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Mathematical modelling of heat conduction in certain functionally graded composites

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The aim of contribution is to formulate a certain extended version of the tolerance modelling technique for functionally graded composites. For the sake of simplicity, the considerations are restricted to the bidirectionally graded heat conductors. It is shown that the proposed approach enables to determine an entire class of mathematical models for which applications can be found in various specific problems.

1 Object of analysis

An object of analysis is a rigid heat conductor which occupies region Ω in the physical space related to the orthogonal Cartesian coordinate system. The conductor is assumed to be homogeneous in Ox^3 direction and functionally graded in directions of Ox^α axes, $\alpha = 1, 2$.

The cross section Ξ of the composite is divided into large number of cells. Let Λ be a set of cells centers $\mathbf{x} \equiv (x^1, x^2)$ in the region Ξ and the decomposition of an arbitrary cell be carried out into four rectangular homogeneous elements $\Delta_{AB}(\mathbf{x})$, $A, B \in \{1, 2\}$.

The composite is assumed to be made of four orthotropic materials. Properties of those materials are represented by heat conduction coefficients k_{AB}^i , $i = 1, 2, 3$ and specific heat c_{AB} . Functional heat conduction coefficients will be denoted by $k^i(\cdot)$ and $c^i(\cdot)$. The heat conduction in the composite is governed by the constitutive equations for heat flux

$$q^1 = -k^1 \partial_1 \theta, \quad q^2 = -k^2 \partial_2 \theta, \quad q^3 = -k^3 \partial_3 \theta, \quad (1)$$

and by heat balance equation

$$\partial_1 q^1 + \partial_2 q^2 + \partial_3 q^3 + c \partial_t \theta = 0, \quad (2)$$

Obviously $k^i, c \in L^2(\Omega)$, $i = 1, 2, 3$ are piecewise constant functions. Since for a composite structure the number of cell $\Delta(\mathbf{x})$, $\mathbf{x} \in \Lambda$, is very large, the direct application of equations (1), (2) to the analysis of specific problem is not possible. Due to this fact, there must be proposed an approximated mathematical model for heat conduction governed by PDEs with smooth coefficients. In this contributions, there is proposed certain new version of the tolerance averaging technique for functionally graded composites. Foundations of the method can be found in [1].

2 Modelling procedure

Let be given the triple of positive reals $d = (l, \delta_0, \delta_1)$. The function f will be called slowly varying function, $f \in SV_d^1(\Xi)$ if and only if for every $(\mathbf{x}, \mathbf{y}) \in \Xi^2 : |\mathbf{x} - \mathbf{y}| \leq l$ then $|f(\mathbf{x}) - f(\mathbf{y})| \leq \delta_0$ and $|\partial_\alpha f(\mathbf{x}) - \partial_\alpha f(\mathbf{y})| \leq \delta_1$.

Subsequently we shall introduce two subclasses of slowly varying functions:

- 1) $f \in VSV_d^1(\Xi)$ iff $f \in SV_d^1(\Xi)$ and for every $\mathbf{x} \in \Xi : |l \partial_\alpha f(\mathbf{x})| \leq \delta_1$,
- 2) $f \in SSV_d^1(\Xi)$ iff $f \in SV_d^1(\Xi)$ and for every $\mathbf{x} \in \Xi : |\partial_\alpha f(\mathbf{x})| \leq \delta_1$.

The modelling procedure is based on the assumption that the temperature field in every problem under consideration has an approximation of the form

$$\tilde{\theta}(\mathbf{x}, t) = \vartheta(\mathbf{x}, t) + h^1(x^1)\psi_1(\mathbf{x}, t) + h^2(x^2)\psi_2(\mathbf{x}, t) \quad (3)$$

and $k_1' \equiv k_{11}^1 = k_{21}^1$, $k_1'' \equiv k_{12}^1 = k_{22}^1$, $k_2' \equiv k_{11}^2 = k_{12}^2$, $k_2'' \equiv k_{21}^2 = k_{22}^2$.

In (3) h^α are solutions to 1-D "cell problems". Decomposition (3) represents approximate distribution of temperature in terms of slowly-varying functions: macro-temperature ϑ and temperature fluctuation amplitude ψ_α , $\alpha = 1, 2$.

