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# GEVREY CLASS SMOOTHING EFFECT FOR THE PRANDTL EQUATION

WEI-XI LI, DI WU AND CHAO-JIANG XU

ABSTRACT. In this paper we study Gevrey smoothing effects of the Cauchy problem for Prandtl boundary layer equation. It is a degenerate parabolic type equation. Under the Oleinik's monotonicity assumption, we prove that any local solution in Sobolev space is belong to some Gevrey class with respect to tangential variables at positive time.

## 1. INTRODUCTION

In this work, we study the regularity of solution for the Prandtl equation which is the foundation of the boundary layer theory introduced by Prandtl in 1904, [20]. It describes the behavior of the flow near a boundary in the small viscosity limit of an incompressible viscous flow with the non-slip boundary condition. We consider the following initial-boundary value problem,

$$(1.1) \quad \begin{cases} u_t + uu_x + vu_y + p_x = u_{yy}, & t > 0, \quad x \in \mathbb{R}, \quad y > 0, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = U(t, x), \\ u|_{t=0} = u_0(x, y), \end{cases}$$

where  $u(t, x, y)$  and  $v(t, x, y)$  represent the tangential and normal velocities of the boundary layer, with  $y$  being the scaled normal variable to the boundary, while  $U(t, x)$  and  $p(t, x)$  are the values on the boundary of the tangential velocity and pressure of the outflow satisfying the Bernoulli law

$$\partial_t U + U \partial_x U + \partial_x p = 0.$$

The well-posedness theories and the justification of the Prandtl equation remain as the challenging problems in the mathematical theory of fluid mechanics. Up to now, there are only a few rigorous mathematical results. Under a monotonic assumption on the tangential velocity of the outflow, Oleinik was the first to obtain the local existence of classical solutions for the initial-boundary value problems, and this result together with some of her works with collaborators were well presented in the monograph [19]. In addition to Oleinik's monotonicity assumption on the velocity field, by imposing a so called favorable condition on the pressure, Xin and Zhang obtained the existence of global weak solutions to the Prandtl equation in [22]. All these well-posedness results were based on the Crocco transformation to overcome the main difficulty caused by degeneracy and mixed type of the equation. Very recently, the well-posedness in the Sobolev space was explored by virtue of energy method instead of the Crocco transformation; see Alexandre et. al [1] and Masmoudi-Wong [17]. There is very few work concerned with the Prandtl equation without the monotonicity assumption; we refer [16, 21, 23] for the existence in the analytic frame, and [14, 8] for the recent works of concerning with the existence in Gevrey class. Recall Gevrey class, denoted by  $G^s$ ,  $s \geq 1$ , is an intermediate space between analytic functions and  $C^\infty$  space. In the whole space, the Gevrey space  $G^s(\mathbb{R}^d)$  is consist of such functions  $f$  that  $f \in C^\infty(\mathbb{R}^d)$  and that

$$\|\partial^\alpha f\|_{L^2(\mathbb{R}^d)} \leq L^{|\alpha|+1}(\alpha!)^s$$

for some constant  $L$  independent of  $\alpha$ . The significant difference between Gevrey and analytic classes is that there exist nontrivial Gevrey functions admitting compact support.

In the aforementioned works, the well-posedness is investigated for the initial data is in some Gevrey class, for instance  $G^{7/4}$  in [8]. As to be seen, the Prandtl equation admits intrinsic subelliptic structure due to the monotonicity condition. So we may expect Gevrey smoothing effect rather than the propagation; that is, given a general initial data belonging to some Sobolev space, the solution will lie in some Gevrey class at positive time, as like heat equation.

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To clarify our proof we only consider here the case of an uniform outflow  $U = 1$ . Then we get the pressure  $p$  being a constant. The form of Prandtl equation is

$$(1.2) \quad \begin{cases} u_t + uu_x + vu_y - u_{yy} = 0, & t > 0, \quad x \in \mathbb{R}, \quad y > 0, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = 1, \\ u|_{t=0} = u_0(x, y), \end{cases}$$

The main result concerned with the Gevrey class regularity can be stated as follows.

**Theorem 1.1.** *Let  $u^s$  be a given shear flow and  $u(t, x, y)$  be a classical solution to Prandtl equation (1.2) with the properties subsequently listed below:*

(i) *there exist two constants  $C_*, \sigma$  with  $C_* > 1, \sigma > 1/2$  such that for any  $y \geq 0$ ,*

$$C_*^{-1}(1+y)^{-\sigma} \leq \partial_y u(y) \leq C_*(1+y)^{-\sigma}$$

and

$$|\partial_y^2 u(y)| + |\partial_y^3 u(y)| \leq C_*(1+y)^{-\sigma-1};$$

(ii)  $e^{2cy} \partial_x u \in L^\infty([0, T]; H^{N_0}(\mathbb{R} \times \mathbb{R}_+))$  for some  $c > 0$  and some integer  $N_0 \geq 7$ .

Then there exists a constant  $A$ , depending only on  $\|e^{cy} \partial_x u\|_{L^\infty([0, T]; H^{N_0}(\mathbb{R} \times \mathbb{R}_+))}$  and the above constants  $C_*, c$  and  $\sigma$ , such that

$$(1.3) \quad \forall m \geq 1, \quad \|t^{9m/4}(T-t)^{9m/4} e^{\tilde{c}y} \partial_x^m u(t)\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} \leq A^{m+1} (m!)^{3+\sigma},$$

where  $\tilde{c}$  is an arbitrary number such that  $c < \tilde{c}$ .

**Remark 1.2.** *Such a solution in the theorem exists; for instance consider the solution  $u(x, y)$  as a perturbation of a shear flow  $u^s(y)$ . Recall a shear flow  $u^s$  is the solution to the following equation*

$$\begin{cases} \partial_t u^s - \partial_{yy} u^s = 0, & t > 0, \quad x \in \mathbb{R}, \quad y > 0, \\ u^s|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u^s = 1, \\ u|_{t=0} = u_0^s(y), \end{cases}$$

with  $u_0^s$  a given initial data which is a function of  $y$ . Precisely we suppose the initial data  $u_0$  can be written as

$$u_0(x, y) = u_0^s(y) + u_0^\epsilon(x, y),$$

where  $C^{-1} \langle y \rangle^{-\sigma/2} \leq u_0^s(y) \leq C \langle y \rangle^{-\sigma/2}$  for some constant  $C \geq 1$ , and  $u_0^\epsilon$  is a small perturbation such that its weighted Sobolev norm  $\|e^{2cy} u_0^\epsilon\|_{H^{N_0}}$  is suitably small. Then using the arguments in [1], the desired solution in Theorem 1.1 follows.

**Remark 1.3.** *As to be seen, we can get a better estimate rather than (1.3), i.e.,*

$$\forall m \geq 1, \quad \|t^{9m/4} e^{\tilde{c}y} \partial_x^m u(t)\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} \leq A^{m+1} (m!)^{3+\sigma}.$$

## 2. SUBELLIPTIC ESTIMATE

In this section we will explore the subelliptic structure, due to the monotonic condition, for the linear Prandtl operator.

**2.1. Notations.** Throughout the paper, we use the notation

$$\langle \cdot \rangle = \left(1 + |\cdot|^2\right)^{1/2},$$

which is the regularization of the modulus  $|\cdot|$ .

Let  $\xi$  be the dual variable of  $x$ . We denote by  $\Lambda^d$  and  $\Lambda_\delta^d$  the Fourier multipliers associate to symbol  $\langle \xi \rangle^d$ , and  $\left(1 + \delta |\xi|^2\right)^{d/2}$ , respectively, i.e.,

$$\forall f \in \mathcal{S}(\mathbb{R}), \quad (\mathcal{F}_x (\Lambda^d f)) (\xi) = \langle \xi \rangle^d (\mathcal{F}_x f) (\xi)$$

and

$$\forall f \in \mathcal{S}(\mathbb{R}), \quad (\mathcal{F}_x (\Lambda_\delta^d f)) (\xi) = \left(1 + \delta |\xi|^2\right)^{d/2} (\mathcal{F}_x f) (\xi),$$

where  $\mathcal{F}_x$  stands for the Fourier transform in  $x$  variable.

Let  $L^2$  be the classical space of square-integrable function, equipped with an inner product and a norm defined by

$$(f, g)_{L^2} = \int f \bar{g} dz, \quad \|f\|_{L^2} = (f, f)_{L^2}^{1/2},$$

where  $\bar{g}$  is the complex conjugate of  $g$ . And we denote by  $\operatorname{Re} (f, g)_{L^2}$  the real part of  $(f, g)_{L^2}$ , i.e.,

$$\operatorname{Re} (f, g)_{L^2} = \frac{1}{2} \left( (f, g)_{L^2} + (g, f)_{L^2} \right).$$

**2.2. Weyl-Hörmander calculus.** We recall here some notations and basic facts of symbolic calculus, and refer to [13, Chapter 18] or [15] for detailed discussions on the pseudo-differential calculus.

Let  $x$  be the tangent variable and  $\xi$  be its dual variable, and let  $a(x, \xi) \in \mathcal{S}'(\mathbb{R}^2)$  be a tempered distribution and let  $\kappa \in \mathbb{R}$ . We define the operator  $\operatorname{op}_\kappa a : \mathcal{S}(\mathbb{R}_x) \rightarrow \mathcal{S}'(\mathbb{R}_x)$ , with  $\mathcal{S}'(\mathbb{R}_x)$  the antidual of  $\mathcal{S}(\mathbb{R}_x)$  (continuous antilinear forms), by the formula

$$\langle (\operatorname{op}_\kappa a) f, h \rangle_{\mathcal{S}', \mathcal{S}} = \langle a, \Omega_{f, h}(\kappa) \rangle_{\mathcal{S}', \mathcal{S}},$$

where

$$\Omega_{f, h}(\kappa)(x, \xi) = \int_{\mathbb{R}} e^{-2i\pi z \cdot \xi} f(x + (1 - \kappa)z) \bar{h}(x - \kappa z) dz.$$

In particular we denote

$$a(x, D_x) = \operatorname{op}_0 a$$

and

$$a^w = \operatorname{op}_{1/2} a.$$

Here  $a(x, D_x)$  is the classical quantization of symbol  $a$  and  $a^w$  is called the Weyl quantization of symbol  $a$ . Moreover  $a(x, D_x)$  and  $a^w$  can be written as the following oscillatory integral

$$(2.1) \quad (a(x, D_x) f)(x) = \int \int e^{2i\pi(x-\tilde{x})\xi} a(x, \xi) f(\tilde{x}) d\tilde{x} d\xi$$

and

$$(2.2) \quad (a^w f)(x) = \int \int e^{2i\pi(x-\tilde{x})\xi} a\left(\frac{x+\tilde{x}}{2}, \xi\right) f(\tilde{x}) d\tilde{x} d\xi.$$

We have the formula of changing quantization (see Proposition 1.1.10 and Lemma 4.1.2 of [15]):

$$(2.3) \quad a^w = (J^{1/2} a)(x, D_x),$$

where  $J^{1/2}$  is defined by the oscillatory integrals

$$(J^{1/2} a)(x, \xi) = 2 \iint e^{-4i\pi z \cdot \xi} a(x + z, \xi + \zeta) dz d\zeta.$$

**Lemma 2.1.** *Let  $a_1(x) \in \mathcal{S}'(\mathbb{R}^2)$  be a function of  $x$  but independent of  $\xi$ , and  $a_1(\xi) \in \mathcal{S}'(\mathbb{R}^2)$  be a function of  $\xi$  but independent of  $x$ . Then*

- (i)  $a_1^w = a_1(x)$ , i.e.,  $a^w$  is the operator of multiplication by  $a(x)$ ;
- (ii)  $a_2^w = a_2(D_x)$ , and moreover  $a_2(D_x)$  is the Fourier multiplier associated to symbol  $a_2(\xi)$ , i.e.,

$$(\mathcal{F}_x (a_2(D_x) f))(\xi) = a_2(\xi) (\mathcal{F}_x f)(\xi).$$

*Proof.* We first prove Conclusion (ii). In view of (2.1) and (2.2), we have

$$a_2^w f = \int \int e^{2i\pi(x-\tilde{x})\xi} a_2(\xi) f(\tilde{x}) d\tilde{x} d\xi = a_2(x, D_x) f.$$

Moreover observe

$$\int \int e^{2i\pi(x-\tilde{x})\xi} a_2(\xi) f(\tilde{x}) d\tilde{x} d\xi = \int \int e^{2i\pi x \xi} a_2(\xi) (\mathcal{F}_x f)(\xi) d\xi,$$

which implies

$$(\mathcal{F}_x (a_2(D_x) f))(\xi) = a_2(\xi) (\mathcal{F}_x f)(\xi).$$

Then the conclusion (ii) follows.

Now we prove (i). To do so, by (2.3) we have

$$\left(J^{1/2}a_1\right)(x, \xi) = 2 \iint e^{-4i\pi z\zeta} a(x+z) dz d\zeta.$$

Moreover observe

$$2 \iint e^{-4i\pi z\zeta} d\zeta = \delta_0(z)$$

with  $\delta_0$  being the Dirac delta function at origin, thus

$$\left(J^{1/2}a_1\right)(x, \xi) = 2 \iint e^{-4i\pi z\zeta} a_1(x+z) dz d\zeta = a_1(x).$$

Similarly,

$$\left(J^{1/2}a_2\right)(x, \xi) = 2 \iint e^{-4i\pi z\zeta} a_2(\xi+\zeta) dz d\zeta = a_2(\xi).$$

As a result, using (2.1) and (2.3) gives

$$a_1^w f = \int \int e^{2i\pi(x-\tilde{x})\xi} \left(J^{1/2}a_1\right)(x, \xi) f(\tilde{x}) d\tilde{x} d\xi = a_1(x) \int \int e^{2i\pi(x-\tilde{x})\xi} f(\tilde{x}) d\tilde{x} d\xi = a_1(x) f(x),$$

that is  $a_1^w = a_1(x)$ , the the operator of multiplication by  $a_1(x)$ . The proof is thus complete.  $\square$

Let

$$g = |dx|^2 + |d\xi|^2$$

be the flat metric. We denote by  $S(\langle\xi\rangle^r, g)$ ,  $r \in \mathbb{R}$ , the space of symbols  $a(x, \xi)$  satisfying that

$$\forall \alpha, \beta \in \mathbb{Z}_+, \forall x, \xi \in \mathbb{R}, \quad \left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} \langle \xi \rangle^r$$

with  $C_{\alpha, \beta}$  a constant depending only on  $\alpha, \beta$ . The space  $S(\langle\xi\rangle^r, g)$  endowed with the semi-norms

$$(2.4) \quad \|a\|_{k; S(\langle\xi\rangle^r, g)} = \max_{0 \leq \alpha + \beta \leq k} \sup_{(x, \xi) \in \mathbb{R}^2} \langle \xi \rangle^{-r} \left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right|,$$

becomes a Fréchet space.

Given  $a_i \in S(1, g)$  we have

$$(2.5) \quad a_1^w a_2^w = (a_1 \# a_2)^w$$

where  $a_1 \# a_2 \in S(1, g)$  admits the expansion, denoting  $Z = (x, \xi)$ ,  $Z_i = (x_i, \xi_i)$ ,  $i = 1, 2$ ,

$$(2.6) \quad a_1 \# a_2(Z) = a_1 a_2(Z) + \int_0^1 \iint_{\mathbb{R}^4} e^{-2i\mu(Z-Z_1, Z-Z_2)/\theta} \frac{1}{2i} \sigma(\partial_{Z_1}, \partial_{Z_2}) a_1(Z_1) a_2(Z_2) dZ_1 dZ_2 d\theta / (\pi\theta)^2,$$

where  $\mu$  is the symplectic form in  $\mathbb{R}^2$  given by

$$\mu\left((z, \zeta), (\tilde{z}, \tilde{\zeta})\right) = \zeta \cdot \tilde{z} - \tilde{\zeta} \cdot z.$$

We also have good estimate for the the last term in (2.6). Precisely, define  $a_1 \#_\theta a_2$  by

$$(2.7) \quad a_1 \#_\theta a_2(Z) = \iint_{\mathbb{R}^4} e^{-2i\mu(Z-Z_1, Z-Z_2)/\theta} \frac{1}{2i} a_1(Z_1) a_2(Z_2) dY_1 dY_2 / (\pi\theta)^2.$$

An elementary property to be used frequently is the  $L^2$  continuity theorem in the class  $S(1, g)$ , see [15, Theorem 2.5.1] for instance, which says that there exists a constant  $C$  and a positive integer  $N$ , both depending only the dimension of  $x$  (and therefore depending only on the dimension  $n = 1$  here), such that if  $a \in S(1, g)$  then

$$(2.8) \quad \forall u \in L^2(\mathbb{R}), \quad \|a^w u\|_{L^2} \leq C \|a\|_{N; S(1, g)} \|u\|_{L^2}$$

with  $\|a\|_{N; S(1, g)}$  is defined by (2.4).

**From now on we always let  $N$  be the integer given in (2.8), which is a fixed quantity depending only on dimension.**

Let  $N$  the positive integer given in (2.8) which depends only on the dimension, and we denote by  $S_N(1, g)$  the space of functions satisfying

$$\forall |\alpha + \beta| \leq N, \forall x, \xi \in \mathbb{R}, \quad \left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq C_{\alpha, \beta}.$$

Then the proof of [15, Theorem 2.5.1] also implies the following

**Lemma 2.2.** *Let  $N$  the positive integer given in (2.8) which depends only on the dimension of  $x$ . If  $a \in S_N(1, g)$  then  $a^w \in \mathcal{B}(L^2(\mathbb{R}))$ , the space of bounded operators in  $L^2(\mathbb{R})$ . Moreover*

$$(2.9) \quad \forall u \in L^2(\mathbb{R}), \quad \|a^w u\|_{L^2} \leq C \|a\|_{N; S(1, g)} \|u\|_{L^2}.$$

**Proposition 2.3** (cf. Theorem 2.3.7 in [15] and Proposition 1.1 in [2]). *Let  $\sharp$  and  $\sharp_\theta$  be defined by (2.6) and (2.7), respectively, and let  $r_1, r_2$  be two real numbers, and let  $N$  be the integer given in (2.8) which is fixed throughout the paper.*

(i) *There exists a positive integer  $\ell_N$  and a constant  $C_N$ , both depending only on the integer  $N$ , such that if  $a_i \in S_{\ell_N}(\langle \xi \rangle^{r_i}, g)$ ,  $i = 1, 2$ , then*

$$(2.10) \quad \|a_1 \sharp a_2\|_{N; S(\langle \xi \rangle^{r_1+r_2}, g)} \leq C_N \|a_1\|_{\ell_N; S(\langle \xi \rangle^{r_1}, g)} \|a_2\|_{\ell_N; S(\langle \xi \rangle^{r_2}, g)}.$$

(ii) *For any integer  $k \geq 0$ , there exists a positive integer  $\tilde{\ell}_N$  and a constant  $\tilde{C}_k$ , both depending only on  $k$ , such that*

$$\|a_1 \sharp_\theta a_2\|_{k; S(\langle \xi \rangle^{r_1+r_2}, g)} \leq C_N \|a_1\|_{\tilde{\ell}_k; S(\langle \xi \rangle^{r_1}, g)} \|a_2\|_{\tilde{\ell}_k; S(\langle \xi \rangle^{r_2}, g)}.$$

*In particular, letting  $\ell_N$  be the integer in (2.10), we can find an integer  $k_0 = k_0(N)$ , depending only on  $\ell_N$  and thus only on  $N$ , such that*

$$(2.11) \quad \|a_1 \sharp_\theta a_2\|_{\ell_N; S(\langle \xi \rangle^{r_1+r_2}, g)} \leq C_N \|a_1\|_{k_0; S(\langle \xi \rangle^{r_1}, g)} \|a_2\|_{k_0; S(\langle \xi \rangle^{r_2}, g)}.$$

**2.3. Estimate on commutator.** Given two operators  $Q_1$  and  $Q_2$ , the commutator between  $Q_1$  and  $Q_2$ , denoting by  $[Q_1, Q_2]$ , is defined by

$$[Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1.$$

Direct verification shows that

$$(2.12) \quad [Q_1, Q_2 Q_3] = Q_2 [Q_1, Q_3] + [Q_1, Q_2] Q_3.$$

Throughout the paper we always let  $N$  the positive integer given in (2.8) which depends only on the dimension of  $x$  (and therefore depends only on the dimension  $n = 1$  here), and let  $\ell_N$  be the constant given in Proposition 2.7 which depends only on  $N$  and thus depends only the dimension  $n = 1$ . This subsection is devoted to proving the following

**Lemma 2.4.** *Let  $N$  be given in (2.8), and let  $\ell_N, k_0 = k_0(N)$  be the integers given in (2.10) and (2.11), respectively. Assume  $\partial_x p \in S_{k_0}(1, g)$  uniformly with respect to  $t$  and  $y$ ; that is*

$$(2.13) \quad \forall \alpha \leq k_0, \forall x \in \mathbb{R}, \forall t \in \mathbb{R}, \forall y \geq 0, \quad |\partial_x^\alpha (\partial_x p)(t, x, y)| \leq c_\alpha$$

*with  $c_\alpha$  a constant depending only on  $\alpha$ . Let  $\lambda(\xi)$  be function of  $\xi$ , the dual variable of  $x$ , such that  $\partial_\xi \lambda \in S_{k_0}(\langle \xi \rangle^{\tau_1} \langle \delta \xi \rangle^{\tau_2}, g)$  for some  $\tau_1, \tau_2 \in \mathbb{R}$ , i.e.,*

$$(2.14) \quad \forall \beta \leq k_0, \forall \xi \in \mathbb{R}, \quad \left| \partial_\xi^\beta (\partial_\xi \lambda)(\xi) \right| \leq \tilde{c}_\beta \langle \xi \rangle^{\tau_1} \langle \delta \xi \rangle^{\tau_2},$$

*where  $\tilde{c}_\beta$  is a constant depending only on  $\beta$ . Then there exists a constant  $\tilde{C}$ , depending only on the dimension of  $N$  but independent of  $\delta$ , such that*

$$\forall f \in L^2(\mathbb{R}_x), \quad \|[p, \lambda(D_x)]f\|_{L^2(\mathbb{R}_x)} \leq C \|\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} f\|_{L^2(\mathbb{R}_x)},$$

*where  $\Lambda^{\tau_1}, \Lambda_\delta^{\tau_2}$  and  $\lambda(D_x)$  stand for the Fourier multipliers associate to symbols  $\langle \xi \rangle^{\tau_1}, \langle \delta \xi \rangle^{\tau_2}$  and  $\lambda(\xi)$ , respectively.*

*Proof. Step 1).* By Lemma 2.1, we have

$$[p, \lambda(D_x)] = (\langle \xi \rangle^\tau)^w [p^w, \lambda^w] = p^w \lambda^w - \lambda^w p^w = (p \sharp \lambda)^w - (\lambda \sharp p)^w,$$

the last equation using (2.5). Moreover the expansion (2.6) gives

$$(p \sharp \lambda)^w - (\lambda \sharp p)^w = (p \sharp \lambda - \lambda \sharp p)^w = (r_1 - r_2)^w,$$

where, recalling the notation  $Z = (x, \xi)$  and  $Z_i = (x_i, \xi_i), i = 1, 2$ ,

$$r_1(Z) = \int_0^1 \iint_{\mathbb{R}^4} e^{-2i\mu(Z-Z_1, Z-Z_2)/\theta} \frac{1}{2i} \mu(\partial_{Z_1}, \partial_{Z_2}) p(Z_1) \lambda(Z_2) dZ_1 dZ_2 d\theta / (\pi\theta)^2.$$

and

$$r_1(Z) = \int_0^1 \iint_{\mathbb{R}^4} e^{-2i\mu(Z-Z_1, Z-Z_2)/\theta} \frac{1}{2i} \mu(\partial_{Z_1}, \partial_{Z_2}) \lambda(Z_1) p(Z_2) dZ_1 dZ_2 d\theta / (\pi\theta)^2.$$

