



HAL
open science

Two-dimensional random interacements and late points for random walks

Francis Comets, Serguei Popov, Marina Vachkovskaia

► **To cite this version:**

Francis Comets, Serguei Popov, Marina Vachkovskaia. Two-dimensional random interacements and late points for random walks. 2015. hal-01116486

HAL Id: hal-01116486

<https://hal.science/hal-01116486>

Preprint submitted on 13 Feb 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Two-dimensional random interlacements and late points for random walks

Francis Comets¹ Serguei Popov²
Marina Vachkovskaia²

February 12, 2015

¹Université Paris Diderot – Paris 7, Mathématiques, case 7012, F-75205 Paris Cedex 13, France

e-mail: `comets@math.univ-paris-diderot.fr`

²Department of Statistics, Institute of Mathematics, Statistics and Scientific Computation, University of Campinas – UNICAMP, rua Sérgio Buarque de Holanda 651, 13083-859, Campinas SP, Brazil

e-mails: `{popov,marinav}@ime.unicamp.br`

Abstract

We define the model of two-dimensional random interlacements using the simple random walk trajectories conditioned on never hitting the origin, and then obtain some properties of this model. Also, for the random walk on a large torus conditioned on not hitting the origin up to some time proportional to the mean cover time, we show that the law of the vacant set around the origin is close to that of random interlacements at the corresponding level. Thus, this new model provides a microscopic description of late points of the covering process.

Keywords: random interlacements, hitting time, simple random walk, Doob's h -transform

AMS 2010 subject classifications: Primary 60K35. Secondary 60G50, 82C41.

1 Introduction

We start by an informal description of our purpose.

1.1 Random interlacements in two dimensions

Random interlacements were introduced by Sznitman in [17], motivated by the problem of disconnection of the discrete torus $\mathbb{Z}_n^d := \mathbb{Z}^d/n\mathbb{Z}^d$ by the trace of simple random walk, in dimension 3 or higher. Detailed accounts can be found in the survey [3] and the recent books [8, 18]. Loosely speaking, the model of random interlacements in \mathbb{Z}^d , $d \geq 3$, is a stationary Poissonian soup of (transient) simple random walk trajectories on the integer lattice. There is an additional parameter $u > 0$ entering the intensity measure of the Poisson process, the larger u is the more trajectories are thrown in. The sites of \mathbb{Z}^d that are not touched by the trajectories constitute the *vacant set* \mathcal{V}^u . The random interlacements are constructed simultaneously for all $u > 0$ in such a way that $\mathcal{V}^{u_1} \subset \mathcal{V}^{u_2}$ if $u_1 > u_2$. In fact, the law of the vacant set at level u can be uniquely characterized by the following identity:

$$\mathbb{P}[A \subset \mathcal{V}^u] = \exp(-u \operatorname{cap}(A)), \quad (1)$$

where $\operatorname{cap}(A)$ is the *capacity* of a finite set $A \subset \mathbb{Z}^d$. Informally, the capacity measures how “big” is the set from the point of view of the walk, see Section 6.5 of [12] for formal definitions, as well as (12)–(13) below.

At first glance, the title of this section seems to be meaningless, just because even a single trajectory of two-dimensional simple random walk a.s. visits all sites of \mathbb{Z}^2 , so the vacant set would be always empty. Nevertheless, there is also a natural notion of capacity in two dimensions (cf. Section 6.6 of [12]), so one may wonder if there is a way to construct a decreasing family $(\mathcal{V}^\alpha, \alpha > 0)$ of random subsets of \mathbb{Z}^2 in such a way that a formula analogous to (1) holds for every finite A . This is, however, clearly not possible since the two-dimensional capacity of one-point sets equals 0. On the other hand, it turns out to be possible to construct such a family so that

$$\mathbb{P}[A \subset \mathcal{V}^\alpha] = \exp(-\pi\alpha \operatorname{cap}(A)) \quad (2)$$

holds *for all sets containing the origin* (the factor π in the exponent is just for convenience, as explained below). We present this construction in Section 2.1. To build the interlacements, we use the trajectories of simple random walks conditioned on never hitting the origin. Of course, the law of the vacant set is no longer translationally invariant, but we show that it has the property of *conditional* translation invariance, cf. Theorem 2.3 below. In addition, we will see that (similarly to the $d \geq 3$ case) the random object we construct

has strong connections to random walks on two-dimensional torus. All this makes us believe that “two-dimensional random interlacements” is the right term for the object we introduce in this paper.

1.2 Cover time and late points of simple random walk on a discrete torus

Consider the simple random walk on the two-dimensional discrete torus \mathbb{Z}_n^2 with the starting point chosen uniformly at random. Let \mathcal{T}_n be the first moment when this random walk visits all sites of \mathbb{Z}_n^2 ; we refer to \mathcal{T}_n as the *cover time* of the torus. It was shown in [5] that $\frac{\mathcal{T}_n}{n^2 \ln^2 n} \rightarrow \frac{4}{\pi}$ in probability; later, this result was refined in [7], and then even finer results on the first correction to this limit were obtained in [2] for the similar problem of covering the (continuous) torus with a Brownian sausage.

The structure of the set of *late points* (i.e., the set of points that are still unvisited up to a given time) of the random walk on the torus is rather well understood in dimensions $d \geq 3$, see [1, 14], and also in the continuous case [10]. On the other hand, much remains to be discovered in two dimensions. In [6] it was shown that this set has interesting fractal-like properties when the elapsed time is a fraction of the expected cover time. This particular behavior is induced by long distance correlations between hitting times due to recurrence. In this paper, we prove that the law of the uncovered set around the origin at time $\frac{4\alpha}{\pi} n^2 \ln^2 n$ *conditioned* on the event that the origin is uncovered, is close to the law of two-dimensional random interlacements at level α . We hope that this result would lead to other advances in understanding the structure of the uncovered set.

As a side note, observe that the two-dimensional random interlacements relate to the simple random walk on the torus at a time proportional to the cover time. In higher dimensions, one starts to observe the “interlacement regime” already at times below the cover time by a factor of $\ln n$.

Organisation of the paper: In Section 2.1 we construct the model of random interlacements and present some of its properties. In Section 2.2 we formulate a result relating this model and the vacant set of the simple random walk on the discrete torus. We prove some results on the spot – when short arguments are available – , postponing the proof of the other ones to Section 4. Section 3 contains a number of auxiliary facts needed for the proof of the main results.

2 Definitions and results

We start by defining the two-dimensional random interlacement process, which involves some potential theoretic considerations.

2.1 Random interacements: definitions, properties

Let $\|\cdot\|$ be the Euclidean norm. Define the (discrete) ball

$$B(x, r) = \{y \in \mathbb{Z}^2 : \|y - x\| \leq r\}$$

(note that x and r need not be necessarily integer), and abbreviate $B(r) := B(0, r)$. We write $x \sim y$ if x and y are neighbours on \mathbb{Z}^2 . The (internal) boundary of $A \subset \mathbb{Z}^2$ is defined by

$$\partial A = \{x \in A : \text{there exists } y \in \mathbb{Z}^2 \setminus A \text{ such that } x \sim y\}.$$

Let $(S_n, n \geq 0)$ be two-dimensional simple random walk. Write P_x for the law of the walk started from x and E_x for the corresponding expectation. Let

$$\tau_0(A) = \inf\{k \geq 0 : S_k \in A\}, \quad (3)$$

$$\tau_1(A) = \inf\{k \geq 1 : S_k \in A\} \quad (4)$$

be the entrance and the hitting time of the set A by simple random walk S . Note that $\inf \emptyset = +\infty$. For a singleton $A = \{x\}$, we will shortly denote $\tau_i(A) = \tau_i(x)$, $i = 0, 1$. Define the potential kernel a by

$$a(x) = \sum_{k=0}^{\infty} (P_0[S_k=0] - P_x[S_k=0]). \quad (5)$$

It can be shown that the above series indeed converges and we have $a(0) = 0$, $a(x) > 0$ for $x \neq 0$, and

$$a(x) = \frac{2}{\pi} \ln \|x\| + \gamma' + O(\|x\|^{-2}) \quad (6)$$

as $x \rightarrow \infty$, cf. Theorem 4.4.4 of [12] (the value of γ' is known¹, but we will not need it in this paper). Also, the function a is harmonic outside the origin, i.e.,

$$\frac{1}{4} \sum_{y: y \sim x} a(y) = a(x) \quad \text{for all } x \neq 0. \quad (7)$$

¹ $\gamma' = \pi^{-1}(2\gamma + \ln 8)$, where $\gamma = 0.5772156 \dots$ is the Euler-Mascheroni constant

Observe that (7) immediately implies that $a(S_{k \wedge \tau_0(0)})$ is a martingale, we will repeatedly use this fact in the sequel. With some abuse of notation, we also consider the function

$$a(r) = \frac{2}{\pi} \ln r + \gamma'$$

of a *real* argument $r \geq 1$. The advantage of using this notation is e.g. that, due to (6), we may write, as $r \rightarrow \infty$,

$$\sum_{y \in \partial B(x,r)} \nu(y) a(y) = a(r) + O(\|x\|/r) \quad (8)$$

for *any* probability measure ν on $\partial B(x,r)$. The *harmonic measure* of a finite $A \subset \mathbb{Z}^2$ is the entrance law “starting at infinity”,

$$\text{hm}_A(x) = \lim_{\|y\| \rightarrow \infty} P_y[S_{\tau_1(A)} = x]. \quad (9)$$

By recurrence of the walk, hm_A is a probability measure on ∂A . The existence of the above limit follows from Proposition 6.6.1 of [12]; also, this proposition together with (6.44) implies that

$$\text{hm}_A(x) = \frac{2}{\pi} \lim_{R \rightarrow \infty} P_x[\tau_1(A) > \tau_1(\partial B(R))] \ln R. \quad (10)$$

Intuitively, the harmonic measure at $x \in \partial A$ is proportional to the probability of escaping from x to a large sphere. Now, for a finite set A containing the origin, we define its capacity by

$$\text{cap}(A) = \sum_{x \in A} a(x) \text{hm}_A(x); \quad (11)$$

in particular, $\text{cap}(\{0\}) = 0$ since $a(0) = 0$. For a set not containing the origin, its capacity is defined as the capacity of a translate of this set that does contain the origin. Indeed, it can be shown that the capacity does not depend on the choice of the translation. A number of alternative definitions is available, cf. Section 6.6 of [12]. Also, observe that, by symmetry, the harmonic measure of any two-point set is uniform, so $\text{cap}(\{x, y\}) = \frac{1}{2}a(y-x)$ for any $x, y \in \mathbb{Z}^2$.

Let us define another random walk $(\widehat{S}_n, n \geq 0)$ on \mathbb{Z}^2 (in fact, on $\mathbb{Z}^2 \setminus \{0\}$) in the following way: the transition probability from x to y equals $\frac{a(y)}{4a(x)}$ for all $x \sim y$ (this definition does not make sense for $x = 0$, but this is not a

problem since the walk \widehat{S} can never enter the origin anyway). The reader will promptly recognise \widehat{S} as the Doob's h -transform of the simple random walk, under condition of not hitting the origin. Note that (7) implies that the random walk \widehat{S} is indeed well defined, and, clearly, it is an irreducible Markov chain on $\mathbb{Z}^2 \setminus \{0\}$. We denote by $\widehat{P}_x, \widehat{E}_x$ the probability and expectation for the random walk \widehat{S} started from $x \neq 0$. Then, it is straightforward to observe that

- the walk \widehat{S} is reversible, with the reversible measure $\mu_x := a^2(x)$;
- in fact, it can be represented as a random walk on the two-dimensional lattice with conductances (or weights) $(a(x)a(y), x, y \in \mathbb{Z}^2, x \sim y)$;
- $(a(x), x \in \mathbb{Z}^2 \setminus \{0\})$ is an excessive measure for \widehat{S} (i.e., for all $y \neq 0$, $\sum_x a(x)\widehat{P}_x(\widehat{S}_1 = y) \leq a(y)$), with equality failing at the four neighbours of the origin. Therefore, by e.g. Theorem 1.9 of Chapter 3 of [16], the random walk \widehat{S} is transient;
- an alternative argument for proving transience is the following: let \mathcal{N} be the set of the four neighbours of the origin. Then, a direct calculation shows that $1/a(\widehat{S}_{k \wedge \widehat{\tau}_0(\mathcal{N})})$ is a martingale. The transience then follows from Theorem 2.2.2 of [9].

Let $\widehat{\tau}_0, \widehat{\tau}_1$ be defined as in (3)–(4), but with \widehat{S} on the place of S . Our next definitions are appropriate for the transient case. For a finite $A \subset \mathbb{Z}^2$, we define the *equilibrium measure*

$$\widehat{e}_A(x) = \mathbf{1}\{x \in A\} \widehat{P}_x[\widehat{\tau}_1(A) = \infty] \mu_x, \quad (12)$$

and the capacity (with respect to \widehat{S})

$$\widehat{\text{cap}}(A) = \sum_{x \in A} \widehat{e}_A(x). \quad (13)$$

Observe that, since $\mu_0 = 0$, it holds that $\widehat{\text{cap}}(A) = \widehat{\text{cap}}(A \cup \{0\})$ for any set $A \subset \mathbb{Z}^2$.

Now, we use the general construction of random interacements on a transient weighted graph introduced in [19]. In the following few lines we

briefly summarize this construction. Let W be the space of all doubly infinite nearest-neighbour transient trajectories in \mathbb{Z}^2 ,

$$W = \{\varrho = (\varrho_k)_{k \in \mathbb{Z}} : \text{the set } \{m : \varrho_m = y\} \text{ is finite for all } y \in \mathbb{Z}^2\}.$$

We say that ϱ and ϱ' are equivalent if they coincide after a time shift, i.e., $\varrho \sim \varrho'$ when there exists k such that $\varrho_{m+k} = \varrho'_m$ for all m . Then, let $W^* = W / \sim$ be the space of trajectories modulo time shift, and define χ^* to be the canonical projection from W to W^* . For a finite $A \subset \mathbb{Z}^2$, let W_A be the set of trajectories in W that intersect A , and we write W_A^* for the image of W_A under χ^* . One then constructs the random interlacements as Poisson point process on $W^* \times \mathbb{R}^+$ with the intensity measure $\nu \otimes du$, where ν is described in the following way. It is the unique sigma-finite measure on W^* such that for every finite A

$$\mathbf{1}_{W_A^*} \cdot \nu = \chi^* \circ Q_A,$$

where the finite measure Q_A on W_A is determined by the following equality:

$$Q_A[(\varrho_k)_{k \geq 1} \in F, \varrho_0 = x, (\varrho_{-k})_{k \geq 1} \in G] = \widehat{e}_A(x) \cdot \widehat{P}_x[F] \cdot \widehat{P}_x[G \mid \widehat{\tau}_1(A) = \infty].$$

The existence and uniqueness of ν was shown in Theorem 2.1 of [19].

