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Nonparametric inference for stationary pairwise interaction point process

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Abstract

Among models allowing to introduce interaction between points, we find the large class of Gibbs models coming from statistical physics. Such models can produce repulsive as well as attractive point pattern. In this paper, we focus on the main class of Gibbs models which is the class of pairwise interaction point processes characterized by the Papangelou conditional intensity. We suggest a new nonparametric estimate of the pairwise interaction function for stationary pairwise interaction point process. Sufficient conditions to strong uniform consistency are obtained for the resulting estimator.

Keywords: Georgii-Nguyen-Zessin formula, kernel estimators, non parametric estimation, pairwise interaction point process, Papangelou conditional intensity, strong uniform consistency, Orlicz spaces.

1 Introduction

The theory of spatial models is a growing field over the last decade, with various applications in several domains such as ecology (Diggle [9]), forestry (Matérn [21]), spatial epidemiology (Lawson [19]) and astrophysics (Neyman and Scott [24]). Gibbs point processes arose in statistical physics as models for interacting particle systems. Gibbs point processes in \mathbb{R}^d can be defined and characterized through the Papangelou conditional intensity (Møller and Waagepetersen [22]) which is a function $\lambda : \mathbb{R}^d \times N_{lf} \rightarrow \mathbb{R}_+$ where N_{lf} is the space of locally finite configurations of points in \mathbb{R}^d . The Papangelou conditional intensity can be interpreted as follows: for any $u \in \mathbb{R}^d$ and $\mathbf{x} \in N_{lf}$, $\lambda(u, \mathbf{x})du$ corresponds to the conditional probability of observing a point in a ball of volume du around u given the rest of the point process is \mathbf{x} . Examples of Markov and non-Markov Gibbs point process models and their conditional intensities are presented in Baddeley et al [1], Møller and Waagepetersen ([22], [23]).

In this paper, we are concerned with nonparametric statistics for stationary pairwise interaction point processes (a special case of a Gibbs process) which describes the interaction between pairs of points by a function (called a pair potential function). They have been introduced in statistical literature by Ripley and Kelly [32], Daley and Vere-Jones [7] and Georgii [14]. Our objective of this work is to study estimating nonparametric interaction function. We suggest a new nonparametric estimate of the pairwise interaction function for stationary pairwise interaction point process. Sufficient conditions to strong uniform consistency are obtained for the resulting estimator. Note also that the main results of this work are obtained via assumptions of belonging to Orlicz spaces induced by exponential Young functions for stationary real random fields which allows us to derive the Kahane-Khintchine inequalities by El Machkouri [20]. The conditions are expressed in terms of a series of conditional expectation. Our results also carry through the most important particular case of Orlicz spaces random fields, we use the inequality follows from a Marcinkiewicz-Zygmund type inequality by Dedecker [8].

Many attempts have been tried to estimate the potential function from point pattern data in a parametric framework: maximization of likelihood approximations (Ogata and Tanemura [27], Ogata and Tanemura [28], Penttinen [30]), pseudolikelihood maximization (Besag et al. [2], Jensen and Møller [16]) and also some ad hoc methods (Strauss [33], Ripley [32], Hanisch and Stoyan [15], Diggle and Gratton [11], Fiksel [12], Takacs [34], Billiot and Goulard [4]).

Nonparametric estimation of the potential function has been largely ignored by researchers. One exception is the suggestion to use the nonparamet-

ric estimation of the pair correlation function and its approximate relation to the pair potential through the Percus-Yevic equation (Diggle et al. [10]). The approximation is a result of a cluster expansion method, and it is accurate only for simulation study. The statistical and asymptotic properties of this estimation techniques have not been studied so far.

The paper is organized as follows. The next section sets up the generic notation and the basic tools of the point processes in \mathbb{R}^d . We present the model for stationary pairwise interaction point process and the assumptions which will be considered in the sequel in Section 3. In Section 4, we present our main results; on both the estimation method and strong uniform consistencies for the resulting estimator. The last section is devoted to the proofs.

2 Generic notation and Basic tools

Let \mathcal{B}^d be the Borel σ -algebra (generated by open sets) in \mathbb{R}^d (the d -dimensional space) and $\mathcal{B}_O^d \subseteq \mathcal{B}^d$ be the system of all bounded Borel sets. A point process \mathbf{X} in \mathbb{R}^d is a locally finite random subset of \mathbb{R}^d , i.e. the number of points $N(\Lambda) = n(\mathbf{X}_\Lambda)$ of the restriction of \mathbf{X} to Λ is a finite random variable whenever Λ is a bounded Borel set of \mathbb{R}^d (see Daley and Vere-Jones [7]). We define the space of locally finite point configurations in \mathbb{R}^d as $N_{lf} = \{\mathbf{x} \subseteq \mathbb{R}^d; n(\mathbf{x}_\Lambda) < \infty, \forall \Lambda \in \mathcal{B}_O^d\}$, where $\mathbf{x}_\Lambda = \mathbf{x} \cap \Lambda$. The volume of a bounded Borel set Λ of \mathbb{R}^d is denoted by $|\Lambda|$ and $o = (0, \dots, 0)$.