Thus

$$(2.15) \quad [p, \lambda(D_x)] = (r_1 - r_2)^w.$$

*Step 2).* In this step we will show that

$$r_1, r_2 \in S_{\ell_N}(\langle \xi \rangle^{\tau_1} \langle \delta \xi \rangle^{\tau_2}, g),$$

where  $\ell_N$  is the positive number given in (2.10) which depending only on  $N$ . For this purpose, observe  $p(Z_1) = p(x_1)$  and  $\lambda(Z_2) = \lambda(\xi_2)$ , and thus

$$\begin{aligned} \mu(\partial_{Z_1}, \partial_{Z_2}) p(Z_1) \lambda(Z_2) &= \left( \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial x_2} - \frac{\partial}{\partial \xi_2} \frac{\partial}{\partial x_1} \right) p(x_1) \lambda(\xi_2) \\ &= -(\partial_{x_1} p(x_1)) \partial_{\xi_2} \lambda(\xi_2) \\ &= -(\partial_x p)(Z_1) (\partial_\xi \lambda)(Z_2). \end{aligned}$$

This gives

$$\begin{aligned} r_1(Z) &= - \int_0^1 \iint_{\mathbb{R}^4} e^{-2i\sigma(Z-Z_1, Z-Z_2)/\theta} \frac{1}{2i} (\partial_x p)(Z_1) (\partial_\xi \lambda)(Z_2) dZ_1 dZ_2 d\theta / (\pi\theta)^2 \\ &= - \int_0^1 [(\partial_x p) \#_\theta (\partial_\xi \lambda)](Z) d\theta, \end{aligned}$$

recalling  $(\partial_x p) \#_\theta (\partial_\xi \lambda)$  is defined in (2.7). Similarly,

$$r_2(Z) = \int_0^1 [(\partial_\xi \lambda) \#_\theta (\partial_x p)](Z) d\theta,$$

In view of the assumptions (2.13) and (2.14), we use the estimate (2.11) in Proposition 2.3 to get

$$\|(\partial_x p) \#_\theta (\partial_\xi \lambda)\|_{\ell_N; S(\langle \xi \rangle^{\tau_1} \langle \delta \xi \rangle^{\tau_2}, g)} \leq C_N \|\partial_x p\|_{k_0; S(1, g)} \|\partial_\xi \lambda\|_{k_0; S(\langle \xi \rangle^{\tau_1} \langle \delta \xi \rangle^{\tau_2}, g)} \leq \tilde{C}_N,$$

where  $\tilde{C}_N = C_N (\max_{\alpha \leq k_0} c_\alpha) (\max_{\beta \leq k_0} \tilde{c}_\beta)$  with  $c_\alpha, \tilde{c}_\beta$  be the constants given in (2.13) and (2.14). This implies  $r_1 \in S_{\ell_N}(\langle \xi \rangle^{\tau_1} \langle \delta \xi \rangle^{\tau_2}, g)$  and

$$\|r_1\|_{\ell_N; S(\langle \xi \rangle^{\tau_1} \langle \delta \xi \rangle^{\tau_2}, g)} \leq \int_0^1 \|(\partial_x p) \#_\theta (\partial_\xi \lambda)\|_{\ell_N; S(\langle \xi \rangle^{\tau_1} \langle \delta \xi \rangle^{\tau_2}, g)} d\theta \leq \tilde{C}_N.$$

Similarly,  $r_2 \in S_{\ell_N}(\langle \xi \rangle^{\tau_1} \langle \delta \xi \rangle^{\tau_2}, g)$  and

$$\|r_2\|_{\ell_N; S(\langle \xi \rangle^{\tau_1} \langle \delta \xi \rangle^{\tau_2}, g)} \leq \tilde{C}_N.$$

Thus we have  $r_1 - r_2 \in S_{\ell_N}(\langle \xi \rangle^{\tau_1} \langle \delta \xi \rangle^{\tau_2}, g)$  and

$$(2.16) \quad \|r_1 - r_2\|_{\ell_N; S(\langle \xi \rangle^{\tau_1} \langle \delta \xi \rangle^{\tau_2}, g)} \leq 2\tilde{C}_N,$$

with  $\tilde{C}_N$  a constant independent of  $\delta$ .

*Step 3).* In view of (2.15) we may write

$$[p, \lambda(D_x)] = (r_1 - r_2)^w = ((r_1 - r_2)^w \Lambda_\delta^{-\tau_2} \Lambda^{-\tau_1}) \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} = \left( (r_1 - r_2) \#(\langle \xi \rangle^{-\tau_1} \langle \delta \xi \rangle^{-\tau_2}) \right)^w \Lambda^{\tau_1} \Lambda_\delta^{\tau_2},$$

the last inequality using (2.5). On the other hand it follows from (??) that  $\langle \xi \rangle^{-\tau_1} \langle \delta \xi \rangle^{-\tau_2} \in S(\langle \xi \rangle^{-\tau_1} \langle \delta \xi \rangle^{-\tau_2}, g)$  and moreover

$$\|\langle \xi \rangle^{-\tau_1} \langle \delta \xi \rangle^{-\tau_2}\|_{\ell_N; S(\langle \xi \rangle^{-\tau_1} \langle \delta \xi \rangle^{-\tau_2}, g)} \leq C'_N$$

with  $C'_N$  a constant depending only on  $N$  but independent of  $\delta$ . As a result we use (2.10) in Proposition 2.3 to conclude that,

$$\begin{aligned} &\| (r_1 - r_2) \#(\langle \xi \rangle^{-\tau_1} \langle \delta \xi \rangle^{-\tau_2}) \|_{N; S(1, g)} \\ &\leq \|r_1 - r_2\|_{\ell_N; S(\langle \xi \rangle^{\tau_1} \langle \delta \xi \rangle^{\tau_2}, g)} \|\langle \xi \rangle^{-\tau_1} \langle \delta \xi \rangle^{-\tau_2}\|_{\ell_N; S(\langle \xi \rangle^{-\tau_1} \langle \delta \xi \rangle^{-\tau_2}, g)} \leq 2C_N \tilde{C}_N C'_N \end{aligned}$$

with  $C_N, \tilde{C}_N$  and  $C'_N$  independent of  $\delta$ . Thus Lemma 2.2 yields

$$\begin{aligned} \|[p, \lambda(D_x)]f\|_{L^2(\mathbb{R}_x)} &= \left\| \left( (r_1 - r_2) \sharp(\langle \xi \rangle^{-\tau_1} \langle \delta \xi \rangle^{-\tau_2}) \right)^w \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} f \right\|_{L^2(\mathbb{R}_x)} \\ &\leq C \left\| (r_1 - r_2) \sharp(\langle \xi \rangle^{-\tau_1} \langle \delta \xi \rangle^{-\tau_2}) \right\|_{N;S(1,g)} \left\| \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} f \right\|_{L^2(\mathbb{R}_x)} \\ &\leq \tilde{C} \left\| \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} f \right\|_{L^2(\mathbb{R}_x)}, \end{aligned}$$

with  $\tilde{C}$  a constant depending only on  $N$  but independent of  $\delta$ . The proof is complete.  $\square$

**Corollary 2.5.** *Suppose that*

$$\forall 1 \leq \alpha \leq k_0 + 2, \quad \|\partial_x^\alpha u\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} + \|\partial_x^\alpha v\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \leq C_\alpha$$

with  $C_\alpha$  a constant depending only on  $\alpha$ . Then for any  $f \in H^1([0,T] \times \mathbb{R}_+^2)$ , we have

$$\sum_{k_1+k_2 \leq 1} \left\| [(\partial_x^{k_1} \partial_y^{k_2} u), \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_x] f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C \left\| \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)},$$

$$\left\| [u \partial_x, \Lambda^{\tau_1} \Lambda_\delta^{\tau_2}] f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C \left\| \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}$$

and

$$\left\| [v \partial_y, \Lambda^{\tau_1} \Lambda_\delta^{\tau_2}] f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C \left\| \partial_y \Lambda^{\tau_1-1} \Lambda_\delta^{\tau_2} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)},$$

where  $C$  is a constant independent of  $\delta$ .

*Proof. Step 1).* We begin with the proof of the following claim

$$(2.17) \quad \forall \beta \geq 1, \quad \left| \partial_\xi^\beta \left( \langle \delta \xi \rangle^d \right) \right| \leq C_{d,\beta} \langle \delta \xi \rangle^d \langle \xi \rangle^{-1},$$

with  $C_{d,\beta}$  a constant depending only on  $\beta$  and  $d$ , but independent of  $\delta$ . To confirm this, we use Faà di Bruno's Formula to get, for any  $\beta > 1$ ,

$$\begin{aligned} &\partial_\xi^\beta \left( \langle \delta \xi \rangle^d \right) \\ &= \sum_{p_1+2p_2+\dots+\beta p_\beta=\beta} \frac{\beta!}{p_1! p_2! \dots p_\beta!} \left( \partial_z^{(p_1+\dots+p_\beta)} \langle z \rangle^{d/2} \right) \Big|_{z=1+\delta^2 \xi^2} \left( \frac{\partial_\xi (1 + \delta^2 \xi^2)}{1!} \right)^{p_1} \dots \left( \frac{\partial_\xi^\beta (1 + \delta^2 \xi^2)}{\beta!} \right)^{p_\beta} \\ &= \sum_{p_1+2p_2=\beta} \frac{\beta!}{p_1! p_2!} \left( \partial_z^{(p_1+p_2)} \langle z \rangle^{d/2} \right) \Big|_{z=1+\delta^2 \xi^2} \left( \frac{\partial_\xi (1 + \delta^2 \xi^2)}{1!} \right)^{p_1} \left( \frac{\partial_\xi^2 (1 + \delta^2 \xi^2)}{2!} \right)^{p_2}. \end{aligned}$$

Since  $p_1 + 2p_2 = \beta \geq 1$ , then at least one of the two nonnegative integers is positive strictly, saying  $p_1 > 0$  without loss of generality. Thus

$$\left| \left( \frac{\partial_\xi (1 + \delta^2 \xi^2)}{1!} \right)^{p_1} \right| \leq \left( 3 (1 + \delta^2 \xi^2) \langle \xi \rangle^{-1} \right)^{p_1} \leq 3^{p_1} (1 + \delta^2 \xi^2)^{p_1} \langle \xi \rangle^{-1}.$$

On the other hand, direct computation shows that

$$\left| \left( \partial_z^{(p_1+p_2)} \langle z \rangle^{d/2} \right) \Big|_{z=1+\delta^2 \xi^2} \right| \leq C_{p_1,p_2,d} (1 + \delta^2 \xi^2)^{\frac{d}{2}-p_1-p_2}$$

with  $C_{p_1,p_2}$  a constant depending only on  $p_1, p_2$  and  $d$  and moreover

$$\left| \left( \frac{\partial_\xi^2 (1 + \delta^2 \xi^2)}{2!} \right)^{p_2} \right| \leq (1 + \delta^2 \xi^2)^{p_2}$$

since  $0 < \delta < 1$ . Combining these inequalities, we have

$$\left| \partial_\xi^\beta \left( \langle \delta \xi \rangle^d \right) \right| \leq C_\beta (1 + \delta^2 \xi^2)^{\frac{d}{2}-p_1-p_2} (1 + \delta^2 \xi^2)^{p_2} (1 + \delta^2 \xi^2)^{p_1} \langle \xi \rangle^{-1} = C_{\beta,d} \langle \delta \xi \rangle^d \langle \xi \rangle^{-1}.$$

with  $C_{\beta,d}$  a constant depending only on  $\beta$  and  $d$ . This is just the desired estimate (2.17).



*Step 2).* It follows from (2.17) and Leibniz's formula that the condition (2.14) is fulfilled, with  $\lambda(\xi) = \langle \xi \rangle^{\tau_1} \langle \delta \xi \rangle^{\tau_2} \xi$ . Moreover (2.13) is fulfilled with  $p = \partial_x^{k_1} \partial_y^{k_2}$ ,  $k_1 + k_2 = 1$ . Thus the first estimate in Corollary 2.5 follows immediately from Lemma 2.4. Now we use using (2.12) to write,

$$\begin{aligned} [u \partial_x, \Lambda^{\tau_1} \Lambda_\delta^{\tau_2}] &= [u, \Lambda^{\tau_1} \Lambda_\delta^{\tau_2}] \partial_x = [u, \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_x] - \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} [u, \partial_x] \\ &= [u, \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_x] + \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} (\partial_x u) \\ &= [u, \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_x] + (\partial_x u) \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} - [(\partial_x u), \Lambda^{\tau_1} \Lambda_\delta^{\tau_2}]. \end{aligned}$$

Thus by Lemma 2.4 we have

$$\| [u \partial_x, \Lambda^{\tau_1} \Lambda_\delta^{\tau_2}] f \|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq C \| \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} f \|_{L^2([0, T] \times \mathbb{R}_+^2)}.$$

Similarly,

$$\| [v \partial_y, \Lambda^{\tau_1} \Lambda_\delta^{\tau_2}] f \|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq C \| \partial_y \Lambda^{\tau_1 - 1} \Lambda_\delta^{\tau_2} f \|_{L^2([0, T] \times \mathbb{R}_+^2)}.$$

Then the proof of Corollary 2.5 is complete.  $\square$

**2.4. Subelliptic estimate for linear Prandtl operator.** In this subsection we consider the linear Prandtl operator which reads

$$(2.18) \quad \mathcal{P} = \partial_t + u \partial_x + v \partial_y - \partial_y^2,$$

where  $u, v$  are classical solution to Prandtl equation (1.1), and thus satisfy the following

$$(2.19) \quad \begin{cases} v(t, x, y) = - \int_0^y \partial_x u(t, x, \tilde{y}) d\tilde{y}, & (t, x, y) \in ]0, T[ \times \mathbb{R}_+^2; \\ u(t, x, 0) = v(t, x, 0) = 0, & (t, x) \in ]0, T[ \times \mathbb{R} \end{cases}$$

Moreover we suppose  $u$  such that satisfies the monotonic condition and that  $\partial_y u$  is bounded; that is,

$$(2.20) \quad \partial_y u(t, x, y) \geq C_* \langle y \rangle^{-\sigma}, \quad \partial_y u \in L^\infty([0, T] \times \mathbb{R}_+^2),$$

where  $\sigma > 1/2$  is a given number, and  $C_*$  is independent of  $t, x$ . As to be seen this monotonicity is crucial to investigate the subelliptic struction of Prandtl operator. In order to estimate the commutators between  $\mathcal{P}$  and Fourier multipliers defined in the previous subsection, we need the following assumption:

$$(2.21) \quad \forall 1 \leq \alpha \leq \ell_N + 2, \quad \| \partial_x^\alpha u \|_{L^\infty([0, T] \times \mathbb{R}_+^2)} + \| \partial_x^\alpha v(t, x, y) \|_{L^\infty([0, T] \times \mathbb{R}_+^2)} \leq C_\alpha$$

with  $C_\alpha$  a constant depending only on  $\alpha$ .

The main result on subelliptic estimate for Prandtl operator can be stated as follows.

**Proposition 2.6.** *Let  $\mathcal{P}$  be the linear Prandtl operator defined by (2.18) with the conditions (2.19)-(2.21) fulfilled, and let  $h, \varphi \in L^2([0, T] \times \mathbb{R}_+^2)$  be given satisfying  $\partial_y h, \partial_y \varphi \in L^2([0, T] \times \mathbb{R}_+^2)$ . Suppose that  $f \in L^2([0, T]; H^2(\mathbb{R}_+^2))$ , with  $\partial_y^3 f \in L^2([0, T] \times \mathbb{R}_+^2)$ , is a classical solution to the equation*

$$(2.22) \quad \mathcal{P}f = \partial_t f + u \partial_x f + v \partial_y f - \partial_y^2 f = h, \quad (t, x, y) \in ]0, T[ \times \mathbb{R}_+^2,$$

with the following initial-boundary conditions fulfilled:

$$(2.23) \quad f(0, x, y) = f(T, x, y) = 0, \quad (x, y) \in \mathbb{R}_+^2$$

and

$$(2.24) \quad \partial_t f(t, x, 0) = (\partial_y^2 f)(t, x, 0) + \varphi(t, x, 0), \quad \partial_y f(t, x, 0) = 0, \quad (t, x) \in \mathbb{R}^2.$$

Then for any  $\varepsilon > 0$  there exists a constant  $C_\varepsilon$ , depending only on  $\max_{\alpha \leq \ell_N + 2} \| \partial_x^\alpha u \|_{L^\infty}$ ,  $\max_{\alpha \leq \ell_N + 2} \| \partial_x^\alpha v \|_{L^\infty}$ ,  $\sigma$ , the constant  $C_*$  given in (2.20) and  $\varepsilon$ , such that

$$\begin{aligned} & \| \langle y \rangle^{-\sigma/2} \Lambda^{1/3} f \|_{L^2([0, T] \times \mathbb{R}_+^2)} + \| \partial_y^2 \Lambda^{-1/3} f \|_{L^2([0, T] \times \mathbb{R}_+^2)} + \| \partial_y^3 \Lambda^{-2/3} f \|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq \varepsilon \| \Lambda^{-2/3} \partial_y h \|_{L^2(\mathbb{R}^2)} + C_\varepsilon \left( \| \Lambda^{-1/3} h \|_{L^2(\mathbb{R}_+^3)} + \| \partial_y f \|_{L^2([0, T] \times \mathbb{R}_+^2)} + \| f \|_{L^2([0, T] \times \mathbb{R}_+^2)} \right) \\ & \quad + C_\varepsilon \left( \| \langle y \rangle^{-\frac{\sigma}{2}} \partial_y \Lambda^{1/6} f \|_{L^2([0, T] \times \mathbb{R}_+^2)} + \| \langle y \rangle \Lambda^{-1/3} \partial_y \varphi \|_{L^2([0, T] \times \mathbb{R}_+^2)} \right). \end{aligned}$$

Recall  $\Lambda^d$  is the Fourier multiplier with symbol  $\langle \xi \rangle^d$ .

*Proof.* To simplify the notation, in the proof we use  $C$  to denote different suitable constants which depend only on  $\max_{\alpha \leq \ell_N + 2} \|\partial_x^\alpha u\|_{L^\infty}$ ,  $\max_{\alpha \leq \ell_N + 2} \|\partial_x^\alpha v\|_{L^\infty}$ ,  $\sigma$  and the  $C_*$  given in (2.20).

Taking the operator  $\Lambda^{-2/3}$  on both sides of (2.22), we see the function  $\Lambda^{-2/3}f$  satisfies the following equation in  $]0, T[ \times \mathbb{R}_+^2$

$$(2.25) \quad \begin{aligned} & \partial_t \Lambda^{-2/3} f + u \partial_x \Lambda^{-2/3} f + v \partial_y \Lambda^{-2/3} f - \partial_y^2 \Lambda^{-2/3} f \\ & = \Lambda^{-2/3} h + [u \partial_x + v \partial_y, \Lambda^{-2/3}] f, \end{aligned}$$

and that

$$(2.26) \quad \Lambda^{-2/3} f|_{t=0} = \Lambda^{-2/3} f|_{t=T} = 0, \quad \Lambda^{-2/3} \partial_y f|_{y=0} = 0$$

due to (2.23) and (2.24), since  $\Lambda^{-2/3}$  is an operator acting only on  $x$  variable. Recall  $[u \partial_x + v \partial_y, \Lambda^{-2/3}]$  stands for the commutator between  $u \partial_x + v \partial_y$  and  $\Lambda^{-2/3}$ .

*Step 1).* We will show in this step that

$$(2.27) \quad \begin{aligned} & \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \leq 2 \left| \operatorname{Re} \left( \partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \right| + C \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \quad + C \left( \|\Lambda^{-1/3} h\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\partial_y f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right). \end{aligned}$$

To do so, we take  $L^2([0, T] \times \mathbb{R}_+^2)$  inner product with the function  $\partial_y \partial_x \Lambda^{-2/3} f \in L^2([0, T] \times \mathbb{R}_+^2)$  on the both sides of eq. (2.25), and then consider the real parts; this gives

$$(2.28) \quad \begin{aligned} & - \operatorname{Re} \left( u \partial_x \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & = \operatorname{Re} \left( \partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} - \operatorname{Re} \left( \partial_y^2 \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \quad + \operatorname{Re} \left( v \partial_y \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} - \operatorname{Re} \left( \Lambda^{-2/3} h, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \quad - \operatorname{Re} \left( [u \partial_x + v \partial_y, \Lambda^{-2/3}] f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)}. \end{aligned}$$

We will treat the terms on both sides. For the term on left hand side we integrate by parts to obtain, observing  $u|_{y=0} = 0$  due to (2.19),

$$\begin{aligned} & - \operatorname{Re} \left( u \partial_x \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & = -\frac{1}{2} \left\{ \left( u \partial_x \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} + \left( \partial_y \partial_x \Lambda^{-2/3} f, u \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \right\} \\ & = \frac{1}{2} \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2. \end{aligned}$$

Next we estimate the terms on the right hand side and have, by Cauchy-Schwarz's inequality ,

$$\begin{aligned} & \left| - \operatorname{Re} \left( \partial_y^2 \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \right| \leq \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\partial_y f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2, \\ & \left| - \operatorname{Re} \left( v \partial_y \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \right| \leq \|v\|_{L^\infty([0, T] \times \mathbb{R}_+^2)} \|\partial_y f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2, \\ & \left| \operatorname{Re} \left( \Lambda^{-2/3} h, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \right| \leq \|\Lambda^{-1/3} h\|_{L^2([0, T] \times \mathbb{R}_+^2)} \|\partial_y f\|_{L^2([0, T] \times \mathbb{R}_+^2)} \end{aligned}$$

and

$$\begin{aligned}
& \left| -\operatorname{Re} \left( [u\partial_x + v\partial_y, \Lambda^{-2/3}]f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\
& \leq \|\Lambda^{1/3} [u\partial_x + v\partial_y, \Lambda^{-2/3}]f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\partial_y \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\
& \leq 2 \left( \|[u\partial_x + v\partial_y, \Lambda^{1/3} \Lambda^{-2/3}]f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|[u\partial_x + v\partial_y, \Lambda^{1/3}] \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right) \\
& \quad + \|\partial_y \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\
& \leq C \left( \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right)
\end{aligned}$$

with the second inequality following from (2.12) and the last inequality using Corollary 2.5. These inequalities, together with (2.28), yields the desired (2.27).