Definition 2.1. For a configuration $\sum_{\lambda} \delta_{(w_{\lambda}^*, u_{\lambda})}$ of the above Poisson process, the process of random interlacements at level α (which will be referred to as $RI(\alpha)$) is defined as the set of trajectories with label less than or equal to $\pi\alpha$, i.e.,

$$\sum_{\lambda: u_{\lambda} \leq \pi\alpha} \delta_{w_{\lambda}^*}.$$

Observe that this definition is a bit unconventional (we used $\pi\alpha$ instead of just α , as one would normally do), but we will see below that it is quite reasonable in two dimensions, since the formulas become generally cleaner.

It is important to have in mind the following “constructive” description of random interlacements at level α “observed” on a finite set $A \subset \mathbb{Z}^2$. Namely,

- take $\text{Poisson}(\pi\alpha \widehat{\text{cap}}(A))$ number of particles;
- place these particles on the boundary of A independently, with distribution $\bar{e}_A = ((\widehat{\text{cap}} A)^{-1} \widehat{e}_A(x), x \in A)$;

- let the particles perform independent \widehat{S} -random walks (since \widehat{S} is transient, each walk only leaves a finite trace on A).

It is also worth mentioning that the FKG inequality holds for random interacements, cf. Theorem 3.1 of [19].

The *vacant set* at level α ,

$$\mathcal{V}^\alpha = \mathbb{Z}^2 \setminus \bigcup_{\lambda: u_\lambda \leq \pi\alpha} \omega_\lambda^*(\mathbb{Z}),$$

is the set of lattice points uncovered by the random interlacement. It contains the origin by definition. In Figure 1 we present a simulation of the vacant set for different values of the parameter.

As a last step, we need to show that we have indeed constructed the object for which (2) is verified. For this, we need to prove the following fact:

Proposition 2.2. *For any finite set $A \subset \mathbb{Z}^2$ such that $0 \in A$ it holds that $\text{cap}(A) = \widehat{\text{cap}}(A)$.*

Proof. Indeed, consider an arbitrary $x \in \partial A$, $x \neq 0$, and (large) r such that $A \subset B(r-2)$. Write using (6)

$$\begin{aligned} \widehat{P}_x[\widehat{\tau}_1(A) > \widehat{\tau}_1(\partial B(r))] &= \sum_{\varrho} \frac{a(\varrho_{\text{end}})}{a(x)} \left(\frac{1}{4}\right)^{|\varrho|} \\ &= (1 + o(1)) \frac{\frac{2}{\pi} \ln r}{a(x)} \sum_{\varrho} \left(\frac{1}{4}\right)^{|\varrho|} \\ &= (1 + o(1)) \frac{\frac{2}{\pi} \ln r}{a(x)} P_x[\tau_1(A) > \tau_1(\partial B(r))], \end{aligned}$$

where the sums are taken over all trajectories ϱ that start at x , end at $\partial B(r)$, and avoid $A \cup \partial B(r)$ in between; $\varrho_{\text{end}} \in \partial B(r)$ stands for the ending point of the trajectory, and $|\varrho|$ is the trajectory's length. Now, we send r to infinity and use (10) to obtain that, if $0 \in A$,

$$a(x) \widehat{P}_x[\widehat{\tau}_1(A) = \infty] = \text{hm}_A(x). \quad (14)$$

Multiplying by $a(x)$ and taking summation in $x \in A$ (recall that $\mu_x = a^2(x)$) we obtain the expressions in (11) and (13) and thus conclude the proof. \square

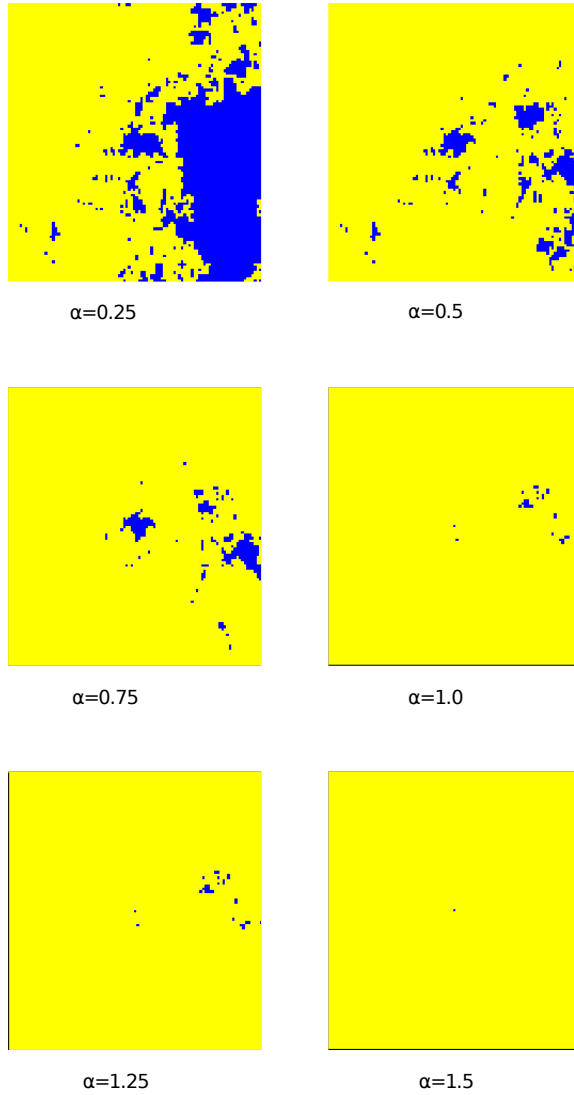


Figure 1: A realization of the vacant set (dark blue) of $RI(\alpha)$ for different values of α . For $\alpha = 1.5$ the only vacant site is the origin. Also, note that the sets are the same for $\alpha = 1$ and $\alpha = 1.25$; this is not surprising since just a few new walks enter the picture when increasing the rate by a small amount.

Together with formula (1.1) of [19], Proposition 2.2 shows the fundamental relation (2) announced in introduction: for all finite subsets A of \mathbb{Z}^2 ,

$$\mathbb{P}[A \subset \mathcal{V}^\alpha] = \exp(-\pi\alpha \operatorname{cap}(A \cup \{0\})).$$

As mentioned before, the law of two-dimensional random interacements is not translationally invariant, although it is of course invariant with respect to reflections/rotations of \mathbb{Z}^2 that preserve the origin. Let us describe some other basic properties of two-dimensional random interacements:

Theorem 2.3. (i) For any $\alpha > 0$, $x \in \mathbb{Z}^2$, $A \subset \mathbb{Z}^2$, it holds that

$$\mathbb{P}[A \subset \mathcal{V}^\alpha \mid x \in \mathcal{V}^\alpha] = \mathbb{P}[-A + x \subset \mathcal{V}^\alpha \mid x \in \mathcal{V}^\alpha]. \quad (15)$$

More generally, for all $\alpha > 0$, $x \in \mathbb{Z}^2 \setminus \{0\}$, $A \subset \mathbb{Z}^2$, and any orthogonal transformation M preserving the lattice and exchanging 0 and x , we have

$$\mathbb{P}[A \subset \mathcal{V}^\alpha \mid x \in \mathcal{V}^\alpha] = \mathbb{P}[MA \subset \mathcal{V}^\alpha \mid x \in \mathcal{V}^\alpha]. \quad (16)$$

(ii) With γ' from (6) we have

$$\mathbb{P}[x \in \mathcal{V}^\alpha] = \exp\left(-\pi\alpha \frac{a(x)}{2}\right) = e^{-\gamma'\pi\alpha/2} \|x\|^{-\alpha} (1 + O(\|x\|^{-2})). \quad (17)$$

(iii) For A such that $0 \in A \subset B(r)$ we have

$$\mathbb{P}[A \subset \mathcal{V}^\alpha \mid x \in \mathcal{V}^\alpha] = \exp\left(-\frac{\pi\alpha}{4} \operatorname{cap}(A) \frac{1 + O\left(\frac{r \ln r \ln \|x\|}{\|x\|}\right)}{1 - \frac{\operatorname{cap}(A)}{2a(x)} + O\left(\frac{r \ln r}{\|x\|}\right)}\right). \quad (18)$$

(iv) For $x, y \neq 0$, $x \neq y$, we have $\mathbb{P}[\{x, y\} \subset \mathcal{V}^\alpha] = \exp(-\pi\alpha \operatorname{cap}(\{0, x, y\}))$, where the expression for $\operatorname{cap}(\{0, x, y\})$ is given in Lemma 3.9 below. Moreover, as $s := \|x\| \rightarrow \infty$, $\ln \|y\| \sim \ln s$ and $\ln \|x - y\| \sim \beta \ln s$ with some $\beta \in [0, 1]$, we have

$$\mathbb{P}[\{x, y\} \subset \mathcal{V}^\alpha] = s^{-\frac{4\alpha}{4-\beta} + o(1)}. \quad (19)$$

(v) Assume that $\ln \|x\| \sim \ln s$, $\ln r \sim \beta \ln s$ with $\beta < 1$. Then, as $s \rightarrow \infty$,

$$\mathbb{P}[B(x, r) \subset \mathcal{V}^\alpha] = s^{-\frac{2\alpha}{2-\beta} + o(1)}. \quad (20)$$

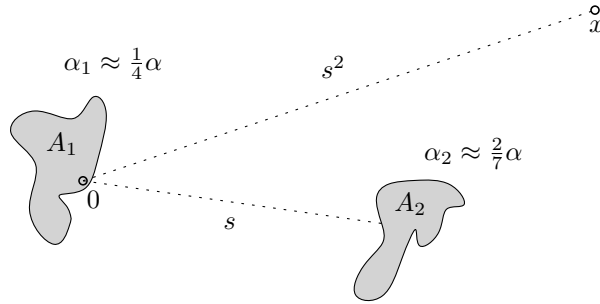


Figure 2: How the “local rate” looks like if we condition in the event that a “distant” site is vacant.

These results invite a few comments.

Remark 2.4. 1. *The statement in (i) describes an invariance property given that a point is vacant. We refer to it as the conditional stationarity of two-dimensional random interacements.*

2. *We can interpret (iii) as follows: the conditional law of $RI(\alpha)$ given that a distant site x is vacant, is similar – near the origin – to the unconditional law of $RI(\alpha/4)$. Combined with (i), the similarity holds near x as well. Moreover, one can also estimate the “local rate” away from the origin, see Figure 2. More specifically, observe from Lemma 3.10 (ii) that $\text{cap}(A_2) \ll \ln s$ with $s = \text{dist}(0, A_2)$ large implies $\text{cap}(\{0\} \cup A_2) = \frac{\alpha(s)}{2}(1 + o(1))$. If x is at a much larger distance, say $\ln \|x\| \sim \ln(s^2)$, then (18) reveals a “local rate” equal to $\frac{2}{7}\alpha$, that is, $\mathbb{P}[A_2 \subset \mathcal{V}^\alpha \mid x \in \mathcal{V}^\alpha] = \exp(-\frac{2}{7}\pi\alpha \text{cap}(\{0\} \cup A_2)(1 + o(1)))$.*

3. *By symmetry, the conclusion of (iv) remains the same in the situation when $\ln \|x\|, \ln \|x - y\| \sim \ln s$ and $\ln \|y\| \sim \ln(s^\beta)$.*

Proof of (i) and (ii). To prove (i), observe that

$$\text{cap}(\{0, x\} \cup A) = \text{cap}(\{0, x\} \cup (-A + x))$$

by symmetry. For the second statement in (i), note that, for $A' = \{0, x\} \cup A$, it holds that $\text{cap}(A') = \text{cap}(MA') = \text{cap}(\{0, x\} \cup MA)$. Item (ii) follows from the above mentioned fact that $\text{cap}(\{0, x\}) = \frac{1}{2}a(x)$ together with (6). \square

We postpone the proof of other parts of Theorem 2.3, since it requires some estimates for capacities of different kind of sets. We now turn estimates on the cardinality of the vacant set.

Theorem 2.5. (i) *We have*

$$\mathbb{E}(|\mathcal{V}^\alpha \cap B(r)|) \sim \begin{cases} \frac{2\pi}{2-\alpha} e^{-\gamma'\pi\alpha/2} \times r^{2-\alpha}, & \text{for } \alpha < 2, \\ 2\pi e^{-\gamma'\pi\alpha/2} \times \ln r, & \text{for } \alpha = 2, \\ \text{const}, & \text{for } \alpha > 2. \end{cases}$$

(ii) *For $\alpha > 1$ it holds that \mathcal{V}^α is finite a.s. Moreover, $\mathbb{P}[\mathcal{V}^\alpha = \{0\}] > 0$ and $\mathbb{P}[\mathcal{V}^\alpha = \{0\}] \rightarrow 1$ as $\alpha \rightarrow \infty$.*

(iii) *For $\alpha \in (0, 1)$, we have $|\mathcal{V}^\alpha| = \infty$ a.s. Moreover,*

$$\mathbb{P}[\mathcal{V}^\alpha \cap (B(r) \setminus B(r/2)) = \emptyset] \leq r^{-2(1-\sqrt{\alpha})^2+o(1)}. \quad (21)$$

Proof of (i) and (ii). Part (i) immediately follows from Theorem 2.3 (ii).

The proof of the part (ii) is easy in the case $\alpha > 2$. Indeed, observe first that $\mathbb{E}|\mathcal{V}^\alpha| < \infty$ implies that \mathcal{V}^α itself is a.s. finite. Also, Theorem 2.3 (ii) actually implies that $\mathbb{E}|\mathcal{V}^\alpha \setminus \{0\}| \rightarrow 0$ as $\alpha \rightarrow \infty$, so $\mathbb{P}[\mathcal{V}^\alpha = \{0\}] \rightarrow 1$ by Chebyshev inequality.