The Papangelou conditional intensity completely characterizes the Gibbs point process in terms of the Georgii-Nguyen-Zessin (GNZ) formula (see Papangelou [29] and Zessin [36] for historical comments and Georgii [13] or Nguyen and Zessin [26] for a general presentation). The GNZ formula states that for any nonnegative measurable function h on $\mathbb{R}^d \times N_{lf}$

$$\mathbb{E} \sum_{u \in \mathbf{X}} h(u, \mathbf{X} \setminus u) = \mathbb{E} \int_{\mathbb{R}^d} h(u, \mathbf{X}) \lambda(u, \mathbf{X}) du. \quad (1)$$

Let $\mathbf{X} \otimes \mathbf{X}$ be the point process on $\mathbb{R}^d \times \mathbb{R}^d$ consisting of all pairs (u, v) of distinct points of \mathbf{X} . It follows immediately from the GNZ formula (1) that $\mathbf{X} \otimes \mathbf{X}$ is a Gibbs point process with (two-point) Papangelou conditional intensity $\lambda(u, v, \mathbf{x}) = \lambda(u, \mathbf{x}) \lambda(v, \mathbf{x} \cup \{u\})$, for $u, v \in \mathbb{R}^d, \mathbf{x} \in N_{lf}$, meaning that the GNZ formula in the form

$$\mathbb{E} \sum_{\substack{\neq \\ u, v \in \mathbf{X}}} h(u, v, \mathbf{X} \setminus \{u, v\}) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{E} h(u, v, \mathbf{X}) \lambda(u, v, \mathbf{X}) du dv \quad (2)$$

is satisfied for any nonnegative measurable function $h(u, v, \mathbf{x})$ on $\mathbb{R}^d \times \mathbb{R}^d \times N_{lf}$.

A Young function ψ is a real convex nondecreasing function defined on \mathbb{R}^+ which satisfies $\lim_{t \rightarrow \infty} \psi(t) = +\infty$ and $\psi(0) = 0$. We define the Orlicz space L_ψ as the space of real random variables Z defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $E[\psi(|Z|/c)] < +\infty$ for some $c > 0$. The Orlicz space L_ψ equipped with the so-called Luxemburg norm $\|\cdot\|_\psi$ defined for any real random variable Z by

$$\|Z\|_\psi = \inf\{c > 0; E[\psi(|Z|/c)] \leq 1\}$$

is a Banach space. For more about Young functions and Orlicz spaces one can refer to Krasnosel'skii and Rutickii [18]. Let $\theta > 0$. We denote by ψ_θ the Young function defined for any $x \in \mathbb{R}^+$ by

$$\psi_\theta(x) = \exp((x + \xi_\theta)^\theta) - \exp(\xi_\theta^\theta) \quad \text{where} \quad \xi_\theta = ((1 - \theta)/\theta)^{1/\theta} \mathbb{1}_{\{0 < \theta < 1\}}.$$

On the lattice \mathbb{Z}^d we define the lexicographic order as follows: if $i = (i_1, \dots, i_d)$ and $j = (j_1, \dots, j_d)$ are distinct elements of \mathbb{Z}^d , the notation $i <_{lex} j$ means that either $i_1 < j_1$ or for some p in $\{2, 3, \dots, d\}$, $i_p < j_p$ and $i_q = j_q$ for $1 \leq q < p$. Let the sets $\{V_i^k; i \in \mathbb{Z}^d, k \in \mathbb{N}^*\}$ be defined as follows:

$$V_i^1 = \{j \in \mathbb{Z}^d; j <_{lex} i\},$$

and for $k \geq 2$

$$V_i^k = V_i^1 \cap \{j \in \mathbb{Z}^d; |i - j| \geq k\} \quad \text{where} \quad |i - j| = \max_{1 \leq l \leq d} |i_l - j_l|.$$

For any subset Γ of \mathbb{Z}^d define $\mathcal{F}_\Gamma = \sigma(\varepsilon_i; i \in \Gamma)$ and set

$$E_{|k|}(\varepsilon_i) = E(\varepsilon_i | \mathcal{F}_{V_i^{|k|}}), \quad k \in V_i^1.$$

Denote $\theta(q) = 2q/(2 - q)$ for $0 < q < 2$ and by convention $1/\theta(2) = 0$.

3 Model

The pairwise interaction point process is characterized by its conditional intensity Papangelou defined by

$$\lambda(u, \mathbf{x}) = g_0(u) \exp\left(-\sum_{v \in \mathbf{x} \setminus u} g_0(\{u, v\})\right).$$

Now we consider a special case where $g_0(u)$ is a constant and $g_0(\{u, v\}) = g(v - u)$ is translation invariant. In this case, the pairwise interaction point process is called stationary.

