepsilon - |\varphi^\varepsilon|^2\varphi^\varepsilon\right)(s)ds \\ &+ i\varepsilon^{-1}\int_{-\infty}^t U_V^\varepsilon(t-s)\mathcal{L}^\varepsilon(s)ds. \end{aligned}$$

Denoting $L^a(]-\infty,t];L^b(\mathbb{R}^3))$ by $L^a_tL^b$, Strichartz estimates yield, for any L^2-admissible pair (q_1,r_1),

$$\varepsilon^{1/q_1}\|w^\varepsilon\|_{L^{q_1}_t L^{r_1}} \lesssim \varepsilon^{3/2-1/q}\left\||\psi^\varepsilon|^2\psi^\varepsilon - |\varphi^\varepsilon|^2\varphi^\varepsilon\right\|_{L^{q'}_t L^{r'}} + \frac{1}{\varepsilon}\|\mathcal{L}^\varepsilon\|_{L^1_tL^2}, \tag{5.3}$$

where (q,r) is the admissible pair chosen in the proof of Proposition 3.3, that is $r = 2\sigma+2$. Since we now have $d=3$ and $\sigma=1$, this means:

$$q = \frac{8}{3}, \quad k = 8,$$

and (5.3) yields

$$\varepsilon^{1/q_1}\|w^\varepsilon\|_{L^{q_1}_t L^{r_1}} \lesssim \varepsilon^{9/8}\left(\|w^\varepsilon\|^2_{L^8_tL^4} + \|\varphi^\varepsilon\|^2_{L^8_tL^4}\right)\|w^\varepsilon\|_{L^{8/3}_tL^4} + \frac{1}{\varepsilon}\|\mathcal{L}^\varepsilon\|_{L^1_tL^2}. \tag{5.4}$$

The strategy is then to first obtain an a priori estimate for w^ε in $L^8_tL^4$, and then to use it in the above estimate. In order to do so, we begin by estimating the source term $\mathcal{L}^\varepsilon$, in the next subsection.

5.4. **Estimating the source term.**

Proposition 5.6. *Let $d=3$, $\sigma=1$, V satisfying Assumption 1.1 with $\mu > 2$, and $u_- \in \Sigma^k$ for some $k \geqslant 7$. Suppose that Assumption 1.2 is satisfied. Let $u \in C(\mathbb{R};\Sigma^k)$ given by Theorem 1.5 and Proposition 5.2. The source term $\mathcal{L}^\varepsilon$ satisfies*

$$\frac{1}{\varepsilon}\|\mathcal{L}^\varepsilon(t)\|_{L^2(\mathbb{R}^3)} \lesssim \frac{\sqrt{\varepsilon}}{\langle t\rangle^{3/2}} \quad \textit{and} \quad \frac{1}{\varepsilon}\|\mathcal{L}^\varepsilon(t)\|_{L^{3/2}(\mathbb{R}^3)} \lesssim \frac{\varepsilon^{3/4}}{\langle t\rangle^{3/2}}, \quad \forall t \in \mathbb{R}.$$

Proof. To ease notation, we note that

$$\frac{1}{\varepsilon}\mathcal{L}^\varepsilon(t,x) = \frac{1}{\varepsilon^{3/4}}\mathcal{S}^\varepsilon(t,y)\Big|_{y=\frac{x-q(t)}{\sqrt{\varepsilon}}} e^{i(S(t)+ip(t)\cdot(x-q(t)))/\varepsilon},$$

where

$$\mathcal{S}^\varepsilon(t,y) = \frac{1}{\varepsilon}\left(V\left(q(t)+y\sqrt{\varepsilon}\right) - V\left(q(t)\right) - \sqrt{\varepsilon}\left\langle \nabla V\left(q(t)\right), y\right\rangle - \frac{\varepsilon}{2}\left\langle Q(t)y, y\right\rangle\right) u(t,y).$$

In particular,

$$\frac{1}{\varepsilon}\|\mathcal{L}^\varepsilon(t)\|_{L^2(\mathbb{R}^3)} = \|\mathcal{S}^\varepsilon(t)\|_{L^2(\mathbb{R}^3)}, \quad \frac{1}{\varepsilon}\|\mathcal{L}^\varepsilon(t)\|_{L^{3/2}(\mathbb{R}^3)} = \varepsilon^{1/4}\|\mathcal{S}^\varepsilon(t)\|_{L^{3/2}(\mathbb{R}^3)}.$$