*Step 2).* In this step will estimate the second term on the right hand side of (2.27) and show that for any  $\varepsilon > 0$ ,

$$\begin{aligned}
(2.29) \quad & \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\
& \leq \varepsilon \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\
& \quad + C_\varepsilon \left( \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\Lambda^{-1/3} h\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right),
\end{aligned}$$

with  $C_\varepsilon$  a constant depending on  $\varepsilon$ . To do so, in view of (2.22) we see the function  $\Lambda^{-1/3} f$  satisfies the equation in  $]0, T[ \times \mathbb{R}_+^2$ ,

$$\begin{aligned}
(2.30) \quad & \partial_t \Lambda^{-1/3} f + (u\partial_x + v\partial_y) \Lambda^{-1/3} f - \partial_y^2 \Lambda^{-2/3} f \\
& = \Lambda^{-1/3} h + [u\partial_x + v\partial_y, \Lambda^{-1/3}] f,
\end{aligned}$$

with the initial-boundary condition

$$(2.31) \quad \Lambda^{-1/3} f|_{t=0} = \Lambda^{-1/3} f|_{t=T} = 0, \quad \Lambda^{-1/3} \partial_y f|_{y=0} = 0.$$

Now we take  $L^2([0, T] \times \mathbb{R}_+^2)$  inner product with the function  $-\partial_y^2 \Lambda^{-1/3} f \in L^2([0, T] \times \mathbb{R}_+^2)$  on both sides of (2.30), and then consider the real parts; this gives

$$(2.32) \quad \|\Lambda^{-1/3} \partial_y^2 f\|_{L^2(\mathbb{R}_+^3)}^2 \leq \sum_{p=1}^4 J_p,$$

where

$$\begin{aligned}
J_1 &= \left| \operatorname{Re} \left( \partial_t \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right|, \\
J_2 &= \left| \operatorname{Re} \left( (u\partial_x + v\partial_y) \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{[0,T] \times L^2([0,T] \times \mathbb{R}_+^2)} \right|, \\
J_3 &= \left| \operatorname{Re} \left( \Lambda^{-1/3} h, \Lambda^{-1/3} \partial_y^2 f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\
J_4 &= \left| \operatorname{Re} \left( [u\partial_x + v\partial_y, \Lambda^{-1/3}] f, \Lambda^{-1/3} \partial_y^2 f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right|.
\end{aligned}$$

Integrating by parts and observing the condition (2.31), we see

$$\begin{aligned}
\left( \partial_t \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} &= - \left( \partial_y \partial_t \Lambda^{-1/3} f, \partial_y \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&= \left( \partial_y \Lambda^{-1/3} f, \partial_y \partial_t \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)}.
\end{aligned}$$

The second equation above implies

$$\operatorname{Re} \left( \partial_y \partial_t \Lambda^{-1/3} f, \partial_y \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} = 0$$

and thus

$$(2.33) \quad J_1 = \left| \operatorname{Re} \left( \partial_y \partial_t \Lambda^{-1/3} f, \partial_y \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| = 0.$$

About  $J_2$  we use again ingrating by parts and observe the boundary condition (2.31), to compute

$$\begin{aligned} & \left( v \partial_y \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= - \left( v \partial_y^2 \Lambda^{-1/3} f, \partial_y \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} - \left( (\partial_y v) \partial_y \Lambda^{-1/3} f, \partial_y \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)}, \end{aligned}$$

which implies

$$-\operatorname{Re} \left( v \partial_y^2 \Lambda^{-1/3} f, \partial_y \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} = \frac{1}{2} \left( (\partial_y v) \partial_y \Lambda^{-1/3} f, \partial_y \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)}.$$

As a result,

$$\begin{aligned} & \operatorname{Re} \left( v \partial_y \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2(\mathbb{R}_+^2)} \\ &= \frac{1}{2} \left( (\partial_y v) \partial_y \Lambda^{-1/3} f, \partial_y \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} - \operatorname{Re} \left( (\partial_y v) \partial_y \Lambda^{-1/3} \partial_y f, \Lambda^{-1/3} \partial_y f \right)_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

Similarly,

$$\begin{aligned} & \operatorname{Re} \left( u \partial_x \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= \frac{1}{2} \left( (\partial_x u) \Lambda^{-1/3} \partial_y f, \Lambda^{-1/3} \partial_y f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} - \operatorname{Re} \left( (\partial_y u) \partial_x \Lambda^{-1/3} f, \Lambda^{-1/3} \partial_y f \right)_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

Thus

$$\begin{aligned} & \operatorname{Re} \left( (u \partial_x + v \partial_y) \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= \frac{1}{2} \left( (\partial_x u + \partial_y v) \Lambda^{-1/3} \partial_y f, \Lambda^{-1/3} \partial_y f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} - \operatorname{Re} \left( (\partial_y u) \partial_x \Lambda^{-1/3} f, \Lambda^{-1/3} \partial_y f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad - \operatorname{Re} \left( (\partial_y v) \partial_y \Lambda^{-1/3} f, \Lambda^{-1/3} \partial_y f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= -\operatorname{Re} \left( (\partial_y u) \partial_x \Lambda^{-1/3} f, \Lambda^{-1/3} \partial_y f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} - \operatorname{Re} \left( (\partial_y v) \partial_y \Lambda^{-1/3} f, \Lambda^{-1/3} \partial_y f \right)_{L^2([0,T] \times \mathbb{R}_+^2)}, \end{aligned}$$

the last equation following from the fact that

$$\partial_x u + \partial_y v = 0,$$

since  $u, v$  satisfy (2.19). This gives

$$\begin{aligned} J_2 &= \left| \operatorname{Re} \left( (u \partial_x + v \partial_y) \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ &\leq \|\Lambda^{-1/3} (\partial_y u) \partial_x \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y v\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2. \end{aligned}$$

On the other hand, we use (2.12) to write

$$\Lambda^{-1/3} (\partial_y u) \partial_x = (\partial_y u) \Lambda^{-1/3} \partial_x + [\Lambda^{-1/3}, \partial_y u] \partial_x;$$

this gives

$$\begin{aligned} & \|\Lambda^{-1/3} (\partial_y u) \partial_x \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq \|(\partial_y u) \Lambda^{-1/3} \partial_x \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|[\Lambda^{-1/3}, \partial_y u] \partial_x \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq \|(\partial_y u)^{1/2}\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \|\Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq C \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)}, \end{aligned}$$

the second inequality using Lemma 2.4. Combining these inequalities, we conclude for any  $\varepsilon > 0$ ,

$$(2.34) \quad J_2 \leq \varepsilon \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C_\varepsilon \left( \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right),$$

where  $C_\varepsilon$  is constant depending on  $\varepsilon$ .

Let  $\tilde{\varepsilon} > 0$  be an arbitrarily small number. Cauchy-Schwarz's inequality gives

$$\begin{aligned} J_3 &= \left| \operatorname{Re} \left( \Lambda^{-1/3} h, \Lambda^{-1/3} \partial_y^2 f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ &\leq \tilde{\varepsilon} \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \left\| \Lambda^{-1/3} h \right\|_{L^2(\mathbb{R}_+^3)}^2, \end{aligned}$$

and for  $J_4$ , Corollary 2.5 implies

$$\begin{aligned} J_4 &= \left| \operatorname{Re} \left( [u \partial_x + v \partial_y, \Lambda^{-1/3}] f, \Lambda^{-1/3} \partial_y^2 f \right)_{L^2(\mathbb{R}_+^2)} \right| \\ &\leq \tilde{\varepsilon} \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \left\| f \right\|_{L^2(\mathbb{R}_+^3)}^2, \end{aligned}$$

where  $\tilde{C}_\varepsilon$  is constant depending on  $\tilde{\varepsilon}$ .

Now the above two estimates for  $J_3$  and  $J_4$ , along with (2.32) and the estimates (2.33) and (2.34), gives

$$\begin{aligned} \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 &\leq \tilde{\varepsilon} \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \varepsilon \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ &\quad + C_\varepsilon \left( \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right) \\ &\quad + C_{\tilde{\varepsilon}} \left( \left\| \Lambda^{-1/3} h \right\|_{L^2(\mathbb{R}_+^3)}^2 + \left\| f \right\|_{L^2(\mathbb{R}_+^3)}^2 \right), \end{aligned}$$

and thus, letting  $\tilde{\varepsilon}$  small sufficiently,

$$\begin{aligned} &\left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ &\leq \varepsilon \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C_\varepsilon \left( \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \Lambda^{-1/3} h \right\|_{L^2(\mathbb{R}_+^3)}^2 \right). \end{aligned}$$

This is just the desired estimate (2.29).

*Step 3).* Combining the estimate (2.27) and (2.29), we obtain, choosing  $\varepsilon$  sufficiently small,

$$\begin{aligned} &\left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ (2.35) \quad &\leq C \left| \operatorname{Re} \left( \partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ &\quad + C \left( \left\| \Lambda^{-1/3} h \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right). \end{aligned}$$

*Step 4)* It remains to treat the first term on the right hand side of (2.35). In this step we will prove that for any  $\varepsilon_1 > 0$  there exists a constant  $C_{\varepsilon_1}$  depending only on  $\varepsilon_1$  such that

$$\begin{aligned} &\left| \operatorname{Re} \left( \partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ (2.36) \quad &\leq \varepsilon_1^{-1} C \left( \left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right) \\ &\quad + \varepsilon_1 C \int_0^T \int_{\mathbb{R}} \left| \left( \partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dt dx + \varepsilon_1 C \left\| \langle y \rangle \Lambda^{-1/3} \partial_y \varphi \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2. \end{aligned}$$

For this purpose we integrate by parts again and observe the boundary condition (2.26), to compute

$$\begin{aligned} &\left( \partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= - \left( \Lambda^{-2/3} f, \partial_t \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= \left( \partial_x \Lambda^{-2/3} f, \partial_t \partial_y \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= - \left( \partial_y \partial_x \Lambda^{-2/3} f, \partial_t \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} + \int_0^T \int_{\mathbb{R}} \left( \partial_t \Lambda^{-2/3} f(t, x, 0) \right) \left( \partial_x \Lambda^{-2/3} f(t, x, 0) \right) dt dx, \end{aligned}$$

which, along with

$$\begin{aligned} & 2 \operatorname{Re} \left( \partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= \left( \partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} + \left( \partial_y \partial_x \Lambda^{-2/3} f, \partial_t \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)}, \end{aligned}$$

yields

$$\begin{aligned} & \left| \operatorname{Re} \left( \partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ &= \frac{1}{2} \left| \int_0^T \int_{\mathbb{R}} \left( \partial_t \Lambda^{-2/3} f(t, x, 0) \right) \left( \partial_x \Lambda^{-2/3} f(t, x, 0) \right) dt dx \right| \\ (2.37) \quad &= \frac{1}{2} \left| \int_0^T \int_{\mathbb{R}} \left( \Lambda^{1/6} \partial_t \Lambda^{-2/3} f(t, x, 0) \right) \left( \Lambda^{-1/6} \partial_x \Lambda^{-2/3} f(t, x, 0) \right) dt dx \right| \\ &\leq \varepsilon_1 \int_0^T \int_{\mathbb{R}} \left( \partial_t \Lambda^{-1/2} f(t, x, 0) \right)^2 dt dx + \varepsilon_1^{-1} \int_0^T \int_{\mathbb{R}} \left( \Lambda^{1/6} f(t, x, 0) \right)^2 dt dx. \end{aligned}$$

Moreover observing

$$\Lambda^{1/6} f(t, x, 0) = \left( \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right) (t, x, 0),$$

it then follows from Sobolev inequality that

$$\begin{aligned} \left| \Lambda^{1/6} f(t, x, 0) \right|^2 &\leq C \left( \left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|_{L^2(\mathbb{R}_+)}^2 + \left\| \partial_y \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|_{L^2(\mathbb{R}_+)}^2 \right) \\ &\leq C \left( \left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|_{L^2(\mathbb{R}_+)}^2 + \left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|_{L^2(\mathbb{R}_+)}^2 \right) \end{aligned}$$

with  $C$  a constant independent of  $t, x$ . And thus

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \left( \Lambda^{1/6} f(t, x, 0) \right)^2 dt dx \\ (2.38) \quad &\leq C \left( \left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right) \\ &\leq C \left( \left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right). \end{aligned}$$

Using the fact that

$$\partial_t \Lambda^{-1/2} f(t, x, 0) = \left( \partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) + \varphi(t, x, 0)$$

due to assumption (2.24), we conclude

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \left( \partial_t \Lambda^{-1/2} f(t, x, 0) \right)^2 dt dx \\ &\leq \int_0^T \int_{\mathbb{R}} \left| \left( \partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dt dx + \int_0^T \int_{\mathbb{R}} \left| \Lambda^{-1/2} \varphi(t, x, 0) \right|^2 dt dx. \end{aligned}$$

Moreover observe

$$\begin{aligned} \left| \Lambda^{-1/2} \varphi(t, x, 0) \right| &= \left| - \int_0^{+\infty} \partial_y \Lambda^{-1/2} \varphi(t, x, \tilde{y}) d\tilde{y} \right| \\ &\leq \left( \int_0^{+\infty} \langle y \rangle^{-2} d\tilde{y} \right)^{1/2} \left( \int_0^{+\infty} \langle y \rangle^2 \left| \Lambda^{-1/2} \partial_y \varphi(t, x, \tilde{y}) \right|^2 d\tilde{y} \right)^{1/2}, \end{aligned}$$

which implies

$$\int_0^T \int_{\mathbb{R}} \left| \Lambda^{-1/2} \varphi(t, x, 0) \right|^2 dt dx \leq C \left\| \langle y \rangle \Lambda^{-1/2} \partial_y \varphi \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \leq C \left\| \langle y \rangle \Lambda^{-1/3} \partial_y \varphi \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2,$$

and thus

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \left( \partial_t \Lambda^{-1/2} f(t, x, 0) \right)^2 dt dx \\ & \leq \int_0^T \int_{\mathbb{R}} \left| \left( \partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dt dx + C \|\langle y \rangle \Lambda^{-1/3} \partial_y \varphi\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2. \end{aligned}$$

This along with (2.37) and (2.38) yields the desired (2.36).

*Step 5)* Combining (2.35) and (2.36), we have, for any  $\varepsilon_1 > 0$ ,

$$\begin{aligned} & \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \leq \varepsilon_1^{-1} C \left( \|\langle y \rangle^{-\sigma/2} \Lambda^{1/6} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\ & \quad + \varepsilon_1 C \int_0^T \int_{\mathbb{R}} \left| \left( \partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dt dx + \varepsilon_1 C \|\langle y \rangle \Lambda^{-1/3} \partial_y \varphi\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \quad + C \left( \|\partial_y f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\Lambda^{-1/3} h\|_{L^2(\mathbb{R}_+^3)}^2 \right). \end{aligned}$$

As for the first term on the hand side, we use the monotonicity condition and interpolation inequality to get, for any  $\varepsilon_2 > 0$

$$\begin{aligned} \|\langle y \rangle^{-\sigma/2} \Lambda^{1/6} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 & \leq \varepsilon_2 \|\langle y \rangle^{-\sigma/2} \Lambda^{1/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \varepsilon_2^{-1} \|\langle y \rangle^{-\sigma/2} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \leq \varepsilon_2 \|\langle y \rangle^{-\sigma/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C_{\varepsilon_2} \|\langle y \rangle^{-\sigma/2} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \leq \varepsilon_2 \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C_{\varepsilon_2} \|f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2. \end{aligned}$$

From the above inequalities, we infer that, choosing  $\varepsilon_2$  small enough,

$$\begin{aligned} & \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ (2.39) \quad & \leq \varepsilon_1 C \int_0^T \int_{\mathbb{R}} \left| \left( \partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dt dx \\ & \quad + C_{\varepsilon_1} \left( \|\langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\langle y \rangle \Lambda^{-1/3} \partial_y \varphi\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\ & \quad + C_{\varepsilon_1} \left( \|\partial_y f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\Lambda^{-1/3} h\|_{L^2(\mathbb{R}_+^3)}^2 \right). \end{aligned}$$

*Step 6)* In this step we treat the first term on the right side of (2.39), and show that

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \left| \left( \partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dt dx \\ (2.40) \quad & \leq C \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \quad + C \left( \|\Lambda^{-2/3} \partial_y h\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\partial_y f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right). \end{aligned}$$

To do so, we integrate by parts to get

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} \left| \left( \partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dt dx & = \operatorname{Re} \left( \partial_y^3 \Lambda^{-1/2} f, \partial_y^2 \Lambda^{-1/2} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & = \operatorname{Re} \left( \partial_y^3 \Lambda^{-2/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)}. \end{aligned}$$

This yields, for any  $\varepsilon_3 > 0$ ,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \left| \left( \partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dt dx \\ (2.41) \quad & \leq \|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \leq 2 \|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2. \end{aligned}$$

the last inequality holding because we can use (2.24) to integrate by parts and then obtain

$$(2.42) \quad \begin{aligned} \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 &\leq \left( \partial_y^3 \Lambda^{-2/3} f, \partial_y f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq \|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

Thus in order to prove (2.40) it suffices to estimate  $\|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}$ . To do so, taking firstly  $\partial_y$  on both side of (2.22) and then taking  $\Lambda_{-2/3}$ , we have the equation

$$\begin{aligned} &\partial_t \Lambda^{-2/3} \partial_y f + u \partial_x \Lambda^{-2/3} \partial_y f + v \partial_y \Lambda^{-2/3} \partial_y f - \partial_y^3 \Lambda^{-2/3} f \\ = &\Lambda^{-2/3} \partial_y h + [u \partial_x + v \partial_y, \Lambda^{-2/3}] \partial_y f - \Lambda^{-2/3} (\partial_y u) \partial_x f - \Lambda^{-2/3} (\partial_y v) \partial_y f, \end{aligned}$$

which implies, by taking  $L^2$  inner product with  $-\partial_y^3 \Lambda^{-2/3} f$  and considering the real parts,

$$(2.43) \quad \begin{aligned} &\|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ = &-\operatorname{Re} \left( \partial_t \Lambda^{-2/3} \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &- \operatorname{Re} \left( u \partial_x \Lambda^{-2/3} \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &- \operatorname{Re} \left( v \partial_y \Lambda^{-2/3} \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &+ \operatorname{Re} \left( \Lambda^{-2/3} \partial_y h, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &+ \operatorname{Re} \left( [u \partial_x + v \partial_y, \Lambda^{-2/3}] \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &- \operatorname{Re} \left( \Lambda^{-2/3} (\partial_y u) \partial_x f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &- \operatorname{Re} \left( \Lambda^{-2/3} (\partial_y v) \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

Next we will treat the terms on the right hand side. Observing

$$\partial_t \Lambda^{-2/3} \partial_y f|_{y=0} = 0$$

due to (2.24), we integrate by part to compute

$$(2.44) \quad \begin{aligned} &-\operatorname{Re} \left( \partial_t \Lambda^{-2/3} \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ = &-\operatorname{Re} \left( \partial_t \partial_y^2 \Lambda^{-2/3} f, \partial_y^2 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ = &0, \end{aligned}$$

the last equation holding because

$$\partial_y^2 \Lambda^{-2/3} f|_{t=0} = \partial_y^2 \Lambda^{-2/3} f|_{t=T} = 0$$



due to (2.23). Since  $u|_{y=0}$  then integrating by parts gives

$$\begin{aligned}
& - \operatorname{Re} \left( u \partial_x \Lambda^{-2/3} \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&= - \operatorname{Re} \left( u \partial_x \Lambda^{-2/3} \partial_y^2 f, \partial_y^2 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&\quad - \operatorname{Re} \left( (\partial_y u) \partial_x \Lambda^{-2/3} \partial_y f, \partial_y^2 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&= \frac{1}{2} \left( (\partial_x u) \Lambda^{-2/3} \partial_y^2 f, \Lambda^{-2/3} \partial_y^2 f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&\quad - \operatorname{Re} \left( (\partial_y u) \partial_x \Lambda^{-2/3} \partial_y f, \partial_y^2 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&\leq \frac{1}{2} \left\| \partial_x u \right\|_{L^\infty} \left\| \Lambda^{-2/3} \partial_y^2 f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&\quad + \left\| \Lambda^{-1/3} (\partial_y u) \partial_x \Lambda^{-2/3} \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \left\| \Lambda^{-1/3} (\partial_y u) \partial_x \Lambda^{-2/3} \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\
&\leq 2 \left\| \Lambda^{-1/3} \partial_x \Lambda^{-2/3} (\partial_y u) \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + 2 \left\| \Lambda^{-1/3} [\partial_y u, \partial_x \Lambda^{-2/3}] \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\
&\leq C \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2.
\end{aligned}$$

Thus

$$\begin{aligned}
& - \operatorname{Re} \left( u \partial_x \Lambda^{-2/3} \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\
(2.45) \quad & \leq C \left( \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right) \\
& \leq \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2,
\end{aligned}$$

where the last inequality using (2.42). Similarly integrating by parts again yields

$$- \operatorname{Re} \left( v \partial_y \Lambda^{-2/3} \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} = -\frac{1}{2} \left( (\partial_y v) \partial_y^2 \Lambda^{-2/3} f, \partial_y^2 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)},$$

which implies

$$\begin{aligned}
& - \operatorname{Re} \left( v \partial_y \Lambda^{-2/3} \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\
(2.46) \quad & \leq C \left\| \partial_y^2 \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \leq \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2.
\end{aligned}$$

Cauchy-Schwarz inequality gives, for any  $\tilde{\varepsilon} > 0$ ,

$$\begin{aligned}
& \operatorname{Re} \left( \Lambda^{-2/3} \partial_y h, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\
(2.47) \quad & \leq \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \tilde{\varepsilon}^{-1} \left\| \Lambda^{-2/3} \partial_y h \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2,
\end{aligned}$$

and

$$\begin{aligned}
& - \operatorname{Re} \left( \Lambda^{-2/3} (\partial_y v) \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\
(2.48) \quad & \leq \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \tilde{\varepsilon}^{-1} \left\| \partial_y v \right\|_{L^\infty} \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2
\end{aligned}$$

and

$$\begin{aligned}
& \operatorname{Re} \left( [u \partial_x + v \partial_y, \Lambda^{-2/3}] \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\
(2.49) \quad & \leq \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \tilde{\varepsilon}^{-1} \left\| [u \partial_x + v \partial_y, \Lambda^{-2/3}] \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\
& \leq \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C \tilde{\varepsilon}^{-1} \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2
\end{aligned}$$

the last inequality using Corollary 2.5. Finally,

$$\begin{aligned}
& - \operatorname{Re} \left( \Lambda^{-2/3} (\partial_y u) \partial_x f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\
& \leq \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C \tilde{\varepsilon}^{-1} \left\| \Lambda^{-2/3} (\partial_y u) \partial_x f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \leq \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \tilde{\varepsilon}^{-1} C \left\| (\partial_y u) \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \quad + \tilde{\varepsilon}^{-1} C \left\| [(\partial_y u) \partial_x, \Lambda^{-2/3}] f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \leq \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \tilde{\varepsilon}^{-1} C \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \quad + \tilde{\varepsilon}^{-1} C \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2.
\end{aligned}$$

This, along with (2.43) -(2.49), yields, for any  $\tilde{\varepsilon} > 0$ ,

$$\begin{aligned}
& \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \leq \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \quad + C_{\tilde{\varepsilon}} \left( \left\| \Lambda^{-2/3} \partial_y h \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right).
\end{aligned}$$

Thus letting  $\tilde{\varepsilon}$  be small enough, we have

$$\begin{aligned}
(2.50) \quad & \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \leq C \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \quad + C \left( \left\| \Lambda^{-2/3} \partial_y h \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right).
\end{aligned}$$

This along with (2.41) yields the desired estimate (2.40).