Throughout this paper, we assume the model is a stationary pairwise interaction point process and its Papangelou conditional intensity at a location u given by

$$\lambda_{\beta^*}(u, \mathbf{x}) = \beta^* \exp\left(-\sum_{v \in \mathbf{x} \setminus u} g(v - u)\right) \quad (3)$$

where β^* is the true value of the Poisson intensity parameter, g represents the nonnegative pairwise interaction potential defined on \mathbb{R}^d and is assumed a nonnegative function and $G = \exp(-g)$ represents the pairwise interaction function. In this semi-parametric model (3), the estimator of the Poisson intensity parameter β^* represented the first step in our procedure in the paper [6], we have established its strong consistency and asymptotic normality. we also considered its finite-sample properties simulation study. Now, we develop a method of nonparametric estimation of the pairwise interaction function G , with the class of model defined by (3), under assumptions as general as possible. The basic assumption throughout this paper is the Papangelou conditional intensity has a finite range R , i.e.

$$\lambda_{\beta^*}(u, \mathbf{x}) = \lambda_{\beta^*}(u, \mathbf{x}_{B(u,R)}), \quad (4)$$

for any $u \in \mathbb{R}^d$, $\mathbf{x} \in N_{lf}$, where $B(u, R)$ is the closed ball in \mathbb{R}^d with centre u and radius R .

4 Main results

Suppose that a single realization \mathbf{x} of a point process \mathbf{X} is observed in a bounded window Λ_n where $(\Lambda_n)_{n \geq 1}$ is a sequence of cubes growing up to \mathbb{R}^d , although more general convex compact sampling regions Λ_n which expands unboundedly in all directions (as $n \rightarrow \infty$) are possible. We face a missing data problem, which in the spatial point process literature is referred to as a problem of edge effects, we can avoid this problem by reducing the window by introducing the $2R$ -interior of the cubes Λ_n , i.e.

$$\Lambda_{n,R} = \{u \in \Lambda_n : B(u, 2R) \subset \Lambda_n\}$$

and assume this has non-zero area. Various edge corrections have been suggested by Ripley [31], Biber [3], Kelly and Ripley [17], Møller and Waagepetersen [22] and Tukey [35]).

For simplicity assume that the support of the interection function $G = \exp(-g)$ is $T = \{t \in \mathbb{R}^d; g(t) > 0, \text{ for } \|t\| < R\}$. With the convention

$c/0 = 0$ for all real c , we consider a nonparametric estimator $\widehat{G}_n(t)$ for $\beta^*G(t)$ for $t \in T$ by

$$\widehat{G}_n(t) = \frac{\widehat{H}_n(t)}{\widehat{F}_n(t)}, \quad (5)$$

where

$$\widehat{H}_n(t) = \frac{1}{b_n^d |\Lambda_{n,R}|} \sum_{\substack{\neq \\ u,v \in \mathbf{X} \\ v-u \in B(o,R)}} \mathbb{1}_{\Lambda_{n,R}}(u)h(u, \mathbf{X} \setminus \{u,v\})h(v, \mathbf{X} \setminus \{u,v\})K\left(\frac{v-u-t}{b_n}\right) \quad (6)$$

is a kernel-type estimator estimating $\beta^{*2}G(t)\bar{F}(o,t)$, where \neq over the summation sign means that the sum runs over all pairwise different points $u; v$ in \mathbf{X} and $K : \mathbb{R}^d \rightarrow \mathbb{R}$ denotes a smoothing kernel function associated with a sequence $(b_n)_{n \geq 1}$ of bandwidths satisfying the below Conditions $K(d, m)$ and \mathcal{Z} .

$$\widehat{F}_n(t) = \frac{1}{|\Lambda_{n,R}|} \sum_{u \in \mathbf{X}_{\Lambda_{n,R}}} h(u, \mathbf{X} \setminus \{u\})h(t-u, \mathbf{X} \setminus \{u\}) \quad (7)$$

meanwhile being an estimator of $\beta^*\bar{F}(o,t)$, where

$$\bar{F}(o,t) = \mathbb{P}(\mathbf{X} \cap B(o,R) = \emptyset, \mathbf{X} \cap B(t,R) = \emptyset). \quad (8)$$

And h is a nonnegative measurable function defined for all $w \in \mathbb{R}^d, \mathbf{x} \in N_{lf}$, by

$$h(w, \mathbf{x}) = \mathbb{1}(\mathbf{x} \cap B(w,R) = \emptyset). \quad (9)$$

Moreover, we have to impose certain natural restrictions on the kernel function K and the sequence $(b_n)_{n \geq 1}$:

Condition $K(d, m)$: The sequence of bandwidths $b_n > 0$ for $n \geq 1$, is chosen such that

$$\lim_{n \rightarrow \infty} b_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n^d |\Lambda_{n,R}| = \infty.$$

The kernel function K is nonnegative, suppose that, for some $0 < R' < \infty$,

$$K(u) = 0 \quad \text{for} \quad \|u\| > R', \quad \int_{\mathbb{R}^d} K(u)du = 1$$

and for $m \in \mathbb{N}^*$

$$\int_{\mathbb{R}^d} z_1^{a_1} \dots z_d^{a_d} K(z)dz = 0$$

with $a_1, \dots, a_d \in \mathbb{N}^*$ such that $0 < \sum_{i=1}^d a_i \leq m$.

To establish uniform consistency over some compact set $T_0 \subset T$ we need a further smoothness condition on the kernel function.