Taylor's formula and Assumption 1.1 yield the pointwise estimate

$$|\mathcal{S}^\varepsilon(t,y)| \lesssim \sqrt{\varepsilon}|y|^3 \int_0^1 \frac{1}{\langle q(t)+\theta y\sqrt{\varepsilon}\rangle^{\mu+3}} d\theta |u(t,y)|.$$

To simplify notations, we consider only positive times. Recall that from Assumption 1.2, $p^+ \neq 0$. Introduce, for $0 < \eta < |p^+|/2$,

$$\Omega = \left\{ y \in \mathbb{R}^3, \quad |y| \geqslant \eta \frac{t}{\sqrt{\varepsilon}} \right\}.$$

Since $q(t) \sim p^+ t$ as $t \to \infty$, on the complement of Ω, we can use the decay of V, (1.12), to infer the pointwise estimate

$$|\mathcal{S}^\varepsilon(t,y)| \lesssim \sqrt{\varepsilon}|y|^3 \frac{1}{\langle t\rangle^{\mu+3}}|u(t,y)| \quad \text{on } \Omega^c. \tag{5.5}$$

Taking the L^2-norm, we have

$$\|\mathcal{S}^\varepsilon(t)\|_{L^2(\Omega^c)} \leqslant \frac{\sqrt{\varepsilon}}{\langle t\rangle^{\mu+3}}\||y|^3 u(t,y)\|_{L^2(\mathbb{R}^3)} \lesssim \frac{\sqrt{\varepsilon}}{\langle t\rangle^{\mu}},$$

where we have used Proposition 5.2. On Ω however, the argument of the potential in Taylor's formula is not necessarily going to infinity, so the decay of the potential is apparently useless. Back to the definition of $\mathcal{L}^\varepsilon$, that is leaving out Taylor's formula, we see that all the terms but the first one can be easily estimated on Ω. Indeed, the definition of Ω implies

$$|V(q(t))u(t,y)| \lesssim \frac{1}{\langle t\rangle^{\mu}}|u(t,y)| \lesssim \frac{1}{\langle t\rangle^{\mu}}\left|\frac{y\sqrt{\varepsilon}}{t}\right|^k |u(t,y)|,$$

where k will be chosen shortly. Taking the L^2 norm, we find

$$\frac{1}{\varepsilon}\|V(q(t))u(t)\|_{L^2(\Omega)} \lesssim \frac{\varepsilon^{k/2-1}}{\langle t\rangle^{\mu+k}}\||y|^k u(t,y)\|_{L^2(\mathbb{R}^3)} \lesssim \frac{\varepsilon^{k/2-1}}{\langle t\rangle^{\mu}},$$

where we have used Proposition 5.2 again. Choosing $k = 3$ yields the expected estimate. The last two terms in $\mathcal{L}^\varepsilon$ can be estimated accordingly. For the first term in $\mathcal{L}^\varepsilon$ however, we face the same problem as above: the argument of V has to be considered as bounded. A heuristic argument goes as follows. In view of Theorem 1.5,

$$u(t,y) \underset{t\to\infty}{\sim} e^{i\frac{t}{2}\Delta}u_+ \underset{t\to\infty}{\sim} \frac{1}{t^{3/2}}\widehat{u}_+\left(\frac{y}{t}\right)e^{i|y|^2/(2t)},$$

where the last behavior stems from standard analysis of the Schrödinger group (see e.g. [50]). In view of the definition of Ω, we have, formally for $y \in \Omega$,

$$|u(t,y)| \lesssim \frac{1}{t^{3/2}}\sup_{|z|\geqslant\eta}\left|\widehat{u}_+\left(\frac{z}{\sqrt{\varepsilon}}\right)\right|.$$

Then the idea is to keep the linear dispersion measured by the factor $t^{-3/2}$ (which is integrable since $d=3$), and use decay properties for $\widehat{u}_+$ to gain powers of ε. To make this argument rigorous, we keep the idea that u must be assessed in L^∞ rather than in L^2, and write

$$\frac{1}{\varepsilon}\|V\left(q(t)+y\sqrt{\varepsilon}\right)u(t,y)\|_{L^2(\Omega)} \leqslant \frac{1}{\varepsilon}\|u(t)\|_{L^\infty(\Omega)}\|V\left(q(t)+y\sqrt{\varepsilon}\right)\|_{L^2(\Omega)}.$$