Now, let us prove that, in general, $\mathbb{P}[|\mathcal{V}^\alpha| < \infty] = 1$ implies that $\mathbb{P}[\mathcal{V}^\alpha = \{0\}] > 0$. Indeed, if \mathcal{V}^α is a.s. finite, then one can find a sufficiently large R such that $\mathbb{P}[|\mathcal{V}^\alpha \cap (\mathbb{Z}^2 \setminus B(R))| = 0] > 0$. Since $\mathbb{P}[x \notin \mathcal{V}^\alpha] > 0$ for any $x \neq 0$, the claim that $\mathbb{P}[\mathcal{V}^\alpha = \{0\}] > 0$ follows from the FKG inequality applied to events $\{x \notin \mathcal{V}^\alpha\}$, $x \in B(R)$ together with $\{|\mathcal{V}^\alpha \cap (\mathbb{Z}^2 \setminus B(R))| = 0\}$. \square

As before, we postpone the proof of part (iii) and the rest of part (ii) of Theorem 2.5. Let us remark that we believe that the right-hand side of (21) gives the correct order of decay of the above probability; we, however, do not have a rigorous argument at the moment. Also, note that the question whether \mathcal{V}^1 is a.s. finite or not, is open. For now, the authors do not have even heuristic arguments in favor of either alternative.

We do have, however, a heuristic explanation about the unusual behaviour of the model for $\alpha \in (1, 2)$: in this non-trivial interval, the vacant set is a.s. finite but its expected size is infinite. The reason for that is the following: the number of \hat{S} -walks that hit $B(r)$ has Poisson law with rate of order $\ln r$. Thus, decreasing this number by a constant factor (with respect to the expectation)

has only a polynomial cost. On the other hand, by doing so, we increase the probability that a site $x \in B(r)$ is vacant for all $x \in B(r)$ at once, which increases the expected size of $\mathcal{V}^\alpha \cap B(r)$ by a polynomial factor. It turns out that this effect causes the actual number of uncovered sites in $B(r)$ to be typically of much smaller order than the expected number of uncovered sites there.

2.2 Simple random walk on a discrete torus and its relationship with random interacements

Now, we state our results for the random walk on torus. Let $(X_k, k \geq 0)$ be the simple random walk on \mathbb{Z}_n^2 with X_0 chosen uniformly at random. Define the entrance time to the site $x \in \mathbb{Z}_n^2$ by

$$T_n(x) = \inf\{t \geq 0 : X_t = x\}, \quad (22)$$

and the *cover time* of the torus by

$$\mathcal{T}_n = \max_{x \in \mathbb{Z}_n^2} T_n(x). \quad (23)$$

Let us also define the *uncovered set* at time t ,

$$U_t^{(n)} = \{x \in \mathbb{Z}_n^2 : T_n(x) > t\}. \quad (24)$$

Denote by $\Upsilon_n : \mathbb{Z}^2 \rightarrow \mathbb{Z}_n^2$, $\Upsilon_n(x, y) = (x \bmod n, y \bmod n)$, the natural projection modulo n . Then we can write $X_k = \Upsilon(S_k)$. Consistently, we will use the same notation for (discrete Euclidean) balls on \mathbb{Z}_n^2 : $B(y, r) \subset \mathbb{Z}_n^2$ is defined by $B(y, r) = \Upsilon_n B(z, r)$, where $z \in \mathbb{Z}^2$ is such that $\Upsilon_n z = y$. Let also

$$t_\alpha := \frac{4\alpha}{\pi} n^2 \ln^2 n.$$

In the following theorem, we prove that, given that 0 is uncovered, the law of the uncovered set around 0 at time t_α is close to that of $\text{RI}(\alpha)$:

Theorem 2.6. *Let $\alpha > 0$ and A is a finite subset of \mathbb{Z}^2 . We have*

$$\lim_{n \rightarrow \infty} \mathbb{P}[\Upsilon_n A \subset U_{t_\alpha}^{(n)} \mid 0 \in U_{t_\alpha}^{(n)}] = \exp(-\pi\alpha \text{cap}(A \cup \{0\})). \quad (25)$$

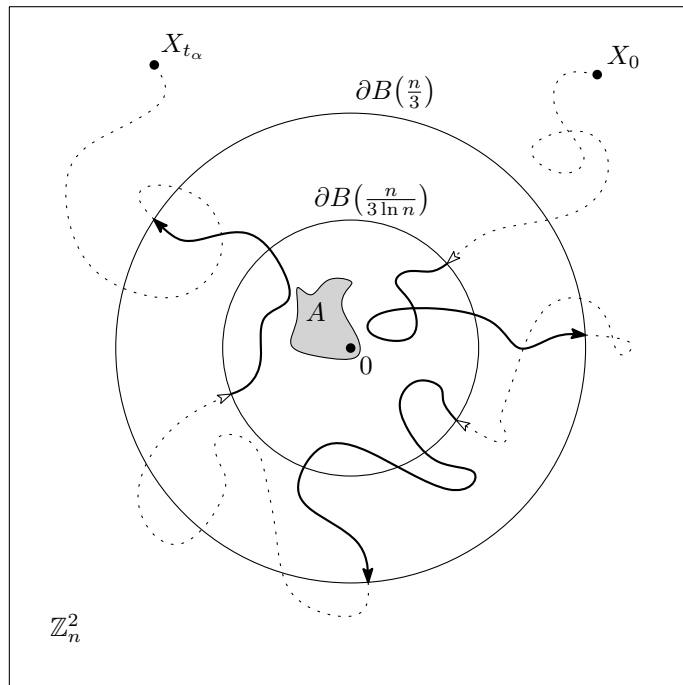


Figure 3: Excursions (depicted as the solid pieces of the trajectory) of the SRW on the torus \mathbb{Z}_n^2

The proof of this theorem will be presented in Section 4. However, at this point we give a heuristic argument for (25). Let us consider the *excursions* of the random walk X between $\partial B(\frac{n}{3\ln n})$ and $\partial B(n/3)$ up to time t_α . We hope that Figure 3 is self-explanatory; formal definitions are presented in Section 3.4. Let N_α be the number of these excursions. It is possible to prove that this (random) number is concentrated around $\frac{2\alpha \ln^2 n}{\ln \ln n}$, with deviation probabilities of subpolynomial order, see Lemma 3.12 below. The result we just mentioned is for unconditional probabilities, but, since the probability of the event $\{0 \in U_{t_\alpha}^{(n)}\}$ is only polynomially small (actually, it is $n^{-2\alpha+o(1)}$), the same holds for the deviation probabilities conditioned on this event. So, let us just assume for now that the number of the excursions is *exactly* $\frac{2\alpha \ln^2 n}{\ln \ln n}$, and see where will it lead us.

Assume without restriction of generality that $0 \in A$. Then, Lemmas 3.1 and 3.3 imply that

- the probability that an excursion hits the origin is roughly $\frac{\ln \ln n}{\ln(n/3)}$;
- provided that $\text{cap}(A) \ll \ln n$, the probability that an excursion hits the set A is roughly $\frac{\ln \ln n}{\ln(n/3)} \left(1 + \frac{\pi \text{cap}(A)}{2 \ln(n/3)}\right)$.

So, the *conditional* probability p_* that an excursion does not hit A given that it does not hit the origin is

$$p_* \approx \frac{1 - \frac{\ln \ln n}{\ln(n/3)} \left(1 + \frac{\pi \text{cap}(A)}{2 \ln(n/3)}\right)}{1 - \frac{\ln \ln n}{\ln(n/3)}} \approx 1 - \frac{\pi \ln \ln n}{2 \ln^2 n} \text{cap}(A),$$

and then we obtain

$$\begin{aligned} \mathbb{P}[\Upsilon_n A \subset U_{t_\alpha}^{(n)} \mid 0 \in U_{t_\alpha}^{(n)}] &\approx p_*^{N_\alpha} \\ &\approx \left(1 - \frac{\pi \ln \ln n}{2 \ln^2 n} \text{cap}(A)\right)^{\frac{2\alpha \ln^2 n}{\ln \ln n}} \\ &\approx \exp\left(-\pi \alpha \text{cap}(A)\right), \end{aligned}$$

which agrees with the statement of Theorem 2.6.

However, turning the above heuristics to a rigorous proof is not an easy task. The reason for this is that, although N_α is indeed concentrated around $\frac{2\alpha \ln^2 n}{\ln \ln n}$, it is not *concentrated enough*: the probability that 0 is not hit during k excursions, where k varies over the “typical” values of N_α , changes too much. Therefore, in the proof of Theorem 2.6 we take a different route by considering the suitable h -transform of the walk, as explained in Section 4.

3 Some auxiliary facts and estimates

In this section we collect lemmas of all sorts that will be needed in the sequel.

3.1 Simple random walk in \mathbb{Z}^2

First, we recall a couple of basic facts for the exit probabilities of simple random walk.

Lemma 3.1. *For all $x, y \in \mathbb{Z}^2$ and $R > 0$ with $x \in B(y, R)$, $\|y\| \leq R - 2$, we have*

$$P_x[\tau_1(0) > \tau_1(\partial B(y, R))] = \frac{a(x)}{a(R) + O(R^{-1}\|y\|)}, \quad (26)$$

and for all $y \in B(r)$, $x \in B(y, R) \setminus B(r)$ with $r + \|y\| \leq R - 2$, we have

$$P_x[\tau_1(\partial B(r)) > \tau_1(\partial B(y, R))] = \frac{a(x) - a(r) + O(r^{-1})}{a(R) - a(r) + O(R^{-1}\|y\| + r^{-1})}, \quad (27)$$

as $r, R \rightarrow \infty$.

Remark 3.2. *Observe that (26) reduces to (27) with $r = 0$ provided that the terms $a(r)$ and $O(r^{-1})$ are taken equal to 0.*

Proof. Both (26) and (27) are easily deduced from the following argument: recall that the sequence $a(S_{k \wedge \tau_0(0)})$, $k \geq 0$, is a martingale, and apply the Optional Stopping Theorem together with (6) and (8). For the second statement, with

$$\tau = \tau_1(\partial B(r)) \wedge \tau_1(\partial B(y, R)) \quad \text{and} \quad q = P_x[\tau_1(\partial B(r)) > \tau_1(\partial B(y, R))],$$

we write

$$\begin{aligned} a(x) &= E_x[a(S_\tau); \tau = \tau_1(\partial B(y, R))] + E_x[a(S_\tau); \tau = \tau_1(\partial B(r))] \\ &= (1 - q)(a(r) + O(r^{-1})) + q(a(R) + O(R^{-1}\|y\|)), \end{aligned}$$

yielding (27). The first statement has a similar proof. \square

We have an estimate for more general sets.

Lemma 3.3. *Let A be a finite subset of \mathbb{Z}^2 such that $A \subset B(r)$. We have for $r + 1 \leq \|x\| \leq R - 2$, $\|x\| + \|y\| \leq R - 1$*

$$P_x[\tau_1(A) > \tau_1(\partial B(y, R))] = \frac{a(x) - \text{cap}(A) + O\left(\frac{r \ln r \ln \|x\|}{\|x\|}\right)}{a(R) - \text{cap}(A) + O\left(R^{-1}\|y\| + \frac{r \ln r \ln \|x\|}{\|x\|}\right)}. \quad (28)$$

Proof. First, observe that, for $u \in A$,

$$P_x[S_{\tau_1(A)} = u] = \text{hm}_A(u) \left(1 + O\left(\frac{r \ln \|x\|}{\|x\|}\right)\right),$$

see e.g. Theorem 2.1.3. in [11]. Then, we can write

$$\begin{aligned} & P_x[S_{\tau_1(A)} = u, \tau_1(A) < \tau_1(\partial B(y, R))] \\ &= P_x[S_{\tau_1(A)} = u] - P_x[S_{\tau_1(A)} = u, \tau_1(A) > \tau_1(\partial B(y, R))] \\ &= \text{hm}_A(u) \left(1 + O\left(\frac{r \ln \|x\|}{\|x\|}\right)\right) - P_x[\tau_1(A) > \tau_1(\partial B(y, R))] \\ &\quad \times \sum_{z \in \partial B(y, R)} P_z[S_{\tau_1(A)} = u] P_x[S_{\tau_1(\partial B(y, R))} = z \mid \tau_1(A) > \tau_1(\partial B(y, R))] \\ &= \text{hm}_A(u) \left(1 + O\left(\frac{r \ln \|x\|}{\|x\|}\right)\right) - P_x[\tau_1(A) > \tau_1(\partial B(y, R))] \text{hm}_A(u) \left(1 + O\left(\frac{r \ln R}{R}\right)\right) \\ &= \text{hm}_A(u) P_x[\tau_1(A) < \tau_1(\partial B(y, R))] \left(1 + O\left(\frac{r \ln \|x\|}{\|x\|}\right)\right), \end{aligned}$$

using twice the above observation and also that $B(\|x\|) \subset B(y, R)$, yielding

$$P_x[S_{\tau_1(A)} = u \mid \tau_1(A) < \tau_1(\partial B(y, R))] = \text{hm}_A(u) \left(1 + O\left(\frac{r \ln \|x\|}{\|x\|}\right)\right). \quad (29)$$

Abbreviating $q := P_x[\tau_1(A) > \tau_1(\partial B(y, R))]$, we have by the Optional Stopping Theorem and (29)

$$\begin{aligned} a(x) &= qa(R) + O(R^{-1}\|y\|) + (1 - q) \sum_{u \in A \setminus \{0\}} a(u) \text{hm}_A(u) \left(1 + O\left(\frac{r \ln \|x\|}{\|x\|}\right)\right) \\ &= qa(R) + O(R^{-1}\|y\|) + (1 - q) \left(1 + O\left(\frac{r \ln \|x\|}{\|x\|}\right)\right) \text{cap}(A), \end{aligned}$$

and (28) follows (observe also that $\text{cap}(A) \leq \text{cap}(B(r)) = \frac{2}{\pi} \ln r + O(1)$). \square

3.2 Simple random walk conditioned on not hitting the origin

Next, we relate the probabilities of certain events for the walks S and \widehat{S} . For $M \subset \mathbb{Z}^2$, let $\Gamma_M^{(x)}$ be the set of all nearest-neighbour finite trajectories that start at $x \in M \setminus \{0\}$ and end when entering ∂M for the first time; denote also $\Gamma_{y,R}^{(x)} = \Gamma_{B(y,R)}^{(x)}$. For $A \subset \Gamma_M^{(x)}$ write $S \in A$ if there exists k such that $(S_0, \dots, S_k) \in A$ (and the same for the conditional walk \widehat{S}). In the next result we show that $P_x[\cdot \mid \tau_1(0) > \tau_1(\partial B(R))]$ and $\widehat{P}_x[\cdot]$ are almost indistinguishable on $\Gamma_{0,R}^{(x)}$ (that is, the conditional law of S almost coincides with the unconditional law of \widehat{S}). A similar result holds for excursions on a “distant” (from the origin) set.