Condition \mathcal{Z} : The kernel function K is a Lipschitz condition, i.e. there exists a constant $\eta > 0$ such that

$$|K(u) - K(v)| \leq \eta \|u - v\| \quad \text{for any } u, v \in T.$$

We also assume additionally that:

Condition $\mathcal{C}(T)$

$\bar{F}(o, t)$ and $G(t)$ are continuous on a compact set $T_0 \subset T$.

The following theorem presents the asymptotic behavior of the expectation of the kernel-type estimator (6) and the rate of convergence.

Theorem 1. *Consider a stationary pairwise interaction point process \mathbf{X} in \mathbb{R}^d with Papangelou conditional intensity (3) satisfy condition (4) and under Condition $K(d, 1)$. We have*

$$\lim_{n \rightarrow \infty} \mathbb{E} \hat{H}_n(t) = \beta^{*2} G(t) \bar{F}(o, t)$$

at any continuity point $t \in T$ of $G\bar{F}$. If Condition $K(d, m)$ is satisfied and $G\bar{F}$ has bounded and continuous partial derivatives of order m in $B^o(t, \delta)$ (for some $\delta > 0$) for $t \in \overset{\circ}{T}$, then

$$\mathbb{E} \hat{H}_n(t) = \beta^{*2} G(t) \bar{F}(o, t) + \mathcal{O}(b_n^m) \quad \text{as } n \rightarrow \infty. \quad (10)$$

The estimator (7) turn out to be unbiased estimator of $\beta^* \bar{F}(o, t)$ and strongly consistent (the uniform strong consistency) as n tends infinity, since a classical ergodic theorem for spatial point processes obtained in [25]. This implies the following:

Proposition 1. *Consider a stationary pairwise interaction point process \mathbf{X} in \mathbb{R}^d with Papangelou conditional intensity (3) satisfy condition (4).*

$$\sup_{t \in T_0} |\hat{G}_n(t) - \beta^* G(t)| \xrightarrow{0} \mathbb{P}\text{-a.s.} \quad \text{iff}$$

$$\sup_{t \in T_0} |\hat{H}_n(t) - \beta^{*2} G(t) \bar{F}(o, t)| \xrightarrow{0} \mathbb{P}\text{-a.s.}$$

Next we list a set of conditions which are needed to obtain (rates of) uniform strong convergence of the estimator $\hat{H}_n(t)$ to the function $\beta^{*2} G(t) \bar{F}(o, t)$.

Now, we decompose $\Lambda_{n,R}$ as $\cup_{i \in \tilde{I}_n} \Lambda_i$ and \tilde{I}_n is a sequence of finite subsets of \mathbb{Z}^d which only satisfies $|\tilde{I}_n|$ goes to infinity as n goes to infinity and Λ_i are the subcubes of $\Lambda_{n,R}$ centered at i . To shorten the notation we introduce the random variables

$$Z_k = \sum_{\substack{\neq \\ u,v \in \mathbf{X} \\ v-u \in B(o,R)}} \mathbb{1}_{\Lambda_k}(u)h(u, \mathbf{X} \setminus \{u,v\})h(v, \mathbf{X} \setminus \{u,v\})K\left(\frac{v-u-t}{b_n}\right).$$

Note for all $k \in \tilde{I}_n$, $\bar{Z}_k = Z_k - \mathbb{E} Z_k$ and $S_n = \sum_{k \in \tilde{I}_n} \bar{Z}_k$.

Theorem 2. *Under Conditions $K(d, m)$, $\mathcal{C}(T)$ and \mathcal{Z} . Further, assume that $G\bar{F}$ has bounded and continuous partial derivatives of order m on T_0 .*

1) *If there exists $0 < q < 2$ such that $\bar{Z}_0 \in \mathbb{L}_{\psi_{\theta(q)}}$ and*

$$\sum_{k \in V_0^1} \left\| \sqrt{|\bar{Z}_k E_{|k|}(\bar{Z}_0)|} \right\|_{\psi_{\theta(q)}}^2 < \infty. \quad (11)$$

Then

$$\sup_{t \in T_o} |\hat{H}_n(t) - \beta^{*2} G(t) \bar{F}(o, t)| = \mathcal{O}_{a.s.} \left(\frac{(\log n)^{1/q}}{(b_n \sqrt{n})^d} \right) + \mathcal{O}(b_n^m) \quad \text{as } n \rightarrow \infty.$$

2) *If $\bar{Z}_0 \in \mathbb{L}^\infty$ and*

$$\sum_{k \in V_0^1} \|\bar{Z}_k E_{|k|}(\bar{Z}_0)\|_\infty < \infty. \quad (12)$$

Then

$$\sup_{t \in T_o} |\hat{H}_n(t) - \beta^{*2} G(t) \bar{F}(o, t)| = \mathcal{O}_{a.s.} \left(\frac{(\log n)^{1/2}}{(b_n \sqrt{n})^d} \right) + \mathcal{O}(b_n^m) \quad \text{as } n \rightarrow \infty.$$