For the last factor, we have

$$\|V\left(q(t)+y\sqrt{\varepsilon}\right)\|_{L^2(\Omega)} \leqslant \varepsilon^{-3/4}\|V\|_{L^2(\mathbb{R}^3)},$$

where the last norm is finite since $\mu > 2$. For the L^∞ norm of u, we use Gagliardo-Nirenberg inequality and the previous vector-fields. To take advantage of the localization in space, introduce a non-negative cut-off function $\chi \in C^\infty(\mathbb{R}^3)$, such that:

$$\chi(z) = \begin{cases} 1 \text{ if } |z| \geqslant \eta, \\ 0 \text{ if } |z| \leqslant \dfrac{\eta}{2}. \end{cases}$$

In view of the definition of Ω,

$$\|u(t)\|_{L^\infty(\Omega)} \leqslant \left\|\chi\left(\frac{y\sqrt{\varepsilon}}{t}\right)u(t,y)\right\|_{L^\infty(\mathbb{R}^3)}.$$

Now with B as defined in the proof of Proposition 5.2, Gagliardo-Nirenberg inequality yields, for any smooth function f (recall that $y\in\mathbb{R}^3$),

$$\|f\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{t^{3/2}}\|f\|^{1/4}_{L^2(\mathbb{R}^3)}\|B^2(t)f\|^{3/4}_{L^2(\mathbb{R}^3)}.$$

We use this inequality with

$$f(t,y) = \chi\left(\frac{y\sqrt{\varepsilon}}{t}\right)u(t,y),$$

and note that

$$B(t)f(t,y) = \chi\left(\frac{y\sqrt{\varepsilon}}{t}\right)B(t)u(t,y) + i\frac{\sqrt{\varepsilon}}{t}W(t)\nabla\chi\left(\frac{y\sqrt{\varepsilon}}{t}\right)\times u(t,y),$$

where $W(t)$ stands for $W_\pm$ or t. Recall that $t\mapsto W(t)/t$ is bounded, so the last term is actually "nice". Proceeding in the same way as above, we obtain

$$\|u(t)\|_{L^2(\Omega)} \lesssim \left\|\left|\frac{y\sqrt{\varepsilon}}{t}\right|^k u(t,y)\right\|_{L^2(\Omega)} \lesssim \varepsilon^{k/2},$$

provided that $u_-\in\Sigma^k$. Similarly,

$$\|B^2(t)u\|_{L^2(\Omega)} \lesssim \varepsilon^{k/2-1},$$

and so

$$\frac{1}{\varepsilon}\|V\left(q(t)+y\sqrt{\varepsilon}\right)u(t,y)\|_{L^2(\Omega)} \lesssim \frac{1}{t^{3/2}}\varepsilon^{-7/4+k/8+3(k/2-1)/4} = \frac{\varepsilon^{k/2-5/2}}{t^{3/2}}.$$

Therefore, the L^2 estimate follows as soon as $k\geqslant 6$. For the $L^{3/2}$-estimate, we resume the same computations, and use the extra estimate: for all $s>1/2$,

$$\|f\|_{L^{3/2}(\mathbb{R}^3)} \lesssim \|f\|^{1-1/2s}_{L^2(\mathbb{R}^3)}\||x|^s f\|^{1/2s}_{L^2(\mathbb{R}^3)}. \tag{5.6}$$

This estimate can easily be proven by writing

$$\|f\|_{L^{3/2}(\mathbb{R}^3)} \leqslant \|f\|_{L^{3/2}(|y|<R)} + \left\| \frac{1}{|x|^s} |x|^s f \right\|_{L^{3/2}(|x|>R)},$$

so Hölder inequality yields, provided that $s > 1/2$ (so that $y \mapsto |y|^{-s} \in L^6(|y| > R)$)

$$\|f\|_{L^{3/2}(\mathbb{R}^3)} \leqslant \sqrt{R} \|f\|_{L^2} + \frac{1}{R^{s-1/2}} \||x|^s f\|_{L^2},$$

and by optimizing in R. Now from (5.5), we have

$$\begin{aligned}
\|\mathcal{S}^\varepsilon(t)\|_{L^{3/2}(\Omega^c)} &\leqslant \frac{\sqrt{\varepsilon}}{\langle t \rangle^{\mu+3}} \||y|^3 u(t,y)\|_{L^{3/2}(\mathbb{R}^d)} \\
&\lesssim \frac{\sqrt{\varepsilon}}{\langle t \rangle^{\mu+3}} \||y|^3 u(t,y)\|_{L^2(\mathbb{R}^d)}^{1/2} \||y|^4 u(t,y)\|_{L^2(\mathbb{R}^d)}^{1/2} \\
&\lesssim \frac{\sqrt{\varepsilon}}{\langle t \rangle^{\mu-1/2}} \lesssim \frac{\sqrt{\varepsilon}}{\langle t \rangle^{3/2}}
\end{aligned}$$

where we have used (5.6) with $s = 1$, Proposition 5.2, and the fact that $\mu > 2$.