*Step 7)* Now we combine (2.39) and (2.40) to conclude for any  $\varepsilon_1 > 0$ ,

$$\begin{aligned}
& \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \leq \varepsilon_1 C \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \quad + \varepsilon_1 C \left( \left\| \Lambda^{-2/3} \partial_y h \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\
& \quad + C_{\varepsilon_1} \left( \left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \langle y \rangle \Lambda^{-1/3} \partial_y \varphi \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\
& \quad + C_{\varepsilon_1} \left( \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \Lambda^{-1/3} h \right\|_{L^2(\mathbb{R}_+^3)}^2 \right),
\end{aligned}$$

which implies, choosing  $\varepsilon_1 > 0$  sufficiently small,

$$\begin{aligned}
& \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \leq C \left( \left\| \Lambda^{-2/3} \partial_y h \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\
& \quad + C \left( \left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \langle y \rangle \Lambda^{-1/3} \partial_y \varphi \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\
& \quad + C \left( \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \Lambda^{-1/3} h \right\|_{L^2(\mathbb{R}_+^3)}^2 \right).
\end{aligned}$$

As a result, the above inequality along with (2.40) gives

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}} \left| \left( \partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dt dx \\
& \leq C \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \quad + C \left( \left\| \Lambda^{-2/3} \partial_y h \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\
& \leq C \left\| \Lambda^{-2/3} \partial_y h \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \quad + C \left( \left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \langle y \rangle \Lambda^{-1/3} \partial_y \varphi \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\
& \quad + C \left( \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \Lambda^{-1/3} h \right\|_{L^2(\mathbb{R}_+^3)}^2 \right).
\end{aligned}$$

Inserting the above estimate into (2.39), we have

$$\begin{aligned}
& \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
(2.51) \quad & \leq \varepsilon_1 C \left\| \Lambda^{-2/3} \partial_y h \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \quad + C_{\varepsilon_1} \left( \left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \langle y \rangle \Lambda^{-1/3} \partial_y \varphi \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\
& \quad + C_{\varepsilon_1} \left( \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \Lambda^{-1/3} h \right\|_{L^2(\mathbb{R}_+^3)}^2 \right).
\end{aligned}$$

and thus, combining (2.29),

$$\begin{aligned}
& \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \leq \varepsilon_1 C \left\| \Lambda^{-2/3} \partial_y h \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \quad + C_{\varepsilon_1} \left( \left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \langle y \rangle \Lambda^{-1/3} \partial_y \varphi \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\
& \quad + C_{\varepsilon_1} \left( \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \Lambda^{-1/3} h \right\|_{L^2(\mathbb{R}_+^3)}^2 \right).
\end{aligned}$$

This implies

$$\begin{aligned}
& \left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \leq \varepsilon_1 C \left\| \Lambda^{-2/3} \partial_y h \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \quad + C_{\varepsilon_1} \left( \left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \langle y \rangle \Lambda^{-1/3} \partial_y \varphi \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\
& \quad + C_{\varepsilon_1} \left( \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \Lambda^{-1/3} h \right\|_{L^2(\mathbb{R}_+^3)}^2 \right),
\end{aligned}$$

since

$$\begin{aligned}
\left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 & \leq C \left\| (\partial_y u)^{1/2} \Lambda^{1/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \leq C \left\| \langle y \rangle^{-\sigma/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2
\end{aligned}$$

due to (2.20). Moreover, by (2.50) and (2.51) we have

$$\begin{aligned}
& \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
& \leq C \left( \left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \langle y \rangle \Lambda^{-1/3} \partial_y \varphi \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\
& \quad + C \left( \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \Lambda^{-1/3} h \right\|_{L^2(\mathbb{R}_+^3)}^2 + \left\| \Lambda^{-2/3} \partial_y h \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right),
\end{aligned}$$

Thus the proof of Proposition 2.6 is complete.  $\square$

## 3. AUXILIARY FUNCTIONS AND WEIGHT FUNCTIONS

Here and throughout the paper we always let  $u$  be the classical solution to the Prandtl equation (1.1). As in [17], in order to overcome the degeneracy in  $x$  direction we work with a family of auxiliary function instead of the original  $u$ . Precisely, With each  $m \geq 1$  we associate a function

$$(3.1) \quad f_m = \partial_x^m \omega - \frac{\partial_y \omega}{\omega} \partial_x^m u = \omega \partial_y \left( \frac{\partial_x^m u}{\omega} \right),$$

where  $\omega = \partial_y u$ . We can verify that, seeing Subsection 6.1 in Appendix, the above  $f_m$  satisfies formally the equation

$$(3.2) \quad \partial_t f_m + u \partial_x f_m + v \partial_y f_m - \partial_y^2 f_m = \mathcal{Z}_m,$$

where

$$(3.3) \quad \begin{aligned} \mathcal{Z}_m = & - \sum_{j=1}^m \binom{m}{j} (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_y f_{m-j}) \\ & - \left[ \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) - 2 \left[ \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \right] f_m. \end{aligned}$$

Moreover if  $\partial_y^j f_m \in L^2$  for all  $j \leq 3$ , then

$$(3.4) \quad \partial_y f_m|_{y=0} = 0.$$

Now we introduce two weight functions to be used later: for each pair  $(m, i) \in \mathbb{Z}_+^2$  with  $m \geq 3$  and  $0 \leq i \leq 3$ ,

$$(3.5) \quad W_m^i = e^{2cy} \left( 1 + \frac{2cy}{(3m+i)\sigma} \right)^{-\frac{(3m+i)\sigma}{2}} (1+cy)^{-1} \Lambda^{\frac{i}{3}}$$

and

$$(3.6) \quad \phi_m^i = \phi^{\frac{9(m-3)+3i}{4}},$$

where  $c$  is the constant given in Theorem 1.1 and  $\Lambda^d$  is the multipliers associate to symbol  $\langle \xi \rangle^d$ , and  $\phi$  is defined by

$$\phi(t) = t(T-t), \quad 0 \leq t \leq T.$$

Without loss of generality, we may assume the above  $T \leq 1$ ; otherwise we instead the above  $\phi$  by

$$\tilde{\phi}(t) = \frac{t}{T} \left( 1 - \frac{t}{T} \right), \quad 0 \leq t \leq T.$$

As a result we have  $0 \leq \phi \leq 1$  and thus

$$(3.7) \quad \phi_{m_1}^{i_1} \leq \phi_{m_2}^{i_2}$$

provided  $3 \leq m_2 \leq m_1$  and  $0 \leq i_2 \leq i_1 \leq 3$ .

Next we list some inequalities for the weight  $W_m^i$ , which is to used frequently later. Observe the function

$$\gamma \longrightarrow \left( 1 + \frac{cy}{\gamma} \right)^{-\gamma}$$

is a monotonically decreasing function as  $\gamma$  varies in the interval  $]0, +\infty[$ . Thus

$$(3.8) \quad \forall f \in L^2(\mathbb{R}_+^2), \forall i \geq 0, \quad \|W_{m_1}^i f\|_{L^2(\mathbb{R}_x)} \leq \|W_{m_2}^i f\|_{L^2(\mathbb{R}_x)}$$

and

$$(3.9) \quad \forall f \in L^2(\mathbb{R}_+^2), \forall i \geq 1, \quad \|W_{m_1}^i f\|_{L^2(\mathbb{R}_x)} \leq \|W_{m_2}^{i-1} \Lambda^{1/3} f\|_{L^2(\mathbb{R}_x)},$$

provided that  $m_1 \geq m_2 > 0$ . Moreover, since

$$\forall \gamma \geq 1, \quad \left| \partial_y^\alpha e^{2cy} \left( 1 + \frac{cy}{\gamma} \right)^{-\gamma} (1+y)^{-1} \right| \leq C_{\alpha, \gamma} e^{2cy} \left( 1 + \frac{cy}{\gamma} \right)^{-\gamma} (1+y)^{-1},$$

with  $C_{\alpha,\gamma}$  a constant depending only on  $\alpha$  and  $\gamma$ . then the following estimates: for any  $f \in L^2(\mathbb{R}_+^2)$

$$(3.10) \quad \|\partial_y, W_m^i f\|_{L^2(\mathbb{R}_+^2)} \leq C_\gamma \|W_m^i f\|_{L^2(\mathbb{R}_+^2)}$$

and

$$(3.11) \quad \|\partial_y^2, W_m^i f\|_{L^2(\mathbb{R}_+^2)} \leq C_\gamma \left( \|W_m^i f\|_{L^2(\mathbb{R}_+^2)} + \|W_m^i \partial_y f\|_{L^2(\mathbb{R}_+^2)} \right)$$

hold for all integers  $m, i$  with  $m \geq 4$  and  $0 \leq i \leq 3$ .

In view of (3.2) we see  $W_m^i f_m$  satisfies equation

$$(3.12) \quad \partial_t W_m^i f_m + u \partial_x W_m^i f_m + v \partial_y W_m^i f_m - \partial_y^2 W_m^i f_m = \mathcal{Z}_{m,i},$$

with

$$\begin{aligned} \mathcal{Z}_{m,i} = & - \sum_{j=1}^m \binom{m}{j} W_m^i (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} W_m^i (\partial_x^j v) (\partial_y f_{m-1}) \\ & - W_m^i \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) - 2 W_m^i \partial_y \left( \frac{\partial_y \omega}{\omega} \right) f_m \\ & + [u \partial_x + v \partial_y - \partial_y^2, W_m^i] f_m \end{aligned}$$

By direct computation we have

$$\partial_y \left( e^{2cy} \left( 1 + \frac{2cy}{(3m+i)\sigma} \right)^{-(3m+i)\sigma/2} (1+cy)^{-1} \right) \Big|_{y=0} = 0$$

and thus in view of (3.4),

$$(\partial_y W_m^i f_m) \Big|_{y=0} = 0,$$

provided  $\partial_y^j f_m \in L^2$  for all  $j \leq 4$ .

**Lemma 3.1.** *Let  $c$  be the constant given in (3.8), and let  $b(D_x)$  be the Fourier multiplier associate with the symbol  $b(\xi) \in S(1, g)$ , which is a bounded operator acting only on  $x$  variable. Then there exists a constant  $C$ , such that for any  $m \geq 1$  and for any  $\tilde{c} > 0$  with  $\tilde{c} < c$  we have*

$$(3.13) \quad \|b(D_x) e^{\tilde{c}y} \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \leq C \|b(D_x) W_m^\ell f_m\|_{L^2(\mathbb{R}_+^2)},$$

and

$$(3.14) \quad \|b(D_x) \partial_x^m v\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{R}_x))} \leq C \|b(D_x) W_m^{\ell+1} f_{m+1}\|_{L^2(\mathbb{R}_+^2)}.$$

*Proof.* We use the notation  $\omega = \partial_y u$ . Observe  $\omega \in L^\infty$  and  $\omega > 0$  then

$$\|b(D_x) e^{\tilde{c}y} \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \leq C \|e^{\tilde{c}y} b(D_x) \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}$$

On the other hand, integrating by parts we have

$$\begin{aligned} \|e^{\tilde{c}y} b(D_x) \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}^2 &= \int_{\mathbb{R}} \int_0^\infty e^{2\tilde{c}y} \left( b(D_x) \frac{\partial_x^m u}{\omega} \right)^2 dx dy \\ &= \frac{1}{2\tilde{c}} \int_{\mathbb{R}} \int_0^\infty (\partial_y e^{2\tilde{c}y}) \left( b(D_x) \frac{\partial_x^m u}{\omega} \right)^2 dx dy \\ &= \frac{1}{\tilde{c}} \int_{\mathbb{R}} \int_0^\infty e^{2\tilde{c}y} \left( b(D_x) \frac{\partial_x^m u}{\omega} \right) \partial_y \left( b(D_x) \frac{\partial_x^m u}{\omega} \right) dx dy \\ &\leq \frac{1}{\tilde{c}} \|e^{\tilde{c}y} b(D_x) \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)} \|e^{\tilde{c}y} b(D_x) \partial_y \left( \frac{\partial_x^m u}{\omega} \right)\|_{L^2(\mathbb{R}_+^2)}, \end{aligned}$$

which implies

$$\begin{aligned} \|e^{\tilde{c}y} \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)} &\leq \|e^{\tilde{c}y} b(D_x) \partial_y \left( \frac{\partial_x^m u}{\omega} \right)\|_{L^2(\mathbb{R}_+^2)} \\ &= 2 \|e^{\tilde{c}y} \omega^{-1} \omega b(D_x) \partial_y \left( \frac{\partial_x^m u}{\omega} \right)\|_{L^2(\mathbb{R}_+^2)} \\ &\leq C \|b(D_x) W_m^\ell f_m\|_{L^2(\mathbb{R}_+^2)}, \end{aligned}$$

the last inequality using the fact that  $f_m = \omega \partial_y \left( \frac{\partial_x^m u}{\omega} \right)$ , and that

$$e^{\tilde{c}y} \omega^{-1} \leq e^{\tilde{c}y} \langle y \rangle^\sigma \leq C e^{2cy} \left( 1 + \frac{2cy}{\gamma} \right)^{\gamma/2}$$

for any  $\gamma \geq 1$ . This is just the desired (3.13). Now we prove (3.14). Recall  $v(x, y, t) = - \int_0^y \partial_x u(x, y', t) dy'$ . Then we have

$$b(D_x) \partial_x^m v = - \int_0^y b(D_x) \partial_x^{m+1} u(x, y') dy'$$

Therefore

$$\begin{aligned} \|b(D_x) \partial_x^m v\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{R}_x))} &\leq \|(1+y)^{-1}\|_{L^2(\mathbb{R}_+)} \|(1+y)b(D_x) \partial_x^{m+1} u\|_{L^2(\mathbb{R}_+^2)} \\ &\leq C \|W_{m+1}^\ell f_{m+1}\|_{L^2(\mathbb{R}_+^2)}, \end{aligned}$$

the last inequality using (3.13). Thus the desired (3.14) follows and the proof of Lemma 3.1 is complete.  $\square$

**Lemma 3.2.** *Given two integers  $m, i$  with  $m \geq 3$  and  $1 \leq i \leq 3$ , we have*

$$\forall f \in H^1(\mathbb{R}_+^2), \quad \|\langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/3} W_m^{i-1} f\|_{L^2(\mathbb{R}_+^2)} \leq \|\partial_y W_m^i f\|_{L^2(\mathbb{R}_+^2)} + \|W_m^i f\|_{L^2(\mathbb{R}_+^2)}.$$

*Proof.* By the definition of  $W_k^{i-1}$ , we can write

$$\Lambda^{1/3} W_m^{i-1} = e^{2cy} \left( 1 + \frac{2cy}{(3m+i-1)\sigma} \right)^{-\frac{(3m+i-1)\sigma}{2}} (1+cy)^{-1} \Lambda^{\frac{i}{3}} = a_{m,i}(y) W_m^i,$$

where

$$a_{m,i}(y) = \left( 1 + \frac{2cy}{(3m+i-1)\sigma} \right)^{-\frac{(3m+i-1)\sigma}{2}} \left( 1 + \frac{2cy}{(3m+i)\sigma} \right)^{\frac{(3m+i)\sigma}{2}}$$

Direct computation gives

$$\begin{aligned} |a_{m,i}(y)| &= \left( 1 + \frac{2cy}{(3m+i-1)\sigma} \right)^{\sigma/2} \left( 1 + \frac{2cy}{(3m+i-1)\sigma} \right)^{-\frac{(3m+i)\sigma}{2}} \left( 1 + \frac{2cy}{(3m+i)\sigma} \right)^{\frac{(3m+i)\sigma}{2}} \\ &\leq \left( 1 + \frac{2cy}{(3m+i-1)\sigma} \right)^{\sigma/2} \leq C \langle y \rangle^{\sigma/2} \end{aligned}$$

and similarly,

$$\begin{aligned} |\partial_y a_{m,i}(y)| &\leq C \left( 1 + \frac{2cy}{(3m+i-1)\sigma} \right)^{\frac{\sigma}{2}-1} \left( 1 + \frac{2cy}{(3m+i-1)\sigma} \right)^{-\frac{(3m+i)\sigma}{2}} \left( 1 + \frac{2cy}{(3m+i)\sigma} \right)^{\frac{(3m+i)\sigma}{2}} \\ &\leq C \langle y \rangle^{\frac{\sigma}{2}-1} \leq C \langle y \rangle^{\sigma/2}. \end{aligned}$$

Thus

$$\begin{aligned} \|\langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/3} W_m^{i-1} f\|_{L^2(\mathbb{R}_+^2)} &= \|\langle y \rangle^{-\sigma/2} \partial_y a_{m,i} W_m^i f\|_{L^2(\mathbb{R}_+^2)} \\ &\leq \|\langle y \rangle^{-\sigma/2} a_{m,i} \partial_y W_m^i f\|_{L^2(\mathbb{R}_+^2)} + \|\langle y \rangle^{-\sigma/2} (\partial_y a_{m,i}) W_m^i f\|_{L^2(\mathbb{R}_+^2)} \\ &\leq C \left( \|\partial_y W_m^i f\|_{L^2(\mathbb{R}_+^2)} + \|W_m^i f\|_{L^2(\mathbb{R}_+^2)} \right). \end{aligned}$$

The proof is complete.  $\square$

**Lemma 3.3.** *We assume that for any  $k \in \mathbb{N}$  and  $i \in \{1, 2, 3\}$  have  $W_k f_k \in L^2(\mathbb{R}_+^2)$  and  $W_{k-1}^i f_{k-1} \in L^2(\mathbb{R}_+^2)$ . Then we have the following estimates: for any  $m \leq k-1$  and any  $0 \leq \ell \leq 3$ ,*

$$(3.15) \quad \|\Lambda^{-1} \phi_m^\ell W_m^0 f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq \|\phi_{k-1}^0 W_{k-1}^0 f_{k-1}\|_{L^2([0,T] \times \mathbb{R}_+^2)},$$

$$(3.16) \quad \|\Lambda^{-2/3} \phi_{k-1}^i W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C \|\phi_{k-1}^i W_{k-1}^i f_{k-1}\|_{L^2([0,T] \times \mathbb{R}_+^2)}$$

and

$$(3.17) \quad \begin{aligned} & \|\Lambda^{-2/3} \partial_y \phi_{k-1}^i W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C \|\partial_y \phi_{k-1}^i W_{k-1}^i f_{k-1}\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \|\phi_{k-1}^i W_{k-1}^i f_{k-1}\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

*Proof.* (i) We first prove (3.15). By (3.7) and (3.8) we have Thus

$$(3.18) \quad \|\Lambda^{-1} \phi_m^\ell W_m^0 f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq \|\phi_{k-1}^0 W_{k-1}^0 \Lambda^{-1} (\partial_x^k \omega - \frac{\partial_y \omega}{\omega} \partial_x^k u)\|_{L^2([0,T] \times \mathbb{R}_+^2)}.$$

Moreover,

$$\begin{aligned} & \|\phi_{k-1}^0 W_{k-1}^0 \Lambda^{-1} (\partial_x^k \omega - \frac{\partial_y \omega}{\omega} \partial_x^k u)\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \|\phi_{k-1}^0 W_{k-1}^0 \Lambda^{-1} \partial_x (\partial_x^{k-1} \omega - \frac{\partial_y \omega}{\omega} \partial_x^{k-1} u)\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \|\phi_{k-1}^0 W_{k-1}^0 \Lambda^{-1} (\partial_x (\partial_y \omega / \omega)) \partial_x^{k-1} u\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \|\phi_{k-1}^0 W_{k-1}^0 f_{k-1}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \|\phi_{k-1}^0 e^{2cy} (1+cy)^{-1} (\partial_x (\partial_y \omega / \omega)) \partial_x^{k-1} u\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \|\phi_{k-1}^0 W_{k-1}^0 f_{k-1}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \|e^{2cy} (1+cy)^{-1-2\sigma} (\partial_x (\partial_y \omega / \omega))\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \|\phi_{k-1}^0 (1+cy)^{2\sigma} \partial_x^{k-1} u\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C \|\phi_{k-1}^0 W_{k-1}^0 f_{k-1}\|_{L^2([0,T] \times \mathbb{R}_+^2)}, \end{aligned}$$

the last inequality following from (3.13), and the fact that

$$\begin{aligned} & \|e^{2cy} (1+cy)^{-1-2\sigma} (\partial_x (\partial_y \omega / \omega))\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \\ & \leq \|e^{2cy} \partial_x \partial_y \omega\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \|(1+cy)^{-1-2\sigma} \omega^{-1}\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \|e^{2cy} \partial_x \omega\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \|(1+cy)^{-1-2\sigma} \omega^{-2} \partial_y \omega\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \\ & \leq C. \end{aligned}$$

Then we get the desired estimate (3.15).

(ii) Now we prove (3.16). We recall that

$$\Lambda^{-2/3} \phi_{k-1}^i W_k^{i-1} f_k = e^{2cy} \left(1 + \frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i-1)\sigma}{2}} (1+cy)^{-1} \Lambda^{\frac{i}{3}} \Lambda^{-1} \left(\partial_x^k \omega - \frac{\partial_y \omega}{\omega} \partial_x^k u\right).$$

Observe

$$\left(1 + \frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i-1)\sigma}{2}} \leq \left(1 + \frac{2cy}{(3(k-1)+i)\sigma}\right)^{-\frac{(3(k-1)+i)\sigma}{2}}$$

since

$$\gamma \longrightarrow \left(1 + \frac{cy}{\gamma}\right)^{-\gamma}$$

is a monotonically decreasing. Thus

$$\|\Lambda^{-2/3} \phi_{k-1}^i W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq \|\phi_{k-1}^i W_{k-1}^i \Lambda^{-1} (\partial_x^k \omega - \frac{\partial_y \omega}{\omega} \partial_x^k u)\|_{L^2([0,T] \times \mathbb{R}_+^2)}.$$

Then arguing similarly as (3.18), we conclude

$$\|\Lambda^{-2/3} \phi_{k-1}^i W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C \|\phi_{k-1}^i W_{k-1}^i f_{k-1}\|_{L^2([0,T] \times \mathbb{R}_+^2)}.$$

Then the desired estimate (3.16) follows.

(iii) Finally we prove (3.17). We have, making use of (3.10),

$$\begin{aligned}
& \left\| \Lambda^{-2/3} \partial_y \phi_{k-1}^i W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
& \leq \left\| \Lambda^{-2/3} \phi_{k-1}^i W_k^{i-1} \partial_y f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \left\| [\partial_y, W_k^{i-1}] \Lambda^{-2/3} \phi_{k-1}^i f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
& \leq \left\| \Lambda^{-2/3} \phi_{k-1}^i W_k^{i-1} \partial_y f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \left\| \Lambda^{-2/3} \phi_{k-1}^i W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
& \leq \left\| \Lambda^{-2/3} \phi_{k-1}^i W_k^{i-1} \partial_y f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \left\| \phi_{k-1}^i W_{k-1}^i f_{k-1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}.
\end{aligned}$$

the last inequality following from (3.16). As for the first term in the last inequality, we have

$$\left\| \Lambda^{-2/3} \phi_{k-1}^i W_k^{i-1} \partial_y f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq \left\| \phi_{k-1}^i W_{k-1}^i \Lambda^{-1} \partial_y (\partial_x^k \omega - \frac{\partial_y \omega}{\omega} \partial_x^k u) \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}.$$

Then repeating the argument in the proof of lamdmin, we have

$$\left\| \phi_{k-1}^i W_{k-1}^i \Lambda^{-1} \partial_y (\partial_x^k \omega - \frac{\partial_y \omega}{\omega} \partial_x^k u) \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq C \left\| \phi_{k-1}^i W_{k-1}^i \partial_y f_{k-1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}.$$

On the other hand

$$\begin{aligned}
& \left\| \phi_{k-1}^i W_{k-1}^i \partial_y f_{k-1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
& \leq \left\| \partial_y \phi_{k-1}^i W_{k-1}^i f_{k-1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \left\| [\partial_y, W_{k-1}^i] \phi_{k-1}^i f_{k-1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
& \leq \left\| \partial_y \phi_{k-1}^i W_{k-1}^i f_{k-1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \left\| \phi_{k-1}^i W_{k-1}^i f_{k-1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}.
\end{aligned}$$

Thus, combining the above inequalities, we obtain the desired (3.17). The proof is then complete.  $\square$

#### 4. CRUCIAL ESTIMATE

We will use  $L^2$  method to prove the Gevrey regularity. To do so we use induction on  $k$  to estimate the weighted  $L^2$  norms of  $\partial_k u$ , and in view of the conclusion in the previous section it suffices to estimate its equivalent form  $f_m$  instead. Recall  $f_m$  is defined in (3.1), i.e.,

$$f_m = \partial_x^m \omega - \frac{\partial_y \omega}{\omega} \partial_x^m (u - 1), \quad m \geq 1.$$

Thus formally  $f_m$  and  $W_m^\ell f_m$  satisfy respectively the equations

$$(\partial_t + u \partial_x + v \partial_y - \partial_y^2) f_m = \mathcal{Z}_m,$$

and

$$(4.1) \quad (\partial_t + u \partial_x + v \partial_y - \partial_y^2) W_m^\ell f_m = \mathcal{Z}_{m, \ell},$$

where

$$\begin{aligned}
\mathcal{Z}_m &= - \sum_{j=1}^m \binom{m}{j} (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_y f_{m-1}) \\
&\quad - \left[ \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) - 2 \left[ \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \right] f_m.
\end{aligned}$$

and

$$\begin{aligned}
(4.2) \quad \mathcal{Z}_{m, \ell} &= - \sum_{j=1}^m \binom{m}{j} W_m^\ell (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} W_m^\ell (\partial_x^j v) (\partial_y f_{m-1}) \\
&\quad - W_m^\ell \left[ \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) - 2 W_m^\ell \left[ \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \right] f_m \\
&\quad + [u \partial_x + v \partial_y - \partial_y^2, W_m^\ell] f_m.
\end{aligned}$$

Moreover if  $\partial_y^j f_m \in L^2$  for all  $j \leq 4$  then

$$(4.3) \quad \partial_y W_m^i f_m \Big|_{y=0} = 0$$

in the sense of trace.



Throughout this section we let  $k \geq 6, 1 \leq i \leq 3$  be two fixed integers. Suppose

$$(4.4) \quad \max_{1 \leq m \leq N_0} \|e^{2cy} f_m\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} < +\infty,$$

where  $N_0$  is a fixed integer, sufficiently large such that the condition (2.21) holds. This is possible due to Sobolev inequalities. Let's first introduce the following induction assumption.