3) *If there exists $p > 2$ such that $\bar{Z}_0 \in \mathbb{L}^p$ and*

$$\sum_{k \in V_0^1} \|\bar{Z}_k E_{|k|}(\bar{Z}_0)\|_{\frac{p}{2}} < \infty. \quad (13)$$

Assume that $b_n = n^{-q_2} (\log n)^{q_1}$ for some $q_1, q_2 > 0$. Let $a, b \geq 0$ be fixed and if $a(p+d) - d^2/2 - q_2 d > 1$ and $b(p+d) + q_1 d > 1$. Then

$$\sup_{t \in T_o} |\hat{H}_n(t) - \beta^{*2} G(t) \bar{F}(o, t)| = \mathcal{O}_{a.s.} \left(\frac{n^a (\log n)^b}{(b_n \sqrt{n})^d} \right) + \mathcal{O}(b_n^m) \quad \text{as } n \rightarrow \infty.$$

Remark 1. *From the Markov property of \mathbf{X} entails that for $i \neq 0$ are not neighborhoods, then \bar{Z}_i et \bar{Z}_o are conditionally independent, i.e $\mathbb{E}[\bar{Z}_0 | (X_{\Lambda_i}; i \neq 0)] = 0$. Since $\sigma(Z_i, i \in V_0^k)$ is contained in $\sigma(X_{\Lambda_i}, i \neq 0)$ for $k > l$, for some integer l , it follows immediately that conditions (11), (12), (13) are satisfied.*

5 Proofs

In the sequel, we denote $H(u, v, \mathbf{X}) = h(u, \mathbf{X})h(v, \mathbf{X})$, $I_R(v - u) = \mathbb{1}(v - u \in B(o, R))$ and keep in mind the function $\bar{F}(o, t)$ defined by (8).

5.1 Proof of Theorem 1

Proof. Follows immediately by expression the $\hat{H}_n(t)$ as a double sum and by means of the GNZ formula (2) with

$$h(u, v, \mathbf{X}) = \mathbb{1}_{\Lambda_{n,R}}(u)I_R(v - u)H(u, v, \mathbf{X})K\left(\frac{v - u - t}{b_n}\right),$$

we get

$$\mathbb{E} \hat{H}_n(t) = \frac{1}{b_n^d |\Lambda_{n,R}|} \mathbb{E} \int_{\mathbb{R}^{2d}} \mathbb{1}_{\Lambda_{n,R}}(u)I_R(v - u)H(u, v, \mathbf{X})K\left(\frac{v - u - t}{b_n}\right) \lambda_{\beta^*}(u, v, \mathbf{X}) du dv.$$

We remember the second order Papangelou conditional intensity by:

$$\lambda_{\beta^*}(u, v, \mathbf{x}) = \lambda_{\beta^*}(u, \mathbf{x})\lambda_{\beta^*}(v, \mathbf{x} \cup \{u\}) \quad \text{for any } u, v \in \mathbb{R}^d \quad \text{and } \mathbf{x} \in N_{lf}.$$

Now, using the finite range property (4) for each function $\lambda_{\beta^*}(u, \mathbf{x})$ and $\lambda_{\beta^*}(v, \mathbf{x} \cup \{u\})$. Hence $\mathbf{x} = \emptyset$, this implies that

$$\lambda_{\beta^*}(u, \emptyset) = \beta^* \quad \text{and} \quad \lambda_{\beta^*}(v, \emptyset \cup \{u\}) = \beta^* G(v - u) \quad \text{for all } u, v \in \mathbb{R}^d.$$

In this way we obtain by stationarity of \mathbf{X}

$$\begin{aligned} \mathbb{E} \hat{H}_n(t) &= \frac{\beta^{*2}}{b_n^d |\Lambda_{n,R}|} \int_{\mathbb{R}^{2d}} \mathbb{1}_{\Lambda_{n,R}}(u)I_R(v - u)\bar{F}(o, v - u)K\left(\frac{v - u - t}{b_n}\right)G(v - u) du dv \\ &= \beta^{*2} \int_{\mathbb{R}^d} I_R(b_n z + t)\bar{F}(o, b_n z + t)K(z)G(b_n z + t) dz. \end{aligned}$$

The continuity of $\bar{F}G$ in t and the boundedness conditions on the kernel function yield the desired result with the first statement of Theorem 1 by dominated convergence theorem.

For the results of the second part of Theorem 1, we consider $B^0(t, \delta)$ an open ball in \mathbb{R}^d , the function $G(t)\bar{F}(o, t)$ has bounded and continuous partial derivatives of order m of in $B^0(t, \delta)$ (for some $\delta > 0$) for $t \in \overset{\circ}{T}$, then for any point z in \mathbb{R}^d , there exists $\theta \in]0, 1[$, such that by Taylor-Lagrange formula, we get

$$G(t + b_n z) = G(t) + \sum_{k=1}^{m-1} \sum_{|\alpha|=k} \frac{(b_n z)^\alpha}{\alpha!} \frac{\partial^k G(t)}{\partial x^\alpha} + b_n^m R_m(z, t)$$

where $R_m(z, t) = \sum_{|\alpha|=m} \frac{z^\alpha}{\alpha!} \frac{\partial^m G}{\partial x^\alpha}(t + \theta b_n z)$.

$$\bar{F}(o, t + b_n z) = \bar{F}(o, t) + \sum_{k=1}^{m-1} \sum_{|\alpha|=k} \frac{(b_n z)^\alpha}{\alpha!} \frac{\partial^k \bar{F}(o, t)}{\partial x^\alpha} + b_n^m R_m(z, t)$$

where $R'_m(z, t) = \sum_{|\alpha|=m} \frac{z^\alpha}{\alpha!} \frac{\partial^m \bar{F}}{\partial x^\alpha}(o, t + \theta b_n z)$.

