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► **To cite this version:**

Yu-Jui Huang, Adrien Nguyen-Huu. Time-consistent stopping under decreasing impatience. 2015. hal-01116414v1

HAL Id: hal-01116414

<https://hal.science/hal-01116414v1>

Preprint submitted on 13 Feb 2015 (v1), last revised 10 Mar 2017 (v4)

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Time-consistent stopping under decreasing impatience

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Abstract: The paper presents a dynamic theory for time-inconsistent problems of optimal stopping. The theory is developed under the paradigm of expected discounted payoff, where the process to stop is continuous and Markovian. We introduce equilibrium stopping policies, which are implementable stopping rules that take into account the change of preferences over time. When the discount function induces decreasing impatience, we establish a constructive method to find equilibrium policies. A new class of stopping problems, involving equilibrium policies, is introduced, as opposed to classical optimal stopping. By studying the stopping of a BES(1) process under hyperbolic discounting, we illustrate our theory in an explicit manner.

MSC 2010 subject classifications: Primary 60G40; secondary 91B06.

Keywords and phrases: Optimal Stopping, Time-inconsistency, Decreasing impatience, Hyperbolic discounting, Markov subgame perfect equilibrium.

1. Introduction

The present article is an attempt to tackle time—or equivalently, dynamic—inconsistency in optimal stopping problems. Time-inconsistency is the lack of conservation in preference ordering through time, which leads to ill-posedness of an inter-temporal optimization problem. This property is of particular interest in Economics, where subjective preferences are given *a priori* in order to compare different alternatives at different dates. Upon axiomatic foundations of time-preferences by Samuelson [34], time-inconsistency has been revealed in the seminal work of Stroz [36], to emphasize limitations in specifying the discount function in the discounted utility paradigm. There are also problems that intrinsically involve time-inconsistency through specific objective functions, from Macroeconomics of Sustainable Development [8, 12] to Finance [5, 9, 19]. We

*Author is partially supported by SFI (07/MI/008 and 08/SRC/FMC1389) and the ERC (278295).

†Author is supported by the RISKERGY project.

refer to [15] for an excellent synthesis on economic modeling and conceptual issues related to time-preferences.

A crucial point to raise with time-inconsistent problems is that they must be rewritten into new questions, which are not unique. Stroz [36] identifies three meaningful behaviors— *consistent planning*, *spendthriftiness* and *commitment*— that are mathematically rigorously defined. Later on, the terminology of *sophisticated* and *naive* strategies has been accepted to designate that the first behavior anticipates, while the second omits, the change of preference through time, see [32]. The second behavior is irrational, in the sense that the related value function for the agent does not corresponds to what is actually reached when the optimization problem is posed at each instant of time. While the former (*sophistication*) is mathematically challenging as dynamic programming arguments do not hold anymore, it reconciles with rationality by proposing time-consistent attainable strategies. Sophisticated strategies have recently emerged as extended solutions of continuous time optimal control problems. In a nutshell, the method considers the controlling agent as a sequence of different agents playing through time an inter-temporal sequential game against his future selves. The optimal control is then replaced by a consistent, reachable, Markov subgame perfect equilibrium strategy. Recent development in continuous time have been made upon this idea, initiated by Karps [24, 22, 23] taken over by Ekeland [11, 14, 13], and gathered into an emerging general theory of time-inconsistent control problems by several authors [3, 4, 17, 42].

To our knowledge, there is no equivalent development for stopping problems. The work of Grenadier and Wang [16] should be mentioned, but the case they consider is very specific so that traditional dynamic programming still applies. In this paper, we deal with the standard stopping problem involving expected discounted non-negative payoffs. We relax the usual assumption of exponential discounting to general discount functions, which results in time-inconsistency. While optimal stopping rules may still be computed in time-inconsistent problems, e.g., Xu and Zhou [41], what does not hold anymore *a priori* is the attainability of this optimal stopping rule along the evolution of the state process. We thus lead a pedestrian tour among counter-examples, rationality and optimization. By setting the problem as an inter-temporal sequential game, our exploration leads to the consideration of the *lifetime* of a stopping policy, and the construction of *equilibrium stopping policies*. The latter are the stopping rules that can be realistically implemented in a dynamical setting.

We restrict ourselves to the setting of expected discounted utility (EDU), when the discount function admits decreasing impatience and the state process is a continuous stochastic Markov process. The EDU paradigm is at core of Economics, but exploited with exponential discounting most of the time. The concept of impatience is strongly related to subjective discounting [25]. Decreasing impatience is a discount pattern that exhibits less discounting over a time interval when the interval is further away from today, see [33]. It encompasses many discount behaviors (hyperbolic, quasi-hyperbolic) that has been strongly supported by Behavioral Economics [37, 1, 26, 2], and can be summarized by the following quote:

In reflecting upon any action which I am to perform a twelvemonth hence, I always resolve to prefer the greater good, whether at that time it will be contiguous or remote. [...] But on a nearer approach [...] a new inclination to the present good springs up, and makes it difficult for me to adhere inflexibly to my first purpose and resolution. (David Hume, *A treatise of Human Nature*, 1738 [18]).

The specific case of Markov diffusion processes allows us to treat a sufficiently nice situation. We expect however that the present treatment will be extended in forthcoming studies.

Our findings are the following. The introduction of equilibrium in stopping problems noticeably differs from its counterpart in control problems. We observe that asymptotic analysis is not suited to stopping problems, and avoid defining equilibrium from local perturbations. The very nature of stopping problems—that the control is singular and a perturbation, e.g. stopping by anticipation, impacts significantly the reward function—leads to a direct definition of equilibrium as a fixed point of a given operator. This has straight important consequences. First, the methodology developed here can benefit other time-inconsistent stopping problems, see [41] and [39] for recent investigations. Moreover, under current EDU setting, we provide a constructive method to obtain equilibrium policies, via iterative application of the aforementioned operator. This construction, however, does not naturally link the equilibrium policy to an analytic or probabilistic characterization. Another drastic consequence is that equilibrium policies do not lead straightly to a solution to the optimization problem. The class of equilibrium policies is not a singleton; consequently, equilibrium policies are not specific candidates for reachable optimality, but merely implementable strategies for which immediate preferences are compatible over time under some commitment mechanism. Put in another way, committing to an equilibrium policy will avoid regrets over time, until the actual stopping moment.

We are thus led to note that there is no natural rewriting of the classical optimal stopping problem, so that in line with [36] and subsequent works: different degrees of acceptance of time-inconsistent preferences by the agent lead to different optimality criteria, and thus different optimal equilibrium solutions. When the discount function is time-consistent, it is shown that there is a natural well-posed problem to which the equilibrium solution is the usual optimal stopping rule. In general, however, this analysis opens the door to a new class of stopping problems, which is to find and characterize equilibrium stopping policies under a given optimality criterion. This theory is illustrated in a simple example: the stopping of an one-dimensional Bessel process under hyperbolic discounting. In this case, decreasing impatience revealed by the discount function allows us to explicitly characterize every equilibrium stopping policy as the hitting time to some threshold lying below the classical optimal stopping boundary.

The remainder proceeds as follows. Section 2 introduces general stopping problems in a dynamic setting, and several situations under which time-inconsistency arises. Section 3 establishes the notion of equilibrium stopping policies, and studies the existence and construction of such policies. A new class of stop-

ping problems involving equilibrium policies is also introduced. Section 4 provides an application and its associated solutions, which illustrate the results in previous sections in an explicit manner.

2. Markov theory of optimal stopping

2.1. Initial setting

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which supports a continuous Markovian process $X : [0, \infty) \times \Omega \mapsto \mathbb{R}^d$. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be the augmentation of the natural filtration generated by X . For any $t \geq 0$, we denote by \mathcal{T}_t the collection of all stopping times τ with $\tau \geq t$ \mathbb{P} -a.s. For any $\omega, \omega' \in \Omega$ and $\pi \in \mathcal{T}_0$, we define

$$(\omega \otimes_{\pi} \tilde{\omega})_u := \omega_u 1_{[0, \pi(\omega))}(u) + \tilde{\omega}_u 1_{[\pi(\omega), \infty)}(u) \quad u \geq 0,$$

and assume that the concatenated path $\omega \otimes_{\pi} \tilde{\omega}$ still belongs to Ω . Note that this is satisfied in particular for $\Omega := (\mathbb{R}^d)^{[0, \infty)}$ and $\Omega := \mathcal{C}([0, \infty)^d)$. We also assume that X admits the following strong Markov property: for any \mathcal{F}_{∞} -measurable random variable Y and any $\pi \in \mathcal{T}_0$,

$$\mathbb{E}[Y \mid \mathcal{F}_{\pi}](\omega) = \mathbb{E}_{\pi(\omega), X_{\pi}(\omega)}[Y(\omega \otimes_{\pi} \cdot)] \quad \text{a.s.} \quad (2.1)$$

Consequently, we will use the notation $X^{t,x}$ for the state process such that $X_t^{t,x} = x$. We will denote by $\mathbb{X} := [0, \infty) \times \mathbb{R}^d$ the space of time and state. We introduce a payoff function $g : \mathbb{R}^d \mapsto \mathbb{R}$, assumed to be nonnegative, continuous, and satisfy

$$\mathbb{E} \left[\sup_{t \geq 0} g(X_t^{0,x}) \right] < \infty, \quad \forall x \in \mathbb{R}^d. \quad (2.2)$$

We finally introduce the discount function $\delta : \mathbb{R}_+ \rightarrow [0, 1]$, which is assumed to be decreasing and verify $\delta(0) = 1$. A standing assumption on δ throughout the paper is the following:

$$\delta(t)\delta(s) \leq \delta(t+s), \quad \forall t, s \geq 0. \quad (2.3)$$

This assumption is related to the notion of *decreasing impatience* (DI) in Behavioral Finance. Under current setting, the general definition of DI [33, Definition 1] reduces to:

$$\text{For any } s \geq 0, \quad t \mapsto \frac{\delta(t+s)}{\delta(t)} \text{ is increasing.}$$

This also corresponds to the formulation in [27, 28]. The above definition of DI in particular implies that $\frac{\delta(t+s)}{\delta(t)} \geq \frac{\delta(0+s)}{\delta(0)} = \delta(s)$ for all $t, s \geq 0$, which is our assumption (2.3). In other words, condition (2.3) is more general than DI: once we pick a discount function which induces decreasing impatience, (2.3) is automatically satisfied.

We introduce the objective function for $(t, x) \in \mathbb{X}$ and $\tau \in \mathcal{T}_t$ by

$$J(t, x; \tau) := \mathbb{E}_{t,x} [\delta(\tau - t)g(X_\tau^{t,x})]. \quad (2.4)$$

The standard optimal stopping problem then consists of finding the value function v , and the associated optimal stopping time $\tilde{\tau}$, defined through

$$v(t, x) := \sup_{\tau \in \mathcal{T}_t} J(t, x; \tau) = J(t, x; \tilde{\tau}). \quad (2.5)$$

The literature on the treatment of this problem is abundant. The following is a very general characterization of its solution, which starts by introducing for all $(t, x) \in \mathbb{X}$ the process $\{Z_s^t\}_{s \geq t}$ defined by

$$Z_s^t := \operatorname{ess\,sup}_{\tau \in \mathcal{T}_s} \mathbb{E}_{s, X_s^{t,x}} [\delta(\tau - t)g(X_\tau^{t,x})], \quad \forall s \geq t. \quad (2.6)$$

Definition 2.1. Let $\tilde{\tau}$ and $\bar{\tau}$ be the \mathbb{F} -stopping times defined by

$$\tilde{\tau}(t, x) := \inf \{s \geq t : \delta(s - t)g(X_s^{t,x}) \geq Z_s^t\} \in \mathcal{T}_t, \quad (2.7)$$

$$\bar{\tau}(t, x) := \inf \{s > t : \delta(s - t)g(X_s^{t,x}) \geq Z_s^t\} \in \mathcal{T}_t. \quad (2.8)$$

Notice that a strict inequality distinguishes $\bar{\tau}$ from $\tilde{\tau}$. In our continuous setting, this distinction might appear superficial for now. It will be more relevant in Section 3.3. We refer to [21, Appendix D] and [31, Chapter I.2] for the demonstration of the following result.