**Definition 4.1** (Assumption  $H_{k,i}(A)$ ). *Let  $A$  be a given constant. We say the sequence  $\{f_m\}_{m \geq 1}$ , which defined by (3.1), satisfies Assumption  $H_{k,i}(A)$  if the following two condition are fulfilled: for any  $6 \leq m \leq k$  and for any  $\ell \in \{0, 1, 2, 3\}$  which satisfy  $m + \frac{\ell}{3} \leq k + \frac{i-1}{3}$ , we have*

$$\begin{aligned} & \|\phi_m^\ell W_m^\ell f_m\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} + \|\partial_y \phi_m^\ell W_m^\ell f_m\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \|\partial_y^2 \Lambda^{-\frac{2}{3}} \phi_m^\ell W_m^\ell f_m\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq A^{m-5+\frac{\ell}{3}} ((m-5)!)^s m^{\frac{\ell s}{3}} \end{aligned}$$

where  $s \geq 3 + \sigma$ .

Now we state the main result of this section.

**Theorem 4.2.** *Let  $u, v$  satisfy (2.19) and (2.21), and let the sequence  $\{f_m\}_{m \geq 1}$ , defined by (3.1), satisfy (4.4). Then there exists a constant  $C_1$ , depending only on*

$$\max_{1 \leq m \leq 5} \|e^{2cy} f_m\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))},$$

such that if and  $\{f_m\}_{m \geq 1}$  satisfies additionally the above Assumption  $H_{k,i}(A)$  with  $A \geq C_1$ , then

$$(4.5) \quad \begin{aligned} & \|\phi_k^i W_k^i f_k\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} + \|\partial_y \phi_k^i W_k^i f_k\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \|\partial_y^2 \Lambda^{-\frac{2}{3}} \phi_k^i W_k^i f_k\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq A^{k-5+\frac{i}{3}} ((k-5)!)^s k^{\frac{is}{3}} \end{aligned}$$

where  $s \geq 3 + \sigma$ .

**4.1. Estimates on nonlinear terms.** In this subsection we estimate the nonlinear terms  $\mathcal{Z}_{m,\ell}$  defined in (4.2), see Proposition 4.3 and Proposition 4.7 below.

**Proposition 4.3.** *Let  $a(D_x)$  be the Fourier multiplier associated to the symbol  $a(\xi) \in S(1, g)$ , let  $u, v$  satisfy (2.19) and (2.21), and let the sequence  $\{f_m\}_{m \geq 1}$ , defined by (3.1), satisfy (4.4) and the above Assumption  $H_{k,i}$ . Then there exists a constant  $C_2$ , depending only on, such that for any  $m \geq k$  and any  $0 \leq \ell \leq i$  we have*

$$\begin{aligned} & \|a(D_x) \phi_m^\ell \mathcal{Z}_{m,\ell}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq C m \|a(D_x) \phi_m^\ell W_m^\ell f_m\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C A^{m-6+\frac{\ell}{3}} ((m-5)!)^s m^{\frac{\ell s}{3}} + C A^{m-5} ((m-5)!)^s. \end{aligned}$$

In view of (4.2), we see

$$(4.6) \quad \begin{aligned} & \|a(D_x) \phi_m^\ell \mathcal{Z}_{m,\ell}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq \sum_{j=1}^m \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell (\partial_x^j u) f_{m+1-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \quad + \sum_{j=1}^{m-1} \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \quad + \sum_{j=1}^{m-1} \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell [\partial_y (\partial_y \omega / \omega)] (\partial_x^j v) (\partial_x^{m-j} u)\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \quad + 2 \|a(D_x) \phi_m^\ell W_m^\ell [\partial_y (\partial_y \omega / \omega)] f_m\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \quad + \|a(D_x) \phi_m^\ell [u \partial_x + v \partial_y - \partial_y^2, W_m^\ell] f_m\|_{L^2([0, T] \times \mathbb{R}_+^2)}. \end{aligned}$$

So in order to prove Proposition 4.3, it suffices to estimate the terms on the right hand side. We will proceed through the following lemmas.

**Lemma 4.4.** *Under the same assumption as in Proposition 4.3, we have*

$$\begin{aligned} & \sum_{j=1}^{m-1} \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell (\partial_x^j v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C_3 m \|a(D_x)\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C_3 A^{m-6+\frac{\ell}{3}} ((m-5)!)^s m^{\frac{\ell s}{3}}. \end{aligned}$$

*Proof.* Denote by  $[p]$  the largest integer less than or equal to  $p$ . We first split the summation as follows:

$$\begin{aligned} & \sum_{j=1}^{m-1} \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell (\partial_x^j v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & = \sum_{j=1}^3 \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell (\partial_x^j v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \sum_{j=m-3}^{m-1} \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell (\partial_x^j v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \sum_{j=4}^{m-4} \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell (\partial_x^j v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

As for last term on the right hand side, we use (3.9) to compute

$$\begin{aligned} & \|a(D_x)\phi_m^\ell W_m^\ell (\partial_x^{m-1} v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \|\phi_m^\ell W_m^0 \Lambda^{\ell/3} (\partial_x^j v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \|\phi_m^\ell W_m^0 \Lambda^1 (\partial_x^j v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \|\phi_m^\ell W_m^0 (\partial_x^j v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\phi_m^\ell W_m^0 \partial_x (\partial_x^j v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \|\phi_m^\ell W_m^0 (\partial_x^j v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \|\phi_m^\ell W_m^0 (\partial_x^j v)(\partial_y \partial_x f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)}, \end{aligned}$$

Thus we have

$$\begin{aligned} & \sum_{j=1}^{m-1} \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell (\partial_x^j v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \sum_{j=1}^3 \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell (\partial_x^j v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \sum_{j=m-3}^{m-1} \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell (\partial_x^{m-1} v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v)(\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j v)(\partial_y \partial_x f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned} \tag{4.7}$$

Next we estimate step by step the terms on the right side of (4.7).

(a) We treat in this step the first and the second terms on the right hand side of (4.7), and prove that

$$\begin{aligned}
(4.8) \quad & \sum_{j=1}^3 \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell (\partial_x^j v) (\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& + \sum_{j=m-3}^{m-1} \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell (\partial_x^{m-1} v) (\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq m \|a(D_x) \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + CA^{j-6+\frac{\ell}{3}} ((m-5)!)^s m^{\frac{\ell s}{3}}.
\end{aligned}$$

To do so, we have

$$\begin{aligned}
& \sum_{j=1}^3 \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell (\partial_x^j v) (\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \sum_{j=1}^3 \binom{m}{j} \|\phi_m^\ell W_m^\ell (\partial_x^j v) (\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \sum_{j=1}^3 \binom{m}{j} \|(\partial_x^j v) \phi_m^\ell W_m^\ell \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \sum_{j=1}^3 \binom{m}{j} \|[W_m^\ell, (\partial_x^j v)] \phi_m^\ell \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq C \sum_{j=1}^3 \binom{m}{j} \|\phi_{m-j}^\ell W_{m-j}^\ell \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)}
\end{aligned}$$

the last inequality using Corollary 2.5. This, along with the estimate:

$$\begin{aligned}
\sum_{j=1}^3 \binom{m}{j} \|\phi_{m-j}^\ell W_{m-j}^\ell \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} & \leq \sum_{j=1}^3 \frac{m!}{j!(m-j)!} A^{m-j-5+\frac{\ell}{3}} ((m-j-5)!)^s (m-j)^{\frac{\ell s}{3}} \\
& \leq CA^{m-6+\frac{\ell}{3}} ((m-5)!)^s m^{\frac{\ell s}{3}}
\end{aligned}$$

due to Hypothesis  $H_{k,i}(A)$  and the fact  $s \geq 2$ , gives

$$(4.9) \quad \sum_{j=1}^3 \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell (\partial_x^j v) (\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq CA^{m-6+\frac{\ell}{3}} ((m-5)!)^s m^{\frac{\ell s}{3}}.$$

Direct computation gives

$$\begin{aligned}
& \sum_{j=m-3}^{m-1} \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell (\partial_x^j v) (\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \sum_{j=m-3}^{m-1} \binom{m}{j} \|a(D_x) \Lambda^{\ell/3} e^{2cy} (1+cy)^{-1} \phi_m^\ell (\partial_x^j v) (\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \sum_{j=m-3}^{m-1} \binom{m}{j} \|e^{2cy} (\partial_y f_{m-j}) a(D_x) \Lambda^{\ell/3} (1+cy)^{-1} \phi_m^\ell \partial_x^j v\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \quad + \sum_{j=m-3}^{m-1} \binom{m}{j} \|[e^{2cy} (\partial_y f_{m-j}), a(D_x) \Lambda^{\ell/3}] (1+cy)^{-1} \phi_m^\ell \partial_x^j v\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \stackrel{\text{def}}{=} J_1 + J_2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
J_1 & \leq \sum_{j=m-3}^{m-1} \binom{m}{j} \|e^{cy} (\partial_y f_{m-j})\|_{L^2(\mathbb{R}_+; L^\infty([0,T] \times \mathbb{R}_x))} \|a(D_x) \phi_m^\ell \Lambda^{\ell/3} \partial_x^j v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))} \\
& \leq C \sum_{j=m-3}^{m-1} \binom{m}{j} \|a(D_x) \phi_{j+1}^\ell \Lambda^{\ell/3} \partial_x^j v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))},
\end{aligned}$$

since  $\phi_m^\ell \leq \phi_{j+1}^\ell$  for  $j \leq m-1$ , and

$$\max_{m-3 \leq j \leq m-1} \|e^{cy}(\partial_y f_{m-j})\|_{L^2(\mathbb{R}_+; L^\infty([0,T] \times \mathbb{R}_x))} \leq C.$$

Moreover using (3.7) gives

$$\begin{aligned} J_2 &\leq C \sum_{j=m-3}^{m-1} \binom{m}{j} \|(1+cy)^{-1} \phi_m^\ell \partial_x^j v\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq C \sum_{j=m-3}^{m-1} \binom{m}{j} \|(1+cy)^{-1}\|_{L^2(\mathbb{R}_+; L^\infty([0,T] \times \mathbb{R}_x))} \|\phi_{j+1}^\ell \partial_x^j v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))} \\ &\leq C \sum_{j=m-3}^{m-1} \binom{m}{j} \|\phi_{j+1}^\ell \partial_x^j v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))} \\ &\leq C \sum_{j=m-3}^{m-1} \binom{m}{j} \|\phi_{j+1}^\ell \Lambda^{\ell/3} \partial_x^j v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))}. \end{aligned}$$

Thus combining these inequalities, we obtain

$$\begin{aligned} &\sum_{j=m-3}^{m-1} \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell (\partial_x^j v) (\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq C \sum_{j=m-3}^{m-1} \|a(D_x) \phi_{j+1}^\ell \Lambda^{\ell/3} \partial_x^j v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))} \\ &\leq C \sum_{j=m-3}^{m-1} \binom{m}{j} \|a(D_x) \phi_{j+1}^\ell W_{j+4}^0 \Lambda^{\ell/3} f_{j+1}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq C \sum_{j=m-3}^{m-1} \binom{m}{j} \|a(D_x) \phi_{j+1}^\ell W_{j+1}^\ell f_{j+1}\|_{L^2([0,T] \times \mathbb{R}_+^2)}, \end{aligned}$$

the second inequality use (3.14) and the last inequality following from (3.9). Now we use Hypothesis  $H_{k,i}(A)$  to calculate, observing  $s \geq 2$ ,

$$\begin{aligned} &\sum_{j=m-3}^{m-1} \binom{m}{j} \|a(D_x) \phi_{j+1}^\ell W_{j+1}^\ell f_{j+1}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq m \|a(D_x) \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + t \sum_{j=m-3}^{m-2} \binom{m}{j} \|\phi_{j+1}^\ell W_{j+1}^\ell f_{j+1}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \\ &\leq m \|a(D_x) \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \sum_{j=m-3}^{m-2} \frac{m!}{j!(m-j)!} A^{j-4+\frac{\ell}{3}} ((j-4)!)^s (j+1)^{\frac{\ell s}{3}} \\ &\leq m \|a(D_x) \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + CA^{j-6+\frac{\ell}{3}} ((m-5)!)^s m^{\frac{\ell s}{3}}. \end{aligned}$$

Thus we have, combining the above inequalities,

$$\begin{aligned} &\sum_{j=m-3}^{m-1} \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell (\partial_x^j v) (\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq m \|a(D_x) \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + CA^{j-6+\frac{\ell}{3}} ((m-5)!)^s m^{\frac{\ell s}{3}}. \end{aligned}$$

This along with (4.9) gives the desired estimate (4.8).

$$\begin{aligned}
& \|a(D_x)\phi_m^\ell W_m^\ell (\partial_x^{m-1}v)(\partial_y f_1)\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
(4.10) \quad & \leq C \left( \|a(D_x)\phi_m^\ell W_{m+3}^0 \Lambda^{\ell/3} f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} + \|\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \right) \\
& \leq C \|a(D_x)\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} + CTA^{m-5+\frac{\ell}{3}} ((m-5)!)^s (m-4)^{\frac{\ell s}{3}},
\end{aligned}$$

the last inequality following from , since

$$\|\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \leq T \|\phi_m^\ell W_m^\ell f_m\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))}$$

for any  $t \in [0, T]$ .

(b) We will estimate in this step the third and the fourth terms on the right hand side of (4.7), and prove that

$$\begin{aligned}
& \sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j v)(\partial_y f_{m-j})\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
(4.11) \quad & + \sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v)(\partial_y f_{m-j})\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
& \leq CA^{m-6} ((m-5)!)^s.
\end{aligned}$$

For this purpose we write

$$\begin{aligned}
& \sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v)(\partial_y f_{m-j})\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
(4.12) \quad & \leq \sum_{j=4}^{\lceil m/2 \rceil} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v)(\partial_y f_{m-j})\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
& + \sum_{j=\lceil m/2 \rceil+1}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v)(\partial_y f_{m-j})\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
& = S_1 + S_2.
\end{aligned}$$

We first treat  $S_1$ . Using the inequality

$$\phi_m^\ell \leq \phi_m^0 \leq \phi_{j+3}^0 \phi_{m-j}^0, \quad W_m^0 \leq W_{m-j}^0 \text{ for } j \geq 1,$$

gives

$$\begin{aligned}
(4.13) \quad S_1 & = \sum_{j=1}^{\lceil m/2 \rceil} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v)(\partial_y f_{m-j})\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
& \leq \sum_{j=1}^{\lceil m/2 \rceil} \binom{m}{j} \|\phi_{j+3}^0 \partial_x^{j+1} v\|_{L^\infty([0,T]\times\mathbb{R}_+^2)} \|\phi_{m-j}^0 W_{m-j}^0 \partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned}$$

By Sobolev inequality, we have

$$\begin{aligned}
\|\phi_{j+3}^0 \partial_x^{j+1} v\|_{L^\infty([0,T]\times\mathbb{R}_+^2)} & \leq \|\phi_{j+3}^0 \partial_x^{j+1} v\|_{L^\infty([0,T]\times\mathbb{R}_+; L^2(\mathbb{R}_x))} + \|\phi_{j+3}^0 \partial_x^{j+2} v\|_{L^\infty([0,T]\times\mathbb{R}_+; L^2(\mathbb{R}_x))} \\
& \leq \|\phi_{j+2}^0 \partial_x^{j+1} v\|_{L^\infty([0,T]\times\mathbb{R}_+; L^2(\mathbb{R}_x))} + \|\phi_{j+3}^0 \partial_x^{j+2} v\|_{L^\infty([0,T]\times\mathbb{R}_+; L^2(\mathbb{R}_x))} \\
& \leq \|\phi_{j+2}^0 W_{j+2}^0 f_{j+2}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \|\phi_{j+3}^0 W_{j+3}^0 f_{j+3}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))},
\end{aligned}$$

the second inequality using (3.7) and the last inequality following from (3.14). Thus

$$\begin{aligned} S_1 &\leq \sum_{j=4}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_{j+2}^0 W_{j+2}^0 f_{j+2}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \|\phi_{m-j}^0 W_{m-j}^0 \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad + \sum_{j=4}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_{j+3}^0 W_{j+3}^0 f_{j+3}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \|\phi_{m-j}^0 W_{m-j}^0 \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

Now we use Hypothesis  $H_{k,i}(A)$  to compute

$$\begin{aligned} &\sum_{j=1}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_{j+3}^0 W_{j+3}^0 f_{j+3}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \|\phi_{m-j}^0 W_{m-j}^0 \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq \sum_{j=4}^{\lfloor m/2 \rfloor} \frac{m!}{j!(m-j)!} A^{j-2} (j-2)!^s \left( A^{m-j-5} ((m-j-5)!)^s \right) \\ &\leq C \sum_{j=4}^{\lfloor m/2 \rfloor} \frac{m!}{j^2(m-j)^5} A^{m-7} (j-2)!^{s-1} ((m-j-5)!)^{s-1} \\ &\leq C \sum_{j=4}^{\lfloor m/2 \rfloor} \frac{m!}{j^2(m/2)^5} A^{m-7} ((m-7)!)^{s-1} \\ &\leq C(m-5)! A^{m-7} ((m-7)!)^{s-1} \\ &\leq C A^{m-6} ((m-5)!)^s. \end{aligned}$$

Repeating the above arguments, we also have

$$\sum_{j=4}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_{j+2}^0 W_{j+2}^0 f_{j+2}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \|\phi_{m-j}^0 W_{m-j}^0 \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C A^{m-6} ((m-5)!)^s.$$

Thus we have

$$(4.14) \quad S_1 \leq C A^{m-6} ((m-5)!)^s.$$

We now treat  $S_2$ . Using the inequality

$$\phi_m^\ell \leq \phi_k^0 \leq \phi_{j+2}^0 \phi_{m-j+1}^0, \quad W_m^0 \leq W_{m-j+1}^0 \quad \text{for } j \geq 1,$$

and thus

$$\begin{aligned} S_2 &= \sum_{j=\lfloor m/2 \rfloor+1}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v) (\partial_y f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ (4.15) \quad &\leq \sum_{j=\lfloor \frac{m}{2} \rfloor+1}^{m-4} \binom{m}{j} \|\phi_{j+2}^0 \partial_x^{j+1} v\|_{L^\infty([0,T] \times \mathbb{R}_+; L^2(\mathbb{R}_x))} \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+; L^\infty(\mathbb{R}_x))} \\ &\leq \sum_{j=\lfloor m/2 \rfloor+1}^{m-4} \binom{m}{j} \|\phi_{j+2}^0 W_{j+2}^0 f_{j+2}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+; L^\infty(\mathbb{R}_x))}, \end{aligned}$$

the last inequality using (3.14). By Sobolev inequality, we have

$$\begin{aligned} &\|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+; L^\infty(\mathbb{R}_x))} \\ &\leq \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y \partial_x f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq \|\phi_{m-j}^0 W_{m-j}^0 \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y \partial_x f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)}, \end{aligned}$$

where the last inequality following from (3.7) and (3.8). Thus

$$\begin{aligned}
S_2 &= \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v) (\partial_y f_{m-j})\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
&\leq \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_{j+2}^0 W_{j+2}^0 f_{j+2}\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} \|\phi_{m-j}^0 W_{m-j}^0 \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
&\quad + \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_{j+2}^0 W_{j+2}^0 f_{j+2}\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y \partial_x f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
&\leq \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_{j+2}^0 W_{j+2}^0 f_{j+2}\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} \|\phi_{m-j}^0 W_{m-j}^0 \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
&\quad + \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_{j+2}^0 W_{j+2}^0 f_{j+2}\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j+1}\|_{L^2([0, T] \times \mathbb{R}_+^2)}.
\end{aligned}$$

Now we use Hypothesis  $H_{k,i}(A)$  to compute

$$\begin{aligned}
&\sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_{j+2}^0 W_{j+2}^0 f_{j+2}\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j+1}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
&\leq \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \frac{m!}{j!(m-j)!} A^{j-3} (j-3)!^s \left( A^{m-j-4} ((m-j-4)!)^s \right) \\
&\leq C \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \frac{m!}{j^3 (m-j)^4} A^{m-7} (j-3)!^{s-1} ((m-j-4)!)^{s-1} \\
&\leq C \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \frac{m!}{(m/2)^3 (m-j)^4} A^{m-7} ((m-7)!)^{s-1} \\
&\leq C(m-3)! A^{m-7} ((m-7)!)^{s-1} \\
&\leq CA^{m-6} ((m-5)!)^s,
\end{aligned}$$

the last inequality using the fact that  $s \geq 2$ . Repeating the above arguments, we also have

$$\sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_{j+2}^0 W_{j+2}^0 f_{j+2}\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} \|\phi_{m-j}^0 W_{m-j}^0 \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq CA^{m-6} ((m-5)!)^s.$$

Thus we have

$$S_2 \leq CA^{m-6} ((m-5)!)^s,$$

which along with (4.14) and (4.12) yields

$$\sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v) (\partial_y f_{m-j})\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq CA^{m-6} ((m-5)!)^s.$$

Repeating the above arguments, we can also deduce that

$$\sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j v) (\partial_y f_{m-j})\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq CA^{m-6} ((m-5)!)^s.$$

Then the desired estimate (4.11) follows.

(c) It remains to prove that

$$(4.16) \quad \sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0(\partial_x^j v)(\partial_y \partial_x f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq CA^{m-6} ((m-5)!)^s.$$

The proof is quite similar as in the previous step. To do so we first write

$$\begin{aligned} & \sum_{j=1}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0(\partial_x^j v)(\partial_y \partial_x f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= \sum_{j=4}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_m^\ell W_m^0(\partial_x^j v)(\partial_y \partial_x f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0(\partial_x^j v)(\partial_y \partial_x f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= \tilde{S}_1 + \tilde{S}_2. \end{aligned}$$

For the term  $\tilde{S}_1$ , we use

$$\phi_m^\ell \leq \phi_k^0 \leq \phi_{j+2}^0 \phi_{m-j+1}^0, \quad W_m^0 \leq W_{m-j+1}^0 \text{ for } j \geq 2,$$

to obtain

$$\begin{aligned} \tilde{S}_1 &\leq \sum_{j=4}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_{j+2}^0 \partial_x^j v\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y \partial_x f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq \sum_{j=4}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_{j+2}^0 \partial_x^j v\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j+1}\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

Then repeating the arguments used in the estimate on the above  $S_1$ , we can deduce that

$$\tilde{S}_1 \leq C(m-5)! A^{m-7} ((m-7)!)^{s-1} \leq CA^{m-6} ((m-6)!)^s.$$

As for  $\tilde{S}_2$ , using the inequality

$$\phi_m^\ell \leq \phi_k^0 \leq \phi_{j+1}^0 \phi_{m-j+2}^0, \quad W_m^0 \leq W_{m-j+2}^0 \text{ for } j \geq 2,$$

gives

$$\begin{aligned} \tilde{S}_2 &= \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0(\partial_x^j v)(\partial_y \partial_x f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_{j+1}^0 \partial_x^j v\|_{L^\infty([0,T] \times \mathbb{R}_+; L^2(\mathbb{R}_x))} \|\phi_{m-j+2}^0 W_{m-j+2}^0 \partial_y \partial_x f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+; L^\infty(\mathbb{R}_x))} \\ &\leq \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_{j+1}^0 W_{j+1}^0 f_{j+1}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \|\phi_{m-j+2}^0 W_{m-j+2}^0 \partial_y f_{m-j+1}\|_{L^2([0,T] \times \mathbb{R}_+; L^\infty(\mathbb{R}_x))}, \end{aligned}$$

the last inequality using (3.14). Then repeating the arguments used to estimate (4.15), we have

$$\tilde{S}_2 \leq C(m-3)! A^{m-7} ((m-7)!)^{s-1} \leq CA^{m-6} ((m-5)!)^s$$

since  $s \geq 2$ . This along with the estimate on  $\tilde{S}_1$  yields (4.16). The proof is thus complete.  $\square$

**Lemma 4.5.** *Under the same assumption as in Proposition 4.3, we have*

$$\begin{aligned} & \sum_{j=1}^m \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell(\partial_x^j u) f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq C_3 m \|a(D_x) \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C_3 A^{m-6+\frac{\ell}{3}} ((m-5)!)^s m^{\frac{\ell s}{3}}. \end{aligned}$$



*Proof.* We first split the summation as follows:

$$\begin{aligned}
& \sum_{j=1}^m \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell(\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&= \sum_{j=1}^3 \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell(\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\quad + \sum_{j=m-3}^m \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell(\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\quad + \sum_{j=4}^{m-4} \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell(\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned}$$

As for last term on the right hand side, we use (3.9) to compute

$$\begin{aligned}
& \|a(D_x)\phi_m^\ell W_m^\ell(\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\leq \|\phi_m^\ell W_m^0 \Lambda^{\ell/3}(\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\leq \|\phi_m^\ell W_m^0 \Lambda^1(\partial_x^j v) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\leq \|\phi_m^\ell W_m^0(\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} + \|\phi_m^\ell W_m^0(\partial_x^{j+1} u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\quad + \|\phi_m^\ell W_m^0(\partial_x^j u) \partial_x f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)},
\end{aligned}$$

Thus we have

$$\begin{aligned}
& \sum_{j=1}^m \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell(\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\leq \sum_{j=1}^3 \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell(\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\quad + \sum_{j=m-3}^m \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell(\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\quad + \sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0(\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\quad + \sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0(\partial_x^{j+1} u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\quad + \sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0(\partial_x^j u) (\partial_x f_{m+1-j})\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned} \tag{4.17}$$

Next we estimate step by step the terms on the right side of (4.17).