So we multiply two such functions, their product equals the product of their m^{th} Taylor polynomials plus terms involving powers of t higher than m . In other words, to compute the m^{th} Taylor polynomial of a product of two functions, find the product of their Taylor polynomials, ignoring powers of t higher than m . So we denote this product by $T(t)(b_n z)^\alpha$, then we have

$$\begin{aligned} \mathbb{E} \hat{H}_n(t) &= \beta^{*2} G(t) \bar{F}(o, t) \int_{\mathbb{R}^d} K(z) dz \\ &\quad + \beta^{*2} T(t) b_n^\alpha \int_{\mathbb{R}^d} z^\alpha K(z) dz \\ &\quad + \mathcal{O}(b_n^m) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Together with condition $K(d, m)$, we get the asserted rate of convergence. \blacksquare

5.2 Preuve du Theorem 2

Proof. To establish rates of the uniform \mathbb{P} - a.s. convergence for the estimator (6), we apply a triangle inequality decomposition allows for

$$\sup_{t \in T_0} |\hat{H}_n(t) - \beta^{*2} G(t) \bar{F}(o, t)| \leq \sup_{t \in T_0} |\hat{H}_n(t) - \mathbb{E} \hat{H}_n(t)| + \sup_{t \in T_0} |\mathbb{E} \hat{H}_n(t) - \beta^{*2} G(t) \bar{F}(o, t)|. \quad (14)$$

Let $(r_n)_{n \geq 1}$ be sequence of positive numbers going to zero. Following Carbon and al. [5], the compact set T_0 can be covered by v_n cubes T_k having sides of length $l_n = r_n b_n^{d+1}$ and center at c_k . Clearly there exists $c > 0$, such that $v_n \leq c/l_n^d$. Define

$$A_1 = \max_{1 \leq k \leq v_n} \sup_{t \in T_k} |\hat{H}_n(t) - \hat{H}_n(c_k)|$$

$$A_2 = \max_{1 \leq k \leq v_n} \sup_{t \in T_k} |\mathbb{E} \hat{H}_n(t) - \mathbb{E} \hat{H}_n(c_k)|$$

$$A_3 = \max_{1 \leq k \leq v_n} |\hat{H}_n(c_k) - \mathbb{E} \hat{H}_n(c_k)|$$

then

$$\sup_{t \in T_o} |\widehat{H}_n(t) - \mathbb{E} \widehat{H}_n(t)| \leq A_1 + A_2 + A_3. \quad (15)$$

We study the following lemmas which significantly improve the desired result.

Lemma 1. *For $j=1,2$, we have*

$$A_j = \mathcal{O}_{p.s.}(r_n).$$

Proof. For any $t \in T_k$, by Condition \mathcal{Z} , we derive that there exists constant $\eta > 0$ such that n sufficiently large

$$\begin{aligned} & \left| \widehat{H}_n(t) - \widehat{H}_n(c_k) \right| \\ & \leq \frac{1}{b_n^{d+1}} \eta \|t - c_k\| \left| \frac{1}{|\Lambda_{n,R}|} \sum_{\substack{u,v \in \mathbf{X} \\ v-u \in B(o,R)}}^{\neq} \mathbb{1}_{\Lambda_{n,R}}(u) h(u, \mathbf{X} \setminus \{u,v\}) h(v, \mathbf{X} \setminus \{u,v\}) \right| \\ & \leq \frac{1}{b_n^{d+1}} \eta r_n b_n^{d+1} \left| \frac{1}{|\Lambda_{n,R}|} \sum_{\substack{u,v \in \mathbf{X} \\ v-u \in B(o,R)}}^{\neq} \mathbb{1}_{\Lambda_{n,R}}(u) h(u, \mathbf{X} \setminus \{u,v\}) h(v, \mathbf{X} \setminus \{u,v\}) \right| \\ & \leq \eta r_n \left| \frac{1}{|\Lambda_{n,R}|} \sum_{\substack{u,v \in \mathbf{X} \\ v-u \in B(o,R)}}^{\neq} \mathbb{1}_{\Lambda_{n,R}}(u) h(u, \mathbf{X} \setminus \{u,v\}) h(v, \mathbf{X} \setminus \{u,v\}) \right|. \end{aligned}$$

The Lemma 1 easily follows from the last inequality and the Nguyen and Zessin [25] ergodic theorem. \blacksquare