Lemma 3.4. (i) Assume $A \subset \Gamma_{0,R}^{(x)}$. We have

$$P_x[S \in A \mid \tau_1(0) > \tau_1(\partial B(R))] = \widehat{P}_x[\widehat{S} \in A] (1 + O((R \ln R)^{-1})). \quad (30)$$

(ii) Assume that $A \subset \Gamma_M^{(x)}$ and suppose that $0 \notin M$, and denote $s = \text{dist}(0, M)$, $r = \text{diam}(M)$. Then, for $x \in M$,

$$P_x[S \in A] = \widehat{P}_x[\widehat{S} \in A] \left(1 + O\left(\frac{r}{s \ln s}\right)\right). \quad (31)$$

Proof. Let us prove part (i). Assume without restricting generality that no trajectory from A passes through the origin. Then, it holds that

$$\widehat{P}_x[\widehat{S} \in A] = \sum_{\varrho \in A} \frac{a(\varrho_{\text{end}})}{a(x)} \left(\frac{1}{4}\right)^{|\varrho|},$$

with $|\varrho|$ the length of ϱ . On the other hand, by (26)

$$P_x[S \in A \mid \tau_1(0) > \tau_1(\partial B(R))] = \frac{a(R) + O(R^{-1})}{a(x)} \sum_{\varrho \in A} \left(\frac{1}{4}\right)^{|\varrho|}.$$

Since $\varrho_{\text{end}} \in \partial B(R)$, we have $a(\varrho_{\text{end}}) = a(R) + O(R^{-1})$, and so (30) follows.

The proof of part (ii) is analogous (observe that $a(y_1)/a(y_2) = 1 + O\left(\frac{r}{s \ln s}\right)$ for any $y_1, y_2 \in M$). \square

As observed in Section 1.1, the random walk \widehat{S} is transient. Next, we estimate the probability that the \widehat{S} -walk avoids a ball centered at the origin:

Lemma 3.5. *Assume $r \geq 1$ and $\|x\| \geq r + 1$. We have*

$$\widehat{P}_x[\widehat{\tau}_1(B(r)) = \infty] = \left(1 - \frac{a(r)}{a(\|x\|)}\right)(1 + O(r^{-1})).$$

Proof. By (26) and Lemma 3.4 (i) we have

$$\widehat{P}_x[\widehat{\tau}_1(B(r)) = \infty] = \lim_{R \rightarrow \infty} P_x[\tau_1(\partial B(r)) > \tau_1(\partial B(R)) \mid \tau_1(0) > \tau_1(\partial B(R))].$$

The claim then follows from (26)–(27). \square

Remark 3.6. *Alternatively, one can deduce the proof of Lemma 3.5 from the fact that $1/a(\widehat{S}_{k \wedge \widehat{\tau}_0(\mathcal{N})})$ is a martingale, together with the Optional Stopping Theorem.*

We will need also an expression for the probability of avoiding *any* finite set containing the origin:

Lemma 3.7. *Assume that $0 \in A \subset B(r)$, and $\|x\| \geq r + 1$. Then*

$$\widehat{P}_x[\widehat{\tau}_1(A) = \infty] = 1 - \frac{\text{cap}(A)}{a(x)} + O\left(\frac{r \ln r \ln \|x\|}{\|x\|}\right). \quad (32)$$

Proof. Indeed, using Lemmas 3.3 and 3.4 (i) together with (26), we write

$$\begin{aligned} \widehat{P}_x[\widehat{\tau}_1(A) = \infty] &= \lim_{R \rightarrow \infty} P_x[\tau_1(A) > \tau_1(\partial B(R)) \mid \tau_1(0) > \tau_1(\partial B(R))] \\ &= \lim_{R \rightarrow \infty} \frac{a(R) + O(R^{-1})}{a(x)} \times \frac{a(x) - \text{cap}(A) + O\left(\frac{r \ln r \ln \|x\|}{\|x\|}\right)}{a(R) - \text{cap}(A) + O\left(R^{-1} + \frac{r \ln r \ln \|x\|}{\|x\|}\right)}, \end{aligned}$$

thus obtaining (32). \square

It is also possible to obtain the exact expressions for one-site escape probabilities, and probabilities of (not) hitting a given site:

$$\widehat{P}_x[\widehat{\tau}_1(y) < \infty] = \frac{a(x) + a(y) - a(x - y)}{2a(x)}, \quad (33)$$

for $x \neq y$, $x, y \neq 0$ and

$$\widehat{P}_x[\widehat{\tau}_1(x) < \infty] = 1 - \frac{1}{2a(x)} \quad (34)$$

for $x \neq 0$. In particular, we recover from (34) the transience of \widehat{S} . Also, observe that (33) implies the following surprising fact: for any $x \neq 0$,

$$\lim_{y \rightarrow \infty} \widehat{P}_x[\widehat{\tau}_1(y) < \infty] = \frac{1}{2}.$$

The above relation leads to the following heuristic explanation for Theorem 2.3 (iii). Since the probability of hitting a distant site is about $1/2$, by conditioning that this distant site is vacant, we kind of throw away three quarters of the trajectories that pass through a neighbourhood of the origin: indeed the double-infinite trajectory has to avoid this distant site two times, before and after reaching that neighborhood.

We temporarily postpone the proof of (33)–(34). Let us first state a couple of other general estimates, for the probability of (not) hitting a given set (which is, typically, far away from the origin), or, more specifically, a disk:

Lemma 3.8. *Assume that $x \notin B(y, r)$ and $\|y\| > 2r \geq 1$. Abbreviate also $\Psi_1 = \|y\|^{-1}r$, $\Psi_2 = \frac{r \ln r \ln \|y\|}{\|y\|}$, $\Psi_3 = r \ln r \left(\frac{\ln \|x-y\|}{\|x-y\|} + \frac{\ln \|y\|}{\|y\|} \right)$.*

(i) *We have*

$$\widehat{P}_x[\widehat{\tau}_1(B(y, r)) < \infty] = \frac{(a(y) + O(\Psi_1))(a(y) + a(x) - a(x-y) + O(r^{-1}))}{a(x)(2a(y) - a(r) + O(r^{-1} + \Psi_1))}. \quad (35)$$

(ii) *Consider now any nonempty set $A \subset B(y, r)$. Then, it holds that*

$$\widehat{P}_x[\widehat{\tau}_1(A) < \infty] = \frac{(a(y) + O(\Psi_1))(a(y) + a(x) - a(x-y) + O(r^{-1} + \Psi_3))}{a(x)(2a(y) - \text{cap}(A) + O(\Psi_2))}. \quad (36)$$

Observe that (35) is *not* a particular case of (36); this is because (27) typically provides a more precise estimate than (28).

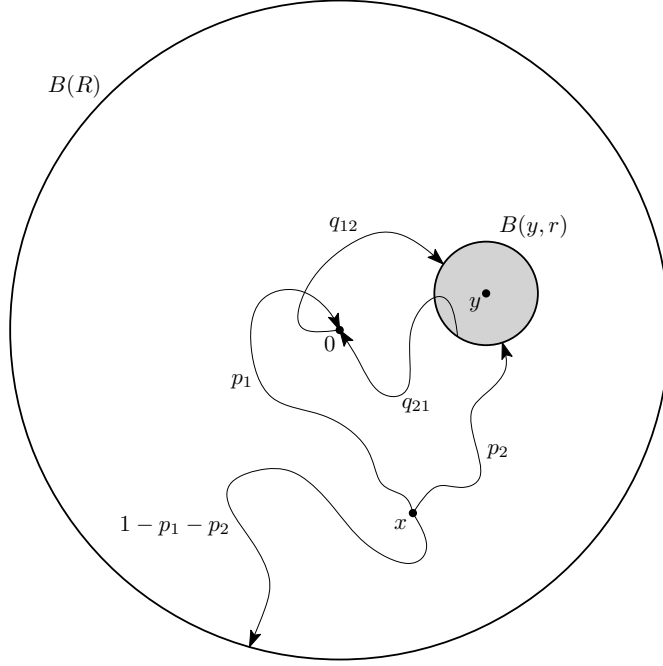


Figure 4: On the proof of Lemma 3.8

Proof. Fix a (large) $R > 0$, such that $R > \max\{\|x\|, s + r\} + 1$. Denote

$$\begin{aligned}
 h_1 &= P_x[\tau_1(0) < \tau_1(\partial B(R))], \\
 h_2 &= P_x[\tau_1(B(y, r)) < \tau_1(\partial B(R))], \\
 p_1 &= P_x[\tau_1(0) < \tau_1(\partial B(R)) \wedge \tau_1(B(y, r))], \\
 p_2 &= P_x[\tau_1(B(y, r)) < \tau_1(\partial B(R)) \wedge \tau_1(0)], \\
 q_{12} &= P_0[\tau_1(B(y, r)) < \tau_1(\partial B(R))], \\
 q_{21} &= P_\nu[\tau_1(0) < \tau_1(\partial B(R))],
 \end{aligned}$$

where ν is the entrance measure to $B(y, r)$ starting from x conditioned on the event $\{\tau_1(B(y, r)) < \tau_1(\partial B(R)) \wedge \tau_1(0)\}$, see Figure 4. Using Lemma 3.1, we obtain

$$h_1 = 1 - \frac{a(x)}{a(R) + O(R^{-1})}, \quad (37)$$

$$h_2 = 1 - \frac{a(x - y) - a(r) + O(r^{-1})}{a(R) - a(r) + O(R^{-1}\|y - x\| + r^{-1})}, \quad (38)$$

and

$$q_{12} = 1 - \frac{a(y) - a(r) + O(r^{-1})}{a(R) - a(r) + O(R^{-1}\|y\| + r^{-1})}, \quad (39)$$

$$q_{21} = 1 - \frac{a(y) + O(\|y\|^{-1}r)}{a(R) + O(R^{-1}\|y\|)}. \quad (40)$$

Then, as a general fact, it holds that

$$h_1 = p_1 + p_2 q_{21},$$

$$h_2 = p_2 + p_1 q_{12}.$$

Solving this system with respect to p_1, p_2 , we obtain

$$p_1 = \frac{h_1 - h_2 q_{21}}{1 - q_{12} q_{21}}, \quad (41)$$

$$p_2 = \frac{h_2 - h_1 q_{12}}{1 - q_{12} q_{21}}, \quad (42)$$

and so, using (37)–(40), we write

$$\begin{aligned} & P_x[\tau_1(B(y, r)) < \tau_1(\partial B(R)) \mid \tau_1(0) > \tau_1(\partial B(R))] \\ &= \frac{p_2(1 - q_{21})}{1 - h_1} \\ &= \frac{(h_2 - h_1 q_{12})(1 - q_{21})}{(1 - h_1)(1 - q_{12} q_{21})} \\ &= \left(\frac{a(x)}{a(R) + O(R^{-1})} + \frac{a(y) - a(r) + O(r^{-1})}{a(R) - a(r) + O(R^{-1}\|y\| + r^{-1})} \right. \\ &\quad \left. - \frac{a(x - y) - a(r) + O(r^{-1})}{a(R) - a(r) + O(R^{-1}\|x - y\| + r^{-1})} + O\left(\frac{\ln \|x\| \ln \|y\|}{\ln^2 R}\right) \right) \\ &\quad \times \frac{a(y) + O(\|y\|^{-1}r)}{a(R) + O(R^{-1}\|y\|)} \times \left(\frac{a(x)}{a(R) + O(R^{-1})} \right)^{-1} \\ &\quad \times \left(\frac{a(y) - a(r) + O(r^{-1})}{a(R) - a(r) + O(R^{-1}\|y\| + r^{-1})} + \frac{a(y) + O(\|y\|^{-1}r)}{a(R) + O(R^{-1}\|y\|)} + O\left(\frac{\ln^2 \|y\|}{\ln^2 R}\right) \right)^{-1}. \end{aligned}$$

Sending R to infinity, we obtain the proof of (35).

To prove (36), we use the same procedure, with $h'_1 = h_1$, q'_{21} expressed in the same way as q_{21} (although they are not necessarily equal, the difference takes place in the error terms $O(\cdot)$) and, by Lemma 3.3,

$$h'_2 = 1 - \frac{a(x-y) - \text{cap}(A) + O\left(\frac{r \ln r \ln \|x-y\|}{\|x-y\|}\right)}{a(R) - \text{cap}(A) + O\left(R^{-1}\|x-y\| + \frac{r \ln r \ln \|x-y\|}{\|x-y\|}\right)},$$

$$q'_{12} = 1 - \frac{a(y) - \text{cap}(A) + O\left(\frac{r \ln r \ln \|y\|}{\|y\|}\right)}{a(R) - \text{cap}(A) + O\left(R^{-1}\|y\| + \frac{r \ln r \ln \|y\|}{\|y\|}\right)}.$$

After the analogous calculations, we obtain (36). \square

Proof of relations (33)–(34). Formula (34) rephrases (14) with $A = \{0, x\}$. Identity (33) follows from the same proof as in Lemma 3.8 (i), taking $r = 0$ and using Remark 3.2. \square

3.3 Harmonic measure and capacities

Next, we need a formula for calculating the capacity of three-point sets:

Lemma 3.9. *Let $x_1, x_2, x_3 \in \mathbb{Z}^2$, and abbreviate $v_1 = x_2 - x_1$, $v_2 = x_3 - x_2$, $v_3 = x_1 - x_3$. Then, the capacity of the set $A = \{x_1, x_2, x_3\}$ is given by the formula*

$$\frac{a(v_1)a(v_2)a(v_3)}{a(v_1)a(v_2) + a(v_1)a(v_3) + a(v_2)a(v_3) - \frac{1}{2}(a^2(v_1) + a^2(v_2) + a^2(v_3))}. \quad (43)$$

Proof. By Proposition 6.6.3 and Lemma 6.6.4 of [12], the inverse capacity of A is equal to the sum of entries of the matrix

$$a_A^{-1} = \begin{pmatrix} 0 & a(v_1) & a(v_3) \\ a(v_1) & 0 & a(v_2) \\ a(v_3) & a(v_2) & 0 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} -\frac{a(v_2)}{a(v_1)a(v_3)} & \frac{1}{a(v_1)} & \frac{1}{a(v_3)} \\ \frac{1}{a(v_1)} & -\frac{a(v_3)}{a(v_1)a(v_2)} & \frac{1}{a(v_2)} \\ \frac{1}{a(v_3)} & \frac{1}{a(v_2)} & -\frac{a(v_1)}{a(v_2)a(v_3)} \end{pmatrix},$$

and this implies (43). \square

Before proceeding, let us notice the following immediate consequence of Lemma 3.7: for any finite $A \subset \mathbb{Z}^2$ such that $0 \in A$, we have

$$\text{cap}(A) = \lim_{\|x\| \rightarrow \infty} a(x) \widehat{P}_x[\tau_1(A) < \infty]. \quad (44)$$

Next, we need estimates for the \widehat{S} -capacity of a “distant” set, and, in particular of a ball which does not contain the origin. Recall the notations $\Psi_{1,2,3}$ from Lemma 3.8.