(a) We treat in this step the first and the second terms on the right hand side of (4.17), and prove that

$$\begin{aligned}
& \sum_{j=1}^3 \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell(\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\quad + \sum_{j=m-3}^m \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell(\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\leq m \|a(D_x)\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} + CA^{j-6+\frac{\ell}{3}} ((m-5)!)^s m^{\frac{\ell s}{3}}.
\end{aligned} \tag{4.18}$$

We first estimate the first term in (4.18), and we have

$$\begin{aligned}
& \sum_{j=1}^3 \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell (\partial_x^j u) f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \sum_{j=1}^3 \binom{m}{j} \|(\partial_x^j u) a(D_x) \phi_m^\ell W_m^\ell f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \quad + \sum_{j=1}^3 \binom{m}{j} \|[a(D_x) W_m^\ell, (\partial_x^j u)] \phi_m^\ell f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq C \sum_{j=1}^3 \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \quad + C \sum_{j=1}^3 \binom{m}{j} \|[a(D_x) W_m^\ell, (\partial_x^j u)] \phi_m^\ell f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)}
\end{aligned}$$

and moreover using Lemma 2.4 with  $\lambda(D_x) = a(D_x) \Lambda^{\ell/3}$ , we have

$$\begin{aligned}
& \sum_{j=1}^3 \binom{m}{j} \|[a(D_x) W_m^\ell, (\partial_x^j u)] \phi_m^\ell f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq C \sum_{j=1}^3 \binom{m}{j} \|\Lambda^{-1} \phi_m^\ell W_m^0 f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq C \sum_{j=1}^3 \binom{m}{j} \|\phi_{m-j}^0 W_{m-j}^0 f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)},
\end{aligned}$$

where the last inequality following from (3.7). As a result,

$$\begin{aligned}
& \sum_{j=1}^3 \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell (\partial_x^j u) f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq C \sum_{j=1}^3 \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \quad + C \sum_{j=1}^3 \binom{m}{j} \|\phi_{m-j}^0 W_{m-j}^0 f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)}.
\end{aligned}$$

This, along with the estimates:

$$\begin{aligned}
& \sum_{j=1}^3 \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \|a(D_x) \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \quad + \sum_{j=2}^3 \binom{m}{j} \|\phi_{m+1-j}^\ell W_{m+1-j}^\ell f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \|a(D_x) \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \quad + \sum_{j=2}^3 \frac{m!}{j!(m-j)!} A^{m-j-4+\frac{\ell}{3}} ((m-j-4)!)^s (m+1-j)^{\frac{\ell s}{3}} \\
& \leq \|a(D_x) \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C A^{m-6+\frac{\ell}{3}} ((m-5)!)^s m^{\frac{\ell s}{3}}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{j=1}^3 \binom{m}{j} \|\phi_{m-j}^0 W_{m-j}^0 f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \sum_{j=1}^3 \frac{m!}{j!(m-j)!} A^{m-j-5} ((m-j-5)!)^s \\
& \leq CA^{m-6} ((m-5)!)^s
\end{aligned}$$

due to Hypothesis  $H_{k,i}(A)$  and the fact  $s \geq 2$ , gives

$$(4.19) \quad \sum_{j=1}^3 \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell (\partial_x^j u) f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq CA^{m-6+\frac{\ell}{3}} ((m-5)!)^s m^{\frac{\ell s}{3}}.$$

Now we estimate the second term in (4.18), and direct computation gives

$$\begin{aligned}
& \sum_{j=m-3}^m \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell (\partial_x^j u) f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \sum_{j=m-3}^m \binom{m}{j} \|a(D_x) \Lambda^{\ell/3} e^{2cy} (1+cy)^{-1} \phi_m^\ell (\partial_x^j u) f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \sum_{j=m-3}^m \binom{m}{j} \|e^{2cy} (f_{m+1-j}) a(D_x) \Lambda^{\ell/3} (1+cy)^{-1} \phi_m^\ell \partial_x^j u\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \quad + \sum_{j=m-3}^m \binom{m}{j} \|[e^{2cy} f_{m+1-j}, a(D_x) \Lambda^{\ell/3}] (1+cy)^{-1} \phi_m^\ell \partial_x^j u\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \stackrel{\text{def}}{=} Y_1 + Y_2.
\end{aligned}$$

On the other hand, using (3.7) and (3.13) we have

$$\begin{aligned}
Y_1 & \leq \sum_{j=m-3}^m \binom{m}{j} \|e^{2cy} f_{m+1-j}\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \|(1+cy)^{-1} a(D_x) \phi_m^\ell \Lambda^{\ell/3} \partial_x^j u\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq C \sum_{j=m-3}^m \binom{m}{j} \|(1+cy)^{-1} a(D_x) \phi_j^\ell \Lambda^{\ell/3} \partial_x^j u\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq C \sum_{j=m-3}^m \binom{m}{j} \|a(D_x) \phi_j^\ell W_m^\ell f_j\|_{L^2([0,T] \times \mathbb{R}_+^2)},
\end{aligned}$$

and

$$\begin{aligned}
Y_2 & \leq C \sum_{j=m-3}^m \binom{m}{j} \|(1+cy)^{-1} \phi_m^\ell \partial_x^j u\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq C \sum_{j=m-3}^m \binom{m}{j} \|\phi_j^0 W_j^0 f_j\|_{L^2([0,T] \times \mathbb{R}_+^2)}.
\end{aligned}$$

Thus combining these inequalities gives

$$\begin{aligned}
& \sum_{j=m-3}^{m-1} \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell (\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
\leq & C \sum_{j=m-3}^m \binom{m}{j} \|a(D_x)\phi_j^\ell W_j^\ell f_j\|_{L^2([0,T]\times\mathbb{R}_+^2)} + C \sum_{j=m-3}^m \binom{m}{j} \|\phi_j^0 W_j^0 f_j\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
\leq & C \|a(D_x)\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} + C \sum_{j=m-3}^{m-1} \frac{m!}{j!(m-j)!} A^{j-5+\frac{\ell}{3}} ((j-5)!)^s j^{\frac{\ell s}{3}} \\
& + C \sum_{j=m-3}^m \frac{m!}{j!(m-j)!} A^{j-5} ((j-5)!)^s \\
\leq & C \|a(D_x)\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} + CA^{m-5} ((m-5)!)^s,
\end{aligned}$$

the second inequality using Hypothesis  $H_{k,i}(A)$ . Thus we have, combining the above inequalities,

$$\begin{aligned}
& \sum_{j=m-3}^{m-1} \binom{m}{j} \|a(D_x)\phi_m^\ell W_m^\ell (\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
\leq & \|a(D_x)\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} + CA^{m-5} ((m-5)!)^s.
\end{aligned}$$

This along with (4.19) gives the desired estimate (4.18).

(b) We will estimate in this step the third and the fourth terms on the right hand side of (4.17), and prove that

$$\begin{aligned}
(4.20) \quad & \sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
& + \sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
\leq & CA^{m-6} ((m-5)!)^s.
\end{aligned}$$

For this purpose we write

$$\begin{aligned}
(4.21) \quad & \sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
\leq & \sum_{j=4}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
& + \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
= & T_1 + T_2.
\end{aligned}$$

We first treat  $T_1$ . Using the inequality

$$\phi_m^\ell \leq \phi_m^0 \leq \phi_{j+2}^0 \phi_{m+1-j}^0, \quad W_m^0 \leq W_{m+1-j}^0 \text{ for } j \geq 1,$$

gives

$$\begin{aligned}
T_1 &= \sum_{j=4}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\leq \sum_{j=4}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_{j+2}^0 \partial_x^{j+1} u\|_{L^\infty([0,T]\times\mathbb{R}_x; L^2(\mathbb{R}_+))} \|\phi_{m+1-j}^0 W_{m+1-j}^0 f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_x; L^\infty(\mathbb{R}_+))}.
\end{aligned}$$

By Sobolev inequality, we have

$$\begin{aligned} & \left\| \phi_{m+1-j}^0 W_{m+1-j}^0 f_{m+1-j} \right\|_{L^2([0,T] \times \mathbb{R}_x; L^\infty(\mathbb{R}_+))} \\ & \leq \left\| \phi_{m+1-j}^0 W_{m+1-j}^0 f_{m+1-j} \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y \phi_{m+1-j}^0 W_{m+1-j}^0 f_{m+1-j} \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \end{aligned}$$

and

$$\begin{aligned} & \left\| \phi_{j+2}^0 \partial_x^{j+1} u \right\|_{L^\infty([0,T] \times \mathbb{R}_x; L^2(\mathbb{R}_+))} \\ & \leq \left\| \phi_{j+2}^0 \partial_x^{j+1} u \right\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \left\| \phi_{j+2}^0 \partial_x^{j+2} u \right\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \\ & \leq \left\| \phi_{j+1}^0 W_{j+1}^0 f_{j+1} \right\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \left\| \phi_{j+2}^0 W_{j+1}^0 f_{j+2} u \right\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))}, \end{aligned}$$

the last inequality using (3.7) and (3.14). Thus we use Hypothesis  $H_{k,i}(A)$  to compute

$$\begin{aligned} (4.22) \quad T_1 & \leq \sum_{j=4}^{\lfloor m/2 \rfloor} \binom{m}{j} \left\| \phi_{j+2}^0 \partial_x^{j+1} u \right\|_{L^\infty([0,T] \times \mathbb{R}_x; L^2(\mathbb{R}_+))} \left\| \phi_{m+1-j}^0 W_{m+1-j}^0 f_{m+1-j} \right\|_{L^2([0,T] \times \mathbb{R}_x; L^\infty(\mathbb{R}_+))} \\ & \leq \sum_{j=4}^{\lfloor m/2 \rfloor} \frac{m!}{j!(m-j)!} \left( A^{j-4} (j-4)!^s + A^{j-3} (j-3)!^s \right) \left( A^{m-j-4} ((m-j-4)!)^s \right) \\ & \leq C \sum_{j=4}^{\lfloor m/2 \rfloor} \frac{m!}{j^3 (m-j)^4} A^{m-7} (j-3)!^{s-1} ((m-j-4)!)^{s-1} \\ & \leq C \sum_{j=4}^{\lfloor m/2 \rfloor} \frac{m!}{j^3 (m/2)^4} A^{m-7} ((m-7)!)^{s-1} \\ & \leq C (m-4)! A^{m-7} ((m-7)!)^{s-1} \\ & \leq C A^{m-5} ((m-5)!)^s, \end{aligned}$$

the last inequality using the fact  $s \geq 3$ . We now treat  $T_2$ . Using the inequality

$$\phi_m^\ell \leq \phi_k^0 \leq \phi_{j+1}^0 \phi_{m-j+2}^0, \quad W_m^0 \leq W_{m-j+2}^0 \text{ for } j \geq 2,$$

and thus

$$\begin{aligned} T_2 & = \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \left\| \phi_m^\ell W_m^0 (\partial_x^{j+1} u) f_{m+1-j} \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \sum_{j=\lfloor \frac{m}{2} \rfloor + 1}^{m-4} \binom{m}{j} \left\| \phi_{j+1}^0 \partial_x^{j+1} u \right\|_{L^2([0,T] \times \mathbb{R}_x; L^\infty(\mathbb{R}_+))} \left\| \phi_{m-j+2}^0 W_{m-j+2}^0 f_{m+1-j} \right\|_{L^\infty([0,T] \times \mathbb{R}_x; L^2(\mathbb{R}_+))}. \end{aligned}$$

Moreover, by Sobolev inequality, we have

$$\begin{aligned} & \left\| \phi_{j+1}^0 \partial_x^{j+1} u \right\|_{L^2([0,T] \times \mathbb{R}_x; L^\infty(\mathbb{R}_+))} \\ & \leq C \left\| \phi_{j+1}^0 \partial_x^{j+1} u \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \phi_{j+1}^0 \partial_x^{j+1} \partial_y u \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C \left\| \phi_{j+1}^0 W_{j+1}^0 f_{j+1} \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \end{aligned}$$

with the last inequality following from (3.13), and

$$\begin{aligned} & \left\| \phi_{m-j+2}^0 W_{m-j+2}^0 f_{m+1-j} \right\|_{L^\infty([0,T] \times \mathbb{R}_x; L^2(\mathbb{R}_+))} \\ & \leq C \left\| \phi_{m-j+2}^0 W_{m-j+2}^0 f_{m+1-j} \right\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + C \left\| \phi_{m-j+2}^0 W_{m-j+2}^0 \partial_x f_{m+1-j} \right\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \\ & \leq C \left\| \phi_{m-j+1}^0 W_{m-j+1}^0 f_{m+1-j} \right\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + C \left\| \phi_{m-j+2}^0 W_{m-j+2}^0 f_{m+2-j} \right\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \end{aligned}$$

with the last inequality using (3.7). Thus we use Hypothesis  $H_{k,i}(A)$  to compute

$$\begin{aligned}
T_2 &\leq \sum_{j=\lfloor \frac{m}{2} \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_{j+1}^0 \partial_x^{j+1} u\|_{L^2([0,T] \times \mathbb{R}_x; L^\infty(\mathbb{R}_+))} \|\phi_{m-j+2}^0 W_{m-j+2}^0 f_{m+1-j}\|_{L^\infty([0,T] \times \mathbb{R}_x; L^2(\mathbb{R}_+))} \\
&\leq C \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \frac{m!}{j!(m-j)!} A^{j-4} (j-4)!^s \left( A^{m-j-4} ((m-j-4)!)^s + A^{m-j-3} ((m-j-3)!)^s \right) \\
&\leq C \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \frac{m!}{j^4 (m-j)^3} A^{m-7} (j-4)!^{s-1} ((m-j-3)!)^{s-1} \\
&\leq C \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \frac{m!}{(m/2)^4 (m-j)^3} A^{m-7} ((m-7)!)^{s-1} \\
&\leq C(m-4)! A^{m-7} ((m-7)!)^{s-1} \\
&\leq C A^{m-6} ((m-5)!)^s,
\end{aligned}$$

the last inequality using the fact that  $s \geq 3$ . This along with (4.21) and (4.22) gives

$$\sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} u) f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C A^{m-6} ((m-5)!)^s$$

Similarly, we can prove that

$$\sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j u) f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C A^{m-6} ((m-5)!)^s.$$

Thus the desired estimate (4.20) follows.

(c) In this step we will prove that

$$(4.23) \quad \sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j u) (\partial_x f_{m+1-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C A^{m-6} ((m-5)!)^s.$$

The proof is quite similar as in the previous step. To do so we first write

$$\begin{aligned}
&\sum_{j=4}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j u) (\partial_x f_{m+1-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&= \sum_{j=4}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j u) (\partial_x f_{m+1-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&\quad + \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j u) (\partial_x f_{m+1-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&= \tilde{T}_1 + \tilde{T}_2.
\end{aligned}$$

For the term  $\tilde{T}_1$ , we use

$$\phi_m^\ell \leq \phi_k^0 \leq \phi_{j+1}^0 \leq \phi_{m+2-j}^0, \quad W_m^0 \leq W_{m+2-j}^0 \quad \text{for } j \geq 2,$$

to obtain

$$\begin{aligned}
\tilde{T}_1 &\leq \sum_{j=4}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_{j+1}^0 \partial_x^j u\|_{L^\infty([0,T] \times \mathbb{R}_x; L^2(\mathbb{R}_+))} \|\phi_{m+2-j}^0 W_{m+2-j}^0 \partial_x f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_x; L^\infty(\mathbb{R}_+))} \\
&\leq \sum_{j=4}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_{j+1}^0 \partial_x^j u\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \|\phi_{m+2-j}^0 W_{m+2-j}^0 f_{m+2-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)}.
\end{aligned}$$

Then repeating the arguments used in the estimate on  $T_1$  (see(4.22)), we can deduce that

$$\tilde{T}_1 \leq C(m-5)!A^{m-7}((m-7)!)^{s-1} \leq CA^{m-5}((m-5)!)^s.$$

As for  $\tilde{T}_2$ , using the inequality

$$\phi_m^\ell \leq \phi_m^0 \leq \phi_j^0 \phi_{m+3-j}^0, \quad W_m^0 \leq W_{m-j+3}^0 \text{ for } j \geq 3,$$

gives

$$\begin{aligned} \tilde{T}_2 &= \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j u) (\partial_x f_{m+1-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_{j+1}^0 \partial_x^j u\|_{L^2([0,T] \times \mathbb{R}_x; L^\infty(\mathbb{R}_+))} \|\phi_{m-j+2}^0 W_{m-j+2}^0 \partial_x f_{m+1-j}\|_{L^\infty([0,T] \times \mathbb{R}_x; L^2(\mathbb{R}_+))} \\ &\leq \sum_{j=\lfloor m/2 \rfloor + 1}^{m-4} \binom{m}{j} \|\phi_{j+1}^0 \partial_x^j u\|_{L^2([0,T] \times \mathbb{R}_x; L^\infty(\mathbb{R}_+))} \|\phi_{m+2-j}^0 W_{m+2-j}^0 f_{m+2-j}\|_{L^\infty([0,T] \times \mathbb{R}_x; L^2(\mathbb{R}_+))}. \end{aligned}$$

Then repeating the arguments used to estimate  $T_2$ , we have

$$\tilde{T}_2 \leq CA^{m-5}((m-5)!)^s.$$

This along with the estimate on  $\tilde{T}_1$  yields the desired (4.23). The proof is thus complete.  $\square$

**Lemma 4.6.** *We have*

$$\begin{aligned} &\sum_{j=1}^{m-1} \binom{m}{j} \|a(D_x) \phi_m^\ell W_m^\ell [\partial_y (\partial_y \omega / \omega)] (\partial_x^j v) (\partial_x^{m-j} u)\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad + 2 \|a(D_x) \phi_m^\ell W_m^\ell [\partial_y (\partial_y \omega / \omega)] f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad + \|a(D_x) \phi_m^\ell [u \partial_x + v \partial_y - \partial_y^2, W_m^\ell] f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq C \|a(D_x) \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + CA^{m-5}((m-5)!)^s \end{aligned}$$

*Proof.* The proof is similar as in the above two lemmas, we omit here.  $\square$

Combing the estimates in Lemma 4.4-Lemma 4.6, we get the desired estimate in Proposition 4.3, completing the proof of Proposition 4.3.

**Proposition 4.7.** *Let  $a(D_x)$  be the Fourier multiplier associated to the symbol  $a(\xi) \in S(1, g)$ , let  $u, v$  satisfy (2.19) and (2.21), and let the sequence  $\{f_m\}_{m \geq 1}$ , defined by (3.1), satisfy (4.4) and the above Assumption  $H_{k,i}$ . Then there exists a constant  $C_2$ , depending only on, such that for any  $m \geq k$  and any  $0 \leq \ell \leq i$  we have*

$$\begin{aligned} &\|\phi_m^\ell \Lambda^{-\frac{2}{3}} \partial_y \mathcal{Z}_{m,\ell}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq Cm \left( \|\Lambda^{-2/3} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda^{-2/3} \partial_y \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\ &\quad + C \| \langle y \rangle^{-\sigma} \phi_m^\ell \Lambda^{1/3} W_m^\ell f_m \|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad + CA^{m-6+\frac{\ell}{3}} ((m-5)!)^s m^{\frac{\ell s}{3}} + CA^{m-5} ((m-5)!)^s. \end{aligned}$$

*Proof.* Recall the definition of  $\mathcal{Z}_{m,\ell}$ ,

$$\begin{aligned} \mathcal{Z}_{m,\ell} &= - \sum_{j=1}^m \binom{m}{j} W_m^\ell (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} W_m^\ell (\partial_x^j v) (\partial_y f_{m-j}) \\ &\quad - W_m^\ell \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) - 2W_m^\ell \partial_y \left( \frac{\partial_y \omega}{\omega} \right) f_m \\ &\quad + [u \partial_x + v \partial_y - \partial_y^2, W_m^\ell] f_m \end{aligned}$$

Therefore

$$\begin{aligned}\Lambda^{-\frac{2}{3}}\partial_y\mathcal{Z}_{m,\ell} &= -\sum_{j=1}^m\binom{m}{j}\partial_y(\Lambda^{-\frac{2}{3}}W_m^\ell(\partial_x^j u)f_{m+1-j})-\sum_{j=1}^{m-1}\binom{m}{j}\partial_y(\Lambda^{-\frac{2}{3}}W_m^\ell(\partial_x^j v)(\partial_y f_{m-j})) \\ &\quad -\partial_y(\Lambda^{-\frac{2}{3}}W_m^\ell\partial_y(\frac{\partial_y\omega}{\omega})\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u))-\partial_y(2\Lambda^{-\frac{2}{3}}W_m^\ell\partial_y(\frac{\partial_y\omega}{\omega})f_m) \\ &\quad +\partial_y\Lambda^{-\frac{2}{3}}([u\partial_x+v\partial_y-\partial_y^2,W_m^\ell]f_m)\end{aligned}$$

We depart the right side into four parts

$$\|\phi_m^\ell\Lambda^{-\frac{2}{3}}\partial_y\mathcal{Z}_{m,\ell}\|_{L^2([0,T]\times\mathbb{R}_+^2)}\leq Q_1+Q_2+Q_3+Q_4,$$

where

$$\begin{aligned}Q_1 &= \|\phi_m^\ell\sum_{j=1}^m\binom{m}{j}\partial_y(\Lambda^{-\frac{2}{3}}W_m^\ell(\partial_x^j u)f_{m+1-j})\|_{L^2([0,T]\times\mathbb{R}_+^2)}, \\ Q_2 &= \|\phi_m^\ell\sum_{j=1}^{m-1}\binom{m}{j}\partial_y(\Lambda^{-\frac{2}{3}}W_m^\ell(\partial_x^j v)(\partial_y f_{m-j}))\|_{L^2([0,T]\times\mathbb{R}_+^2)}, \\ Q_3 &= \|\phi_m^\ell\partial_y(\Lambda^{-\frac{2}{3}}W_m^\ell\partial_y(\frac{\partial_y\omega}{\omega})\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u))\|_{L^2([0,T]\times\mathbb{R}_+^2)}, \\ Q_4 &= 2\|\phi_m^\ell\Lambda^{-\frac{2}{3}}W_m^\ell\partial_y(\frac{\partial_y\omega}{\omega})f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ Q_5 &= \|\phi_m^\ell\Lambda^{-\frac{2}{3}}\partial_y[u\partial_x+v\partial_y-\partial_y^2,W_m^j]f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}.\end{aligned}$$

The terms  $Q_1$ - $Q_4$  can be treated similarly as in Lemma 4.4-Lemma 4.6, and we have

$$\begin{aligned}\sum_{j=1}^4 Q_j &\leq Cm\left(\|\Lambda^{-2/3}\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}+\|\Lambda^{-2/3}\partial_y\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}\right) \\ &\quad +CA^{m-6+\frac{\ell}{3}}((m-5)!)^s m^{\frac{\ell s}{3}}+CA^{m-5}((m-5)!)^s.\end{aligned}$$

So it remains to estimate  $Q_5$ . Write

$$\begin{aligned}Q_5 &= \|\phi_m^\ell\Lambda^{-\frac{2}{3}}\partial_y[u\partial_x+v\partial_y-\partial_y^2,W_m^j]f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\leq \|\phi_m^\ell[u\partial_x+v\partial_y-\partial_y^2,\Lambda^{-\frac{2}{3}}W_m^j]\Lambda^{-\frac{2}{3}}\partial_y f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\quad +\|\phi_m^\ell[[u\partial_x+v\partial_y-\partial_y^2,\Lambda^{-\frac{2}{3}}W_m^j],\Lambda^{-\frac{2}{3}}\partial_y]f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}.\end{aligned}$$

Moreover, using the arguments in Proposition 4.3 we have

$$\begin{aligned}&\|\phi_m^\ell[u\partial_x+v\partial_y-\partial_y^2,\Lambda^{-\frac{2}{3}}W_m^j]\Lambda^{-\frac{2}{3}}\partial_y f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\leq C\left(\|\Lambda^{-2/3}\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}+\|\Lambda^{-2/3}\partial_y\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}\right).\end{aligned}$$

On the other hand, direct verification shows that

$$\begin{aligned}&\|\phi_m^\ell[[u\partial_x+v\partial_y-\partial_y^2,\Lambda^{-\frac{2}{3}}W_m^j],\Lambda^{-\frac{2}{3}}\partial_y]f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\leq C\langle y\rangle^{-\sigma}\phi_m^\ell\Lambda^{1/3}W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}.\end{aligned}$$

Thus we have

$$\begin{aligned}Q_5 &\leq C\langle y\rangle^{-\sigma}\phi_m^\ell\Lambda^{1/3}W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\quad +C\left(\|\Lambda^{-2/3}\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}+\|\Lambda^{-2/3}\partial_y\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}\right).\end{aligned}$$

This along with the estimates on  $Q_1$ - $Q_4$  gives the desired estimate, completing the proof of Proposition 4.7.  $\square$



**4.2. Upper bound of the terms with highest order.** In this subsection we treat the terms with  $W_k^i f_k$  involved, we will make use of subelliptic to estimate these terms. Precisely we will prove the following proposition.