Lemma 2. *Assume that either (11) holds for some $0 < q < 2$ such that $\bar{Z}_0 \in \mathbb{L}_{\psi_{\theta(q)}}$ and $r_n = (\log n)^{1/q} / (b_n \sqrt{n})^d$ or (12) holds such that $\bar{Z}_0 \in \mathbb{L}^\infty$ and $r_n = (\log n)^{1/2} / (b_n \sqrt{n})^d$, then*

$$A_3 = \mathcal{O}_{p.s.}(r_n).$$

Proof. In the sequel, the letter C denotes any generic positive constant. For $\varepsilon > 0$, using Markov's inequality, we get

$$\begin{aligned} & \mathbb{P} \left(|\widehat{H}_n(t) - \mathbb{E} \widehat{H}_n(t)| > \varepsilon r_n \right) = \mathbb{P} \left(|S_n| > \varepsilon r_n (nb_n)^d \right) \\ & \leq \exp \left[- \left(\frac{\varepsilon r_n (nb_n)^d}{\|S_n\|_{\psi_{\theta(q)}}} + \xi_q \right)^q \right] \mathbb{E} \exp \left[\left(\frac{|S_n|}{\|S_n\|_{\psi_{\theta(q)}}} + \xi_q \right)^q \right]. \end{aligned}$$

Therefore, we assume that there exists a real $0 < q < 2$, such that $\bar{Z}_0 \in \mathbb{L}_{\psi_{\theta(q)}}$ and using Kahane-Khintchine inequalities (cf. El Machkouri [20], Theorem 1), we have

$$\begin{aligned} \mathbb{P}\left(|\hat{H}_n(t) - \mathbb{E}\hat{H}_n(t)| > \varepsilon r_n\right) &\leq \mathbb{P}\left(|S_n| > \varepsilon r_n (nb_n)^d\right) \\ &\leq (1 + e^{\xi_q^q}) \exp\left[-\left(\frac{\varepsilon r_n (nb_n)^d}{M(\sum_{i \in \tilde{I}_n} b_{i,q}(\bar{Z}))^{1/2}} + \xi_q\right)^q\right] \end{aligned}$$

denote

$$b_{i,q}(\bar{Z}) = \|\bar{Z}_0\|_{\psi_{\theta(q)}}^2 + \sum_{k \in V_0^1} \left\| \sqrt{|\bar{Z}_k E_{|k|}(\bar{Z}_0)|} \right\|_{\psi_{\theta(q)}}^2.$$

where M is a positive constant depending only on q . We derive that if condition (11) holds, then there exist constant $C > 0$ and so if $r_n = (\log n)^{1/q}/(b_n \sqrt{n})^d$, such that

$$\begin{aligned} \sup_{t \in T_0} \mathbb{P}\left(|\hat{H}_n(t) - \mathbb{E}\hat{H}_n(t)| > \varepsilon r_n\right) &\leq (1 + e^{\xi_q^q}) \exp\left[-\left(\frac{\varepsilon r_n (\sqrt{n} b_n)^d}{C} + \xi_q\right)^q\right] \\ &\leq (1 + e^{\xi_q^q}) \exp\left[-\frac{\varepsilon^q \log n}{C^q}\right]. \end{aligned} \quad (16)$$

Now, we will accomplish the second step the proof of Theorem 2. Using Kahane-Khintchine inequalities (cf. El Machkouri [20], Theorem 1) with $q = 2$, such that $\bar{Z}_0 \in \mathbb{L}^\infty$, we have

$$\mathbb{P}\left(|\hat{H}_n(t) - \mathbb{E}\hat{H}_n(t)| > \varepsilon r_n\right) \leq 2 \exp\left[-\left(\frac{\varepsilon r_n (nb_n)^d}{M(\sum_{i \in \tilde{I}_n} b_{i,2}(\bar{Z}))^{1/2}}\right)^2\right]$$

denote

$$b_{i,2}(\bar{Z}) = \|\bar{Z}_0\|_\infty^2 + \sum_{k \in V_0^1} \|\bar{Z}_k E_{|k|}(\bar{Z}_0)\|_\infty.$$

We derive that if condition (12) holds and so if $r_n = (\log n)^{1/2}/(b_n \sqrt{n})^d$, there existe $C > 0$ such that

$$\sup_{t \in T_0} \mathbb{P}\left(|\hat{H}_n(t) - \mathbb{E}\hat{H}_n(t)| > \varepsilon r_n\right) \leq 2 \exp\left[-\frac{\varepsilon^2 \log n}{C^2}\right]. \quad (17)$$

On the other hand, we have

$$\mathbb{P}(|A_3| > \varepsilon r_n) \leq v_n \sup_{t \in T_0} \mathbb{P}\left(|\hat{H}_n(t) - \mathbb{E}\hat{H}_n(t)| > \varepsilon r_n\right).$$

Using (16) and (17), choosing ε sufficiently large, therefore, it follows with Borel-Cantelli's lemma

$$\mathbb{P}(\limsup_{n \rightarrow \infty} |A_3| > \varepsilon r_n) = 0.$$

■

Now, we will accomplish the last step the proof of Theorem 2.