Lemma 3.10. *Assume that $\|y\| > 2r \geq 1$.*

(i) *We have*

$$\text{cap}(\{0\} \cup B(y, r)) = \frac{(a(y) + O(\Psi_1))(a(y) + O(r^{-1}))}{2a(y) - a(r) + O(r^{-1} + \Psi_1)}. \quad (45)$$

(ii) *Suppose that $A \subset B(y, r)$. Then*

$$\text{cap}(\{0\} \cup A) = \frac{(a(y) + O(\Psi_1))(a(y) + O(r^{-1} + \Psi_2))}{2a(y) - \text{cap}(A) + O(\Psi_2)}. \quad (46)$$

Proof. This immediately follows from (44) and Lemma 3.8 (observe that $a(x) - a(x - y) \rightarrow 0$ as $x \rightarrow \infty$ and Ψ_3 becomes Ψ_2). \square

We also need to compare the harmonic measure on a set (distant from the origin) to the entrance measure of the \widehat{S} -walk started far away from that set.

Lemma 3.11. *Assume that A is a finite subset of \mathbb{Z}^2 , $0 \notin A$, $x \neq 0$, and also that $2 \text{diam}(A) < \text{dist}(x, A) < \frac{1}{4} \text{dist}(0, A)$. Abbreviate $u = \text{diam}(A)$, $s = \text{dist}(x, A)$. Then, for $y \in A$, it holds that*

$$\widehat{P}_x[\widehat{S}_{\widehat{\tau}_1(A)} = y \mid \widehat{\tau}_1(A) < \infty] = \text{hm}_A(y) \left(1 + O\left(\frac{u \ln s}{s}\right)\right). \quad (47)$$

Proof. Let $z_0 \in A$ be such that $\|z_0 - x\| = s$. Define the discrete circle $L = \partial B(z_0, s)$; observe that $x \in L$ and $\text{dist}(z', A) \geq s/2$ for all $z' \in L$. Let

$$\sigma = \sup \{0 \leq k \leq \widehat{\tau}_1(A) : \widehat{S}_k \in L\}$$

be the *last* moment before $\widehat{\tau}_1(A)$ when the trajectory passes through L . Note also that for all $z \in L$

$$P_z[S_{\tau_1(A)} = y \mid \tau_1(A) < \tau_1(L)] = \text{hm}_A(y) \left(1 + O\left(\frac{u \ln s}{s}\right)\right); \quad (48)$$

the proof of this is actually the same as the proof of (29).

Using the Markov property of \widehat{S} , we write

$$\begin{aligned}
& \widehat{P}_x[\widehat{\tau}_1(A) < \infty, \widehat{S}_{\widehat{\tau}_1(A)} = y] \\
&= \sum_{k \geq 0, z \in L} \widehat{P}_x[\widehat{\tau}_1(A) < \infty, \sigma = k, \widehat{S}_\sigma = z, \widehat{S}_{\widehat{\tau}_1(A)} = y] \\
&= \sum_{k \geq 0, z \in L} \widehat{P}_x[\widehat{S}_k = z, \widehat{S}_\ell \notin A \text{ for all } \ell \leq k] \\
&\quad \times \widehat{P}_z[\widehat{\tau}_1(A) < \infty, \widehat{S}_{\widehat{\tau}_1(A)} = y, \widehat{S}_\ell \notin L \text{ for all } \ell \leq \widehat{\tau}_1(A)]. \quad (49)
\end{aligned}$$

Now, observe that the last term in (49) only involves trajectories that lie in $B(z_0, s)$, and we have $\text{dist}(0, B(z_0, s)) \geq r/2$. So, we can use Lemma 3.4 (ii) together with (48) to write, with $r = \text{dist}(0, A)$,

$$\begin{aligned}
& \widehat{P}_z[\widehat{\tau}_1(A) < \infty, \widehat{S}_{\widehat{\tau}_1(A)} = y, \widehat{S}_\ell \notin L \text{ for all } \ell \leq \widehat{\tau}_1(A)] \\
&= P_z[S_{\tau_1(A)} = y, S_\ell \notin L \text{ for all } \ell \leq \tau_1(A)] \left(1 + O\left(\frac{u}{r \ln r}\right)\right) \\
&= P_z[S_{\tau_1(A)} = y \mid \tau_1(A) < \tau_1(L)] P_z[\tau_1(A) < \tau_1(L)] \left(1 + O\left(\frac{u}{r \ln r}\right)\right) \\
&= \text{hm}_A(y) \widehat{P}_z[\widehat{\tau}_1(A) < \widehat{\tau}_1(L)] \left(1 + O\left(\frac{u}{r \ln r} + \frac{u \ln s}{s}\right)\right).
\end{aligned}$$

Inserting this back to (49), we obtain

$$\begin{aligned}
& \widehat{P}_x[\widehat{\tau}_1(A) < \infty, \widehat{S}_{\widehat{\tau}_1(A)} = y] \\
&= \text{hm}_A(y) \left(1 + O\left(\frac{u}{r \ln r} + \frac{u \ln u}{s}\right)\right) \\
&\quad \times \sum_{k \geq 0, z \in L} \widehat{P}_x[\widehat{S}_k = z, \widehat{S}_\ell \notin A \text{ for all } \ell \leq k] \widehat{P}_z[\widehat{\tau}_1(A) < \widehat{\tau}_1(L)] \\
&= \text{hm}_A(y) \widehat{P}_z[\widehat{\tau}_1(A) < \infty] \left(1 + O\left(\frac{u}{r \ln r} + \frac{u \ln s}{s}\right)\right),
\end{aligned}$$

and this concludes the proof of Lemma 3.11 (observe that the first term in $O(\cdot)$ is of smaller order than the second one). \square

3.4 Random walk on the torus and its excursions

First, we define the entrance time to a set $A \subset \mathbb{Z}_n^2$ by

$$T_n(A) = \min_{x \in A} T_n(x).$$

Now, consider two sets $A \subset A' \subset \mathbb{Z}_n^2$, and suppose that we are only interested in the trace left by the random walk on the set A . Then, (apart from the initial piece of the trajectory until hitting $\partial A'$ for the first time) it is enough to know what are the excursions of the random walk between the boundaries of A and A' . By definition, an excursion ϱ is a simple random walk path that starts at ∂A and ends on its first visit to $\partial A'$, i.e., $\varrho = (\varrho_0, \varrho_1, \dots, \varrho_m)$, where $\varrho_0 \in \partial A$, $\varrho_m \in \partial A'$, $\varrho_k \notin \partial A'$ and $\varrho_k \sim \varrho_{k+1}$ for $k < m$. With some abuse of notation, we denote by $\varrho_{\text{st}} := \varrho_0$ and $\varrho_{\text{end}} := \varrho_m$ the starting and the ending points of the excursion. To define these excursions, consider the following sequence of stopping times:

$$\begin{aligned} D_0 &= T_n(\partial A'), \\ J_1 &= \inf\{t > D_0 : X_t \in \partial A\}, \\ D_1 &= \inf\{t > J_1 : X_t \in \partial A'\}, \end{aligned}$$

and

$$\begin{aligned} J_k &= \inf\{t > D_{k-1} : X_t \in \partial A\}, \\ D_k &= \inf\{t > J_k : X_t \in \partial A'\}, \end{aligned}$$

for $k \geq 2$. Then, denote by $Z^{(i)} = (X_{J_i}, \dots, X_{D_i})$ the i th excursion of X between ∂A and $\partial A'$, for $i \geq 1$. Also, let $Z_0 = (X_0, \dots, X_{D_0})$ be the “initial” excursion (it is possible, in fact, that it does not intersect the set A at all). Recall that $t_\alpha := \frac{4\alpha}{\pi} n^2 \ln^2 n$ and define

$$N_\alpha = \max\{k : J_k \leq t_\alpha\}, \quad (50)$$

$$N'_\alpha = \max\{k : D_k \leq t_\alpha\} \quad (51)$$

to be the number of incomplete (respectively, complete) excursions up to time t_α .

Next, we need also to define the excursions of random interlacements in an analogous way. Assume that the trajectories of the \widehat{S} -walks that intersect A are enumerated according to their u -labels (recall the construction in Section 2.2). For each trajectory from that list (say, the j th one, denoted $\widehat{S}^{(j)}$ and time-shifted in such a way that $\widehat{S}_k^{(j)} \notin A$ for all $k \leq -1$ and $\widehat{S}_0^{(j)} \in A$) define the stopping times

$$\widehat{J}_1 = 0,$$

$$\hat{D}_1 = \inf\{t > \hat{J}_1 : \widehat{S}_t^{(j)} \in \partial A'\},$$

and

$$\begin{aligned} \hat{J}_k &= \inf\{t > \hat{D}_{k-1} : \widehat{S}_t^{(j)} \in \partial A\}, \\ \hat{D}_k &= \inf\{t > \hat{J}_k : \widehat{S}_t^{(j)} \in \partial A'\}, \end{aligned}$$

for $k \geq 2$. Let $\ell_j = \inf\{k : \hat{J}_k = \infty\} - 1$ be the number of excursions corresponding to the j th trajectory. The excursions of $\text{RI}(\alpha)$ between ∂A and $\partial A'$ are then defined by

$$\hat{Z}^{(i)} = (\widehat{S}_{\hat{J}_m}^{(j)}, \dots, \widehat{S}_{\hat{D}_m}^{(j)}),$$

where $i = m + \sum_{k=1}^{j-1} \ell_k$, and $m = 1, 2, \dots, \ell_j$. We let R_α to be the number of trajectories intersecting A and with labels less than $\alpha\pi$, and denote $\hat{N}_\alpha = \sum_{k=1}^{R_\alpha} \ell_k$ to be the total number of excursions of $\text{RI}(\alpha)$ between ∂A and $\partial A'$.

Observe also that the above construction makes sense with $\alpha = \infty$ as well; we then obtain an infinite sequence of excursions of RI ($=\text{RI}(\infty)$) between ∂A and $\partial A'$.

Finally, let us recall a result of [6] on the number of excursions for the simple random walk on \mathbb{Z}_n^2 .

Lemma 3.12. *Consider the random variables J_k, D_k defined in this section with $A = B(\frac{n}{3 \ln n})$, $A' = B(\frac{n}{3})$. Then, there exist positive constants δ_0, c_0 such that, for any δ with $\frac{c_0}{\ln n} \leq \delta \leq \delta_0$, we have*

$$\mathbb{P}\left[J_k \in \left((1 - \delta) \frac{2n^2 \ln \ln n}{\pi} k, (1 + \delta) \frac{2n^2 \ln \ln n}{\pi} k\right)\right] \geq 1 - \exp(-c\delta^2 k), \quad (52)$$

and the same result holds with D_k on the place of J_k .

Proof. This is, in fact, a particular case of Lemma 3.2 of [6]. \square

A useful consequence of Lemma 3.12 is

$$\mathbb{P}\left[(1 - \delta) \frac{2\alpha \ln^2 n}{\ln \ln n} \leq N_\alpha \leq (1 + \delta) \frac{2\alpha \ln^2 n}{\ln \ln n}\right] \geq 1 - \exp\left(-c\delta^2 \frac{\ln^2 n}{\ln \ln n}\right), \quad (53)$$

and the same result holds with N'_α on the place of N_α , where N_α, N'_α are defined as in (50)–(51) with $A = B(\frac{n}{3 \ln n})$, $A' = B(\frac{n}{3})$.

4 Proofs of the main results

First of all, we apply some results of Section 3 to finish the proof of Theorem 2.3.

Proof of Theorem 2.3, parts (iii)–(v). Recall the fundamental formula (2) for the random interlacement and the relation (6). Then, the statement (iv) follows from Lemma 3.9, while (v) is a consequence of Lemma 3.10 (i).

Finally, observe that, by symmetry and Lemma 3.10 (ii) we have

$$\begin{aligned} \mathbb{P}[A \subset \mathcal{V}^\alpha \mid x \in \mathcal{V}^\alpha] &= \exp\left(-\pi\alpha(\text{cap}(A \cup \{x\}) - \text{cap}(\{0, x\}))\right) \\ &= \exp\left(-\pi\alpha\left(\frac{(a(x) + O(\frac{r \ln r \ln \|x\|}{\|x\|}))^2}{2a(x) - \text{cap}(A) + O(\frac{r \ln r \ln \|x\|}{\|x\|})} - \frac{a(x)}{2}\right)\right) \\ &= \exp\left(-\frac{\pi\alpha}{4} \text{cap}(A) \frac{1 + O(\frac{r \ln r \ln \|x\|}{\|x\|})}{1 - \frac{\text{cap}(A)}{2a(x)} + O(\frac{r \ln r}{\|x\|})}\right), \end{aligned}$$

thus proving the part (iii). \square

Proof of Theorem 2.5 (iii). We start by stating an elementary fact: let N be a Poisson random variable with parameter λ , and Y_1, Y_2, Y_3, \dots be independent (also of N) random variables with exponential distribution $\mathcal{E}(p)$ with parameter p . Let $\Theta = \sum_{j=1}^N Y_j$ be the corresponding compound Poisson random variable. Its Cramér transform $b \mapsto \lambda(\sqrt{b} - 1)^2$ is easily computed, and the Chernov's bound writes, for all $b > 1$,

$$\mathbb{P}[\Theta \geq b\lambda p^{-1}] \leq \exp(-\lambda(\sqrt{b} - 1)^2). \quad (54)$$

Now, assume that $\alpha < 1$. Fix $\beta \in (0, 1)$, which will be later taken close to 1, and fix some set of non-intersecting disks $B'_1 = B(x_1, r^\beta), \dots, B'_{k_r} = B(x_{k_r}, r^\beta) \subset B(r) \setminus B(r/2)$, with cardinality $k_r = \frac{1}{4}r^{2(1-\beta)}$. Denote also $B_j := B(x_j, \frac{r^\beta}{\ln^3 r^\beta})$, $j = 1, \dots, k_r$.

Before going to the heart of the matter we briefly sketch the strategy of proof. We start to show that at least a half of these balls B_j will receive at most $\frac{2\alpha \ln^2 r^\beta}{3\beta^2 \ln \ln r^\beta}$ excursions from ∂B_j to $\partial B'_j$ up to time t_α . Moreover, using soft local times, we couple such excursions from $RI(\alpha)$ with a slightly larger number of independent excursions from the \hat{S} -walk: with overwhelming probability, the trace on $\cup_j B_j$ of the latter excursion process contains the trace

of the former, so the vacant set \mathcal{V}^α restricted to balls B_j is smaller than the set of unvisited points by the independent process. Now, by independence, it will be possible to estimate the probability for leaving that many balls partially uncovered, and this will conclude the proof.