On Ω, we can repeat the computations from the L^2-estimate (up to incorporating (5.6)): for the last term, we note that

$$\frac{1}{\varepsilon} \|V\left(q(t) + y\sqrt{\varepsilon}\right) u(t,y)\|_{L^{3/2}(\Omega)} \leqslant \frac{1}{\varepsilon} \|u(t)\|_{L^\infty(\Omega)} \|V\left(q(t) + y\sqrt{\varepsilon}\right)\|_{L^{3/2}(\Omega)},$$

and that

$$\|V\left(q(t) + y\sqrt{\varepsilon}\right)\|_{L^{3/2}(\Omega)} \leqslant \varepsilon^{-1} \|V\|_{L^{3/2}(\mathbb{R}^3)},$$

where the last norm is finite since $\mu > 2$. Up to taking u in Σ^7, we conclude

$$\|\mathcal{S}^\varepsilon(t)\|_{L^{3/2}(\mathbb{R}^3)} \lesssim \frac{\sqrt{\varepsilon}}{\langle t \rangle^{3/2}},$$

and the proposition follows. □

5.5. **A priori estimate for the error in the critical norm.** In this subsection, we prove:

Proposition 5.7. *Under the assumptions of Theorem 5.1, the error $w^\varepsilon = \psi^\varepsilon - \varphi^\varepsilon$ satisfies the a priori estimate, for any $\dot{H}^{1/2}$-admissible pair (q,r),*

$$\varepsilon^{\frac{1}{q}} \|w^\varepsilon\|_{L^q(\mathbb{R};L^r(\mathbb{R}^3))} \lesssim \varepsilon^{1/4}.$$

Proof. The reason for considering $\dot{H}^{1/2}$-admissible pairs is that the cubic three-dimensional Schrödinger equation is $\dot{H}^{1/2}$-critical; see e.g. [14]. The proof of Proposition 5.7 is then very similar to the proof of [38, Proposition 2.3].

An important tool is the known estimate for the approximate solution φ^ε: we have, in view of the fact that $u, Bu \in L^\infty L^2$,

$$\|\varphi^\varepsilon(t)\|_{L^r(\mathbb{R}^3)} \lesssim \left(\frac{1}{\langle t \rangle \sqrt{\varepsilon}} \right)^{3\left(\frac{1}{2} - \frac{1}{r}\right)}, \quad 2 \leqslant r \leqslant 6. \tag{5.7}$$

Note that for an $\dot{H}^{1/2}$ admissible pair, we infer

$$\|\varphi^\varepsilon(t)\|_{L^q(\mathbb{R};L^r(\mathbb{R}^3))} \lesssim \varepsilon^{-\frac{3}{2}\left(\frac{1}{2} - \frac{1}{r}\right)} = \varepsilon^{-\frac{1}{q} - \frac{1}{4}},$$

so Proposition 5.7 shows a $\sqrt{\varepsilon}$ gain for w^ε compared to φ^ε, which is the order of magnitude we eventually prove in $L^\infty L^2$, and stated in Theorem 5.1. Let $0<\eta\ll 1$, and set

$$\|w^\varepsilon\|_{\mathcal{N}^\varepsilon(I)} := \sup_{\substack{(q,r)\ \dot H^{1/2}-\text{admissible}\\ 3\leqslant r\leqslant 6-\eta}} \varepsilon^{\frac{1}{q}}\|w^\varepsilon\|_{L^q(I;L^r(\mathbb{R}^3))}.$$

Duhamel's formula for (5.2) reads, given $w^\varepsilon_{|t=-\infty}=0$,

$$w^\varepsilon(t) = -i\varepsilon^{3/2}\int_{-\infty}^t U_V^\varepsilon(t-s)\left(|\psi^\varepsilon|^2\psi^2-|\varphi^\varepsilon|^2\varphi^\varepsilon\right)(s)ds + i\varepsilon^{-1}\int_{-\infty}^t U_V^\varepsilon(t-s)\mathcal{L}^\varepsilon(s)ds.$$

Since we have the point-wise estimate

$$\left||\psi^\varepsilon|^2\psi^2-|\varphi^\varepsilon|^2\varphi^\varepsilon\right| \lesssim \left(|w^\varepsilon|^2+|\varphi^\varepsilon|^2\right)|w^\varepsilon|,$$