Proposition 2.1. For all $(t, x) \in \mathbb{X}$, $\tilde{\tau}$ and $\bar{\tau}$ are optimal, in the sense that for any $s \geq t$

$$J(t, x; \tilde{\tau}(t, x)) = J(t, x; \bar{\tau}(t, x)) = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} J(t, x; \tau). \quad (2.9)$$

Proposition 2.1 does not provide uniqueness explicitly, but Definition 2.1 states that $\tilde{\tau}$ (respectively $\bar{\tau}$) is the first time that is better than any other posterior time, from the point of view (t, x) . Accordingly, $\tilde{\tau}$ (respectively $\bar{\tau}$) is uniquely defined. In full generality however, the optimal stopping time $\tilde{\tau}$ is a function of (t, x) as stated in the definition. This is the triggering point of our investigation, and leads to the following natural extension of stopping times.

Definition 2.2. A map $\tau : \mathbb{X} \mapsto \mathcal{T}_0$ is a stopping policy if for any $(t, x) \in \mathbb{X}$, $\tau(t, x) \in \mathcal{T}_t$. We denote by $\mathcal{T}(\mathbb{X})$ the collection of all stopping policies.

From now on, we consider $\tilde{\tau}$ and $\bar{\tau}$ defined in Definition 2.1 as elements in $\mathcal{T}(\mathbb{X})$. For a given (t, x) , we label $\tilde{\tau}(t, x)$ the (t, x) -optimal stopping time.

2.2. Time-inconsistency and lifetime of a stopping policy

Consider the problem given by (2.4)-(2.5), and its general solution $\tilde{\tau} \in \mathcal{T}(\mathbb{X})$ given by Proposition 2.1. Assume that for some $(t, x) \in \mathbb{X}$,

$$\tilde{\tau}(t, x) = \bar{\tau}(t, x) > t.$$

Then there exists $t' > t$ such that $\mathbb{P}[\tilde{\tau}(t, x) \geq t'] > 0$. Time-consistency of the stopping problem (2.5) informally translates into the property that the optimal behavior at time t is still optimal from the point of view at time t' , or equivalently,

$$\tilde{\tau}(t, x) = \tilde{\tau}(t', X_{t'}^{t,x}) \quad \text{on} \quad \{\tilde{\tau}(t, x) \geq t'\}. \quad (2.10)$$

In full generality, equation (2.10) may not hold on some subset of $\{\tilde{\tau}(t, x) \geq t'\}$. This can be observed by noting that the criterion for stopping in (2.7) has t dependence in $\delta(\cdot - t)$. It conveys the situation where the current (t, x) -optimal stopping time $\tilde{\tau}(t, x)$ (i.e., from the point of view at time t) is not $(t', X_{t'}^{t,x})$ -optimal anymore (i.e., from the point of view at time $t' > t$). Put differently, there is no guarantee that the (t, x) -optimal stopping time can be preserved and reached as soon as $\tilde{\tau}(t, x) > t$. This leads to the following concept of actual lifetime of a stopping policy.

Definition 2.3. We define the map $\mathcal{L} : \mathcal{T}(\mathbb{X}) \rightarrow \mathcal{T}(\mathbb{X})$, for any $(t, x) \in \mathbb{X}$ and $\tau \in \mathcal{T}(\mathbb{X})$, by $\mathcal{L}\tau(t, x) := \inf \{s \geq t : \tau(s, X_s^{t,x}) = s\}$.

By convention, $\inf \emptyset = +\infty$. Definition 2.3 does not restrict to $\tilde{\tau}$ for the specific problem (2.4)-(2.5). It embeds straightforward properties that are expected of any stopping policy. Namely, \mathcal{L} is a projector on $\mathcal{T}(\mathbb{X})$, i.e., $\mathcal{L}\mathcal{L}\tau(t, x) = \mathcal{L}\tau(t, x)$ \mathbb{P} -a.s. for any $\tau \in \mathcal{T}(\mathbb{X})$ and $(t, x) \in \mathbb{X}$. If we introduce

$$\mathcal{R}(\mathbb{X}) := \{\rho \in \mathcal{T}(\mathbb{X}) : \mathcal{L}\rho(t, x) = \rho(t, x) \text{ } \mathbb{P}\text{-a.s. } \forall (t, x) \in \mathbb{X}\},$$

the set of fixed points of \mathcal{L} , then $\mathcal{R}(\mathbb{X})$ is clearly not empty and equal to $\mathcal{L}\mathcal{T}(\mathbb{X})$. Additionally, and in reference to (2.10), we have that for all $\tau \in \mathcal{T}(\mathbb{X})$ and $(t, x) \in \mathbb{X}$:

$$\mathcal{L}\tau(s, X_s^{t,x}) = \mathcal{L}\tau(t, x) \quad \text{on} \quad \{\mathcal{L}\tau(t, x) \geq s\}. \quad (2.11)$$

A policy belonging to $\mathcal{R}(\mathbb{X})$ is thus rational in the sense that the eventual time to stop is not being *altered* over time, and the dependency on (t, x) can be omitted without ambiguity. To confirm this insight, given $\tau \in \mathcal{T}(\mathbb{X})$, consider the stopping criterion $T_\tau(t, x)$ associated with each (t, x) , i.e., $\tau(t, x) = \inf \{s \geq t : (s, X_s^{t,x}) \in T_\tau(t, x)\}$. If $\tau \in \mathcal{R}(\mathbb{X})$, a nice formulation of τ can be provided via the *stopping region* R_τ given by

$$R_\tau := \{(t, x) \in \mathbb{X} : (t, x) \in T_\tau(t, x)\}. \quad (2.12)$$

That is, $\tau(t, x) = \mathcal{L}\tau(t, x) = \inf \{s \geq t : (s, X_s^{t,x}) \in R_\tau\}$. Note that R_τ , as expressed above, does not depend on (t, x) anymore. It is easy to provide rational stopping policies, by defining a stopping rule independent of the time-space values (t, x) in consideration. A particular example of rational stopping policy is the immediate stopping policy $(t, x) \mapsto \iota(t, x) := t$. It is also easy to provide stopping policies that are irrational, and compute their actual lifetime from Definition 2.3 as a first step toward rationality.

Example 2.1 (Procrastination). For any $(t, x) \in \mathbb{X}$, define the policy $\pi \in \mathcal{T}(\mathbb{X})$ by $\pi(t, x) = \inf \{s \geq t : s \geq t + 1\} = t + 1$. Then obviously $\mathcal{L}\pi(t, x) = +\infty$ \mathbb{P} -a.s. for every $(t, x) \in \mathbb{X}$.

Consider the problem (2.4)-(2.5) and the solution given by $\tilde{\tau} \in \mathcal{T}(\mathbb{X})$ of Definition 2.1. Then for all $(t, x) \in \mathbb{X}$,

$$\mathcal{L}\tilde{\tau}(t, x) = \inf \left\{ s \geq t : g(X_s^{t,x}) \geq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_s} J(s, X_s^{t,x}; \tau) \right\}. \quad (2.13)$$

Characterization (2.13) holds our attention. One can see that it conveys some notion of optimality, that it is the first time s which is better than any other posterior time, from the point of view $(s, X_s^{t,x})$. The following example shows that, in a setting different from EDU, all above reasoning applies, and exhibits that it is inappropriate to apply the optimal stopping rule because of time-inconsistency.

Example 2.2 (Stopping under probability distortion). *We introduce the time-inconsistent problem of Xu and Zhou [41]. Namely, $X^{t,x}$ is a process defined as*

$$dX_s^{t,x} = X_s^{t,x} \left(\frac{\sigma^2}{2} ds + \sigma dW_s \right) \text{ for } s \geq t, \quad X_t^{t,x} = x,$$

with a standard Brownian motion $(W_s)_{s \geq t}$. The objective function is given by

$$J(t, x; \tau) := \int_0^\infty w(\mathbb{P}[U(X_\tau^{t,x}) > u]) du, \quad (2.14)$$

where $U : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nondecreasing and continuous, and $w : [0, 1] \rightarrow [0, 1]$ a strictly increasing continuous function with $w(0) = 0$ and $w(1) = 1$. Equation (2.14) refers to a Choquet expectation under capacity w . It conveys subjective probability distortion, whereas $w(u) := u$ refers to standard expectation. Considering the particular non-trivial case where U is convex, it is shown [41, Th. 4.4] that the optimal stopping time is

$$\tilde{\tau}(t, x) = t \mathbf{1}_{\{a^*(x) = b^*(x)\}} + \inf \{ s \geq t : X_s^{t,x} \notin (a^*(x), b^*(x)) \} \mathbf{1}_{\{a^*(x) > b^*(x)\}}$$

where

$$(a^*, b^*)(x) = \arg \max_{0 \leq a \leq x \leq b} \left(\left(1 - w \left(\frac{x-a}{b-a} \right) \right) U(a) + w \left(\frac{x-a}{b-a} \right) U(b) \right).$$

We can then easily show that

$$\mathcal{L}\tilde{\tau}(t, x) = \inf \{ s \geq t : X_s^{t,x} = 0 \text{ or } +\infty \} = +\infty.$$

Indeed, one convinces himself that if x is in the interior of the support of the process $(X_t^{0,x})_{t \geq 0}$, then $a^* < x < b^*$ in the maximization problem above. This confirms Theorem 3 in [10].

Following our investigation, the question that arises is whether a stopping policy is reachable (implementable). In Example 2.2, the application of $\mathcal{L}\tilde{\tau}(t, x)$ translates into never stopping, which leads to $J(t, x; +\infty) = 0$ by convention. On the contrary, stopping immediately provides $U(x)$ which, for non-negative

utility function U , is better than the rational projection of the optimal stopping policy. This corresponds to the situation

$$J(t, x; \tilde{\tau}(t, x)) > J(t, x; t) \geq J(t, x; \mathcal{L}\tilde{\tau}(t, x)). \quad (2.15)$$

Then one faces a dilemma if he wants to reach optimality: choosing $\tilde{\tau}$ only leads to applying $\mathcal{L}\tilde{\tau}$ (Thus, $\tilde{\tau}$ is not implementable); however, one would not really apply $\mathcal{L}\tilde{\tau}$ because it is worse than immediate stopping; then, one similarly finds that immediate stopping is worse than following $\tilde{\tau}$, which brings us back to the start of the cycle. We then realize that none of $\tilde{\tau}$, $\mathcal{L}\tilde{\tau}$, and immediate stopping are reachable (implementable).

We must then develop a new methodology under which reachable (implementable) stopping policies can be found.

3. Equilibrium stopping policies

3.1. Definition

In optimal control problems, time-inconsistency is managed through the concept of equilibrium policies. It consists of a *status quo* policy applied to every $(t, x) \in \mathbb{X}$, from which there is no incentive to differ locally. This last property is mathematically translated by considering a small interval $[t, t + \varepsilon]$ over which it is possible to freely select the control, whereas the status quo policy takes over after $t + \varepsilon$. If the limiting optimal control obtained at each time t , as ε vanishes, coincides with the status quo policy, such a policy is called an equilibrium policy, which is naturally a fixed point of certain operator.

In our pure stopping setting, we will modify a stopping policy instantaneously at any given time t , instead of over a small interval $[t, t + \varepsilon]$. This is because an instantaneous modification already changes drastically the expected discounted payoff. Notice that, for a stopping policy, only two kinds of modifications are available, namely to stop when it is expected to defer, and *vice versa*. This naturally leads to instantaneous comparison between payoffs obtained from stopping and deferring.

To facilitate the instantaneous comparison, we need to first introduce a modified operator \mathcal{L}^* from Definition 2.3 to reflect the act of deferring at a given time t .

Definition 3.1. We define the map $\mathcal{L}^* : \mathcal{T}(\mathbb{X}) \rightarrow \mathcal{T}(\mathbb{X})$, for any $\tau \in \mathcal{T}(\mathbb{X})$ and $(t, x) \in \mathbb{X}$, by $\mathcal{L}^*\tau(t, x) := \inf \{s > t : \tau(s, X_s^{t,x}) = s\}$.

Note that $\mathcal{L}^*\tau(t, x)$ reflects the situation where the agent chooses to defer at time t , while all his future selves will follow the policy τ . The policy $\mathcal{L}^*\tau$ does not necessarily belong to $\mathcal{R}(\mathbb{X})$. If $\mathcal{L}^*\tau(t, x) > t$, then $\mathcal{L}^*\tau(t, x) = \mathcal{L}\tau(t, x)$. However, it is possible that $\mathcal{L}^*\tau(t, x) > \mathcal{L}\tau(t, x) = t$. In our continuous diffusion case, this might happen with a reflection phenomenon on the boundary of the stopping region R_τ , defined in (2.12). This will be crucial in the proof of the main Theorem below.