**Proposition 4.8.** *Let  $u, v$  satisfy (2.19) and (2.21), and let the sequence  $\{f_m\}_{m \geq 1}$ , defined by (3.1), satisfy (4.4) and Assumption  $H_{k,i}(A)$ . Then we have*

$$\begin{aligned}
(4.24) \quad & \left\| \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y^2 \Lambda^{-2/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \varepsilon k^{-1/2} \left( \left\| \partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
& \quad + C_\varepsilon k^{1+\frac{\sigma}{2}} \left( A^{k-5+\frac{i-2}{3}} ((k-5)!)^s k^{\frac{(i-2)s}{3}} \right) \\
& \quad + C_\varepsilon k^{(1+2\sigma)/2} \left( A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{\frac{(i-1)s}{3}} \right).
\end{aligned}$$

*Proof.* (a) In this step we will prove the following two estimate:

$$\begin{aligned}
(4.25) \quad & \left\| \partial_y^2 \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \left\| \partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{(i-1)/3}.
\end{aligned}$$

and

$$\begin{aligned}
(4.26) \quad & \left\| \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq C k^{\sigma/2} \left\| (1+y)^{-\frac{\sigma}{2}} \Lambda^{1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}.
\end{aligned}$$

To prove (4.25), we write

$$\begin{aligned}
& \left\| \partial_y^2 \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \left\| \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_k^i W_k^i \partial_y^2 f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| [\partial_y^2, W_k^i] \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \left\| \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_k^i W_k^i \partial_y^2 f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \left\| W_k^i \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \quad + C \left\| W_k^i \partial_y \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \left\| \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i W_k^{i-1} \partial_y^2 f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \left\| W_k^{i-1} \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \quad + C \left\| W_k^{i-1} \partial_y \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}
\end{aligned}$$

the second inequality following from (3.11) and the last inequality following from (3.9). Moreover for the first term on the right hand side, arguing similarly, we have

$$\begin{aligned}
& \left\| \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i W_k^{i-1} \partial_y^2 f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \left\| \partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i W_k^{i-1} f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| [\partial_y^2, W_k^{i-1}] \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \left\| \partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \left\| W_k^{i-1} \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \quad + C \left\| W_k^{i-1} \partial_y \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}.
\end{aligned}$$

the last inequality holding due to (3.11) and the fact that

$$\phi_k^i \leq \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}}.$$

Combining these inequalities gives

$$\begin{aligned}
& \left\| \partial_y^2 \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq C \left\| \partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \left\| W_k^{i-1} \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \quad + C \left\| W_k^{i-1} \partial_y \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}.
\end{aligned}$$

Observe the last two terms on the right hand side is bounded from above by

$$C A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{(i-1)/3}$$

due to Hypothesis  $H_{k,i}(A)$ , since

$$\|W_k^{i-1}\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^i f_k\|_{L^2([0,T]\times\mathbb{R}_+^2)} \leq \|W_k^{i-1}\phi_k^{i-1} f_k\|_{L^2([0,T]\times\mathbb{R}_+^2)}$$

and

$$\begin{aligned} & \|W_k^{i-1}\partial_y\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^i f_k\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \leq C\|\partial_y W_k^{i-1}\phi_k^{i-1} f_k\|_{L^2([0,T]\times\mathbb{R}_+^2)} + C\|W_k^{i-1}\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_k^{i-1} f_k\|_{L^2([0,T]\times\mathbb{R}_+^2)} \end{aligned}$$

due to (3.7) and (3.10). Thus the desired (4.25) follows.

Now we prove (4.26). Observe

$$\begin{aligned} (1+y)^{-\frac{\sigma}{2}} &= \left(\frac{(3k+i-1)\sigma}{2c}\right)^{-\frac{\sigma}{2}} \left(\frac{2c}{(3k+i-1)\sigma} + \frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{\sigma}{2}} \\ &\geq C\left(\frac{(3k+i-1)\sigma}{2c}\right)^{-\frac{\sigma}{2}} \left(1 + \frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{\sigma}{2}} \\ &\geq Ck^{-\frac{\sigma}{2}} \left(1 + \frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{\sigma}{2}} \end{aligned}$$

Then

$$(4.27) \quad (1+y)^{-\frac{\sigma}{2}} \left(1 + \frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i-1)\sigma}{2}} \geq Ck^{-\frac{\sigma}{2}} \left(1 + \frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}}$$

Moreover we find

$$\begin{aligned} \left(1 + \frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}} &= ((3k+i-1)\sigma)^{\frac{(3k+i)\sigma}{2}} ((3k+i-1)\sigma + 2cy)^{-\frac{(3k+i)\sigma}{2}} \\ &= \left(\frac{(3k+i)\sigma}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}} \left(\frac{(3k+i-1)}{(3k+i)} + \frac{2cy}{(3k+i)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}} \\ &\geq \left(\frac{(3k+i)\sigma}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}} \left(1 + \frac{2cy}{(3k+i)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}}, \end{aligned}$$

and

$$\begin{aligned} \left(\frac{(3k+i)\sigma}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}} &= \left(1 + \frac{1}{3k+i-1}\right)^{-\frac{(3k+i)\sigma}{2}} = \left(1 + \frac{1}{3k+i-1}\right)^{-\frac{(3k+i-1)\sigma}{2}} \frac{3k+i-1}{3k+i} \\ &\geq e^{-\frac{\sigma}{2}}. \end{aligned}$$

Thus

$$\left(1 + \frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}} \geq e^{-\frac{\sigma}{2}} \left(1 + \frac{2cy}{(3k+i)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}},$$

which along with (4.27) gives

$$(1+y)^{-\frac{\sigma}{2}} \left(1 + \frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i-1)\sigma}{2}} \geq Ck^{-\frac{\sigma}{2}} \left(1 + \frac{2cy}{(3k+i)\sigma}\right)^{-\frac{(3k+i)\sigma}{2}}.$$

As a result, recalling

$$(1+y)^{-\frac{\sigma}{2}}\Lambda^{1/3}W_k^{i-1} = (1+y)^{-\frac{\sigma}{2}}e^{2cy}\left(1 + \frac{2cy}{(3k+i-1)\sigma}\right)^{-\frac{(3k+i-1)\sigma}{2}}(1+cy)^{-1}\Lambda^{\frac{i}{3}}$$

we have

$$\|(1+y)^{-\frac{\sigma}{2}}\Lambda^{1/3}\Lambda_\delta^{-2}\phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}}W_k^{i-1}f_k\|_{L^2([0,T]\times\mathbb{R}_+^2)} \geq Ck^{-\sigma/2}\|\Lambda_\delta^{-2}\phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}}W_k^i f_k\|_{L^2([0,T]\times\mathbb{R}_+^2)}.$$

This is just the desired estimate (4.26).

(b) Now we will use Proposition 4.8, with  $f = \Lambda_\delta^{-2}\phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}}W_k^{i-1}f_k$ , to estimate the terms on the right hand sides of (3.11) and (4.26).

Observe

$$\Lambda_\delta^{-2} : L^2(\mathbb{R}_x) \longrightarrow H^2(\mathbb{R}_x)$$

is a bounded operator, then  $\Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k \in L^2([0, T]; H^2(\mathbb{R}_+^2))$  due to Hypotheisi  $H_{k,i}(A)$ , since

$$\begin{aligned} & \left\| \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T]; H^2(\mathbb{R}_+^2))} \\ & \leq C_\delta \left\| \phi_k^{i-1} W_k^{i-1} f_k \right\|_{L^2([0, T]; H^2(\mathbb{R}_+^2))} + C_\delta \left\| \partial_y \phi_k^{i-1} W_k^{i-1} f_k \right\|_{L^2([0, T]; H^2(\mathbb{R}_+^2))} \\ & \quad + C_\delta \left\| \partial_y^2 \Lambda^{-2/3} \phi_k^{i-1} W_k^{i-1} f_k \right\|_{L^2([0, T]; H^2(\mathbb{R}_+^2))}, \end{aligned}$$

with  $C_\delta$  a constant depending only  $\delta$ .

Now we check that the initial-boundary conditions in Proposition 2.2 are fulfilled. By (4.3), we have

$$\partial_y \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k \Big|_{y=0} = 0.$$

And it is clear that

$$\Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k \Big|_{t=0} = \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k \Big|_{t=T} = 0.$$

Now we compute, using the fact that  $\partial_t f_m \Big|_{y=0} = \partial_y^2 f_m \Big|_{y=0}$ ,

$$\begin{aligned} & \partial_t \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k \Big|_{y=0} \\ & = \Lambda_\delta^{-2} \left( \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} \right)' W_k^{i-1} f_k \Big|_{y=0} + \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} \partial_t f_k \Big|_{y=0} \\ & = \Lambda_\delta^{-2} \left( \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} \right)' \Lambda^{(i-1)/3} f_k \Big|_{y=0} + \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} \partial_y^2 f_k \Big|_{y=0} \\ & = \Lambda_\delta^{-2} \left( \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} \right)' \Lambda^{(i-1)/3} f_k \Big|_{y=0} - \left( \frac{2c^2}{(2k+i-1)\sigma} + c^2 \right) \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} \Lambda^{(i-1)/3} f_k \Big|_{y=0} \\ & \quad + \partial_y^2 \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k \Big|_{y=0}, \end{aligned}$$

where the last equality holds because

$$\Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} \partial_y^2 f_k \Big|_{y=0} = \partial_y^2 \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k \Big|_{y=0} - [\partial_y^2, W_k^{i-1}] \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} f_k \Big|_{y=0}$$

and, using the fact that  $\partial_y \Lambda_\delta^{-2} f_k \Big|_{y=0} = 0$ ,

$$-[\partial_y^2, W_k^{i-1}] \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} f_k \Big|_{y=0} = - \left( \frac{2c^2}{(2k+i-1)\sigma} + c^2 \right) \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} \Lambda^{(i-1)/3} f_k \Big|_{y=0}.$$

So we obtain

$$\begin{aligned} & \partial_t \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k \Big|_{y=0} - \partial_y^2 \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k \Big|_{y=0} \\ & = \Lambda_\delta^{-2} \left( \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} \right)' \Lambda^{(i-1)/3} f_k \Big|_{y=0} - \left( \frac{2c^2}{(2k+i-1)\sigma} + c^2 \right) \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} \Lambda^{(i-1)/3} f_k \Big|_{y=0} \\ & \stackrel{\text{def}}{=} \varphi \Big|_{y=0} \end{aligned}$$

with

$$(4.28) \quad \varphi = \Lambda_\delta^{-2} \left( \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} \right)' \Lambda^{(i-1)/3} f_k - \left( \frac{2c^2}{(2k+i-1)\sigma} + c^2 \right) \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} \Lambda^{(i-1)/3} f_k$$

Moreover, in view of (4.1) we see  $\phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k$  satisfies the following equation

$$\begin{aligned} & (\partial_t + u\partial_x + v\partial_y - \partial_y^2) \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k \\ & = \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} \mathcal{Z}_{k,i-1} + \Lambda_\delta^{-2} \left( \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}}(t) \right)' W_k^{i-1} f_k + [u\partial_x + v\partial_y, \Lambda_\delta^{-2}] \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k. \end{aligned}$$

Then by Proposition 2.6 we conclude

$$(4.29) \quad \begin{aligned} & \left\| \langle y \rangle^{-\frac{\sigma}{2}} \Lambda^{1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & + \left\| \partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq \varepsilon (R_2 + R_4 + R_6) + C_\varepsilon (R_1 + R_3 + R_5 + R_7 + R_8 + R_9 + R_{10}) \end{aligned}$$

with

$$(4.30) \quad \begin{aligned} R_1 &= \left\| \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} \mathcal{Z}_{k, i-1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ R_2 &= \left\| \Lambda^{-2/3} \partial_y \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} \mathcal{Z}_{k, i-1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ R_3 &= \left\| \Lambda^{-1/3} \Lambda_\delta^{-2} \left( \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} \right)' W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ R_4 &= \left\| \Lambda^{-2/3} \partial_y \Lambda_\delta^{-2} \left( \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} \right)' W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ R_5 &= \left\| \Lambda^{-1/3} [u \partial_x + v \partial_y, \Lambda_\delta^{-2}] \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ R_6 &= \left\| \Lambda^{-2/3} \partial_y [u \partial_x + v \partial_y, \Lambda_\delta^{-2}] \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ R_7 &= \left\| \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ R_8 &= \left\| \partial_y \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ R_9 &= \left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ R_{10} &= \left\| \langle y \rangle \Lambda^{-1/3} \varphi \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}, \end{aligned}$$

where  $\varphi$  is given in (4.39).

(c) In this step we treat  $R_j$  defined above. Using the estimate

$$\phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} \leq \phi_k^{i-1},$$

we have,

$$\left\| \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} \mathcal{Z}_{k, i-1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq \left\| \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^{i-1} \mathcal{Z}_{k, i-1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)},$$

which, along with the inequalities

$$\begin{aligned} & \left\| \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^{i-1} \mathcal{Z}_{k, i-1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq k \left\| \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^{i-1} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C A^{k-6 + \frac{(i-1)}{3}} ((k-5)!)^s k^{\frac{(i-1)s}{3}} \end{aligned}$$

due to Proposition 4.3, yields

$$(4.31) \quad \begin{aligned} R_1 &= \left\| \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} \mathcal{Z}_{k, i-1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq k \left\| \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^{i-1} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C A^{k-6 + \frac{(i-1)}{3}} ((k-5)!)^s k^{\frac{(i-1)s}{3}} \\ & \leq k \left\| \Lambda^{-1/3} \phi_k^{i-2} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C A^{k-6 + \frac{(i-1)}{3}} ((k-5)!)^s k^{\frac{(i-1)s}{3}}, \end{aligned}$$

since  $\phi_k^{i-1} \leq \phi_k^{i-2}$ . Similarly, using Proposition 4.7, we have

$$(4.32) \quad \begin{aligned} R_2 &= \left\| \Lambda^{-2/3} \partial_y \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} \mathcal{Z}_{k, i-1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq C \left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/3} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \quad + k \left\| \Lambda^{-2/3} \partial_y \phi_k^{i-2} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C A^{k-6 + \frac{(i-1)}{3}} ((k-5)!)^s k^{\frac{(i-1)s}{3}}. \end{aligned}$$

By virtue of the inequality

$$\left( \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} \right)' \leq 3k \phi^{\frac{9(k-4)+3i}{4} - \frac{3}{2}} = 3k \phi_k^{i-2},$$

it follows that

$$(4.33) \quad R_3 = \left\| \Lambda^{-\frac{1}{3}} \Lambda_\delta^{-2} \left( \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} \right)' W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq 3k \left\| \Lambda^{-\frac{1}{3}} \phi_k^{i-2} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}$$

and

$$(4.34) \quad R_4 \leq 3k \left\| \Lambda^{-\frac{2}{3}} \partial_y \phi_k^{i-2} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}.$$

As for (4.30), using (2.12) gives

$$\begin{aligned} R_5 &= \left\| \Lambda^{-1/3} [u \partial_x + v \partial_y, \Lambda_\delta^{-2}] \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ &\leq \left\| [u \partial_x + v \partial_y, \Lambda^{-1/3} \Lambda_\delta^{-2}] \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ &\quad + \left\| [u \partial_x + v \partial_y, \Lambda^{-1/3}] \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}, \end{aligned}$$

which along with Corollary 2.5 yields

$$(4.35) \quad R_5 \leq C \left\| \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq C \left\| \Lambda^{-\frac{1}{3}} \partial_y \phi_k^{i-2} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}.$$

Now we estimate  $R_6$ . It follows from (2.12) that

$$\begin{aligned} R_6 &= \left\| \Lambda^{-2/3} \partial_y [u \partial_x + v \partial_y, \Lambda_\delta^{-2}] \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ &\leq \left\| [u \partial_x + v \partial_y, \Lambda^{-2/3} \partial_y \Lambda_\delta^{-2}] \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ &\quad + \left\| [u \partial_x + v \partial_y, \Lambda^{-2/3} \partial_y] \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ &\leq \left\| [u \partial_x + v \partial_y, \Lambda^{-2/3} \Lambda_\delta^{-2}] \partial_y \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ &\quad + \left\| \Lambda^{-2/3} \Lambda_\delta^{-2} [u \partial_x + v \partial_y, \partial_y] \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ &\quad + \left\| [u \partial_x + v \partial_y, \Lambda^{-2/3}] \partial_y \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ &\quad + \left\| \Lambda^{-2/3} [u \partial_x + v \partial_y, \partial_y] \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}. \end{aligned}$$

On the other hand, by Corollary 2.5 we calculate

$$\begin{aligned} &\left\| [u \partial_x + v \partial_y, \Lambda^{-2/3} \Lambda_\delta^{-2}] \partial_y \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ &\quad + \left\| [u \partial_x + v \partial_y, \Lambda^{-2/3}] \partial_y \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ &\leq C \left\| \Lambda^{-2/3} \partial_y \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ &\leq C \left\| \Lambda^{-2/3} \partial_y \phi_k^{i-2} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}, \end{aligned}$$

and

$$\begin{aligned}
& \left\| \Lambda^{-2/3} \Lambda_\delta^{-2} [u \partial_x + v \partial_y, \partial_y] \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
\leq & \left\| \Lambda^{-2/3} \Lambda_\delta^{-2} (\partial_y u) \partial_x \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
& + \left\| \Lambda^{-2/3} \Lambda_\delta^{-2} (\partial_y v) \partial_y \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
\leq & \left\| (\partial_y u) \Lambda^{-2/3} \Lambda_\delta^{-2} \partial_x \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
& + \left\| [(\partial_y u), \Lambda^{-2/3} \Lambda_\delta^{-2}] \partial_x \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
& + \left\| (\partial_y v) \Lambda^{-2/3} \Lambda_\delta^{-2} \partial_y \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
& + \left\| [(\partial_y v), \Lambda^{-2/3} \Lambda_\delta^{-2}] \partial_y \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
\leq & C \left\| \langle y \rangle^{-\sigma/2} \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
& + C \left\| \Lambda^{-2/3} \partial_y \phi_k^{i-2} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \left\| \Lambda^{-1/3} \phi_k^{i-2} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \left\| \Lambda^{-2/3} [u \partial_x + v \partial_y, \partial_y] \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
\leq & C \left\| \langle y \rangle^{-\sigma/2} \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
& + C \left\| \Lambda^{-2/3} \partial_y \phi_k^{i-2} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \left\| \Lambda^{-1/3} \phi_k^{i-2} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}.
\end{aligned}$$

Combining the above inequalities, we get

$$\begin{aligned}
(4.36) \quad R_6 \leq & C \left\| \langle y \rangle^{-\sigma/2} \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
& + C \left\| \Lambda^{-2/3} \partial_y \phi_k^{i-2} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \left\| \Lambda^{-1/3} \phi_k^{i-2} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}.
\end{aligned}$$

Observe

$$\phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} \leq \phi_k^{i-1},$$

and thus

$$\begin{aligned}
(4.37) \quad R_7 + R_8 = & \left\| \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \left\| \partial_y \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
\leq & \left\| \phi_k^{i-1} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \left\| \partial_y \phi_k^{i-1} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
\leq & T \left\| \phi_k^{i-1} W_k^{i-1} f_k \right\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} + \left\| \partial_y \phi_k^{i-1} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
\leq & CA^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{\frac{(i-1)s}{3}},
\end{aligned}$$

the last inequality using Hypothesis  $H_{k,i}(A)$  (see Definition 4.1). As for  $R_9$ , we have

$$\begin{aligned}
R_9 = & \left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda_\delta^{-2} \Lambda^{1/6} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
= & \left( \langle y \rangle^{-\sigma} \partial_y \Lambda_\delta^{-2} \Lambda^{1/6} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k, \partial_y \Lambda_\delta^{-2} \Lambda^{1/6} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\
= & \left( \langle y \rangle^{-\sigma} \partial_y \Lambda_\delta^{-2} \Lambda^{1/3} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k, \partial_y \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^{i-1} f_k \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\
\leq & \left\| \langle y \rangle^{-\sigma} \partial_y \Lambda_\delta^{-2} \Lambda^{1/3} \phi_k^i W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \left\| \partial_y \Lambda_\delta^{-2} \phi_k^{i-1} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
\leq & C \left( \left\| \partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \right) \left\| \partial_y \phi_k^{i-1} W_k^{i-1} f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)},
\end{aligned}$$

the last inequality following from Lemma 3.2. This implies, for any  $\varepsilon > 0$ ,

$$\begin{aligned}
R_9 &\leq \varepsilon k^{-(1+\sigma)/2} \left( \|\partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
&\quad + C_\varepsilon k^{(1+\sigma)/2} \|\partial_y \phi_k^{i-1} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
(4.38) \quad &\leq \varepsilon k^{-(1+\sigma)/2} \left( \|\partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
&\quad + C_\varepsilon k^{(1+\sigma)/2} \left( A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{\frac{(i-1)s}{3}} \right),
\end{aligned}$$

the last inequality using Hypothesis  $H_{k,i}(A)$  (see Definition 4.1). It remains to estimate  $R_{10}$ . Recall

$$(4.39) \quad \varphi = \Lambda_\delta^{-2} \left( \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} \right)' \Lambda^{(i-1)/3} f_k - \left( \frac{2c^2}{(2k+i-1)\sigma} + c^2 \right) \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} \Lambda^{(i-1)/3} f_k.$$