Lemma 5.5 yields, with $(q_2,r_2)=(\frac{10}{7},5)$ for the first term of the right hand side, and with $(q_2,r_2)=(2,3)$ for the second term,

$$\begin{aligned}\|w^\varepsilon\|_{\mathcal{N}^\varepsilon(-\infty,t)} &\lesssim \varepsilon^{3/2-7/10}\left\|\left(|w^\varepsilon|^2+|\varphi^\varepsilon|^2\right)w^\varepsilon\right\|_{L_t^{10/3}L^{5/4}} + \varepsilon^{-3/2}\|\mathcal{L}^\varepsilon\|_{L_t^2L^{3/2}}\\ &\lesssim \varepsilon^{4/5}\left(\|w^\varepsilon\|^2_{L_t^{20}L^{10/3}}+\|\varphi^\varepsilon\|^2_{L_t^{20}L^{10/3}}\right)\|w^\varepsilon\|_{L_t^5L^5}+\varepsilon^{-3/2}\|\mathcal{L}^\varepsilon\|_{L_t^2L^{3/2}},\end{aligned}$$

where we have used Hölder inequality. Note that the pairs $(20,\frac{10}{3})$ and $(5,5)$ are $\dot H^{1/2}$-admissible. Denote by

$$\omega(t)=\frac{1}{\langle t\rangle^{3/5}}.$$

This function obviously belongs to $L^{20}(\mathbb{R})$. The estimate (5.7) and the definition of the norm $\mathcal{N}^\varepsilon$ yield

$$\|w^\varepsilon\|_{\mathcal{N}^\varepsilon(-\infty,t)} \lesssim \sqrt{\varepsilon}\|w^\varepsilon\|^3_{\mathcal{N}^\varepsilon(-\infty,t)} + \|\omega\|^2_{L^{20}(-\infty,t)}\|w^\varepsilon\|_{\mathcal{N}^\varepsilon(-\infty,t)} + \varepsilon^{-3/2}\|\mathcal{L}^\varepsilon\|_{L_t^2L^{3/2}}.$$

Taking $t\ll -1$, we infer

$$\|w^\varepsilon\|_{\mathcal{N}^\varepsilon(-\infty,t)} \lesssim \sqrt{\varepsilon}\|w^\varepsilon\|^3_{\mathcal{N}^\varepsilon(-\infty,t)} + \varepsilon^{-3/2}\|\mathcal{L}^\varepsilon\|_{L_t^2L^{3/2}} \lesssim \sqrt{\varepsilon}\|w^\varepsilon\|^3_{\mathcal{N}^\varepsilon(-\infty,t)} + \varepsilon^{1/4},$$

where we have use Proposition 5.6. We can now use a standard bootstrap argument, as recalled in Section 4. We infer that for $t_1\ll -1$,

$$\|w^\varepsilon\|_{\mathcal{N}^\varepsilon(-\infty,t_1)} \lesssim \varepsilon^{1/4}.$$

Using Duhamel's formula again, we have

$$\begin{aligned}U_V^\varepsilon(t-t_1)w^\varepsilon(t_1) = -i\varepsilon^{3/2}&\int_{-\infty}^{t_1} U_V^\varepsilon(t-s)\left(|\psi^\varepsilon|^2\psi^2-|\varphi^\varepsilon|^2\varphi^\varepsilon\right)(s)ds\\ &+ i\varepsilon^{-1}\int_{-\infty}^{t_1} U_V^\varepsilon(t-s)\mathcal{L}^\varepsilon(s)ds,\end{aligned}$$

so we infer

$$\begin{aligned}\|U_V^\varepsilon(t-t_1)w^\varepsilon(t_1)\|_{\mathcal{N}^\varepsilon(\mathbb{R})} &\lesssim \sqrt{\varepsilon}\|w^\varepsilon\|^3_{\mathcal{N}^\varepsilon(-\infty,t_1)} + \|\omega\|^2_{L^{20}(-\infty,t_1)}\|w^\varepsilon\|_{\mathcal{N}^\varepsilon(-\infty,t_1)}\\ &\quad + \varepsilon^{-3/2}\|\mathcal{L}^\varepsilon\|_{L^2((-\infty,t_1];L^{3/2})}\\ &\leqslant C_0\varepsilon^{1/4}.\end{aligned}$$