Let us turn back to the instantaneous modification of a policy. Suppose the agent is given a policy $\tau \in \mathcal{T}(\mathbb{X})$ to apply. At each time-state (t, x) , he can carry out the following game-theoretic thinking: given that all my future selves will follow $\tau \in \mathcal{T}(\mathbb{X})$, what is my best stopping strategy at current time t ? Since the agent at t has only two possible actions, to stop and to defer, he just needs to compare the payoffs resulting from the two actions. If he chooses to stop at time t , he will get $g(x)$ immediately. If he chooses to defer at time t , since all his future selves will follow $\tau \in \mathcal{T}(\mathbb{X})$, the agent will eventually stop at the moment $\mathcal{L}^*\tau(t, x)$; the resulting payoff is therefore $J(t, x; \mathcal{L}^*\tau(t, x))$. It follows that if (t, x) belongs to the region

$$S_\tau := \{(t, x) \in \mathbb{X} : g(x) > J(t, x; \mathcal{L}^*\tau(t, x))\}, \quad (3.1)$$

it is optimal to stop immediately (even if $\tau(t, x) > t$). If (t, x) belongs to the region

$$C_\tau := \{(t, x) \in \mathbb{X} : g(x) < J(t, x; \mathcal{L}^*\tau(t, x))\}, \quad (3.2)$$

it is optimal to defer at time t (even if $\tau(t, x) = t$). We are left with the situation where (t, x) belongs to

$$I_\tau := \{(t, x) \in \mathbb{X} : g(x) = J(t, x; \mathcal{L}^*\tau(t, x))\}. \quad (3.3)$$

In this case, the agent is indifferent between to stop and to defer at time t . Two possibilities are reasonable. One is to keep the *status quo* policy, i.e. follow the given $\tau(t, x) \in \mathcal{T}_t$, as there is no incentive to deviate from it at time t . The agent then rationally applies $\mathcal{L}\tau(t, x)$. The other one is to stop immediately, inspired by the classical rationale of optimal stopping (once we reach the largest possible payoff, we stop). Notice that they make no difference when $\tau(t, x) = t$. Here, we particularly choose to keep the *status quo* policy $\tau(t, x)$, which will be justified in Remark 3.1 below.

Definition 3.2. Let $\tau \in \mathcal{T}(\mathbb{X})$ and $(t, x) \in \mathbb{X}$. We define the map $\Theta : \mathcal{T}(\mathbb{X}) \rightarrow \mathcal{T}(\mathbb{X})$ by

$$\Theta\tau(t, x) := \begin{cases} t & \text{for } (t, x) \in S_\tau \\ \mathcal{L}\tau(t, x) & \text{for } (t, x) \in I_\tau \\ \mathcal{L}^*\tau(t, x) & \text{for } (t, x) \in C_\tau \end{cases}. \quad (3.4)$$

The definition (3.4) summarizes our previous reasoning, and $\Theta\tau(t, x)$ can be viewed as the best stopping strategy at time t given that all future selves will follow $\tau \in \mathcal{T}(\mathbb{X})$. It is worth noting that in general $\Theta\tau \notin \mathcal{R}(\mathbb{X})$, for $\tau \in \mathcal{T}(\mathbb{X})$. For instance, given $(t, x) \in \mathbb{X}$, suppose $t_0 := \mathcal{L}^*\tau(t, x) > t$, $(s, X_s^{t,x}) \in C_\tau$ for all $s \in [t, t_0 + \varepsilon]$ for some $\varepsilon > 0$, and $\mathcal{L}^*\tau(t_0, X_{t_0}^{t,x}) > t_0$. Then $\Theta\tau(t, x) = t_0 < \mathcal{L}\Theta\tau(t, x)$. Similarly to $\mathcal{R}(\mathbb{X})$, we introduce the set of fixed points of Θ

$$\mathcal{E}(\mathbb{X}) := \{\tau \in \mathcal{T}(\mathbb{X}) : \Theta\tau(t, x) = \tau(t, x) \mathbb{P} - \text{a.s.}, \forall (t, x) \in \mathbb{X}\} \quad (3.5)$$

as the set of *equilibrium policies*. These are, by construction, Markov subgame perfect equilibrium strategies.

Remark 3.1. When $(t, x) \in I_\tau$ and $\tau(t, x) > t$, it is preferable to stay with $\tau(t, x)$, instead of immediate stopping, because this expresses a minimal modification of a policy under the use of Θ , and accordingly lower requirement for being a fixed point of it. We can then expect a broader set $\mathcal{E}(\mathbb{X})$.

Note that an equilibrium policy $\tau \in \mathcal{E}(\mathbb{X})$ naturally belongs to $\mathcal{R}(\mathbb{X})$.

Proposition 3.1. $\mathcal{E}(\mathbb{X}) \subset \mathcal{R}(\mathbb{X})$.

Proof. For any $\tau \in \mathcal{E}(\mathbb{X})$, it is clear if $(t, x) \in S_\tau$, then $\tau(t, x) = \Theta\tau(t, x) = t$. Then $\mathcal{L}\tau(t, x) = t = \tau(t, x)$. If $(t, x) \in C_\tau$, then $\tau(t, x) = \mathcal{L}^*\tau(t, x) > t$ (Note that if $\mathcal{L}^*\tau(t, x) = t$, then we would have $(t, x) \in I_\tau$, a contradiction). Therefore, $\mathcal{L}\tau(t, x) = \mathcal{L}^*\tau(t, x) = \tau(t, x)$. Finally, for $(t, x) \in I_\tau$, by definition $\tau(t, x) = \mathcal{L}\tau(t, x)$. Altogether, $\tau(t, x) = \mathcal{L}\tau(t, x)$ for all $(t, x) \in \mathbb{X}$. \square

3.2. Existence and construction

The existence of at least one policy in $\mathcal{E}(\mathbb{X})$ is ensured by the following trivial result.

Remark 3.2. Recall $\iota : (t, x) \rightarrow t \in \mathcal{R}(\mathbb{X})$. We have $R_\iota = T_\iota = \mathbb{X}$. Since $\mathcal{L}^*\iota(t, x) = t$, $I_\iota = \mathbb{X}$. We thus have $\Theta\iota(t, x) = t = \iota(t, x)$ for all $(t, x) \in \mathbb{X}$, i.e. $\iota \in \mathcal{E}(\mathbb{X})$.

The operator Θ is not as easy to apply as \mathcal{L} . *A fortiori*, it is not trivial to characterize fixed points of Θ in a direct manner. The present section presents an iterative application of Θ which leads to an equilibrium policy in the limit, under certain regularity assumptions. The following result relies on the decreasing impatience assumption (2.3).

Theorem 3.1. Let $\tau \in \mathcal{T}(\mathbb{X})$. Assume that

$$\mathcal{L}^*\Theta\tau(t, x) \leq \mathcal{L}^*\tau(t, x) \quad \mathbb{P} - a.s., \quad \forall (t, x) \in \mathbb{X}. \quad (3.6)$$

Then, for any $(t, x) \in \mathbb{X}$,

$$\widehat{\tau}(t, x) := \lim_{n \rightarrow \infty} \mathcal{L}\Theta^n\tau(t, x)$$

converges $\mathbb{P} - a.s.$ Moreover, $\widehat{\tau} \in \mathcal{E}(\mathbb{X})$.

The proof of this result is demonstrated in Appendix A. Notice that if we take $\Theta(t, x) = t$ for $(t, x) \in I_\tau$ in Definition 3.2, the proof of this theorem does not hold anymore, which corresponds to Remark 3.1.

In the following, we in particular show that when applying this iterative method to the standard optimal stopping policy $\tilde{\tau} \in \mathcal{T}(\mathbb{X})$ defined in (2.7), we can indeed obtain a non-trivial equilibrium policy.

Lemma 3.1. Assumption (3.6) holds for $\tilde{\tau} \in \mathcal{T}(\mathbb{X})$ of Definition 2.1.

Proof. Denote $\tau_0(t) := \mathcal{L}\tilde{\tau}(t, x)$, $\tau_1(t) := \mathcal{L}\Theta\tilde{\tau}(t, x)$, $\tau_0^*(t) := \mathcal{L}^*\tilde{\tau}(t, x)$ and $\tau_1^*(t) := \mathcal{L}^*\Theta\tilde{\tau}(t, x)$ for any $(t, x) \in \mathbb{X}$, policies in $\mathcal{T}(\mathbb{X})$. We aim to show that for any $(t, x) \in \mathbb{X}$, $\tau_1(t) \leq \tau_0(t)$ \mathbb{P} -a.s. Given $(t, x) \in \mathbb{X}$, recall the process $\{Z_s^t\}_{s \geq t}$ defined in (2.6). We consider two cases, namely $\tau_0^*(t) > t$ and $\tau_0^*(t) = t$. The time $\tau_0^*(t)$ being a \mathbb{F} -stopping time, $\{\tau_0^* > t\}$ is \mathcal{F}_t -measurable. We thus have two cases.

1. Assume $\tau_0^*(t) > t$ and

$$g\left(X_{\tau_0^*(t)}^{t,x}\right) = Z_{\tau_0^*(t)}^{\tau_0^*(t)} \mathbb{P} - \text{a.s.}$$

This implies $\Theta\tilde{\tau}\left(\tau_0^*(t), X_{\tau_0^*(t)}^{t,x}\right) = \tau_0^*(t)$ \mathbb{P} -a.s. and

$$g\left(X_{\tau_0^*(t)}^{t,x}\right) \geq \mathbb{E}_{\tau_0^*(t), X_{\tau_0^*(t)}^{t,x}} \left[\delta\left(\tau_0^*(\tau_0^*(t)) - \tau_0^*(t)\right) g\left(X_{\tau_0^*(\tau_0^*(t))}^{t,x}\right) \right] \mathbb{P} - \text{a.s.}$$

It follows that $\left(\tau_0^*(t), X_{\tau_0^*(t)}^{t,x}\right) \notin C_{\tilde{\tau}}$ with probability one, and thus

$$\Theta\tilde{\tau}\left(\tau_0^*(t), X_{\tau_0^*(t)}^{t,x}\right) = \begin{cases} \tau_0^*(t) & \text{for } \left(\tau_0^*(t), X_{\tau_0^*(t)}^{t,x}\right) \in S_{\tilde{\tau}} \\ \tau_0^*(\tau_1^*(t)) & \text{for } \left(\tau_0^*(t), X_{\tau_0^*(t)}^{t,x}\right) \in I_{\tilde{\tau}} \end{cases} = \tau_0^*(t).$$

This shows that $\tau_1^*(t) \leq \tau_0^*(t)$ \mathbb{P} -a.s.