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Setting aside calculations, we shall rather restrict to find system of equation for ϑ and ψ_α . Under denotations $k^{\alpha\beta} = k^\alpha$ if $\alpha = \beta$ and 0 if $\alpha \neq \beta$, $\kappa_\alpha^\beta = \partial_\alpha h^\beta$, we finally obtain

$$\begin{aligned}\partial_\alpha(\langle k^{\alpha\beta} \rangle \partial_\beta \vartheta) + \langle k^3 \rangle \partial_3^2 \vartheta + \partial_\alpha(\langle k^{\alpha\beta} \kappa_\beta^\gamma \rangle \psi_\gamma) - \langle c \rangle \partial_t \vartheta &= 0, \\ \partial_\alpha(\langle k^{\alpha\beta} h^\gamma h^\delta \rangle \partial_\beta \psi_\gamma) + \langle k^3 h^\gamma h^\delta \rangle \partial_3^2 \psi_\gamma - \langle k^{\alpha\beta} \kappa_\beta^\gamma \kappa_\alpha^\delta \rangle \psi_\gamma - \langle k^{\alpha\beta} \kappa_\alpha^\delta \rangle \partial_\beta \vartheta &= 0.\end{aligned}\quad (4)$$

Symbol $\langle \cdot \rangle(\mathbf{x})$ stands for the averaging of an arbitrary function over a cell $\Delta(\mathbf{x})$.

3 Overview of tolerance models

Equations (4) were obtained under assumptions that functions $\theta(\cdot)$, $\psi_\alpha(\cdot)$ are slowly-varying in Ξ without any specification. If $\vartheta, \psi_\alpha \in VSV_d^1(\Xi)$, there can be shown that the tolerance modelling procedure leads to equations

$$\begin{aligned}\partial_\alpha(\langle k^{\alpha\beta} \rangle \partial_\beta \vartheta) + \langle k^3 \rangle \partial_3^2 \vartheta - \langle c \rangle \partial_t \vartheta + \partial_\alpha(\langle k^{\alpha\beta} \kappa_\beta^\gamma \rangle \psi_\gamma) &= 0, \\ \langle k^3 h^\gamma h^\delta \rangle \partial_3^2 \psi_\gamma - \langle k^{\alpha\beta} \kappa_\beta^\gamma \kappa_\alpha^\delta \rangle \psi_\gamma - \langle k^{\alpha\beta} \kappa_\alpha^\delta \rangle \partial_\beta \vartheta &= 0.\end{aligned}\quad (5)$$

This form of model equations was obtained independently in recent contribution on tolerance modelling, cf [1] and the paper related. Hence, the equations (5) will be referred to as *a standard model equations*.

Now assume that $\vartheta, \psi_\alpha \in VSV_d^1(\Xi)$, than

$$\begin{aligned}\partial_\alpha(\langle k^{\alpha\beta} \rangle \partial_\beta \vartheta + \langle k^{\alpha\beta} \kappa_\beta^\gamma \rangle \psi_\gamma) + \langle k^3 \rangle \partial_3^2 \vartheta - \langle c \rangle \partial_t \vartheta &= 0, \\ \langle k^{\alpha\beta} \kappa_\beta^\gamma \kappa_\alpha^\delta \rangle \psi_\gamma = - \langle k^{\alpha\beta} \kappa_\alpha^\delta \rangle \partial_\beta \vartheta.\end{aligned}$$

The characteristic feature of this model (*locally homogenized model*) is that it has the same form as the model obtained by the asymptotic modelling procedure.

Under assumption $\vartheta, \psi_\alpha \in SSV_d^1(\Xi)$ the modelling procedure leads to two independent equations which are coupled only by boundary conditions:

$$\begin{aligned}\langle k^3 \rangle \partial_3^2 \vartheta - \langle c \rangle \partial_t \vartheta &= 0, \\ \langle k^3 h^\gamma h^\delta \rangle \partial_3^2 \psi_\gamma - \langle k^{\alpha\beta} \kappa_\beta^\gamma \kappa_\alpha^\delta \rangle \psi_\gamma &= 0.\end{aligned}$$

It can be observed that the equations for ψ_α describe a phenomenon which can be called *a boundary-layer effect*, [2].

Under assumption $\vartheta \in VSV_d^1(\Omega)$, $\psi_\alpha \in SSV_d^1(\Xi)$ we obtain equations which will be called *a simple boundary-layer model* given by

$$\begin{aligned}\partial_\alpha(k_0^{\alpha\beta} \partial_\beta \vartheta) + \langle k^3 \rangle \partial_3^2 \vartheta - \langle c \rangle \partial_t \vartheta &= 0, \\ \langle k^3 h^\gamma h^\delta \rangle \partial_3^2 \psi_\gamma - \langle k^{\alpha\beta} \kappa_\beta^\gamma \kappa_\alpha^\delta \rangle \psi_\gamma &= 0, \quad \delta = 1, 2.\end{aligned}$$

4 Final remarks

It can be easy observed that the standard model is able to describe the physical phenomena strictly related to the effect of microstructure length on the overall behavior of the heat conduction in composites under consideration.

On the other hand, locally homogenized model is deprived of this feature. This model determines quantities which can be called effective properties of heat conductor. The fluctuation amplitudes ψ_α in this model are determined only by the spatial gradient of the macro-temperature field θ . In this tolerance approach we obtain a heat conductor as a certain new material. The boundary-layer model has relatively simple form which enables to satisfy boundary conditions which are imposed not only on macro-temperature but also on the whole temperature field including its disturbances.

References

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