We now rewrite Duhamel's formula with some initial time t_j:

$$w^\varepsilon(t) = U_V^\varepsilon(t-t_j)w^\varepsilon(t_j) - i\varepsilon^{3/2}\int_{t_j}^t U_V^\varepsilon(t-s)\left(|\psi^\varepsilon|^2\psi^2 - |\varphi^\varepsilon|^2\varphi^\varepsilon\right)(s)ds \\ + i\varepsilon^{-1}\int_{t_j}^t U_V^\varepsilon(t-s)\mathcal{L}^\varepsilon(s)ds.$$

For $t \geqslant t_j$ and $I = [t_j, t]$, the same estimates as above yield

$$\|w^\varepsilon\|_{\mathcal{N}^\varepsilon(I)} \leqslant \|U_V^\varepsilon(\cdot - t_j)w^\varepsilon(t_j)\|_{\mathcal{N}^\varepsilon(I)} + C\sqrt{\varepsilon}\|w^\varepsilon\|^3_{\mathcal{N}^\varepsilon(I)} + C\|\omega\|^2_{L^{20}(I)}\|w^\varepsilon\|_{\mathcal{N}^\varepsilon(I)} \\ + C\varepsilon^{-3/2}\|\mathcal{L}^\varepsilon\|_{L^2(I;L^{3/2})},$$

where the above constant C is independent of ε, t_j and t. We split $\mathbb{R}_t$ into finitely many intervals

$$\mathbb{R} = (-\infty, t_1] \cup \bigcup_{j=1}^N [t_j, t_{j+1}] \cup [t_N, \infty) =: \bigcup_{j=0}^{N+1} I_j,$$

on which

$$C\|\omega\|^2_{L^{20}(I_j)} \leqslant \frac{1}{2},$$

so that we have

$$\|w^\varepsilon\|_{\mathcal{N}^\varepsilon(I_j)} \leqslant 2\|U_V^\varepsilon(\cdot - t_j)w^\varepsilon(t_j)\|_{\mathcal{N}^\varepsilon(I_j)} + 2C\sqrt{\varepsilon}\|w^\varepsilon\|^3_{\mathcal{N}^\varepsilon(I_j)} + 2C\varepsilon^{-3/2}\|\mathcal{L}^\varepsilon\|_{L^2(I_j;L^{3/2})} \\ \leqslant 2\|U_V^\varepsilon(\cdot - t_j)w^\varepsilon(t_j)\|_{\mathcal{N}^\varepsilon(I_j)} + 2C\sqrt{\varepsilon}\|w^\varepsilon\|^3_{\mathcal{N}^\varepsilon(I_j)} + \tilde{C}\varepsilon^{1/4}\left\|\langle t\rangle^{-3/2}\right\|_{L^2(I_j)},$$

where we have used Proposition 5.6 again. Since we have

$$\|U_V^\varepsilon(t-t_1)w^\varepsilon(t_1)\|_{\mathcal{N}^\varepsilon(\mathbb{R})} \leqslant C_0\varepsilon^{1/4},$$

the bootstrap argument shows that at least for $\varepsilon \leqslant \varepsilon_1$ ($\varepsilon_1 > 0$),

$$\|w^\varepsilon\|_{\mathcal{N}^\varepsilon(I_1)} \leqslant 3\|U_V^\varepsilon(\cdot - t_1)w^\varepsilon(t_1)\|_{\mathcal{N}^\varepsilon(I_1)} + \frac{3}{2}\tilde{C}\varepsilon^{1/4}\left\|\langle t\rangle^{-3/2}\right\|_{L^2(I_1)}.$$

On the other hand, Duhamel's formula implies

$$U_V^\varepsilon(t-t_{j+1})w^\varepsilon(t_{j+1}) = U_V^\varepsilon(t-t_j)w^\varepsilon(t_j) + i\varepsilon^{-1}\int_{t_j}^{t_{j+1}} U_V^\varepsilon(t-s)\mathcal{L}^\varepsilon(s)ds \\ - i\varepsilon^{3/2}\int_{t_j}^{t_{j+1}} U_V^\varepsilon(t-s)\left(|\psi^\varepsilon|^2\psi^2 - |\varphi^\varepsilon|^2\varphi^\varepsilon\right)(s)ds.$$

Therefore, we infer

$$\|U_V^\varepsilon(t-t_{j+1})w^\varepsilon(t_{j+1})\|_{\mathcal{N}^\varepsilon(\mathbb{R})} \leqslant \|U_V^\varepsilon(t-t_j)w^\varepsilon(t_j)\|_{\mathcal{N}^\varepsilon(\mathbb{R})} + +C\sqrt{\varepsilon}\|w^\varepsilon\|^3_{\mathcal{N}^\varepsilon(I_j)} \\ + C\|\omega\|^2_{L^{20}(I_j)}\|w^\varepsilon\|_{\mathcal{N}^\varepsilon(I_j)} + C\varepsilon^{-3/2}\|\mathcal{L}^\varepsilon\|_{L^2(I_j;L^{3/2})}.$$