2. Assume $\tau_0^*(t) = t$ or

$$g\left(X_{\tau_0^*(t)}^{t,x}\right) < Z_{\tau_0^*(t)}^{\tau_0^*(t)}.$$

By definition of $\tau_0^*(t)$, there exists a decreasing sequence of stopping times $(t_n)_{n \geq 0}$ converging almost surely to t , such that $g\left(X_{t_n}^{t,x}\right) = Z_{t_n}^{t_n} \mathbb{P}$ -a.s. This implies $\tilde{\tau}(t_n, X_{t_n}^{t,x}(\omega)) = t_n$ and

$$g\left(X_{t_n}^{t,x}\right) \geq \mathbb{E}_{t_n, X_{t_n}^{t,x}} \left[\delta\left(\tau_0^*(t_n) - t_n\right) g\left(X_{\tau_0^*(t_n)}^{t,x}\right) \right] \mathbb{P} - \text{a.s.}$$

It follows that $(t_n, X_{t_n}^{t,x}) \notin C_{\tilde{\tau}}$ with probability one, and thus

$$\Theta\tilde{\tau}(t_n, X_{t_n}^{t,x}) = \begin{cases} t_n & \text{for } (t_n, X_{t_n}^{t,x}) \in S_{\tilde{\tau}} \\ \tau_1(t_n) & \text{for } (t_n, X_{t_n}^{t,x}) \in I_{\tilde{\tau}} \end{cases} = t_n \mathbb{P} - \text{a.s.}$$

This shows that $\tau_1^*(t) \leq t_n$ \mathbb{P} -a.s. Sending $n \rightarrow \infty$, we get $\tau_1^*(t) \leq \tau_0^*(t)$ \mathbb{P} -a.s. \square

Remark 3.3. *The demonstration of the above result relies on a fundamental property for $\Theta\tau$ for arbitrary $\tau \in \mathcal{T}(\mathbb{X})$, i.e., they are bounded by $\mathcal{L}\tilde{\tau}$. Indeed, one can see that on the stopping region $R_{\mathcal{L}\tilde{\tau}}$, we are either on S_{τ} or $I_{\tau} \cap R_{\tau}$ for any $\tau \in \mathcal{T}(\mathbb{X})$. We can assert in general that*

$$\Theta\tau(t, x) \leq \mathcal{L}\tilde{\tau}(t, x) \mathbb{P} - \text{a.s.}, \quad \forall \tau \in \mathcal{T}(\mathbb{X}) \text{ and } (t, x) \in \mathbb{X}. \quad (3.7)$$

Remark 3.4. Lemma 3.1 and Theorem 3.1 together show that

$$\hat{\tau}(t, x) := \lim_{n \rightarrow \infty} \Theta^n \tilde{\tau}(t, x) \quad \forall (t, x) \in \mathbb{X}$$

is an equilibrium policy. This admits nice economic interpretation: At first, the agent intends to apply over time the policy $\tilde{\tau}$, obtained from the classical optimal stopping theory. Then, from the game-theoretic analysis in Section 3.1, the agent at time t realizes his best stopping strategy is actually $\Theta \tilde{\tau}(t, x)$. Since the initial time t is arbitrarily chosen, it turns out that the agent wants to switch the entire policy from $\tilde{\tau}$ to $\Theta \tilde{\tau}$. With the intention to apply $\Theta \tilde{\tau}$ over time, the agent can carry out the same game-theoretic analysis and realize that it is better to employ $\Theta^2 \tilde{\tau}$. This procedure continues until the agent finally obtains τ_0 , which cannot be “improved” anymore by Θ , i.e. $\Theta \hat{\tau} = \hat{\tau}$.

Assumption (3.6) is in general non-trivial to obtain for a generic policy, in particular for semi-continuous processes. We will show, nonetheless, in Section 4 that it holds for a simple example.

3.3. Time-consistent stopping and time-consistent discounting

To solve a stopping problem, the most important thing is to specify a reasonable objective. The standard objective in the literature is simple: at each time-state $(t, x) \in \mathbb{X}$, the agent aims to maximize his current expected discounted payoff $J(t, x; \tau)$ over $\tau \in \mathcal{T}_t$. Under a general discount function, it turns out that this objective, in general, is never attained over time. This is because at each (t, x) , in order to attain the objective, the agent needs to follow $\tilde{\tau}(t, x) \in \mathcal{T}_t$ over time. Yet, as shown in Section 2, $\tilde{\tau}(t, x)$ in general will not be honored by the agent’s future selves.

To overcome this, one has to specify a new objective for stopping problems. Suppose an agent at time t decides to choose some $\tau \in \mathcal{T}(\mathbb{X})$ and commit to it over time. While deciding which $\tau \in \mathcal{T}(\mathbb{X})$ to employ, he takes into account his future selves’ feeling as follows: given the fact that a policy $\tau \in \mathcal{T}(\mathbb{X})$ is being enforced over time, will myself at time $s > t$ be satisfied with his current stopping strategy $\tau(s, X_s^{t,x})$, or unhappy because there are other better ones? (While the future self cannot deviate from $\tau(s, X_s^{t,x})$, as τ is being enforced, there is no way preventing him from feeling bad). The new objective for stopping problems is therefore to find a stopping policy $\tau \in \mathcal{T}(\mathbb{X})$ such that when it is enforced over time, every future self is satisfied with his current stopping strategy $\tau(s, X_s^{t,x})$, under the knowledge that $\tau \in \mathcal{T}(\mathbb{X})$ is being enforced.

To attain this new objective, the agent should employ an equilibrium policy $\tau \in \mathcal{E}(X)$. By definition of $\mathcal{E}(X)$, the agent will not find himself in a situation where his preferences (discount and utility) push him to deviate from τ , i.e., to stop when τ suggests him to continue, or *vice versa*. It thus provides a reasonable and reachable stopping policy to commit to.

In general, an equilibrium policy is not a solution to problem (2.5), nor does it provide a natural alternative formulation of the problem. Indeed, one might

suggest to pose the problem

$$v(t, x) := \sup_{\tau \in \mathcal{E}(\mathbb{X})} J(t, x; \tau(t, x)), \quad (3.8)$$

which is well-defined in the sense that $\mathcal{E}(\mathbb{X}) \neq \emptyset$. However, nothing guarantees *a priori* that an equilibrium policy which is optimal, under (3.8), at (t, x) will still be optimal at a later time. We will, nevertheless, present an example in which an equilibrium policy $\tau^* \in \mathcal{E}(\mathbb{X})$ solves the problem (3.8) for all $(t, x) \in \mathbb{X}$; see Section 4.3. We also want to emphasize that the characterization of $\mathcal{E}(\mathbb{X})$ is a sufficient challenge that shall be pursued in further research.

Time-consistency in our specific problem is given by the following special case of (2.3):

$$\delta(t)\delta(s) = \delta(t + s), \quad \forall 0 \leq t \leq s; \quad (3.9)$$

see [35] and [7]. It corresponds to the exponential discount function. Under time-consistency, $\tilde{\tau}$ obtained in the classical optimal stopping literature is already an equilibrium policy.

Proposition 3.2. *Suppose (3.9). Then $\tilde{\tau}(t, x) \in \mathcal{E}(\mathbb{X})$.*

Proof. The fact that $\tilde{\tau}(t, x) \in \mathcal{R}(\mathbb{X})$ follows from (3.9) applied to the very definition of $\tilde{\tau}(t, x)$, which provides (2.13), by dividing both sides of the equality by $\delta(s - t)$. As a direct consequence, $\mathcal{L}^* \tilde{\tau}(t, x) = \bar{\tau}(t, x)$ \mathbb{P} -a.s. for all $(t, x) \in \mathbb{X}$, where $\bar{\tau}$ is defined in (2.8). Considering (3.1)-(3.2)-(3.3)-(2.12), we trivially have that $I_{\tilde{\tau}} \subseteq R_{\tilde{\tau}}$ and by Proposition 2.1, $S_{\tilde{\tau}} = \emptyset$. Since $\tilde{\tau} \in \mathcal{R}(\mathbb{X})$, $\tilde{\tau}(t, x) = t$ on $R_{\tilde{\tau}}$ and $\tilde{\tau}(t, x) > t$ on $C_{\tilde{\tau}}$, or equivalently, $\tilde{\tau}(t, x) = \bar{\tau}(t, x) = \mathcal{L}^* \tilde{\tau}(t, x)$ on $C_{\tilde{\tau}}$. Altogether, this shows that $\tilde{\tau} \in \mathcal{E}(\mathbb{X})$. \square

A straight consequence to Proposition 3.2 is that for any $(t, x) \in \mathbb{X}$,

$$J(t, x; \tilde{\tau}(t, x)) = \sup_{\tau \in \mathcal{T}_t} J(t, x; \tau) \geq \sup_{\tau \in \mathcal{E}(\mathbb{X})} J(t, x; \tau(t, x)).$$

Under time-consistency, since $\tilde{\tau}$ is an equilibrium policy, the above inequality shows that $\tilde{\tau}$ attains the maximum on the right hand side for all $(t, x) \in \mathbb{X}$. Problem (3.8) is therefore making sense, and just reduces to the classical optimal stopping problem (2.5). Similarly to the equilibrium treatment of control problems, see [11], the dynamic equilibrium and the committed optimal solution coincide under time-consistency.

4. A case study with hyperbolic discounting

4.1. Setting and optimal stopping

We consider Ω the Wiener space of continuous functions of time, on \mathbb{R}_+ . In this section, we take $(X_s^{t,x})_{s \geq t}$ a one-dimensional Brownian motion starting at value x at time t . It can be derived from the canonical process (shifted) associated to the Wiener measure on Ω . We consider the associated natural

filtration $\mathbb{F} := (\mathcal{F}_s)_{s \geq t}$ with $\mathcal{F}_r := \sigma\{X_s^{t,x} : t \leq s \leq r\}$, completed by null sets on the Wiener space. We consider the *hyperbolic* discount function

$$\delta(s-t) := \frac{1}{1 + \beta(s-t)}, \quad \forall 0 \leq t \leq s.$$

for some fixed parameter $\beta > 0$. The function satisfies (2.3). We eventually consider the payoff function $g(x) := |x|$, being continuous and non-negative. We thus fall into the framework of the paper.

Given initial time and state $(t, x) \in \mathbb{X} := \mathbb{R}_+ \times \mathbb{R}$, the optimal stopping problem in the classical framework is given by

$$U(t, x) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_{t,x} [\delta(\tau-t)g(X_\tau^{t,x})]. \quad (4.1)$$

Under current setting, the process $(\delta(s-t)U(s, X_s^{t,x}))_{s \geq t}$ needs not be a supermartingale, as $\delta(s-t)\delta(\tau-s) > \delta(\tau-t)$ for $t < s < \tau$. We thus can not directly apply standard methods to solve (4.1) and find the optimal stopping time. To overcome this, we introduce an auxiliary value function. For any fixed $t \geq 0$, we define

$$\begin{aligned} V(t, s, x) &:= \sup_{\tau \in \mathcal{T}_s} \mathbb{E}_{s,x} [\delta(\tau-t)g(X_\tau^{s,x})] \\ &= \sup_{\tau \in \mathcal{T}_s} \mathbb{E}_{s,x} \left[\frac{|X_\tau^{s,x}|}{1 + \beta(\tau-t)} \right], \quad \text{for } (s, x) \in [t, \infty) \times \mathbb{R}. \end{aligned} \quad (4.2)$$

By definition, for any fixed $t \geq 0$, $V(t, t, x) = U(t, x)$. However, the stopped process $(V(t, s, X_s^{t,x}))_{s \geq t}$ is a supermartingale, with the associated variational inequality given, for $(s, x) \in [t, \infty) \times \mathbb{R}$, by

$$\min \left\{ v_s(t, s, x) + \frac{1}{2}v_{xx}(t, s, x), v(t, s, x) - \frac{|x|}{1 + \beta(s-t)} \right\} = 0. \quad (4.3)$$

This allows to provide the following classical solution, by use of the classical method of time-change [30].

Proposition 4.1. *For any $(t, x) \in \mathbb{X}$, the optimal stopping time of problem (4.2) is uniquely defined by*

$$\tilde{\tau}(t, x) := \inf \left\{ s \geq t : |X_s^{t,x}| \geq \sqrt{1/\beta + (s-t)} \right\}.$$

Proof. According to [38], V is a viscosity solution of (4.3). The later equation can be rewritten as a free boundary problem, with a boundary supposedly of the form $s \mapsto b(t, s)$:

$$\begin{cases} v_s(t, s, x) + \frac{1}{2}v_{xx}(t, s, x) = 0, & v(t, s, x) > \frac{|x|}{1 + \beta(s-t)}, & \text{for } |x| < b(t, s); \\ v(t, s, x) = \frac{|x|}{1 + \beta(s-t)}, & & \text{for } |x| = b(t, s). \end{cases} \quad (4.4)$$

Following [30], we propose to study the ansatz

$$v(t, s, x) = \frac{1}{\sqrt{1 + \beta(s-t)}} h\left(\frac{x}{\sqrt{1 + \beta(s-t)}}\right).$$

Equation (4.4) then becomes a one-dimensional free boundary problem for h :

$$\begin{cases} -\beta y h'(y) + h''(y) = \beta h(y), & h(y) > |y|, & \text{for } |y| < \frac{b(t,s)}{\sqrt{1+\beta(s-t)}}; \\ h(y) = |y|, & & \text{for } |y| = \frac{b(t,s)}{\sqrt{1+\beta(s-t)}}. \end{cases} \quad (4.5)$$

Since (t, s) does not appear in the ODE for h , we seek for a constant selective criterion for $|y|$, that is, $b(t, s) = \alpha\sqrt{1 + \beta(s-t)}$ for some $\alpha \geq 0$. The second-order ODE of (4.5) yields a general solution of the form

$$h(y) = e^{\frac{\beta}{2}y^2} \left(c_1 + c_2 \sqrt{\frac{2}{\beta}} \int_0^{\sqrt{\frac{\beta}{2}}y} e^{-u^2} du \right), \quad (c_1, c_2) \in \mathbb{R}^2. \quad (4.6)$$