We start with an observation: the number of \widehat{S} -walks in $RI(\alpha)$ intersecting a given disk B_{i_0} has Poisson law with parameter $\lambda = (1 + o(1)) \frac{2\alpha}{2-\beta} \ln r$. Indeed, the law is Poisson by construction, the parameter $\pi\alpha \text{cap}(B_{i_0} \cup \{0\})$ is found in (2) and then estimated using Lemma 3.10 (i).

Next, by Lemma 3.8 (i), the probability that the walk \widehat{S} started from any $y \in \partial B'_{i_0}$ does not hit B_{i_0} is $(1 + o(1)) \frac{3 \ln \ln r^\beta}{(2-\beta) \ln r}$. This depends on the starting point, however each \widehat{S} -walk generates a number of excursions between ∂B_{i_0} and $\partial B'_{i_0}$ which is dominated by a geometric law $G(p')$ with success parameter $p' = (1 + o(1)) \frac{3 \ln \ln r^\beta}{(2-\beta) \ln r}$. Recall also that the integer part of $\mathcal{E}(u)$ is geometric $G(1 - e^{-u})$. So, with $p = -\ln(1 - p')$, the total number $\widehat{N}_\alpha^{(i_0)}$ of excursions between ∂B_{i_0} and $\partial B'_{i_0}$ in $RI(\alpha)$ can be dominated by a compound Poisson law with $\mathcal{E}(p)$ terms in the sum with expectation

$$\lambda p^{-1} = (1 + o(1)) \frac{2\alpha \ln^2 r^\beta}{3\beta^2 \ln \ln r^\beta}.$$

Then, using (54), we obtain for $b > 1$

$$\begin{aligned} \mathbb{P} \left[\widehat{N}_\alpha^{(i_0)} \geq b \frac{2\alpha \ln^2 r^\beta}{3\beta^2 \ln \ln r^\beta} \right] &\leq \exp \left(- (1 + o(1)) (\sqrt{b} - 1)^2 \frac{2\alpha}{\beta(2-\beta)} \ln r^\beta \right) \\ &= r^{-(1+o(1))(\sqrt{b}-1)^2 \frac{2\alpha}{2-\beta}}. \end{aligned} \quad (55)$$

Now, let W_b be the set

$$W_b = \left\{ j \leq k_r : \widehat{N}_\alpha^{(j)} < b \frac{2\alpha \ln^2 r^\beta}{3\beta^2 \ln \ln r^\beta} \right\}.$$

Combining (55) with Markov inequality, we get

$$\frac{k_r}{2} \mathbb{P} [|W_b^c| > k_r/2] \leq \mathbb{E} |W_b^c| \leq k_r r^{-(1+o(1))(\sqrt{b}-1)^2 \frac{2\alpha}{2-\beta}},$$

so

$$\mathbb{P} [|W_b| \geq k_r/2] \geq 1 - 2r^{-(1+o(1))(\sqrt{b}-1)^2 \frac{2\alpha}{2-\beta}}. \quad (56)$$

Assume that $1 < b < \alpha^{-1}$ and $\beta \in (0, 1)$ is close enough to 1, so that $\frac{b\alpha}{\beta^2} < 1$. As in Section 3.4, we denote by $\hat{Z}^{(1),j}, \dots, \hat{Z}^{(\hat{N}_\alpha^{(j)}),j}$ the excursions of $\text{RI}(\alpha)$ between ∂B_j and $\partial B'_j$. Also, let $\tilde{Z}^{(1),j}, \tilde{Z}^{(2),j}, \tilde{Z}^{(3),j}, \dots$ be a sequence of i.i.d. \widehat{S} -excursions between ∂B_j and $\partial B'_j$, started at hm_{B_j} . In the following, we assume that both \hat{Z} - and \tilde{Z} -excursions are constructed on the same probability space using the *soft local times*, as described in Section 2 of [4].

Let $\delta > 0$ be such that $(1 + \delta)\frac{b\alpha}{\beta^2} < 1$ and abbreviate $m = b\frac{2\alpha \ln^2 r^\beta}{3\beta^2 \ln r^\beta}$. Define the events

$$\tilde{\mathcal{G}}_j = \{B'_j \text{ is completely covered by } \tilde{Z}^{(1),j} \cup \dots \cup \tilde{Z}^{((1+\delta)m),j}\}.$$

Then, for all $j \leq k_r$ it holds that

$$\mathbb{P}[\tilde{\mathcal{G}}_j] \leq \frac{1}{5} \quad (57)$$

for all large enough r . Indeed, if the \tilde{Z} 's were SRW-excursions, the above inequality (with any fixed constant in the right-hand side) could be obtained in a completely analogous way as the proof of Lemma 3.2 of [4]. On the other hand, Lemma 3.4 (ii) implies that the first $(1 + \delta)m$ \widehat{S} -excursions can be coupled with SRW-excursions with high probability, so (57) holds for \widehat{S} -excursions as well.

Define the set

$$\widetilde{W} = \{j \leq k_r : \tilde{\mathcal{G}}_j^c \text{ occurs}\}.$$

Since the events $(\tilde{\mathcal{G}}_j, j \leq k_r)$ are independent, by (57) we have (recall that $k_r = \frac{1}{4}r^{2(1-\beta)}$)

$$\mathbb{P}\left[|\widetilde{W}| \geq \frac{3}{5}k_r\right] \geq 1 - \exp(-Cr^{2(1-\beta)}) \quad (58)$$

for all r large enough.

Next, we consider the events

$$D_j = \left\{ \left\{ \hat{Z}^{(1),j}, \dots, \hat{Z}^{(\hat{N}_\alpha^{(j)}),j} \right\} \subset \left\{ \tilde{Z}^{(1),j}, \dots, \tilde{Z}^{((1+\delta)m),j} \right\} \right\}$$

(recall that the excursions are coupled using soft local times), and

$$D = \bigcap_{j \leq k_r} D_j.$$

Lemma 2.1 of [4] implies that (observe that, by Lemma 3.11, the parameter v in Lemma 2.1 of [4] can be anything exceeding $O(1/\ln^2 r)$, so we choose e.g. $v = (\ln \ln r)^{-1}$)

$$\mathbb{P}[D_j^{\mathfrak{C}}] \leq \exp\left(-C' \frac{\ln^2 r}{(\ln \ln r)^2}\right),$$

so we obtain by the union bound the subpolynomial estimate

$$\mathbb{P}[D^{\mathfrak{C}}] \leq \frac{1}{4} r^{2(1-\beta)} \exp\left(-C' \frac{\ln^2 r}{(\ln \ln r)^2}\right). \quad (59)$$

Observe that, by construction, on the event D we have $\mathcal{V}^\alpha \cap B'_j \neq \emptyset$ for all $j \in \widetilde{W} \cap W_b$. So, using (56), (58), and (59), we obtain

$$\begin{aligned} & \mathbb{P}\left[\mathcal{V}^\alpha \cap (B(r) \setminus B(r/2)) = \emptyset\right] \\ & \leq \mathbb{P}\left[|W_b| < \frac{k_r}{2}\right] + \mathbb{P}[D^{\mathfrak{C}}] + \mathbb{P}\left[|\widetilde{W}| < \frac{3k_r}{5}\right] \\ & \leq 2r^{-(1+o(1))(\sqrt{b-1})^2 \frac{2\alpha}{2-\beta}} + \frac{1}{4} r^{2(1-\beta)} \exp\left(-C' \frac{\ln^2 r}{(\ln \ln r)^2}\right) + \exp(-Cr^{2(1-\beta)}). \end{aligned}$$

Since $b \in (1, \alpha^{-1})$ can be arbitrarily close to α^{-1} and $\beta \in (0, 1)$ can be arbitrarily close to 1, this concludes the proof of (21). \square

Proof of Theorem 2.5 (ii). To complete the proofs in Section 2.1, it remains to prove that $|\mathcal{V}^\alpha| < \infty$ a.s. for $\alpha > 1$. First, we establish the following elementary fact. For $x \in \partial B(2r)$ and $y \in B(r) \setminus B(r/2)$, it holds

$$\widehat{P}_x[\widehat{\tau}_1(y) < \widehat{\tau}_1(r \ln r)] = \frac{\ln \ln r}{\ln r} (1 + o(1)). \quad (60)$$

Indeed, define the events

$$\begin{aligned} G_0 &= \{\tau_1(0) < \tau_1(r \ln r)\}, \\ G_1 &= \{\tau_1(y) < \tau_1(r \ln r)\}; \end{aligned}$$

then, Lemma 3.1 implies that

$$P_x[G_0] = \frac{\ln \ln r}{\ln r} (1 + o(1)) = P_x[G_1] (1 + o(1)).$$

Observe that

$$\begin{aligned}
P_x[G_0 \cap G_1] &= P_x[G_0 \cup G_1]P_x[G_0 \cap G_1 \mid G_0 \cup G_1] \\
&\leq (P_x[G_0] + P_x[G_1])(P_0[G_1] + P_y[G_0]) \\
&\leq \left(\frac{\ln \ln r}{\ln r}\right)^2 (1 + o(1)).
\end{aligned}$$

So,

$$\begin{aligned}
P_x[G_1 \mid G_0^c] &= \frac{P_x[G_1] - P_x[G_0 \cap G_1]}{1 - P_x[G_0]} \\
&= \frac{\ln \ln r}{\ln r} (1 + o(1)),
\end{aligned}$$

and we use Lemma 3.4 to conclude the proof of (60).

Now, the goal is to prove that, for $\alpha > 1$

$$\mathbb{P}[\text{there exists } y \in B(r) \setminus B(r/2) \text{ such that } y \in \mathcal{V}^\alpha] \leq r^{-\frac{\alpha}{2}(1-\alpha^{-1})^2(1+o(1))}. \quad (61)$$

This would clearly imply that the set \mathcal{V}^α is a.s. finite, since

$$\{\mathcal{V}^\alpha \text{ is infinite}\} = \{\mathcal{V}^\alpha \cap (B(2^n) \setminus B(2^{n-1})) \neq \emptyset \text{ for infinitely many } n\}, \quad (62)$$

and the Borel-Cantelli lemma together with (61) imply that the probability of the latter event equals 0.

Let $N_{\alpha,r}$ be the number of \widehat{S} -excursions of $\text{RI}(\alpha)$ between $\partial B(r)$ and $\partial B(r \ln r)$. Analogously to (55), it is straightforward to show that, for $b < 1$,

$$\mathbb{P}\left[N_{\alpha,r} \leq b \frac{2\alpha \ln^2 r}{\ln \ln r}\right] \leq r^{-2\alpha(1-\sqrt{b})^2(1+o(1))}. \quad (63)$$

Now, (60) implies that for $y \in B(r) \setminus B(r/2)$

$$\mathbb{P}\left[y \text{ is uncovered by first } b \frac{2\alpha \ln^2 r}{\ln \ln r} \text{ excursions}\right] \leq \left(1 - \frac{\ln \ln r}{\ln r} (1 + o(1))\right)^{\frac{2\alpha \ln^2 r}{\ln \ln r}} = r^{-2b\alpha(1+o(1))}, \quad (64)$$

so, using the union bound,

$$\mathbb{P}\left[\exists y \in B(r) \setminus B(r/2) : y \in \mathcal{V}^\alpha, N_{\alpha,r} > b \frac{2\alpha \ln^2 r}{\ln \ln r}\right] \leq r^{-2(b\alpha-1)(1+o(1))}. \quad (65)$$

Using (63) and (65) with $b = \frac{1}{4}(1 + \frac{1}{\alpha})^2$ we conclude the proof of (61) and of Theorem 2.5 (ii). \square

Proof of Theorem 2.6. Abbreviate $\delta_{n,\alpha} = C\alpha\sqrt{\frac{\ln \ln n}{\ln n}}$ and

$$I_{\delta_{n,\alpha}} = \left[(1 - \delta_{n,\alpha}) \frac{2\alpha \ln^2 n}{\ln \ln n}, (1 + \delta_{n,\alpha}) \frac{2\alpha \ln^2 n}{\ln \ln n} \right].$$

Let N_α be the number of excursions between $\partial B(\frac{n}{3 \ln n})$ and $\partial B(n/3)$ up to time t_α . It holds that $\mathbb{P}[0 \in U_{t_\alpha}^{(n)}] = n^{-2\alpha+o(1)}$, see e.g. (1.6)–(1.7) in [4]. Then, observe that Lemma 3.12 implies that

$$\mathbb{P}[N_\alpha \notin I_\delta \mid 0 \in U_{t_\alpha}^{(n)}] \leq \frac{\mathbb{P}[N_\alpha \notin I_\delta]}{\mathbb{P}[0 \in U_{t_\alpha}^{(n)}]} \leq n^{2\alpha+o(1)} \times n^{-C'\alpha^2},$$

so, if C is large enough, for some $c'' > 0$ it holds that

$$\mathbb{P}[N_\alpha \in I_{\delta_{n,\alpha}} \mid 0 \in U_{t_\alpha}^{(n)}] \geq 1 - n^{-c''\alpha}. \quad (66)$$

Define

$$h(t, x) = P_x[T_n(0) > t].$$

To simplify the notations, let us also assume that t_α is integer. As mentioned in the end of Section 2.2, we will represent the conditioned random walk as a time-dependent Markov chain, using the Doob's h -transform. Indeed, it is well known and easily checked that the simple random walk on \mathbb{Z}_n^2 conditioned on the event $\{0 \in U_{t_\alpha}^{(n)}\}$ is a time-dependent Markov chain \tilde{X} with transition probabilities given by

$$\mathbb{P}[\tilde{X}_{s+1} = y \mid \tilde{X}_s = x] = \frac{h(t_\alpha - s - 1, y)}{h(t_\alpha - s, x)} \times \frac{1}{4}, \quad (67)$$

if $\|y - x\|_1 = 1$, and equal to 0 otherwise. For simpler notations, we do not indicate the dependence on t_α in the notation \tilde{X} . In order to proceed, we need the following fact, which proof can be skipped in a first reading.

Lemma 4.1. *For all $\lambda \in (0, 1/5)$, there exist $c_1 > 0, n_1 \geq 2$ (depending on λ) such that for all $n \geq n_1, \beta \geq 1, \|x\|, \|y\| \geq \lambda n, |r| \leq \beta n^2$ and all $s \geq 0$,*

$$\left| \frac{h(s, x)}{h(s+r, y)} - 1 \right| \leq \frac{c_1 \beta}{\ln n}. \quad (68)$$

Proof. Denote

$$h(t, \mu) := P_\mu[T_n(0) > t],$$

where $P_\mu[\cdot]$ is the probability for the simple random walk on \mathbb{Z}_n^2 starting from the initial distribution μ .