By induction (carrying over finitely many steps), we conclude

$$\|U_V^\varepsilon(t-t_j)w^\varepsilon(t_j)\|_{\mathcal{N}^\varepsilon(\mathbb{R})} = \mathcal{O}\left(\varepsilon^{1/4}\right), \quad 0 \leqslant j \leqslant N+1,$$

and $\|w^\varepsilon\|_{\mathcal{N}^\varepsilon(\mathbb{R})} = \mathcal{O}\left(\varepsilon^{1/4}\right)$ as announced. □

5.6. **End of the argument.** Resume the estimate (5.4) with the L^2-admissible pair $(q_1, r_1) = (\frac{8}{3}, 4)$:

$$\varepsilon^{3/8}\|w^\varepsilon\|_{L_t^{8/3}L^4} \lesssim \varepsilon^{3/4}\left(\|w^\varepsilon\|_{L_t^8L^4}^2 + \|\varphi^\varepsilon\|_{L_t^8L^4}^2\right)\varepsilon^{3/8}\|w^\varepsilon\|_{L_t^{8/3}L^4} + \frac{1}{\varepsilon}\|\mathcal{L}^\varepsilon\|_{L_t^1L^2}.$$

From Proposition 5.7 (the pair $(8,4)$ is $\dot H^{1/2}$-admissible),

$$\|w^\varepsilon\|_{L^8(\mathbb{R};L^4)} \lesssim \varepsilon^{1/8},$$

and we have seen in the course of the proof that

$$\|\varphi^\varepsilon\|_{L^8(\mathbb{R};L^4)} \lesssim \varepsilon^{-3/8}.$$

Therefore, we can split $\mathbb{R}_t$ into finitely many intervals, in a way which is independent of ε, so that

$$\varepsilon^{3/4}\left(\|w^\varepsilon\|_{L^8(I;L^4)}^2 + \|\varphi^\varepsilon\|_{L^8(I;L^4)}^2\right) \leqslant \eta$$

on each of these intervals, with η so small that we infer

$$\varepsilon^{3/8}\|w^\varepsilon\|_{L^{8/3}(\mathbb{R};L^4)} \lesssim \frac{1}{\varepsilon}\|\mathcal{L}^\varepsilon\|_{L^1(\mathbb{R};L^2)} \lesssim \sqrt{\varepsilon},$$

where we have used Proposition 5.6. Plugging this estimate into (5.4) and now taking (q_1, r_1), Theorem 5.1 follows.

6. Superposition

In this section, we sketch the proof of Corollary 1.8. This result heavily relies on the (finite time) superposition principle established in [11], in the case of two initial coherent states with different centers in phase space. We present the argument in the case of two initial wave packets, and explain why it can be generalized to any finite number of initial coherent states.

Following the proof of [11, Proposition 1.14], we introduce the approximate evolution of each individual initial wave packet:

$$\varphi_j^\varepsilon(t,x) = \varepsilon^{-3/4}u_j\left(t, \frac{x-q_j(t)}{\sqrt{\varepsilon}}\right)e^{i(S_j(t)+p_j(t)\cdot(x-q_j(t)))/\varepsilon},$$

where u_j solves (1.11) with initial datum a_j. In the proof of [11, Proposition 1.14], the main remark is that all that is needed is the control of a new source term, corresponding to the interactions of the approximate solutions. Set

$$w^\varepsilon = \psi^\varepsilon - \varphi_1^\varepsilon - \varphi_2^\varepsilon.$$

It solves

$$i\varepsilon\partial_t w^\varepsilon + \frac{\varepsilon^2}{2}\Delta w^\varepsilon = Vw^\varepsilon - \mathcal{L}^\varepsilon + \mathcal{N}_I^\varepsilon + \mathcal{N}_s^\varepsilon \quad ; \quad w^\varepsilon_{|t=0} = 0,$$

where the linear source term is the same as in Section 5 (except than now we consider the sums of two such terms), $\mathcal{N}_s^\varepsilon$ is the semilinear term