The boundary condition is given by $h(\alpha) = \alpha$, whereas smooth pasting condition implies that $h'(\alpha) = 1$. We must however find three parameters c_1, c_2 and α by matching the first moments. This is done by abandoning the ansatz and take

$$v(t, s, x) = \frac{e^{\frac{\beta x^2}{2(1+\beta(s-t))}}}{\sqrt{1 + \beta(s-t)}} \left(c_1 + c_2 \sqrt{\frac{2}{\beta}} \int_0^{\sqrt{\frac{\beta}{2}} \frac{x}{\sqrt{1+\beta(s-t)}}} e^{-u^2} du \right) \quad (4.7)$$

for $|x| < \alpha\sqrt{1 + \beta(s-t)}$ and

$$v(t, s, x) = \frac{|x|}{1 + \beta(s-t)} \quad \text{for } |x| \geq \alpha\sqrt{1 + \beta(s-t)}. \quad (4.8)$$

Equating the functions (4.7) and (4.8), and their partial derivatives in x and s provides the following equations

$$\alpha = e^{\frac{\beta}{2}\alpha^2} \left(c_1 + c_2 \sqrt{\frac{2}{\beta}} \int_0^{\sqrt{\frac{\beta}{2}}\alpha} e^{-u^2} du \right) \quad \text{and} \quad \text{sgn}(x) - c_2 = \text{sgn}(x)\alpha^2\beta.$$

The last equation implies $c_2 = 0$, to which it follows that $\alpha = 1/\sqrt{\beta}$ and $c_1 = \alpha e^{-1/2}$. Finally, we obtain

$$v(t, s, x) = \frac{1}{\sqrt{\beta}\sqrt{1 + \beta(s-t)}} \exp\left(\frac{1}{2} \left(\frac{\beta x^2}{1 + \beta(s-t)} - 1 \right)\right) \quad (4.9)$$

on the domain $\{(t, x) \in \mathbb{R}^2 : |x| \leq \sqrt{1/\beta + (t-s)}\}$ and (4.8) outside of it. Notice that $v(t, \cdot)$ is $\mathcal{C}^{1,2}$ on this domain, and $\mathcal{C}^{1,1}$ on $[t, +\infty) \times \mathbb{R}$. Additionally, from (4.8),

$$v_s(t, s, x) + \frac{1}{2}v_{xx}(t, s, x) < 0 \quad \text{for } |x| > \sqrt{1/\beta + (s-t)}.$$

We therefore conclude from the standard verification theorem [29, Theorem 3.2] that $V = v$ is a regular solution of (4.3). This implies that $V(t, \cdot \wedge \tilde{\tau}(t, x), X_{\cdot \wedge \tilde{\tau}(t, x)}^{t, x})$ is a martingale for $s \geq t$. The rest of the proof is standard. \square

4.2. Construction of equilibrium policies

The stopping time $\tilde{\tau}(t, x)$ is naive as it depends on the initial time t . More precisely, $\tilde{\tau}(t, x)$ is different from

$$\mathcal{L}\tilde{\tau} = \inf \left\{ s \geq t : X_s^{t, x} \notin \left(-\sqrt{\frac{1}{\beta}}, \sqrt{\frac{1}{\beta}} \right) \right\}.$$

Indeed, $|x| \geq 1/\sqrt{\beta}$ leads to immediate stopping, and is a necessary condition too. The rational policy is simple, and moreover

$$\mathcal{L}^*\tilde{\tau}(t, x) = \mathcal{L}\tilde{\tau}(t, x) \mathbb{P} - \text{a.s.}, \quad \forall t \in \mathbb{R}_+, \forall x \neq -1/\sqrt{\beta} \text{ or } 1/\sqrt{\beta}.$$

The specification of policies of Section 2 are thus considerably simplified in the present case. Without loss of generality, stopping policies defined by

$$\tau_a(t, x) := \inf \{ s \geq t : |X_s^{t, x}| \geq a \}, \quad \forall (t, x) \in \mathbb{X},$$

with $0 \leq a$, are in $\mathcal{R}(\mathbb{X})$, i.e., rational policies.

Remark 4.1. *In fact, policies of the form $\{\tau_a : a \geq 0\}$ are the only relevant ones. First, the symmetry of the Brownian motion implies that the same rule shall be given for $\pm X^{t, x}$ and $\pm X^{t, -x}$. This implies to define a hitting region for $|X|$ only. Second, by taking a hitting region of the form $H = [a, b] \cup [c, d]$, with $0 \leq a < b < c < d$, and $\tau(t, x) = \inf \{ s \geq t : |X^{t, x}| \in H \}$, we have for $x \in (b, c)$*

$$\begin{aligned} \mathbb{E} [\delta(\tau(t, x) - t) | X_\tau^{t, x}] &= (\mathbb{P} [|X_\tau^{t, x}| = b] b + \mathbb{P} [|X_\tau^{t, x}| = c] c) \mathbb{E} [\delta(\tau(t, x) - t)] \\ &< \mathbb{P} [|X_\tau^{t, x}| = b] b + (1 - \mathbb{P} [|X_\tau^{t, x}| = b]) c = x \end{aligned}$$

The latter is deduced from Doob's Optional Stopping Theorem, which provides

$$\mathbb{P} [|X_\tau^{t, x}| = b] = \mathbb{P} [X_\tau^{t, 0} - x = b - x] = \frac{x - b}{c - b}.$$

For $x \leq b$ or $x \in [c, d]$, $\tau(t, x)$ makes no difference with τ_a . For $x > d$, it is clear that $\mathbb{E} [\delta(\tau - t)d] < x = \mathbb{E} [\delta(\tau_a - t) | X_\tau^{t, x}]$. Altogether, the policy τ_a is better than τ . By iteration, it is possible to generalize to any Lebesgue set of \mathbb{R}_+ . The distribution of $|X^{t, x}|$ being absolutely continuous with respect to Lebesgue measure, the conclusion holds for any stopping rule.

According to Remark 4.1, we reduce to the study of τ_a with $a \leq 1/\sqrt{\beta}$. The reason is that equilibrium policies are necessarily bounded by $\mathcal{L}\tilde{\tau} \equiv \tau_{1/\sqrt{\beta}}$, see Remark 3.3. This family of stopping policies will serve as a base for construction

of equilibrium policies. Consider the policy $\tau_a \in \mathcal{T}(\mathbb{X})$ for $a > 0$. Studying the sets (3.1)-(3.2)-(3.3), we notice that

$$\{(t, x) \in \mathbb{X} : |x| > a\} \subseteq I_{\tau_a} \quad (4.10)$$

as $\mathcal{L}^* \tau_a(t, x) = \mathcal{L} \tau_a(t, x) = t$ if $x > a$. In the particular case $x = a$,

$$J(t, a; \mathcal{L}^* \tau_a(t, x)) = a \mathbb{E}_{t,x} \left[\mathbf{1}_{\{\mathcal{L}^* \tau_a = t\}} + \frac{\mathbf{1}_{\{\mathcal{L}^* \tau_a(t, x) > t\}}}{1 + \beta(\mathcal{L}^* \tau_a(t, x) - t)} \right].$$

The fact that on the right of t , $\{X_s^{t,a} < a : s \in (t, \mathcal{L}^* \tau_a(t, x))\}$ is of positive probability implies that the above quantity is strictly lower than a , and

$$\{(t, x) \in \mathbb{X} : |x| = a\} \subseteq S_{\tau_a}. \quad (4.11)$$

Eventually, for $|x| < a$, we need to study

$$J(t, x; \mathcal{L}^* \tau_a) = \mathbb{E}_{t,x} \left[\frac{a}{1 + \beta(\tau_a - t)} \right].$$

Lemma 4.1. *On the space $\{(x, a) \in \mathbb{R}_+^2 : a \geq x\}$, define the function*

$$\eta(x, a) := \mathbb{E}_{t,x} \left[\frac{a}{1 + \beta(\tau_a - t)} \right].$$

- (i) *Given $a \geq 0$, $x \mapsto \eta(x, a)$ is strictly increasing and strictly convex on $[0, a]$, and satisfies $0 < \eta(0, a) < a$ and $\eta(a, a) = a$.*
- (ii) *Given $x \geq 0$, $\eta(x, a) \rightarrow 0$ as $a \rightarrow \infty$.*
- (iii) *There exists a unique $a^* \in (0, 1/\sqrt{\beta})$ such that for any $a > a^*$, there is a unique solution $x^*(a) \in (0, a^*)$ of $\eta(x, a) = x$. Hence, $\eta(x, a) > x$ for $x < x^*(a)$ and $\eta(x, a) < x$ for $x > x^*(a)$. Otherwise, $a \leq a^*$ implies that $\eta(x, a) > x$ for all $x \in (0, a)$.*

Proof. (i) Given $a \geq 0$, it is obvious from definition that $\eta(0, a) \in (0, a)$ and $\eta(a, a) = a$. Fix $x \in (0, a)$, and let f_a^x denote the density of $T_a^x := \tau_a - t$. We obtain

$$\begin{aligned} \mathbb{E}_{t,x} \left[\frac{1}{1 + \beta T_a^x} \right] &= \int_0^\infty \frac{1}{1 + \beta t} f_a^x(t) dt = \int_0^\infty \int_0^\infty e^{-(1+\beta t)s} f_a^x(t) ds dt \\ &= \int_0^\infty e^{-s} \left(\int_0^\infty e^{-\beta s t} f_a^x(t) dt \right) ds = \int_0^\infty e^{-s} \mathbb{E}_{t,x} [e^{-\beta s T_a^x}] ds. \end{aligned} \quad (4.12)$$

Notice that T_a^x is the first hitting time to a by an one-dimensional Bessel process. We thus compute its Laplace transform from the formula in [6]:

$$\mathbb{E}_{t,x} \left[e^{-\frac{\lambda^2}{2} T_a^x} \right] = \frac{\sqrt{x} I_{-\frac{1}{2}}(x\lambda)}{\sqrt{a} I_{-\frac{1}{2}}(a\lambda)} = \cosh(x\lambda) \operatorname{sech}(a\lambda), \quad \text{for } x \leq a.$$

Here, I_ν denotes the modified Bessel function of the first kind. Thanks to the above formula with $\lambda = \sqrt{2\beta s}$, we obtain from (4.12) that

$$\eta(x, a) = a \int_0^\infty e^{-s} \cosh(x\sqrt{2\beta s}) \operatorname{sech}(a\sqrt{2\beta s}) ds. \quad (4.13)$$

It is then obvious that $x \mapsto \eta(x, a)$ is strictly increasing. Moreover,

$$\eta_{xx}(x, a) = 2a\beta^2 \int_0^\infty e^{-s} s \cosh(x\sqrt{2\beta s}) \operatorname{sech}(a\sqrt{2\beta s}) ds > 0 \quad \text{for } x \in [0, a],$$

which shows the strict convexity.

(ii) By using (4.13) and the dominated convergence theorem, we get the desired convergence.

(iii) The proof here divides into two parts. We will first prove the desired result with $x^*(a) \in (0, a)$, and then upgrade it to $x^*(a) \in (0, a^*)$ in the second step. Fix $a \geq 0$. In view of the properties in part (i), we observe that the two curves $y = \eta(x, a)$ and $y = x$ intersect at some $x^*(a) \in (0, a)$ if and only if $\eta_x(a, a) > 1$. Define $k(a) := \eta_x(a, a)$. From (4.13), we have

$$k(a) = a \int_0^\infty e^{-s} \sqrt{2\beta s} \tanh(a\sqrt{2\beta s}) ds.$$

Thus, we see that $k(0) = 0$ and $k(a)$ is strictly increasing on $(0, 1)$, since for any $a > 0$,

$$k'(a) = \int_0^\infty e^{-s} \sqrt{2s} \left(\tanh(a\sqrt{2s}) + \frac{a\sqrt{2s}}{\cosh^2(a\sqrt{2s})} \right) ds > 0.$$

By numerical computation, we find

$$k(1/\sqrt{\beta}) = \int_0^\infty e^{-s} \sqrt{2s} \tanh(\sqrt{2s}) ds \approx 1.07461 > 1.$$

It follows that there must exist $a^* \in (0, 1/\sqrt{\beta})$ such that $k(a^*) = \eta_x(a^*, a^*) = 1$. Monotonicity of $k(a)$ then gives the desired result.