Let us define the set \mathcal{M} of probability measures on \mathbb{Z}_n^2 in the following way:

$$\mathcal{M} = \{\nu : \nu(B(j)) \leq 7j^2n^{-2} \text{ for all } j \leq \lambda n\};$$

observe that any probability measure concentrated on a one-point set $\{x\}$ with $\|x\| \geq \lambda n$ belongs to \mathcal{M} . Assume from now on that n is odd, so that the simple random walk on \mathbb{Z}_n^2 is aperiodic (the case of even n can be treated essentially in the same way, with some obvious modifications). Recall that the uniform measure μ_0 on \mathbb{Z}_n^2 , i.e., $\mu_0(x) = n^{-2}$ for all $x \in \mathbb{Z}_n^2$, is the invariant law of simple random walk X on \mathbb{Z}_n^2 .

Since the mixing time of X is of order n^2 (e.g., Theorem 5.5 in [13]), it is clear that for any $\delta \in (0, 1)$ one can find large enough c' (in fact, $c' = O(\ln \delta^{-1})$) such that for any probability measure ν it holds that

$$\nu P^{(s)} = (1 - \delta)\mu_0 + \delta\nu', \quad \text{with } \nu' \in \mathcal{M} \quad (69)$$

for all $s \geq c'n^2$. In fact, the argument only shows that ν' , defined in the above equality, is a probability measure. However, using that the s -step transition probability for X from x to y in \mathbb{Z}_n^2 is equal to $P_x(S_s \in \{y\} + n\mathbb{Z}^2)$, the local limit theorem implies that ν' belongs to the set \mathcal{M} for large enough c' .

Then, we are going to obtain that there exist some $c_2 > 0, b_0 > 0$, such that for all $b \in \{b_0, 2b_0, 3b_0, \dots\}$ and all $\nu \in \mathcal{M}$,

$$h(bn^2, \nu) = P_\nu[T_n(0) > bn^2] \geq 1 - \frac{bc_2}{\ln n}. \quad (70)$$

To prove (70), let us first show that there exists $c_3 = c_3(\lambda) > 0$ such that

$$P_\nu[T_n(0) < T_n(\partial B(\lambda n))] \leq \frac{c_3}{\ln n} \quad (71)$$

for all $\nu \in \mathcal{M}$. Abbreviate $W_j = B(\frac{\lambda n}{2j-1}) \setminus B(\frac{\lambda n}{2j})$ and write, using (26)

$$\sum_{x \in W_j} \nu(x) P_x[T_n(0) < T_n(\partial B(\lambda n))] \leq \sum_{x \in W_j} \nu(x) \left(1 - \frac{a(x)}{a(\lambda n) + O(n^{-1})} \right)$$

$$\begin{aligned}
&\leq 7n^{-2} \times \frac{\lambda^2 n^2}{2^{2(j-1)}} \times \left(1 - \frac{\ln \lambda n - j \ln 2}{\ln \lambda n + O(n^{-1})}\right) \\
&\leq \frac{1}{\ln \lambda n} \times \frac{7j\lambda^2 \ln 2}{2^{2(j-1)}}.
\end{aligned}$$

Let j_0 be such that $\frac{\lambda n}{2^{j_0}} \leq \frac{n}{\sqrt{\ln n}}$. Write

$$\begin{aligned}
&P_\nu[T_n(0) < T_n(\partial B(\lambda n))] \\
&\leq \nu\left(B\left(\frac{n}{\sqrt{\ln n}}\right)\right) + \sum_{j=1}^{j_0} \sum_{x \in W_j} \nu(x) P_x[T_n(0) < T_n(\partial B(\lambda n))] \\
&\leq \frac{7}{\ln n} + \frac{1}{\ln \lambda n} \sum_{j=1}^{j_0} \frac{7j\lambda^2 \ln 2}{2^{2(j-1)}}
\end{aligned}$$

for $\nu \in \mathcal{M}$, which proves (71). To obtain (70), let us first fix $b_0 > 0$ such that $\nu P^{(b_0 n^2)} \in \mathcal{M}$ for any $\nu \in \mathcal{M}$. Observe that the number of excursions by time $b_0 n^2$ between $\partial B(\lambda n)$ and $\partial B(n/3)$ is stochastically bounded by a Geometric random variable with expectation of constant order. Since (again by (26)) for any $x \in \partial B(\lambda n)$

$$P_x[T_n(0) < T_n(\partial B(n/3))] \leq \frac{c_4}{\ln n},$$

and, considering a random sum of a geometric number of independent Bernoulli with parameter $c_4/\ln n$, it is not difficult to obtain that

$$P_\nu[T_n(0) \leq b_0 n^2] \leq \frac{c_5}{\ln n}. \quad (72)$$

The inequality (70) then follows from (72) and the union bound,

$$P_\nu[T_n(0) \leq kb_0 n^2] \leq \sum_{j=0}^{k-1} P_{\nu P^{(j b_0 n^2)}}[T_n(0) \leq b_0 n^2] \leq \frac{k c_5}{\ln n}.$$

Now, let $c' \in b_0 \mathbb{N}^*$ be such that $\delta < 1/3$ in (69). Assume also that n is sufficiently large so that $\left(1 - \frac{c' c_2}{\ln n}\right)^{-1} \leq 2$. Then, (70) implies that for all $s \leq c' n^2$ and $\nu \in \mathcal{M}$

$$\frac{h(s, \mu)}{h(s, \nu)} - 1 \leq \frac{1}{1 - \frac{c' c_2}{\ln n}} - 1 = \left(1 - \frac{c' c_2}{\ln n}\right)^{-1} \times \frac{c' c_2}{\ln n},$$

and therefore

$$\frac{h(s, \mu)}{h(s, \nu)} - 1 \leq \frac{3c'c_2}{\ln n} \quad \text{for any } \nu \in \mathcal{M}, \quad (73)$$

$$1 - \frac{h(s, \mu)}{h(s, \nu)} \leq \frac{3c'c_2}{\ln n} \quad \text{for any } \mu \in \mathcal{M} \quad (74)$$

(note that the above inequalities hold with factor 2 in the right-hand side, but we intentionally put a larger factor there for reasons that will become clear later).

We now extend by induction that (73)–(74) from times $s \leq s_0 = c'n^2$ to all times. Let $s_k = (k+1)c'n^2$, and consider the recursion hypothesis

$$(H_k) : \quad (73) \text{ and } (74) \text{ hold for } s \leq s_k,$$

that we just have proved for $k = 0$. Assume now (H_k) for some k . Define the event $G_{r,s} = \{X_j \neq 0 \text{ for all } r+1 \leq j \leq s\}$, and write

$$\begin{aligned} h(s+t, \mu) &= P_\mu[G_{t,s+t}]P_\mu[T_n(0) > t \mid G_{t,s+t}] \\ &= h(s, \mu P^{(t)})P_\mu[T_n(0) > t \mid G_{t,s+t}]. \end{aligned} \quad (75)$$

Abbreviate $t = c'n^2$ for the rest of the proof of the Lemma. Let us estimate the second term in the right-hand side of (75). Let $\Gamma_{[0,t]}$ be the set of all nearest-neighbour trajectories on \mathbb{Z}_n^2 of length t . For $\varrho \in \Gamma_{[0,t]}$ we have $P_\mu[\varrho] = \mu(\varrho_0)\left(\frac{1}{4}\right)^{|\varrho|}$ and

$$P_\mu[\varrho \mid G_{t,s+t}] = \mu(\varrho_0)\left(\frac{1}{4}\right)^{|\varrho|} \times \frac{h(s, \varrho_{\text{end}})}{h(s, \mu P^{(t)})} \leq \mu(\varrho_0)\left(\frac{1}{4}\right)^{|\varrho|} \left(1 + \frac{3c'c_2}{\ln n}\right)$$

using the relation (73) for $s \leq s_k$. Summing over ϱ such that $T_n(0) \leq t$ and using (70), we obtain, for $\mu \in \mathcal{M}$,

$$P_\mu[T_n(0) > t \mid G_{t,s+t}] \geq 1 - \frac{c'c_2}{\ln n} \left(1 + \frac{3c'c_2}{\ln n}\right). \quad (76)$$

Now, we use (69) and (75) to obtain that, with μ', ν' defined in (69),

$$\frac{h(t+s, \mu)}{h(t+s, \nu)} = \frac{h(s, \mu P^{(c'n^2)})P_\mu[T_n(0) > c'n^2 \mid G_{c'n^2, t+s}]}{h(s, \nu P^{(c'n^2)})P_\nu[T_n(0) > c'n^2 \mid G_{c'n^2, t+s}]}$$

$$= \frac{(1 - \delta + \delta \frac{h(s, \mu')}{h(s, \mu_0)}) P_\mu [T_n(0) > c'n^2 \mid G_{c'n^2, t+s}]}{(1 - \delta + \delta \frac{h(s, \nu')}{h(s, \mu_0)}) P_\nu [T_n(0) > c'n^2 \mid G_{c'n^2, t+s}]},$$

for $s \leq s_k$. We now use (H_k) for the two ratios of h 's in the above expression, we also use (76) for the conditional probability in the denominator – simply bounding it by 1 in the numerator – to obtain

$$\frac{h(t+s, \mu)}{h(t+s, \nu)} \leq \frac{(1 - \delta + \delta(1 + \frac{3c'c_2}{\ln n}))}{(1 - \delta + \delta(1 - \frac{3c'c_2}{\ln n}))(1 - \frac{c'c_2}{\ln n}(1 + \frac{3c'c_2}{\ln n}))},$$

that is,

$$\frac{h(t+s, \mu)}{h(t+s, \nu)} - 1 \leq (6\delta + 1) \frac{c'c_2}{\ln n} + o((\ln n)^{-1})$$

for $\nu \in \mathcal{M}$. Since $\delta < 1/3$, for large enough n we obtain that (73) also holds for all $s \leq s_{k+1}$. In the same way, we prove the validity of (74) for $s \leq s_{k+1}$.

This proves the recursion, which in turn, implies (68) for the case $r = 0$. To conclude the proof of Lemma 4.1, we observe that the general case follows from (70). \square

We continue the proof of Theorem 2.6. We assume that the set A is fixed, so that $\text{cap}(A) = O(1)$ and $\text{diam}(A) = O(1)$. In addition, assume without restricting generality that $0 \in A$. Recall that with (66) we control the number of excursions between $\partial B(\frac{n}{3 \ln n})$ and $\partial B(n/3)$ up to time t_α . Now, we estimate the (conditional) probability that an excursion hits the set A . For this, observe that Lemmas 3.1, 3.3 and 3.4 imply that, for any $x \in \partial B(\frac{n}{3 \ln n})$

$$\begin{aligned} & \widehat{P}_x[\widehat{\tau}_1(A) > \widehat{\tau}_1(\partial B(n/3))] \\ &= \frac{P_x[\tau_1(A) > \tau_1(\partial B(n/3)), \tau_1(0) > \tau_1(\partial B(n/3))]}{P_x[\tau_1(0) > \tau_1(\partial B(n/3))]} (1 + O((n \ln n)^{-1})) \\ &= \frac{a(x) - \text{cap}(A) + O(\frac{\ln n}{n})}{a(n/3) - \text{cap}(A) + O(\frac{\ln n}{n})} \times \frac{a(n/3) + O(n^{-1})}{a(x)} (1 + O((n \ln n)^{-1})) \\ &= \frac{1 - \frac{\text{cap}(A)}{a(x)}}{1 - \frac{\text{cap}(A)}{a(n/3)}} (1 + O(n^{-1})) \\ &= 1 - \frac{\pi}{2} \text{cap}(A) \frac{\ln \ln n}{\ln^2 n} (1 + o(1)). \end{aligned} \tag{77}$$

Now, we need the following fact: there exists $c_5 > 0$ such that

$$h(s, x) \geq \exp\left(-\frac{c_5 s}{n^2 \ln n}\right) \quad (78)$$

for all $s \geq n^2 \ln n$ and all $x \in \mathbb{Z}_n^2$ such that $\|x\| \geq \frac{n}{3 \ln n}$ (in fact, it is straightforward to obtain that (78) holds if $\|x\| \geq n^{\varepsilon_0}$, where $\varepsilon_0 \in (0, 1)$ is some fixed number). To prove (78) it is enough to observe that

- the number of (possibly incomplete) excursions between $\partial B\left(\frac{n}{3 \ln n}\right)$ and $\partial B(n/3)$ until time s does not exceed $\frac{3s}{n^2 \ln \ln n}$ with probability at least $1 - \exp\left(-\frac{c_6 s}{n^2 \ln \ln n}\right)$, by (53);
- regardless of the past, each excursion hits 0 with probability $\frac{\ln \ln n}{\ln n}(1 + o(1))$, by (26).

To extract from (77) the corresponding formula for the \tilde{X} -excursion, we first observe that, for $x \in \partial B\left(\frac{n}{3 \ln n}\right)$ and $s \geq n^2 \sqrt{\ln n}$

$$\begin{aligned} h(s, x) &= P_x[T_n(0) > T_n(\partial B(n/3))] \times P_x[T_n(0) > s \mid T_n(0) > T_n(\partial B(n/3))] \\ &\quad + P_x[T_n(\partial B(n/3)) \geq T_n(0) > s] \\ &= \frac{a(x)}{a(n/3) + O(n^{-1})} P_x[T_n(0) > s \mid T_n(0) > T_n(\partial B(n/3))] + \psi_{s,n} \\ &= \frac{a(x)}{a(n/3) + O(n^{-1})} \sum_{\substack{y \in \partial B(n/3), \\ k \geq 1}} h(s - k, y) \ell_{y,k} + \psi_{s,n}, \end{aligned} \quad (79)$$

where $\ell_{y,k} = P_x[X_{T_n(\partial B(n/3))} = y, T_n(\partial B(n/3)) = k \mid T_n(0) > T_n(\partial B(n/3))]$ and $0 \leq \psi_{s,n} \leq e^{-Cs/n^2}$.

Recall the notation $\Gamma_{0,R}^{(x)}$ from the beginning of Section 3.2. Then, for a fixed $x \in \partial B\left(\frac{n}{3 \ln n}\right)$ let us define the set of paths

$$\Lambda_j = \{\varrho \in \Gamma_{0,n/3}^{(x)} : (j-1)n^2 < |\varrho| \leq jn^2\},$$

It is straightforward to obtain that

$$\max(P_x[\Lambda_j], \widehat{P}_x[\Lambda_j]) \leq e^{-cj} \quad (80)$$

for some $c > 0$.