$$\mathcal{N}_s^\varepsilon = \varepsilon^{5/2}\left(|w^\varepsilon + \varphi_1^\varepsilon + \varphi_2^\varepsilon|^2(w^\varepsilon + \varphi_1^\varepsilon + \varphi_2^\varepsilon) - |\varphi_1^\varepsilon + \varphi_2^\varepsilon|^2(\varphi_1^\varepsilon + \varphi_2^\varepsilon)\right),$$

and $\mathcal{N}_I^\varepsilon$ is precisely the new interaction term,

$$\mathcal{N}_I^\varepsilon = \varepsilon^{5/2}\left(|\varphi_1^\varepsilon + \varphi_2^\varepsilon|^2(\varphi_1^\varepsilon + \varphi_2^\varepsilon) - |\varphi_1^\varepsilon|^2\varphi_1^\varepsilon - |\varphi_2^\varepsilon|^2\varphi_2^\varepsilon\right).$$

In [11], it is proven that if $(q_{01}, p_{01}) \neq (q_{02}, p_{02})$, then the possible interactions between φ_1^ε and φ_2^ε are negligible on every finite time interval, in the sense that

$$\frac{1}{\varepsilon}\|\mathcal{N}_I^\varepsilon\|_{L^1(0,T;L^2)} \leqslant C(T,\gamma)\varepsilon^\gamma,$$

for every $\gamma < 1/2$. We infer that $\|w^\varepsilon\|_{L^\infty(0,T;L^2)} = \mathcal{O}(\varepsilon^\gamma)$ for every $T > 0$. For $t \geqslant T$, we have

$$\begin{aligned}\frac{1}{\varepsilon}\|\mathcal{N}_I^\varepsilon(t)\|_{L^2} &\lesssim \sum_{\ell_1,\ell_2\geqslant 1,\ \ell_1+\ell_2=3} \left\| u_1^{\ell_1}\left(t, y - \frac{q_1(t)-q_2(t)}{\sqrt{\varepsilon}}\right) u_2^{\ell_2}(t,y)\right\|_{L^2}\\ &\lesssim \sum_{\ell_1,\ell_2\geqslant 1,\ \ell_1+\ell_2=3} \|u_1(t)\|_{L^\infty}^{\ell_1}\|u_2(t)\|_{L^\infty}^{\ell_2-1}\|u_2(t)\|_{L^2} \lesssim \frac{1}{t^3}.\end{aligned}$$

Similarly, resuming the same estimates as in the proof of Proposition 5.6,

$$\frac{1}{\varepsilon}\|\mathcal{N}_I^\varepsilon(t)\|_{L^{3/2}} \lesssim \frac{\varepsilon^{1/4}}{t^{5/2}}.$$

By resuming the proof of Theorem 5.1 on the time interval $[T,\infty)$, we infer

$$\|w^\varepsilon\|_{L^\infty(0,\infty;L^2)} \leqslant C(T,\gamma)\varepsilon^\gamma + \frac{C}{T^2}.$$

Therefore,

$$\limsup_{\varepsilon\to 0} \|w^\varepsilon\|_{L^\infty(0,\infty;L^2)} \lesssim \frac{1}{T^2},$$

for all $T > 0$, hence the result by letting $T \to \infty$.

In the case of more than two initial coherent states, the idea is that the nonlinear interaction term, $\mathcal{N}_I^\varepsilon$, always contains the product of two approximate solutions corresponding to different trajectories in phase space. This is enough for the proof of [11, Proposition 1.14] to go through: we always have

$$\begin{aligned}&\frac{1}{\varepsilon}\|\mathcal{N}_I^\varepsilon(t)\|_{L^2}\\ &\lesssim \sum_{\substack{j\neq k,\ \ell_j,\ell_k\geqslant 1\\ \ell_j+\ell_k+\ell_m=3}} \left\| u_j^{\ell_j}\left(t, y - \frac{q_j(t)-q_k(t)}{\sqrt{\varepsilon}}\right) u_k^{\ell_k}(t,y) u_m^{\ell_m}\left(t, y - \frac{q_m(t)-q_k(t)}{\sqrt{\varepsilon}}\right)\right\|_{L^2}\\ &\lesssim \sum_{\substack{j\neq k,\ \ell_j,\ell_k\geqslant 1\\ \ell_j+\ell_k+\ell_m=3}} \|u_m(t)\|_{L^\infty}^{\ell_m} \left\| u_j^{\ell_j}\left(t, y - \frac{q_j(t)-q_k(t)}{\sqrt{\varepsilon}}\right) u_k^{\ell_k}(t,y)\right\|_{L^2},\end{aligned}$$

so the last factor is exactly the one considered in [11] and above.

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