Now, for any $a > a^*$, we intend to upgrade the previous result to $x^*(a) \in (0, a^*)$. Fix $x \geq 0$. By the definition of η and part (ii), on the domain $a \in [x, \infty)$, the map $a \mapsto \eta(x, a)$ must either first increase and then decrease to 0, or directly decrease down to 0. From (4.13), we have

$$\eta_a(x, x) = 1 - x \int_0^\infty e^{-s} \sqrt{2\beta s} \tanh(x\sqrt{2\beta s}) ds = 1 - k(x),$$

where k is defined in part (iii). Recalling that $k(a^*) = 1$, we have $\eta_a(a^*, a^*) = 0$. Notice that

$$\begin{aligned} \eta_{aa}(a^*, a^*) &= -\frac{2}{a^*} k(a^*) - 2\beta a^* + a^* \int_0^\infty 4\beta s e^{-s} \tanh^2(a^* \sqrt{2\beta s}) ds \\ &\leq -\frac{2}{a^*} + 2\beta a^* < 0, \end{aligned}$$

where the second line follows from $\tanh(x) \leq 1$ for $x \geq 0$ and $a^* \in (0, 1/\sqrt{\beta})$. Since $\eta_a(a^*, a^*) = 0$ and $\eta_{aa}(a^*, a^*) < 0$, we conclude that on the domain $a \in [a^*, \infty)$, the map $a \mapsto \eta(a^*, a)$ decreases down to 0. Now, for any $a > a^*$, since $\eta(a^*, a) < \eta(a^*, a^*) = a^*$, we must have $x^*(a) < a^*$. \square

Remark 4.2. To find $a^* \in (0, 1)$ in Lemma 4.1 (ii), we need to solve in $0 < a < 1$

$$(\eta^a)'(a) = a \int_0^\infty e^{-s} \sqrt{2\beta s} \tanh(a\sqrt{2\beta s}) ds = 1$$

for any $\beta > 0$. One however notice that similarly to the proof of Lemma 4.1, one can change a for $a\sqrt{\beta}$ in the above equation. Using $\beta = 1$, numerical computation shows that the general solution is $a^* \approx 0.946475/\sqrt{\beta} < 1/\sqrt{\beta}$.

A numerical illustration of Lemma 4.1.(iii) is given by taking $a = 1/\sqrt{\beta}$. To find $x^*(a)$, we need to solve for x

$$\eta^{1/\sqrt{\beta}}(x) = \frac{1}{\sqrt{\beta}} \int_0^\infty e^{-s} \cosh(x\sqrt{2\beta s}) \operatorname{sech}(\sqrt{2s}) ds = x.$$

for a given β . Similarly as above, we search for $x\sqrt{\beta}$. Numerical computations for $\beta = 1$ show that the general solution is $x^* \approx 0.92195/\sqrt{\beta}$. Accordingly, $x^*(1/\sqrt{\beta}) < a^*$, for any $\beta > 0$.

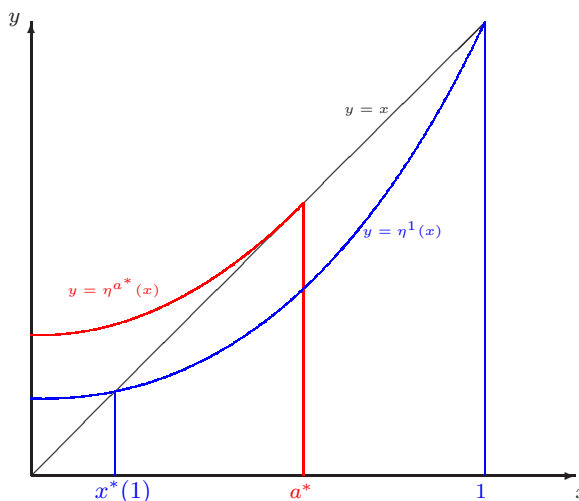


FIG 1. Illustration of properties of η^a for $a = a^*$ (red) and $a = 1$ (blue), with $\beta = 1$.

The behavior of the above function η can be illustrated by figure 4.2, which discriminates two cases. First observe that $J(t, x; \mathcal{L}^* \tau_a(t, x)) > a$ for $x < a \leq a^*$, and

$$\{(t, x) \in \mathbb{X} : |x| < a\} \subseteq C_{\tau_a}. \tag{4.14}$$

By complementarity of sets in \mathbb{X} , we must have equality in (4.11)-(4.10)-(4.14). Looking back at (4.10), we obtain $I_{\tau_a} \subset R_{\tau_a}$. The operator Θ thus provides

$$\Theta \tau_a(t, x) = \inf \{s \geq t : |X_s^{t,x}| \geq a\} = \tau_a(t, x).$$

We therefore have a straight equilibrium policy by considering low values of a .

Consider now $a > a^*$. Then according to Lemma 4.1, any starting point in absolute value $|x|$ below $x^*(a)$ incites to wait, whereas any starting point between $x^*(a)$ and a will push to anticipate stopping. We thus specify three subsets of $\{(t, x) : |x| < a\}$, given by

$$\begin{cases} S^* := \{(t, x) \in \mathbb{X} : x^*(a) < |x| < a\} \\ I^* := \{(t, x) \in \mathbb{X} : |x| = x^*(a)\} \\ C^* := \{(t, x) \in \mathbb{X} : |x| < x^*(a)\} \end{cases}.$$

Altogether,

$$\Theta\tau_a(t, x) = \inf \{s \geq t : |X_s^{t,x}| > x^*(a)\}$$

since at $|x| = x^*(a)$, we supposedly use $\mathcal{L}\tau_a(t, x) = \tau_a(t, x) > t$. At the other singular point $|x| = a$, we are on S_{τ_a} . Therefore, $\tau_a \notin \mathcal{E}(\mathbb{X})$, and $\Theta\tau_a = \mathcal{L}^*\tau_{x^*(a)}$. Notice then that $x^*(a) < a$ implies that assumption (3.6) holds here, so that Theorem 3.1 can be applied. Now from Lemma 4.1, $x^*(a) < a^*$, so that after this round, we fall on the first case, and

$$\Theta\tau_a(t, x) = \Theta\mathcal{L}^*\tau_{x^*(a)}(t, x) = \tau_{x^*(a)}(t, x), \quad \forall (t, x) \in \mathbb{X}.$$

Notice now that the whole reasoning above does not depend on the value of β . We summarize these findings in the following proposition.

Proposition 4.2. *Let $a \in (0, 1/\sqrt{\beta}]$. Then*

$$\widehat{\tau}_a(t, x) := \lim_n \Theta^n \tau_a(t, x), \quad \forall (t, x) \in \mathbb{X},$$

is an equilibrium policy of the form τ_b for $b \in (0, a^]$ where $a^* \propto \beta^{-1/2}$.*

According to Remark 4.1, it is clear that if a stopping region H is a Lebesgue set of \mathbb{R}_+ for some stopping policy, it will be improved by taking τ_a with $a = \inf H$, as a consequence of applying Θ . Therefore, we exhaust equilibrium policies:

$$\mathcal{E}(\mathbb{X}) = \{\tau \in \mathcal{T}(\mathbb{X}) : \tau = \tau_a \text{ for } a \in (0, a^*]\}. \quad (4.15)$$

Remark 4.3. *Notice that if the property $x^*(a) < a^*$ for $a \in (0, 1/\sqrt{\beta})$ did not hold, it would still be possible to show that equilibrium policies are given by (4.15). It suffices to apply the iterative construction principle of Section 3.2.*

The probability that the Brownian motion $X_{\tau_{x^(a)}}^{t,x}$ reflects on the boundary $x^*(a)$ is null for $x < x^*(a)$, and therefore*

$$\tau_{x^*(a)}(t, x) = \mathcal{L}^*\tau_{x^*(a)}(t, x) \mathbb{P} - a.s., \quad \forall x < x^*(a).$$

We can thus apply Lemma 4.1, with $x^(a)$ instead of a .*

Applying the above reasoning recursively, we can set $a_0 = a$, and $a_{n+1} = x^(a_n)$ for $n \geq 0$, in order to obtain a sequence of stopping policies $\tau_{a_n} \in \mathcal{T}(\mathbb{X})$:*

$$\tau_{a_{n+1}} := \Theta\tau_{a_n}.$$

Assume that the sequence $(a_n)_{n \geq 0}$ is bounded from below by a^* . Since it is decreasing, it converges to some $a_\infty \geq a^*$. The function η_a in Lemma 4.1 being \mathcal{C}^2 in x and parameter a on $(a^*, 1)$, the function x^* is continuous and

$$x^*(a_\infty) = a_\infty,$$

which according to Lemma 4.1 again, implies that $a_\infty \leq a^*$. The convergence must then be toward a^* . Another possible situation is that there exists $N \geq 0$ such that $a_N \leq a^*$. Then according to Lemma 4.1,

$$\Theta\tau_{a_N}(t; x) = \inf \{s \geq t : |X_s^{t,x}| \geq a_N\} = \tau_{a_N}(t, x)$$

and we stop the sequence which converges to that equilibrium policy.

4.3. Defining a new class of stopping problems

Let $(t, x) \in \mathbb{X}$. Recall Problem (3.8), which is to

$$\text{find } \tau^* \in \mathcal{E}(\mathbb{X}) \text{ such that } J(t, x; \tau^*(t, x)) = \sup_{\tau \in \mathcal{E}(\mathbb{X})} J(t, x; \tau(t, x)). \quad (4.16)$$

According to the development above,

$$\sup_{\tau \in \mathcal{E}(\mathbb{X})} \mathbb{E}_{t,x} \left[\frac{|X_{\tau(t,x)}^{t,x}|}{1 + \beta(\tau(t, x) - t)} \right] = \sup_{a \in [x, a^* \vee x]} \mathbb{E}_{t,x} \left[\frac{a}{1 + \beta(\tau_a(t, x) - t)} \right]. \quad (4.17)$$

We now wonder if this problem makes sense. The following Lemma shows that the present problem is sufficiently nice to possess a good solution: τ_{a^*} , for all $(t, x) \in \mathbb{X}$.

Lemma 4.2. *Given $t \in [0, \infty)$ and $0 \leq x \leq a < a^*$,*

$$\mathbb{E}_{t,x} \left[\frac{a^*}{1 + \beta(\tau_{a^*}(t, x) - t)} \right] > \mathbb{E}_{t,x} \left[\frac{a}{1 + \beta(\tau_a(t, x) - t)} \right].$$

Proof. Since $x \leq a \leq a^*$, $\tau_a(t, x) \leq \tau_{a^*}(t, x)$ \mathbb{P} -a.s. We then have

$$\begin{aligned} \mathbb{E}_{t,x} \left[\frac{a^*}{1 + \beta(\tau_{a^*}(t, x) - t)} \right] &\geq \mathbb{E}_{t,x} \left[\frac{1}{1 + \beta(\tau_a(t, x) - t)} \frac{a^*}{1 + \beta(\tau_{a^*}(t, x) - t)} \right] \\ &= \mathbb{E}_{t,x} \left[\frac{1}{1 + \beta(\tau_a(t, x) - t)} \mathbb{E}_{\tau_a(t,x), a} \left[\frac{a^*}{1 + \beta(\tau_{a^*}(\tau_a(t, x), a) - \tau_a(t, x))} \right] \right] \\ &> \mathbb{E}_{t,x} \left[\frac{a}{1 + \beta(\tau_a(t, x) - t)} \right], \end{aligned}$$

where the last inequality follows from Lemma 4.1. \square

Since Lemma 4.2 holds for arbitrary $(t, x) \in \mathbb{X}$ with $x < a^*$, we realize that $\tau_{a^*} \in \mathcal{E}(\mathbb{X})$ solves (4.17) for all $(t, x) \in \mathbb{X}$. Problem (4.16) is then *a posteriori* dynamically consistent: committing to the equilibrium policy τ_{a^*} , chosen to be the optimal at some initial state (t, x) , remains optimal in $\mathcal{E}(\mathbb{X})$ when the Brownian motion evolves. It is not only a time-consistent policy, but maximizes each future self's expected discounted payoff as well.

Appendix A: Proof of Theorem 3.1

Detailed analysis will be carried out to show that the proposed procedure in Section 3.1 indeed leads to equilibrium policies. We start with a regularity property of the operator Θ in the diffusion case. We then use it in a core proposition asserting the monotonicity of a special operator involving Θ on stopping policy. By noticing the boundedness from below of policies, we can then construct the limit of iterative application of that operator. We then conclude that the limit is a fixed point of Θ , and thus an equilibrium policy.