Lemma 3. *Assume (13) holds for some $p > 2$ such that $\bar{Z}_0 \in \mathbb{L}^p$ and $b_n = n^{-q_2}(\log n)^{q_1}$ for some $q_1, q_2 > 0$. Let $a, b \geq 0$ be fixed and denote $r_n = n^a(\log n)^b / (b_n \sqrt{n})^d$. If*

$$a(p+d) - d^2/2 - q_2d > 1 \quad \text{and} \quad b(p+d) + q_1d > 1,$$

then

$$A_3 = \mathcal{O}_{p,s}(r_n).$$

Proof. Let $p > 2$ be fixed, such that $\bar{Z}_0 \in \mathbb{L}^p$ and for any $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}(|\widehat{H}_n(t) - \mathbb{E} \widehat{H}_n(t)| > \varepsilon r_n) &= \mathbb{P}(|S_n| > \varepsilon r_n (nb_n)^d) \\ &\leq \frac{\varepsilon^{-p} \mathbb{E} |S_n|^p}{r_n^p (nb_n)^{pd}} \\ &\leq \frac{\varepsilon^{-p}}{r_n^p (nb_n)^{pd}} \left(2p \sum_{i \in \tilde{I}_n} c_i(\bar{Z}) \right)^{p/2}. \end{aligned}$$

The last inequality follows from a Marcinkiewicz-Zygmund type inequality by Dedecker [8], where

$$c_i(\bar{Z}) = \|\bar{Z}_i\|_p^2 + \sum_{k \in V_i^1} \|\bar{Z}_k E_{|k-i|}(\bar{Z}_i)\|_{\frac{p}{2}}.$$

Under assumption (13) and with the stationarity of \mathbf{X} , we derive that there exists $C > 0$ such that

$$\begin{aligned} \mathbb{P}(|A_3| > \varepsilon r_n) &\leq v_n \sup_{t \in \tilde{T}_0} \mathbb{P}(|\widehat{H}_n(t) - \mathbb{E} \widehat{H}_n(t)| > \varepsilon r_n) \\ &\leq v_n \frac{C \varepsilon^{-p}}{r_n^p (b_n \sqrt{n})^{pd}}. \end{aligned}$$

As $v_n \leq c/l_n^d$ and $l_n = r_n b_n^{1+d}$, therefore for $r_n = n^a (\log n)^b / (b_n \sqrt{n})^d$, it follows for n sufficiently large,

$$\begin{aligned} \mathbb{P}(|A_3| > \varepsilon r_n) &\leq \frac{C\varepsilon^{-p}}{n^{a(p+d)-d^2/2}(\log n)^{b(p+d)}b_n^d} \\ &\leq \frac{C\varepsilon^{-p}}{n^{a(p+d)-d^2/2-q_2d}(\log n)^{b(p+d)+q_1d}}. \end{aligned}$$

For $a(p+d) - d^2/2 - q_2d > 1$ and $b(p+d) + q_1d > 1$, we have

$$\sum_{n \geq 1} \mathbb{P}(|A_3| > \varepsilon r_n) < \infty.$$

The proof of Lemma 3 is complete. ■

By strengthening the uniform continuity assumption of $G(t)\bar{F}(o, t)$, one hand, by Theorem 1, we have for a conventional calculation

$$\sup_{t \in T_o} |\mathbb{E} \hat{H}_n(t) - \beta^{*2} G(t)\bar{F}(o, t)| = \mathcal{O}(b_n^m).$$

We conclude the proof of Theorem 2 by combining inequality (14) and inequality (15). ■

5.3 Proof of Proposition 1

Proof. Our proofs will be expressed using constant α_1 et α_2 . Denote $\tilde{F}(t) = \beta^* \bar{F}(o, t)$ and $\tilde{H}(t) = \beta^{*2} G(t)\bar{F}(o, t)$. For $t \in T_0$, we assume that

$$\exists \alpha_1 > 0, \quad \tilde{F}(t) \geq \alpha_1 \quad \text{and} \quad \exists \alpha_2 > 0, \quad \tilde{H}(t) \leq \alpha_2,$$

implies,

$$|\hat{G}_n(t) - \beta^* G(t)| \leq \frac{|\hat{H}_n(t) - \tilde{H}(t)|}{\alpha_1 - |\hat{F}_n(t) - \tilde{F}(t)|} + \alpha_2 \alpha_1^{-1} |\hat{F}_n(t) - \tilde{F}(t)|. \quad (18)$$

Ergodic theorem (Nguyen and Zessin [25]) immediately gives, as $n \rightarrow \infty$,

$$|\hat{F}_n(t) - \tilde{F}(t)| \xrightarrow{a.s.} 0. \quad (19)$$

Using the monotony of functions \tilde{F} and \hat{F}_n , we can approach the functions \tilde{F} and \hat{F}_n by their values in a finite number of points. Bringing this remark and the result (19), we have as $n \rightarrow \infty$,

$$\sup_{t \in T_o} |\hat{F}_n(t) - \tilde{F}(t)| \xrightarrow{a.s.} 0. \quad (20)$$

By Theorem 2, the expression (20) and the expression (18), we complete the proof. ■

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