Now, going back to (79) and setting $a_j^{(n)} = \left(1 + \frac{c_1}{\ln n}\right)^j - 1$ with c_1 from Lemma 4.1, observe that for *any* $y_0 \in \partial B(n/3)$

$$\begin{aligned}
\sum_{\substack{y \in \partial B(n/3), \\ k \geq 1}} h(s-k, y) \ell_{y,k} &= \sum_{j \geq 1} \sum_{\substack{y \in \partial B(n/3), \\ (j-1)n^2 < k \leq jn^2}} h(s-k, y) \ell_{y,k} \\
&= \sum_{j \geq 1} \sum_{\substack{y \in \partial B(n/3), \\ (j-1)n^2 < k \leq jn^2}} \ell_{y,k} h(s, y_0) \left(1 + O(a_j^{(n)})\right) \\
&= h(s, y_0) \left(1 + O\left(\sum_{j \geq 1} e^{-cj} a_j^{(n)}\right)\right) \\
&= h(s, y_0) \left(1 + O\left(\frac{1}{\ln n}\right)\right),
\end{aligned}$$

due to Lemma 4.1 and (80). We plug this into (79), divide by $h(s, y_0)$, and use (78) to obtain, for $x \in \partial B\left(\frac{n}{3 \ln n}\right)$ and $s \geq n^2 \sqrt{\ln n}$

$$\frac{h(s, x)}{h(s, y_0)} = \frac{a(x)}{a(n/3) + O(n^{-1})} \left(1 + O\left(\frac{1}{\ln n}\right)\right) + \psi'_{s,n},$$

where $|\psi'_{s,n}| \leq \exp\left(-\frac{s}{n^2}\left(c - \frac{c_5}{\ln n}\right)\right)$. Equivalently,

$$\frac{h(s, y_0)}{h(s, x)} = \frac{a(n/3)}{a(x)} \left(1 + O\left(\frac{1}{\ln n}\right)\right), \tag{81}$$

a relation which can be compared to Lemma 4.1.

For $A \subset \mathbb{Z}^2$, let us define also the hitting times of the corresponding set on the torus for the \tilde{X} -walk *after* a given moment s :

$$\tilde{T}_n^{(s)}(A) = \min\{k \geq s : \tilde{X}_k \in \Upsilon_n A\};$$

we abbreviate also $\tilde{T}_n(A) = \tilde{T}_n^{(0)}(A)$. Then, in the same way we obtain

$$\begin{aligned}
&\mathbb{P}[\tilde{T}_n^{(s)}(A) > \tilde{T}_n^{(s)}(\partial B(n/3)) \mid \tilde{X}_s = x] \\
&= \sum_{\varrho} \frac{h(t_\alpha - s - |\varrho|, \varrho_{\text{end}})}{h(t_\alpha - s, x)} \left(\frac{1}{4}\right)^{|\varrho|} \\
&= \sum_{\varrho} \frac{a(n/3)}{a(x)} \left(\frac{1}{4}\right)^{|\varrho|} \left(1 + O\left(\frac{1}{\ln n}\right)\right)
\end{aligned}$$

$$= \widehat{P}_x[\widehat{\tau}_1(A) > \widehat{\tau}_1(\partial B(n/3))] \left(1 + O\left(\frac{1}{\ln n}\right)\right),$$

where the sum is over all paths ϱ that begin in x and end on the first visit to $\partial B(n/3)$. So, using (77), we obtain for all $s \leq t_\alpha - n^2\sqrt{\ln n}$ and all $x \in \partial B(\frac{n}{3\ln n})$,

$$\mathbb{P}[\widetilde{T}_n^{(s)}(A) > \widetilde{T}_n^{(s)}(\partial B(n/3)) \mid \widetilde{X}_s = x] = 1 - \frac{\pi}{2} \text{cap}(A) \frac{\ln \ln n}{\ln^2 n} (1 + o(1)). \quad (82)$$

Before we are able to conclude the proof of Theorem 2.6, we need another step to take care of times close to t_α . Consider any $x \in \partial B(\frac{n}{3\ln n})$ and any $s \geq 0$ such that $t_\alpha - s \leq n^2\sqrt{\ln n}$. Then, (78) together with the fact that $h(\cdot, \cdot)$ is nondecreasing with respect to the first (temporal) argument imply that

$$\begin{aligned} \mathbb{P}_x[(\widetilde{X}_0, \dots, \widetilde{X}_{|\varrho|}) = \varrho] &= \frac{h(t_\alpha - s - |\varrho|, \varrho_{\text{end}})}{h(t_\alpha - s, x)} \mathbb{P}_x[(X_0, \dots, X_{|\varrho|}) = \varrho] \\ &\leq c_7 \mathbb{P}_x[(X_0, \dots, X_{|\varrho|}) = \varrho] \end{aligned} \quad (83)$$

for any path ϱ with $\varrho_0 = x$. Then, similarly to Section 3.4, define $\widetilde{J}_k, \widetilde{D}_k$ to be the starting and ending times of k th excursion of \widetilde{X} between $\partial B(\frac{n}{3\ln n})$ and $\partial B(n/3)$, $k \geq 1$. Let

$$\zeta = \min\{k : \widetilde{J}_k \geq t_\alpha - n^2\sqrt{\ln n}\}$$

be the index of the first \widetilde{X} -excursion that starts before time $t_\alpha - n^2\sqrt{\ln n}$. Let $\xi'_1, \xi'_2, \xi'_3, \dots$ be a sequence of i.i.d. Bernoulli random variables independent of everything, with

$$\mathbb{P}[\xi'_k = 1] = 1 - \mathbb{P}[\xi'_k = 0] = 1 - \frac{\pi}{2} \text{cap}(A) \frac{\ln \ln n}{\ln^2 n}.$$

For $k \geq 1$ define two sequences of random variables

$$\xi_k = \begin{cases} \mathbf{1}\{\widetilde{X}_j \notin A \text{ for all } j \in [\widetilde{J}_k, \widetilde{D}_k]\}, & \text{for } j < \zeta, \\ \xi'_k, & \text{for } j \geq \zeta, \end{cases}$$

and

$$\eta_k = \mathbf{1}\{\widetilde{X}_j \notin A \text{ for all } j \in [\widetilde{J}_{\zeta+k-1}, \widetilde{D}_{\zeta+k-1}]\}.$$

Now, observe that (82) and the strong Markov property imply that

$$\mathbb{P}[\xi_k = 1 \mid \xi_1, \dots, \xi_{k-1}] = 1 - \frac{\pi}{2} \text{cap}(A) \frac{\ln \ln n}{\ln^2 n} (1 + o(1)). \quad (84)$$

Also, the relation (83) together with Lemma 3.3 imply that

$$\mathbb{P}[\eta_k = 0 \mid \eta_1, \dots, \eta_{k-1}] \leq c_7 \frac{\ln \ln n}{\ln n}. \quad (85)$$

Denote

$$\zeta' = \max\{k : \tilde{J}_k \leq t_\alpha\}.$$

Then, (83) and Lemma 3.12 imply that (note that $\pi/2 < 3$)

$$\mathbb{P}\left[\zeta' - \zeta \geq \frac{3\sqrt{\ln n}}{\ln \ln n}\right] \leq \exp\left(-c_8 \frac{\sqrt{\ln n}}{\ln \ln n}\right). \quad (86)$$

Recall the notation $\delta_{n,\alpha}$ from the beginning of the proof of this theorem. We can write

$$\begin{aligned} & \mathbb{P}\left[\xi_k = 1 \text{ for all } k \leq (1 - \delta_{n,\alpha}) \frac{2\alpha' \ln^2 n}{\ln \ln n} - \frac{3\sqrt{\ln n}}{\ln \ln n}\right] \\ & \geq \mathbb{P}\left[\tilde{T}_n(A) > t_\alpha, N'_\alpha \geq (1 - \delta_{n,\alpha}) \frac{2\alpha \ln^2 n}{\ln \ln n}, \zeta' - \zeta \leq \frac{3\sqrt{\ln n}}{\ln \ln n}\right] \end{aligned}$$

so

$$\begin{aligned} \mathbb{P}[\tilde{T}_n(A) > t_\alpha] & \leq \mathbb{P}\left[\xi_k = 1 \text{ for all } k \leq (1 - \delta_{n,\alpha}) \frac{2\alpha' \ln^2 n}{\ln \ln n} - \frac{3\sqrt{\ln n}}{\ln \ln n}\right] \\ & \quad + \mathbb{P}\left[N'_\alpha < (1 - \delta_{n,\alpha}) \frac{2\alpha \ln^2 n}{\ln \ln n}\right] + \mathbb{P}\left[\zeta' - \zeta > \frac{3\sqrt{\ln n}}{\ln \ln n}\right]. \end{aligned} \quad (87)$$

Also,

$$\begin{aligned} & \mathbb{P}[\tilde{T}_n(A) > t_\alpha] \\ & \geq \mathbb{P}\left[\xi_k = 1 \text{ for all } k \leq (1 + \delta_{n,\alpha}) \frac{2\alpha \ln^2 n}{\ln \ln n}, N_\alpha \leq (1 + \delta_{n,\alpha}) \frac{2\alpha \ln^2 n}{\ln \ln n}, \right. \\ & \quad \left. \eta_k = 1 \text{ for all } k \leq \frac{3\sqrt{\ln n}}{\ln \ln n}, \zeta' - \zeta \leq \frac{3\sqrt{\ln n}}{\ln \ln n}\right] \end{aligned}$$

$$\begin{aligned}
&\geq \mathbb{P}\left[\xi_k = 1 \text{ for all } k \leq (1 + \delta_{n,\alpha}) \frac{2\alpha \ln^2 n}{\ln \ln n}\right] - \mathbb{P}\left[N_\alpha > (1 + \delta_{n,\alpha}) \frac{2\alpha \ln^2 n}{\ln \ln n}\right] \\
&\quad - \mathbb{P}\left[\left(\eta_k = 1 \text{ for all } k \leq \frac{3\sqrt{\ln n}}{\ln \ln n}\right)^c\right] - \mathbb{P}\left[\zeta' - \zeta > \frac{3\sqrt{\ln n}}{\ln \ln n}\right]. \tag{88}
\end{aligned}$$

Using (84), we obtain that the first terms in the right-hand sides of (87)–(88) are both equal to

$$(1 + o(1)) \left(1 - \frac{\pi}{2} \text{cap}(A) \frac{\ln \ln n}{\ln^2 n}\right)^{\frac{2\alpha \ln^2 n}{\ln \ln n}} = (1 + o(1)) \exp\left(-\pi\alpha \text{cap}(A \cup \{0\})\right).$$

The other terms in the right-hand sides of (87)–(88) are $o(1)$ due to (66), (85), and (86). Since, as observed just before Lemma 4.1

$$\mathbb{P}[\Upsilon_n A \subset U_{t_\alpha}^{(n)} \mid 0 \in U_{t_\alpha}^{(n)}] = \mathbb{P}[\tilde{T}_n(A) > t_\alpha],$$

the proof of Theorem 2.6 is concluded. \square

Acknowledgments

The authors are grateful to Caio Alves and Darcy Cunha for pointing out to us that $\frac{1}{a}$ is a martingale for the conditioned walk, and to David Belius and Augusto Teixeira for useful discussions. Darcy Cunha also did the simulations of the model presented on Figure 1.

Also, the authors thank the financial support from Franco-Brazilian Scientific Cooperation program. S.P. and M.V. were partially supported by CNPq (grants 300886/2008–0 and 301455/2009–0). The last two authors thank FAPESP (2009/52379–8, 2014/06815–9, 2014/06998–6) for financial support. F.C. is partially supported by CNRS, UMR 7599 LPMA.

References

- [1] D. BELIUS (2013) Gumbel fluctuations for cover times in the discrete torus. *Probab. Theory Relat. Fields* **157** (3-4), 635–689.
- [2] D. BELIUS, N. KISTLER (2014) The subleading order of two dimensional cover times. arXiv:1405.0888

- [3] J. ČERNÝ, A. TEIXEIRA (2012) From random walk trajectories to random interlacements. *Ensaaios Matemáticos [Mathematical Surveys]* **23**. Sociedade Brasileira de Matemática, Rio de Janeiro.
- [4] F. COMETS, C. GALLESKO, S. POPOV, M. VACHKOVSKAIA (2013) On large deviations for the cover time of two-dimensional torus. *Electr. J. Probab.*, **18**, article 96.
- [5] A. DEMBO, Y. PERES, J. ROSEN, O. ZEITOUNI (2004) Cover times for Brownian motion and random walks in two dimensions. *Ann. Math. (2)* **160** (2), 433–464.
- [6] A. DEMBO, Y. PERES, J. ROSEN, O. ZEITOUNI (2006) Late points for random walks in two dimensions. *Ann. Probab.* **34** (1), 219–263.
- [7] J. DING (2012) On cover times for 2D lattices. *Electr. J. Probab.* **17** (45), 1–18.
- [8] A. DREWITZ, B. RÁTH, A. SAPOZHNIKOV (2014) *An introduction to random interlacements*. Springer.
- [9] G. FAYOLLE, V.A. MALYSHEV AND M.V. MENSHIKOV (1995) *Topics in the Constructive Theory of Countable Markov Chains*. Cambridge University Press.
- [10] J. GOODMAN, F. DEN HOLLANDER (2014) Extremal geometry of a Brownian porous medium. *Probab. Theory Relat. Fields.* **160** (1-2), 127–174.
- [11] G. LAWLER (1996) *Intersections of random walks*. Birkhäuser/Springer, New York.
- [12] G. LAWLER, V. LIMIC (2010) *Random walk: a modern introduction*. Cambridge Studies in Advanced Mathematics, **123**. Cambridge University Press, Cambridge.
- [13] D. LEVIN, Y. PERES, E. WILMER (2009) *Markov chains and mixing times*. American Mathematical Society.
- [14] J. MILLER, P. SOUSI (2013) Uniformity of the late points of random walk on \mathbb{Z}_n^d for $d \geq 3$. arXiv:1309.3265

- [15] S. POPOV, A. TEIXEIRA (2015) Soft local times and decoupling of random interlacements. To appear in: *J. European Math. Soc.*
- [16] D. REVUZ (1984) *Markov chains*. North-Holland Publishing Co., Amsterdam, 2nd edition.
- [17] A.-S. SZNITMAN (2010) Vacant set of random interlacements and percolation. *Ann. Math. (2)*, **171** (3), 2039–2087.
- [18] A.-S. SZNITMAN (2012) *Topics in occupation times and Gaussian free fields*. Zurich Lect. Adv. Math., European Mathematical Society, Zürich.
- [19] A. TEIXEIRA (2009) Interlacement percolation on transient weighted graphs. *Electr. J. Probab.* **14**, 1604–1627.