Lemma A.1. *Given $\tau \in \mathcal{T}(\mathbb{X})$, we denote $\tau^*(s) := \mathcal{L}^*\tau(s, X_s)$, $\tau_1(s) := \Theta\tau(s, X_s)$, and $\tau_1^*(s) := \mathcal{L}^*\tau_1(s)$. Assume that $\tau_1^*(t) \leq \tau^*(t)$ \mathbb{P} -a.s. for any $(t, x) \in \mathbb{X}$. Then for $(t, x) \in \mathbb{X}$, on the set $\{t < \tau_1^*(t)\}$, we have*

$$g(X_{\tau_1^*(t)}^{t,x}) \leq \mathbb{E}_{\tau_1^*(t), X_{\tau_1^*(t)}^{t,x}} \left[\delta(\tau^*(\tau_1^*(t)) - \tau_1^*(t)) g \left(X_{\tau^*(\tau_1^*(t))}^{t,x} \right) \right].$$

Proof. On $\{t < \tau_1^*(t)\}$, we have $\tau_1(s) > s$ for any $t < s < \tau_1^*(t)$. This implies that for such s , $(s, X_s^{t,x}) \in S_\tau \cup I_\tau$, and together with monotonicity of δ and non-negativity of g , we have

$$\begin{aligned} g(X_s^{t,x}) &\leq \mathbb{E}_{s, X_s^{t,x}} \left[\delta(\tau^*(s) - s) g \left(X_{\tau^*(s)}^{t,x} \right) \right] \\ &\leq \mathbb{E}_{s, X_s^{t,x}} \left[\delta(\tau^*(s) - \tau_1^*(s)) g \left(X_{\tau^*(s)}^{t,x} \right) \right] \\ &\leq \mathbb{E} \left[\delta(\tau^*(t) - \tau_1^*(t)) g \left(X_{\tau^*(t)}^{t,x} \right) \mid \mathcal{F}_s \right] \end{aligned} \quad (\text{A.1})$$

The third line follows from the strong Markov property (2.1). Now, let us denote $Y := \delta(\tau^*(t) - \tau_1^*(t)) g \left(X_{\tau^*(t)}^{t,x} \right)$. We claim that the process $s \mapsto \mathbb{E}[Y \mid \mathcal{F}_s]$ is left-continuous, if we select a version of $\mathbb{E}[Y \mid \mathcal{F}_s]$ appropriately for all $s > 0$. Let \hat{M} be a martingale with \hat{M}_s defined as an arbitrary version of $\mathbb{E}[Y \mid \mathcal{F}_s]$ for all $s \geq 0$. By [21, Proposition 1.3.14], there exists $\Omega^* \in \mathcal{F}$ with $\mathbb{P}(\Omega^*) = 1$ such that for any $s > 0$,

$$M_s(\omega) := \lim_{r \uparrow s, r \in \mathbb{Q}} \hat{M}_r(\omega) \quad \text{exists} \quad \forall \omega \in \Omega^*.$$

Recall that the continuity of X implies that the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is not only right-continuous, but continuous; see e.g. [20, Corollary 2.7.8]. Thanks to Levy's upward theorem (see e.g. [40, Theorem 14.2]) and the left-continuity of \mathbb{F} , M_s is again a version of $\mathbb{E}[Y \mid \mathcal{F}_t]$ for all $s > 0$. Since the process M is by definition left-continuous, the claim is proved. Given $\omega \in \Omega^* \cap \{t < \tau_1^*(t)\}$, take $\{k_n\} \subset \mathbb{R}_+$ such that $k_n \uparrow \tau_1^*(t)(\omega)$. By (A.1), we have

$$g(X_{k_n}^{t,x}(\omega)) \leq M_{k_n}(\omega) \quad \forall n \in \mathbb{N}.$$

Sending n to infinity gives

$$\begin{aligned} g(X_{\tau_1^*(t)}^{t,x}(\omega)) &\leq M_{\tau_1^*(t)}(\omega) \\ &= \mathbb{E}_{\tau_1^*(t)(\omega), X_{\tau_1^*(t)}^{t,x}(\omega)} \left[\delta(\tau^*(\tau_1^*(t)) - \tau_1^*(t)) g\left(X_{\tau^*(\tau_1^*(t))}^{t,x}\right) \right], \end{aligned}$$

where the last equality follows from the strong Markov property (2.1) again. \square

Proposition A.1. *Let $\tau \in \mathcal{T}(\mathbb{X})$ satisfy (3.6). Then*

$$\mathcal{L}^* \Theta \tau(t, x) \leq \mathcal{L}^* \Theta \tau(t, x) \mathbb{P} - a.s., \quad \forall (t, x) \in \mathbb{X}.$$

Proof. Let us denote $\tau^*(s) := \mathcal{L}^* \tau(s, X_s)$, $\tau_1(s) := \Theta \tau(s, X_s)$, $\tau_1^*(s) := \mathcal{L}^* \Theta \tau(s, X_s)$, $\tau_2(s) := \Theta \tau_1(s)$ and $\tau_2^*(s) := \mathcal{L}^* \tau_2(s)$. Recall assumption (3.6):

$$\tau_1^*(t) \leq \tau^*(t) \mathbb{P} - a.s. \quad \forall (t, x) \in \mathbb{X}.$$

We want to show that $\tau_2^*(t) \leq \tau_1^*(t) \mathbb{P} - a.s.$ for all $(t, x) \in \mathbb{X}$. Fix $(t, x) \in \mathbb{X}$. We will consider three distinct cases which, following (3.6), cover Ω . As in the proof of Lemma 3.1, we assume without loss of generality that each case is of probability one, and treat the general case by conditioning.

1. Denote for sake of simplicity $\tau_{11}^* := \tau_1^*(\tau_1^*(t))$ and $\tau_{01}^* := \tau^*(\tau_1^*(t))$ in this part of the proof. Assume $\mathbb{P}[t < \tau_1^*(t) < \tau_{11}^*] = 1$. Suppose that there exists a decreasing sequence of stopping times $(t_n)_{n \in \mathbb{N}} \subset \mathcal{T}_{\tau_1}$ such that

$$\lim_n t_n = \tau_1^*(t) \mathbb{P} - a.s. \quad \text{and} \quad \tau_1(t_n) = t_n \mathbb{P} - a.s. \quad \forall n \in \mathbb{N}.$$

Then one must have $\tau_1^*(\tau_1^*(t)) = \tau_1^*(t)$, which is not possible. Therefore, since $t < \tau_1^*(t)$ we have

$$\tau_1(\tau_1^*(t)) = \tau_1^*(t) \mathbb{P} - a.s. \quad (\text{A.2})$$

Since $\tau_1 = \Theta \tau$, (A.2) implies

$$g\left(X_{\tau_1^*(t)}^{t,x}\right) \geq \mathbb{E}_{\tau_1^*(t), X_{\tau_1^*(t)}^{t,x}} \left[\delta\left(\tau_{01}^* - \tau_1^*(t)\right) g\left(X_{\tau_{01}^*}^{t,x}\right) \right]. \quad (\text{A.3})$$

Recall from (3.6) that $\tau_{11}^* \leq \tau_{01}^* \mathbb{P} - a.s.$ According to (2.3),

$$\delta(\tau_{01}^* - \tau_1^*(t)) \geq \delta(\tau_{11}^* - \tau_1^*(t)) \delta(\tau_{01}^* - \tau_{11}^*). \quad (\text{A.4})$$

According to (2.1), (A.4) and non-negativity of g , inequality (A.3) becomes

$$g(X_{\tau_1^*(t)}^{t,x}) \geq \mathbb{E} \left[\delta(\tau_{11}^* - \tau_1^*(t)) \mathbb{E} \left[\delta(\tau_{01}^* - \tau_{11}^*) g(X_{\tau_{01}^*}^{t,x}) \mid \mathcal{F}_{\tau_{11}^*} \right] \mid \mathcal{F}_{\tau_1^*(t)} \right] \quad (\text{A.5})$$

with probability one. Now according to Lemma A.1, if $\tau_{11}^* < \tau_{01}^*$,

$$g(X_{\tau_{11}^*}^{t,x}) \leq \mathbb{E} \left[\delta(\tau_{01}^* - \tau_{11}^*) g(X_{\tau_{01}^*}^{t,x}) \mid \mathcal{F}_{\tau_{11}^*} \right],$$

and if $\tau_{11}^* = \tau_{01}^*$, the above inequality holds trivially (as an equality). Altogether, these inequalities imply

$$g(X_{\tau_1^*(t)}^{t,x}) \geq \mathbb{E} \left[\delta(\tau_{11}^* - \tau_1^*(t))g(X_{\tau_1^*(t)}^{t,x}) \mid \mathcal{F}_{\tau_1^*(t)} \right] \mathbb{P} - \text{a.s.}$$

which implies that $(\tau_1^*(t), X_{\tau_1^*(t)}^{t,x}) \notin C_{\tau_1}$. Consequently,

$$\tau_2(\tau_1^*(t)) = \begin{cases} \tau_1^*(t) & \text{for } (\tau_1^*(t), X_{\tau_1^*(t)}^{t,x}) \in S_{\tau_1} \\ \mathcal{L}\tau_1(\tau_1^*(t)) & \text{for } (\tau_1^*(t), X_{\tau_1^*(t)}^{t,x}) \in I_{\tau_1} \end{cases} = \tau_1^*(t) \mathbb{P} - \text{a.s.}$$

It follows that $\tau_2^*(t) \leq \tau_1^*(t)$.

2. Assume now that $\mathbb{P}[t < \tau_1^*(t) = \tau_1^*(\tau_1^*(t))] = 1$. Accordingly, there exists a decreasing sequence of stopping times $(t_n)_{n \in \mathbb{N}}$ such that

$$\lim_n t_n = \tau_1^*(t) \mathbb{P} - \text{a.s.} \quad \text{and} \quad \tau_1(t_n) = t_n \mathbb{P} - \text{a.s.} \quad \forall n \in \mathbb{N}.$$

We claim that $\tau_2^*(t) \leq t_n$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Note that if $\tau_1^*(t_n) = t_n$ on some subset $A \subseteq \Omega$, then straightforwardly $(t_n, X_{t_n}^{t,x}) \in I_{\tau_1}$ and $\mathcal{L}\tau_1(t_n) = t_n$. Since $\tau_2 = \Theta\tau_1$,

$$\tau_2(t_n) = \mathcal{L}\tau_1(t_n) = t_n \text{ on } A.$$

This implies that $\tau_2^*(t) \leq t_n$ on A . Now on the set A^c , since $\tau_1^*(t_n) > t_n$, we can argue as in step 1 above, with t_n replacing $\tau_1^*(t)$, to show that $\tau_2^*(t) \leq t_n$ on A^c . Sending n to infinity provides

$$\tau_2^*(t) \leq \tau_1^*(t) \mathbb{P} - \text{a.s.}$$

3. Assume that $\mathbb{P}[\tau_1^*(t) = t] = 1$. Then one can argue as in step 2, with t replacing $\tau_1^*(t)$, by choosing a properly decreasing sequence of stopping times converging to t $\mathbb{P} - \text{a.s.}$ The same reasoning leads to $\tau_2^*(t) \leq \tau_1^*(t) \mathbb{P} - \text{a.s.}$ \square

Corollary A.1. *Let $\tau \in \mathcal{T}(\mathbb{X})$ satisfy (3.6). Assume $\Theta\tau(t, x) = t$ for some $(t, x) \in \mathbb{X}$. Then $\Theta\Theta\tau(t, x) = t$.*

Proof. By (3.4), $\Theta\tau(t, x) = t$ implies that $(t, x) \notin C_\tau$, i.e.

$$g(x) \geq \mathbb{E}_{t,x}[\delta(\mathcal{L}^*\tau(t, x) - t)g(X_{\mathcal{L}^*\tau(t,x)}^{t,x})].$$

If $\mathcal{L}^*\Theta\tau(t, x) = \mathcal{L}^*\tau(t, x)$, the above inequality immediately yields $(t, x) \notin C_{\Theta\tau}$, and thus

$$\Theta\Theta\tau(t, x) = \begin{cases} t & \text{for } (t, x) \in S_{\Theta\tau} \\ \Theta\tau(t, x) & \text{for } (t, x) \in I_{\Theta\tau} \end{cases} = t. \quad (\text{A.6})$$

If $t < \mathcal{L}^*\Theta\tau(t, x) < \mathcal{L}^*\tau(t, x)$, we may argue as in case **1.** in the previous proof of Proposition A.1 to show that

$$g(x) \geq \mathbb{E}_{t,x}[\delta(\mathcal{L}^*\tau(t, x) - t)g(X_{\mathcal{L}^*\tau(t,x)}^{t,x})] \geq \mathbb{E}_{t,x}[\delta(\mathcal{L}^*\Theta\tau(t, x) - t)g(X_{\mathcal{L}^*\Theta\tau(t,x)}^{t,x})].$$

This again gives (A.6). Finally, if $\mathcal{L}^*\Theta\tau(t, x) = t$, then $(t, x) \in I_{\Theta\tau}$ trivially, and thus $\Theta\Theta\tau(t, x) = \Theta\tau(t, x) = t$. \square

Remark A.1. The above Corollary applies to the immediate stopping policy ι defined by

$$\iota(t, x) = t, \quad \forall (t, x) \in \mathbb{X}.$$

Indeed, since $\iota \in \mathcal{E}(\mathbb{X})$, we have equality in Proposition A.1 as soon as $\tau(t, x) = \iota(t, x)$ for some $(t, x) \in \mathbb{X}$. This is of course an irrelevant equilibrium policy, but it ensures a lower bound for convergence of iterative applications of Θ .

Corollary A.2. Let $\tau \in \mathcal{T}(\mathbb{X})$ satisfy (3.6). Define

$$\hat{\tau}(t, x) := \lim_n \mathcal{L}\Theta^n \tau(t, x), \quad \forall (t, x) \in \mathbb{X}.$$

Then $\hat{\tau} \in \mathcal{T}(\mathbb{X})$ and $\mathcal{L}^* \hat{\tau}(t, x) = \lim_n \mathcal{L}^* \Theta^n \tau(t, x)$ \mathbb{P} -a.s. for all $(t, x) \in \mathbb{X}$. Moreover, if $\Theta^n \tau(t, x) > t$ for all $n \geq 1$, then $\hat{\tau}(t, x) = \mathcal{L}^* \hat{\tau}(t, x)$.

Proof. For the sake of clarity, denote $\tau_n := \Theta^n \tau \in \mathcal{T}(\mathbb{X})$ for $n \in \mathbb{N}$. Fix $(t, x) \in \mathbb{X}$. If $\tau_n(t, x) = t$ for some $n \in \mathbb{N}$, then by Corollary A.1, $\hat{\tau}(t, x) = t$. If $\tau_n(t, x) > t$ for all $n \in \mathbb{N}$, then $\mathcal{L}\Theta^n \tau(t, x) = \mathcal{L}^* \Theta^n \tau(t, x)$ for all $n \in \mathbb{N}$ and according to Proposition A.1,

$$\hat{\tau}(t, x) = \lim_n \mathcal{L}^* \Theta^n \tau(t, x) \text{ converges.}$$

Thus, $\hat{\tau}$ is well-defined and belongs to $\mathcal{T}(\mathbb{X})$. Now, let us denote

$$\hat{\tau}^*(t, x) := \lim_n \mathcal{L}^* \Theta^n \tau(t, x).$$

Being a \mathbb{F} -stopping time, $\hat{\tau}^*(t, x)$ must fall under the next two cases in \mathcal{F}_t .

1. Assume that $\hat{\tau}^*(t, x) = t$. We claim that there exists a decreasing sequence of stopping times $(t_n)_{n \in \mathbb{N}}$ converging to t \mathbb{P} -a.s., such that $\tau_n(t_n, X_{t_n}^{t, x}) = t_n$ \mathbb{P} -a.s.

First, suppose that $\mathcal{L}^* \tau_n(t, x) = t$ for n large enough, and take $n = 1$ without loss of generality. It thus hold for all $n \in \mathbb{N}$. For $n = 1$, we can find a decreasing sequence of stopping times $(t_{1, k})_{k \in \mathbb{N}}$ such that $\tau_1(t_{1, k}, X_{t_{1, k}}^{t, x}) = t_{1, k}$ \mathbb{P} -a.s. for all $k \in \mathbb{N}$. Similarly for any $n \geq 1$, we can find a decreasing sequence $(t_{n, k})_{k \in \mathbb{N}}$, additionally verifying $t_{n, 1} < t_{1, n-1}$ \mathbb{P} -a.s., such that $\tau_n(t_{n, k}, X_{t_{n, k}}^{t, x}) = t_{n, k}$ \mathbb{P} -a.s. for all $k \in \mathbb{N}$. The claim then follows by taking $t_n := t_{n, 1}$ for all $n \in \mathbb{N}$.

Next, assume that $\mathcal{L}^* \tau_n(t, x) > t$ for all $n \in \mathbb{N}$. For any $n \geq 1$, this implies that either

$$\tau_n(\mathcal{L}^* \tau_n(t, x), X_{\mathcal{L}^* \tau_n(t, x)}^{t, x}) = \mathcal{L}^* \tau_n(t, x) \text{ } \mathbb{P}\text{-a.s.}$$

or there exists a decreasing sequence of stopping times $(t_{n, k})_{k \in \mathbb{N}}$, converging to $\mathcal{L}^* \tau_n(t, x)$ \mathbb{P} -a.s. as above, verifying $t_{n, 1} < t_{1, n-1}$ \mathbb{P} -a.s. and $\tau_n(t_{n, k}, X_{t_{n, k}}^{t, x}) = t_{n, k}$ \mathbb{P} -a.s. for all $k \in \mathbb{N}$. The claim then follows by taking $t_n := \mathcal{L}^* \tau_n(t, x)$ in the first case or $t_n := t_{n, 1}$ in the second case.

By the claim and Corollary A.1, $\hat{\tau}(t_n, X_{t_n}^{t, x}) = t_n$ for all $n \in \mathbb{N}$, which gives $\mathcal{L}^* \hat{\tau}(t, x) = t$. Moreover, the claim implies that $\hat{\tau}(t, x) = \lim_n \mathcal{L} \tau_n(t, x) \leq \lim_n t_n = t$. It follows that $\hat{\tau}(t, x) = \mathcal{L}^* \hat{\tau}(t, x) = t = \hat{\tau}^*(t, x)$.

2. Assume that $\hat{\tau}^*(t, x) > t$. Similarly to the proof of Proposition A.1, observe that $\tau_n(s, X_s^{t,x}) > s$ and $\mathcal{L}\tau_n(s, X_s^{t,x}) = \mathcal{L}^*\tau_n(s, X_s^{t,x}) = \mathcal{L}^*\tau_n(t, x)$ for all $n \in \mathbb{N}$ and $s \in (t, \hat{\tau}^*(t, x))$ \mathbb{P} -a.s. Then, for the same s ,

$$\hat{\tau}(s, X_s^{t,x}) = \lim_n \mathcal{L}^*\tau_n(s, X_s^{t,x}) = \lim_n \mathcal{L}^*\tau_n(t, x) = \hat{\tau}^*(t, x) > s. \quad (\text{A.7})$$

Now, we claim that $\hat{\tau}(\hat{\tau}^*(t, x), X_{\hat{\tau}^*(t,x)}^{t,x}) = \hat{\tau}^*(t, x)$ \mathbb{P} -a.s.. This, together with (A.7), would imply $\mathcal{L}^*\hat{\tau}(t, x) = \hat{\tau}^*(t, x)$, as desired. Here, we have two cases that, as in preceding proofs, can be assumed to be of probability one separately. By Corollary A.1, the claim above is trivial if

$$\tau_n(\hat{\tau}^*(t, x), X_{\hat{\tau}^*(t,x)}^{t,x}) = \hat{\tau}^*(t, x) \quad \mathbb{P} - \text{a.s.}$$

for some $n \in \mathbb{N}$. We thus assume that

$$\tau_n(\hat{\tau}^*(t, x), X_{\hat{\tau}^*(t,x)}^{t,x}) > \hat{\tau}^*(t, x) \quad \mathbb{P} - \text{a.s.} \quad \text{for all } n \in \mathbb{N}.$$

If $\mathcal{L}^*\tau_n(t, x) = \hat{\tau}^*(t, x)$ \mathbb{P} -a.s. for some $n \in \mathbb{N}$, then, since $\tau_n(s, X_s^{t,x}) > s$ for all $s \in (t, \hat{\tau}^*(t, x)]$ \mathbb{P} -a.s.,

$$\mathcal{L}^*\tau_n(\hat{\tau}^*(t, x), X_{\hat{\tau}^*(t,x)}^{t,x}) = \hat{\tau}^*(t, x) \quad \mathbb{P} - \text{a.s.}$$

It immediately follows that

$$\hat{\tau}(\hat{\tau}^*(t, x), X_{\hat{\tau}^*(t,x)}^{t,x}) = \lim_n \mathcal{L}^*\tau_n(\hat{\tau}^*(t, x), X_{\hat{\tau}^*(t,x)}^{t,x}) = \hat{\tau}^*(t, x) \quad \mathbb{P} - \text{a.s.}$$

If on the contrary, $\mathcal{L}^*\tau_n(t, x) > \hat{\tau}^*(t, x)$ \mathbb{P} -a.s. for all $n \in \mathbb{N}$, then the same calculation in (A.7) directly gives $\hat{\tau}(\hat{\tau}^*(t, x), X_{\hat{\tau}^*(t,x)}^{t,x}) = \hat{\tau}^*(t, x)$. \square

We can now turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. Consider $(\tau_n)_{n \in \mathbb{N}}$ as in the proof of Corollary A.2. Fix $(t, x) \in \mathbb{X}$.

1. Assume that $\tau_{n_0}(t, x) = t$ for some $n_0 \in \mathbb{N}$. Then, following Remark A.1, $\hat{\tau}(t, x) = t$ and thus

$$\Theta\hat{\tau}(t, x) = \begin{cases} t & \text{for } (t, x) \in S_{\hat{\tau}} \\ t & \text{for } (t, x) \in I_{\hat{\tau}} \\ \mathcal{L}^*\hat{\tau}(t, x) & \text{for } (t, x) \in C_{\hat{\tau}} \end{cases}$$

Thanks to Corollary A.2, for any $n \geq n_0$, $\tau_n(t, x) = t$, and thus $(t, x) \notin C_{\tau_n}$. Accordingly

$$g(x) \geq \mathbb{E}_{t,x} \left[\delta(\mathcal{L}^*\tau_n(t, x) - t)g(X_{\mathcal{L}^*\tau_n(t,x)}^{t,x}) \right], \quad \forall n > n_0.$$

Sending n to infinity, and using Corollary A.2 again,

$$g(x) \geq \mathbb{E}_{t,x} \left[\delta(\mathcal{L}^*\hat{\tau}(t, x) - t)g(X_{\mathcal{L}^*\hat{\tau}(t,x)}^{t,x}) \right],$$

which means that $(t, x) \notin C_{\hat{\tau}}$. Consequently, $\Theta\hat{\tau}(t, x) = t = \hat{\tau}(t, x)$.

2. Assume now that $\tau_n(t, x) > t$ for all $n \in \mathbb{N}$. This implies together with Corollary A.2 that

$$\Theta\hat{\tau}(t, x) = \begin{cases} t & \text{for } (t, x) \in S_{\hat{\tau}} \\ \hat{\tau}(t, x) & \text{for } (t, x) \in I_{\hat{\tau}} \\ \hat{\tau}(t, x) & \text{for } (t, x) \in C_{\hat{\tau}} \end{cases}$$

This also implies that, for any $n \in \mathbb{N}$, $(t, x) \notin S_{\tau_n}$, i.e.,

$$g(x) \leq \mathbb{E}_{t,x} \left[\delta(\mathcal{L}^* \tau_n(t, x) - t) g(X_{\mathcal{L}^* \tau_n(t,x)}^{t,x}) \right], \quad \forall n \geq 0. \quad (\text{A.8})$$

Again, by sending n to infinity, with help of Corollary A.2, we obtain that $(t, x) \notin S_{\hat{\tau}}$, which finishes the proof. \square

Acknowledgements

Authors would like to express special gratitude to Traian Pirvu for launching this research project. They would also like to thank Ivar Ekeland, Paolo Guasoni, René Carmona and Ronnie Sircar for fruitful advice and comments.

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