Then

$$R_{10} = \|\langle y \rangle \Lambda^{-1/3} \varphi\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq Ck \|\Lambda^{-1/3} \phi_k^{i-2} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)},$$

since

$$\langle y \rangle \leq e^{2cy} \left( 1 + \frac{2cy}{(3k+i-1)\sigma} \right)^{-(3k+i-1)\sigma/2} (1+cy)^{-1}.$$

This along with (4.29)-(4.38) gives, for any  $\varepsilon, \varepsilon_1 > 0$ ,

$$\begin{aligned}
&\|\langle y \rangle^{-\sigma/2} \Lambda^{1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&\quad + \|\partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
\leq &\varepsilon_1 \|\langle y \rangle^{-\sigma/2} \Lambda^{1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&+ C_{\varepsilon_1} \left\{ k \|\Lambda^{-1/3} \phi_k^{i-2} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + k \|\Lambda^{-2/3} \partial_y \phi_k^{i-2} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right. \\
&+ \varepsilon k^{-(1+\sigma)/2} \left( \|\partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
&\left. + C_\varepsilon k^{(1+\sigma)/2} \left( A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{\frac{(i-1)s}{3}} \right) \right\},
\end{aligned}$$

and thus, letting  $\varepsilon_1$  be small enough,

$$\begin{aligned}
&\|\langle y \rangle^{-\sigma/2} \Lambda^{1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&\quad + \|\partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
\leq &C \left\{ k \|\Lambda^{-1/3} \phi_k^{i-2} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + k \|\Lambda^{-2/3} \partial_y \phi_k^{i-2} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right. \\
&+ \varepsilon k^{-(1+\sigma)/2} \left( \|\partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
&\left. + C_\varepsilon k^{(1+\sigma)/2} \left( A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{\frac{(i-1)s}{3}} \right) \right\}.
\end{aligned}$$

As a result, we obtain, along with (4.26) and (4.25),

$$\begin{aligned}
&\|\Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y^2 \Lambda^{-2/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
\leq &\varepsilon C k^{-1/2} \left( \|\partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
(4.40) \quad &+ C_\varepsilon k^{1+\frac{\sigma}{2}} \|\Lambda^{-1/3} \phi_k^{i-2} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&+ C_\varepsilon k^{1+\frac{\sigma}{2}} \|\Lambda^{-2/3} \partial_y \phi_k^{i-2} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
&+ C_\varepsilon k^{(1+2\sigma)/2} \left( A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{\frac{(i-1)s}{3}} \right)
\end{aligned}$$

(d) We treat the second and the third terms on the right hand side of (4.40). To do so we use the relation

$$\phi_k^{i-2} \phi_k^{i-2} = \phi^{\frac{18(k-4)+6(i-2)}{4}} = \phi^{\frac{9((k-1)-4)+3i}{4}} \phi^{\frac{9(k-4)+3(i-1)}{4}} = \phi_{k-1}^i \phi_k^{i-1} \leq \phi_{k-1}^i \phi_{k-1}^i,$$

to write

$$\begin{aligned} \|\Lambda^{-1/3} \phi_k^{i-2} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 &= \left( \Lambda^{-2/3} \phi_{k-1}^i W_k^{i-1} f_k, \phi_k^{i-1} W_k^{i-1} f_k \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq \|\Lambda^{-2/3} \phi_{k-1}^i W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \|\phi_k^{i-1} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)}, \end{aligned}$$

On the other hand, it follows from Lemma 3.3 that

$$\|\Lambda^{-2/3} \phi_{k-1}^i W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C \|\phi_{k-1}^i W_{k-1}^i f_{k-1}\|_{L^2([0,T] \times \mathbb{R}_+^2)}.$$

Thus,

$$\begin{aligned} \|\Lambda^{-1/3} \phi_k^{i-2} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 &\leq C \|\phi_{k-1}^i W_{k-1}^i f_{k-1}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \|\phi_k^{i-1} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq C A^{k-6+\frac{i}{3}} ((k-6)!)^s k^{\frac{is}{3}} \left( A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{\frac{(i-1)s}{3}} \right) \\ &\leq C \left( A^{k-5+\frac{i-2}{3}} ((k-5)!)^s k^{\frac{(i-2)s}{3}} \right)^2, \end{aligned}$$

that is,

$$(4.41) \quad \|\Lambda^{-1/3} \phi_k^{i-2} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C A^{k-5+\frac{i-2}{3}} ((k-5)!)^s k^{\frac{(i-2)s}{3}}.$$

Similarly by virtue of Lemma 3.3 it follows that

$$\begin{aligned} &\|\Lambda^{-2/3} \partial_y \phi_k^{i-2} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq \|\Lambda^{-2/3} \partial_y \phi_{k-1}^i W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq C \left( \|\partial_y \phi_{k-1}^i W_{k-1}^i f_{k-1}\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\phi_k^{i-1} W_{k-1}^i f_{k-1}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right), \end{aligned}$$

and thus using again Hypothesis  $H_{k,i}(A)$  gives

$$\|\Lambda^{-2/3} \partial_y \phi_k^{i-2} W_k^{i-1} f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C A^{k-6+\frac{i}{3}} ((k-6)!)^s k^{\frac{is}{3}} \leq C A^{k-5+\frac{i-2}{3}} ((k-5)!)^s k^{\frac{(i-2)s}{3}}.$$

This along with (4.40) and (4.41) gives

$$\begin{aligned} &\|\Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_m^i f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y^2 \Lambda^{-2/3} \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4}-\frac{1}{2}} W_m^i f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq \varepsilon C k^{-1/2} \left( \|\partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\ &\quad + C_\varepsilon k^{1+\frac{\sigma}{2}} \left( A^{k-5+\frac{i-2}{3}} ((k-5)!)^s k^{\frac{(i-2)s}{3}} \right) \\ &\quad + C_\varepsilon k^{(1+2\sigma)/2} \left( A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{\frac{(i-1)s}{3}} \right) \end{aligned}$$

This is just the desired estimate, since  $\varepsilon$  is an arbitrary number. The proof is thus complete.  $\square$

**4.3. Proof of Theorem 4.2.** Now we prove Theorem 4.2, the main result of this section. We will proceed through the following steps.

*Step 1)* In view of (4.1) we see the function  $\Lambda_\delta^{-2} \phi_k^i W_k^i f_k$  satisfies the equation

$$(4.42) \quad \begin{aligned} &(\partial_t + u \partial_x + v \partial_y - \partial_y^2) \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \\ &= \Lambda_\delta^{-2} \phi_k^i \mathcal{Z}_{k,i} + \Lambda_\delta^{-2} \frac{d\phi_k^i}{dt} W_k^i f_k + [u \partial_x + v \partial_y, \Lambda_\delta^{-2}] \phi_k^i W_k^i f_k. \end{aligned}$$

in the sense of distribution. In this step we will show that all the terms in the above equation belong to  $L^2([0,T] \times \mathbb{R}_+^2)$ .

Observe

$$\Lambda_\delta^{-2} : L^2(\mathbb{R}_x) \longrightarrow H^2(\mathbb{R}_x)$$



is a bounded operator and thus

$$u\partial_x\Lambda_\delta^{-2}\phi_k^iW_k^if_k, \quad v\partial_y\Lambda_\delta^{-2}\phi_k^iW_k^if_k, \quad \partial_y^2\Lambda_\delta^{-2}\phi_k^iW_k^if_k \in L^2([0, T] \times \mathbb{R}_+^2),$$

due to Assumption  $H_{k,i}(A)$  (see Definition 4.1), since

$$\|u\partial_x\Lambda_\delta^{-2}\phi_k^iW_k^if_k\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq C_\delta \|u\|_{L^\infty([0, T] \times \mathbb{R}_+^2)} \|\phi_k^0W_k^0f_k\|_{L^2([0, T] \times \mathbb{R}_+^2)},$$

$$\|v\partial_y\Lambda_\delta^{-2}\phi_k^iW_k^if_k\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq C_\delta \|v\|_{L^\infty([0, T] \times \mathbb{R}_+^2)} \|\partial_y\phi_k^0W_k^0f_k\|_{L^2([0, T] \times \mathbb{R}_+^2)}$$

and

$$\|\partial_y^2\Lambda_\delta^{-2}\phi_k^iW_k^if_k\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq C_\delta \|\Lambda^{-2/3}\partial_y^2\phi_k^0W_k^0f_k\|_{L^2([0, T] \times \mathbb{R}_+^2)},$$

with  $C_\delta$  a constant depending only on  $\delta$ , where we used (3.7). Similarly, observe

$$\left| \frac{d\phi_k^i}{dt} \right| \leq \phi_k^0$$

and thus

$$\|\Lambda_\delta^{-2}\frac{d\phi_k^i}{dt}W_k^if_k\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq C_\delta \|\phi_k^0W_k^0f_k\|_{L^2([0, T] \times \mathbb{R}_+^2)} < +\infty,$$

which gives

$$\Lambda_\delta^{-2}\frac{d\phi_k^i}{dt}W_k^if_k \in L^2([0, T] \times \mathbb{R}_+^2).$$

Finally it follows from Proposition 4.3 and Corollary 2.5 that

$$\|\Lambda_\delta^{-2}\phi_k^i\mathcal{Z}_{k,i}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq Ck \|\Lambda_\delta^{-2}\phi_k^iW_k^if_k\|_{L^2([0, T] \times \mathbb{R}_+^2)} + CA^{k-6+\frac{i}{3}}((k-5)!)^s k^{\frac{is}{3}}$$

and

$$(4.43) \quad \|[u\partial_x + v\partial_y, \Lambda_\delta^{-2}]\phi_k^iW_k^if_k\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq \|\Lambda_\delta^{-2}\phi_k^iW_k^if_k\|_{L^2([0, T] \times \mathbb{R}_+^2)};$$

this implies

$$\Lambda_\delta^{-2}\phi_k^i\mathcal{Z}_{k,i}, \quad [u\partial_x + v\partial_y, \Lambda_\delta^{-2}]\phi_k^iW_k^if_k \in L^2([0, T] \times \mathbb{R}_+^2).$$

since

$$\|\Lambda_\delta^{-2}\phi_k^iW_k^if_k\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq C_\delta \|\phi_k^0W_k^0f_k\|_{L^2([0, T] \times \mathbb{R}_+^2)} < \infty.$$

By now we have shown that all the terms except  $\partial_t\Lambda^{i/3}\Lambda_\delta^{-2}\phi_k^if_k$  are in  $L^2([0, T] \times \mathbb{R}_+^2)$ , and thus so is  $\partial_t\Lambda^{i/3}\Lambda_\delta^{-2}\phi_k^if_k$  by equation (4.42). Moreover

$$(4.44) \quad \partial_y\Lambda_\delta^{-2}\phi_k^iW_k^if_k|_{y=0} = 0$$

due to (4.3), since  $\partial_y^2\Lambda^{i/3}\Lambda_\delta^{-2}\phi_k^if_k \in L^2([0, T] \times \mathbb{R}_+^2)$  and  $\partial_y\Lambda^{i/3}\Lambda_\delta^{-2}\phi_k^if_k|_{y=0} = 0$ .

*Step 2)* Let  $t \in [0, T]$  be an arbitrary number. The conclusion in the previous step enables us to take  $L^2([0, t] \times \mathbb{R}_+^2)$  inner product over with  $\Lambda_\delta^{-2}\phi_k^iW_k^if_k$  on both sides ; this gives

$$\begin{aligned} & \operatorname{Re} (\partial_t\Lambda_\delta^{-2}\phi_k^iW_k^if_k, \Lambda_\delta^{-2}\phi_k^iW_k^if_k)_{L^2([0, t] \times \mathbb{R}_+^2)} \\ & + \operatorname{Re} ((u\partial_x + v\partial_y)\Lambda_\delta^{-2}\phi_k^iW_k^if_k, \Lambda_\delta^{-2}\phi_k^iW_k^if_k)_{L^2([0, t] \times \mathbb{R}_+^2)} \\ & - \operatorname{Re} (\partial_y^2\Lambda_\delta^{-2}\phi_k^iW_k^if_k, \Lambda_\delta^{-2}\phi_k^iW_k^if_k)_{L^2([0, t] \times \mathbb{R}_+^2)} \\ = & \operatorname{Re} (\Lambda_\delta^{-2}\phi_k^i\mathcal{Z}_{k,i}, \Lambda_\delta^{-2}\phi_k^iW_k^if_k)_{L^2([0, t] \times \mathbb{R}_+^2)} + \operatorname{Re} \left( \Lambda_\delta^{-2}\frac{d\phi_k^i}{dt}W_k^if_k, \Lambda_\delta^{-2}\phi_k^iW_k^if_k \right)_{L^2([0, t] \times \mathbb{R}_+^2)} \\ & + \operatorname{Re} ([u\partial_x + v\partial_y, \Lambda_\delta^{-2}]\phi_k^iW_k^if_k, \Lambda_\delta^{-2}\phi_k^iW_k^if_k)_{L^2([0, t] \times \mathbb{R}_+^2)}. \end{aligned}$$

On the other hand , we have

$$\operatorname{Re} (\partial_t\Lambda_\delta^{-2}\phi_k^iW_k^if_k, \Lambda_\delta^{-2}\phi_k^iW_k^if_k)_{L^2([0, t] \times \mathbb{R}_+^2)} = \frac{1}{2} \|\Lambda_\delta^{-2}\phi_k^i(t)W_k^if_k(t)\|_{L^2(\mathbb{R}_+^2)}^2$$

by integrating by parts, since

$$\Lambda_\delta^{-2}\phi_k^iW_k^if_k|_{t=0} = 0$$

due to the fact that  $\phi_k^i(0) = 0$ . Similarly observing  $u|_{y=0} = v|_{y=0} = 0$ , we integrate by parts to get

$$\begin{aligned} & \operatorname{Re} \left( (u\partial_x + v\partial_y) \Lambda_\delta^{-2} \phi_k^i W_k^i f_k, \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right)_{L^2([0,t] \times \mathbb{R}_+^2)} \\ &= -\frac{1}{2} \left( (\partial_x u + \partial_y v) \Lambda_\delta^{-2} \phi_k^i W_k^i f_k, \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right)_{L^2([0,t] \times \mathbb{R}_+^2)} \\ &= 0, \end{aligned}$$

the last inequality using again the fact that  $\partial_x u + \partial_y v = 0$ . Moreover in view of (4.44) we have,

$$-\operatorname{Re} \left( \partial_y^2 \Lambda_\delta^{-2} \phi_k^i W_k^i f_k, \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right)_{L^2([0,t] \times \mathbb{R}_+^2)} = \left\| \partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,t] \times \mathbb{R}_+^2)}^2.$$

Combining these equations we conclude, for any  $t \in [0, T]$ ,

$$\begin{aligned} & \frac{1}{2} \left\| \Lambda_\delta^{-2} \phi_k^i(t) W_k^i f_k(t) \right\|_{L^2(\mathbb{R}_+^2)}^2 + \int_0^t \left\| \partial_y \Lambda_\delta^{-2} \phi_k^i(t) W_k^i f_k(t) \right\|_{L^2(\mathbb{R}_+^2)}^2 dt \\ & \leq \left| \left( \Lambda_\delta^{-2} \phi_k^i \mathcal{Z}_{k,i}, \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right)_{L^2([0,t] \times \mathbb{R}_+^2)} \right| + \left| \left( \Lambda_\delta^{-2} \frac{d\phi_k^i}{dt} W_k^i f_k, \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right)_{L^2([0,t] \times \mathbb{R}_+^2)} \right| \\ & \quad + \left| \left( [u\partial_x + v\partial_y, \Lambda_\delta^{-2}] \phi_k^i W_k^i f_k, \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right)_{L^2([0,t] \times \mathbb{R}_+^2)} \right| \\ & \leq \left\| \Lambda_\delta^{-2} \phi_k^i \mathcal{Z}_{k,i} \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + 3k \left\| \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \quad + C \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \end{aligned}$$

the last inequality using (4.43) the fact that

$$\left| \frac{d\phi_k^i}{dt} \right| \leq 3k \phi^{\frac{9(k-4)+3i}{4} - 1},$$

recalling  $\phi_k^i = \phi^{\frac{9(k-4)+3i}{4}}$  with  $\phi(t) = t(T-t)$ . Moreover using Proposition 4.3 gives

$$\left\| \Lambda_\delta^{-2} \phi_k^i \mathcal{Z}_{k,i} \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq Ck \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + CA^{k-6+\frac{i}{3}} ((k-5)!)^s k^{\frac{is}{3}}$$

and thus

$$\begin{aligned} & \left\| \Lambda_\delta^{-2} \phi_k^i \mathcal{Z}_{k,i} \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq Ck \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \left( A^{k-6+\frac{i}{3}} ((k-5)!)^s k^{\frac{is}{3}} \right) \\ & \leq Ck \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C \left( A^{k-6+\frac{i}{3}} ((k-5)!)^s k^{\frac{is}{3}} \right)^2. \end{aligned}$$

As a result we obtain

$$\begin{aligned} & \frac{1}{2} \left\| \Lambda_\delta^{-2} \phi_k^i(t) W_k^i f_k(t) \right\|_{L^2(\mathbb{R}_+^2)}^2 + \int_0^t \left\| \partial_y \Lambda_\delta^{-2} \phi_k^i(t) W_k^i f_k(t) \right\|_{L^2(\mathbb{R}_+^2)}^2 dt \\ & \leq Ck \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C \left( A^{k-6+\frac{i}{3}} ((k-5)!)^s k^{\frac{is}{3}} \right)^2 \\ & \quad + 3k \left\| \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \leq Ck \left\| \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C \left( A^{k-6+\frac{i}{3}} ((k-5)!)^s k^{\frac{is}{3}} \right)^2, \end{aligned}$$

where the last inequality holds because

$$\phi_k^i = \phi^{\frac{9(k-4)+3i}{4}} \leq \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}}$$

since  $\phi \leq 1$ . Then, observing  $t \in [0, T]$  is arbitrary,

$$\begin{aligned} & \left\| \Lambda_\delta^{-2} \phi_k^i W_k^i f_k \right\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \left\| \Lambda_\delta^{-2} \partial_y \phi_k^i W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq Ck^{1/2} \left\| \Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^i f_k \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + CA^{k-6+\frac{i}{3}} ((k-5)!)^s k^{\frac{is}{3}}, \end{aligned}$$

and thus

$$\begin{aligned}
& \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \|\Lambda_\delta^{-2} \partial_y \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \quad + \|\partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
(4.45) \quad & \leq Ck^{1/2} \|\Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \quad + CA^{k-6+\frac{i}{3}} ((k-5)!)^s k^{\frac{is}{3}} \\
& \leq Ck^{1/2} \left( \|\Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
& \quad + CA^{k-6+\frac{i}{3}} ((k-5)!)^s k^{\frac{is}{3}},
\end{aligned}$$

Moreover using Proposition 4.8 gives

$$\begin{aligned}
& k^{1/2} \left( \|\Lambda_\delta^{-2} \phi^{\frac{9(k-4)+3i}{4} - \frac{1}{2}} W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
& \leq Ck^{1/2} \left\{ \varepsilon k^{-1/2} \left( \|\partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \right. \\
& \quad + C_\varepsilon k^{1+\frac{\sigma}{2}} \left( A^{k-5+\frac{i-2}{3}} ((k-5)!)^s k^{\frac{(i-2)s}{3}} \right) \\
& \quad \left. + C_\varepsilon k^{(1+2\sigma)/2} \left( A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{\frac{(i-1)s}{3}} \right) \right\} \\
& \leq \varepsilon C \left( \|\partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
& \quad + C_\varepsilon k^{\frac{3+\sigma}{2}} A^{k-5+\frac{i-2}{3}} ((k-5)!)^s k^{\frac{(i-2)s}{3}} + C_\varepsilon k^{1+\sigma} A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{\frac{(i-1)s}{3}} \\
& \leq \varepsilon C \left( \|\partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
& \quad + C_\varepsilon A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{\frac{is}{3}},
\end{aligned}$$

the last inequality holding because  $s = 3 + 3\sigma > (9 + 3\sigma)/4$  and thus

$$k^{\frac{3+\sigma}{2}} \leq k^{2s/3}, \quad k^{1+\sigma} \leq k^{s/3}.$$

This along with (4.45) gives

$$\begin{aligned}
& \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \|\Lambda_\delta^{-2} \partial_y \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \quad + \|\partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq \varepsilon C \left( \|\partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
& \quad + C_\varepsilon A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{\frac{is}{3}} \\
& \leq \varepsilon C \left( \|\partial_y \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \right) \\
& \quad + C_\varepsilon A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{\frac{is}{3}}
\end{aligned}$$

and thus, letting  $\varepsilon$  be small sufficiently,

$$\begin{aligned}
& \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \|\Lambda_\delta^{-2} \partial_y \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq C_* A^{k-5+\frac{i-1}{3}} ((k-5)!)^s k^{\frac{is}{3}},
\end{aligned}$$

where  $C_*$  is a constant depending only on, but independent of  $\delta$ . Thus if we choose  $A$  in such a way that

$$A \geq C_*^3,$$

then we obtain

$$\begin{aligned}
& \|\Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \|\Lambda_\delta^{-2} \partial_y \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
& \leq A^{k-5+\frac{i}{3}} ((k-5)!)^s k^{\frac{is}{3}}.
\end{aligned}$$

Since  $A$  doesn't depend on  $\delta$ , the above inequality implies, letting  $\delta \rightarrow 0$ ,

$$\begin{aligned} & \|\phi_k^i W_k^i f_k\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \|\partial_y \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y^2 \Lambda^{-1/3} \phi_k^i W_k^i f_k\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq A^{k-5+\frac{1}{3}} ((k-5)!)^s k^{\frac{ks}{3}}. \end{aligned}$$

The is just the desired estimate. So the proof of Theorem 4.2 is thus complete.

## 5. PROOF OF THE MAIN RESULT

This section is devoted to the proof Theorem, the main result on Gevrey smoothing effect. By Theorem 4.2, we can use induction on  $k$  to claim that

$$\begin{aligned} & \|\phi_m^\ell W_m^\ell f_m\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \|\phi_m^\ell \partial_y W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda^{-\frac{2}{3}} \phi_m^\ell \partial_y^2 W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq A^{m-5+\frac{\ell}{3}} ((m-5)!)^s m^{\frac{\ell s}{3}} \end{aligned}$$

holds for all integers  $k, i$  with  $k \geq 6$  and  $0 \leq i \leq 3$ . Here  $s = 3 + 3\sigma$ . As a result by (3.13) we have, for any  $m \geq 6$ ,

$$\begin{aligned} & \|\phi_m^0 e^{\tilde{c}y} \partial_x^m u\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \leq C \|\phi_m^0 W_m^0 f_m\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \\ & \leq CA^{m-5} ((m-5)!)^s \leq CA^m (m!)^s. \end{aligned}$$

Now we choose  $A$  such that

$$A \geq \sum_{1 \leq m \leq 5} \|e^{\tilde{c}y} \partial_x^m u\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))}.$$

Then

$$\forall m \geq 1, \quad \|\phi_m^0 e^{\tilde{c}y} \partial_x^m u\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \leq CA^m (m!)^s.$$

Observe  $\phi_m^0 \geq t^{9m/4} (T-t)^{9m/4}$  and thus the desired estimate (1.3) in Theorem 1.1 follows. We complete the proof of Theorem 1.1.

## 6. APPENDIX

**6.1. Equation for  $f_m$ .** Let  $u$  be the classical solution to the Prandtl equation (1.1), and recall

$$f_m = \partial_x^m \omega - \frac{\partial_y \omega}{\omega} \partial_x^m u = \omega \partial_y \left( \frac{\partial_x^m u}{\omega} \right),$$

where  $\omega = \partial_y u$ . Here we derive the equation satisfied by  $f_m$ . Taking  $\partial_y$  on the equation

$$(6.1) \quad u_t + uu_x + vu_y - u_{yy} = 0,$$

we have, using the condition  $u_x + v_y = 0$ ,

$$(6.2) \quad \omega_t + u\omega_x + v\omega_y - \omega_{yy} = 0.$$

Then taking  $\partial_x^m$  on both equations (6.1) and (6.2), we infer that

$$\partial_t f_m + u \partial_x f_m + v \partial_y f_m - \partial_y^2 f_m = \mathcal{Z}_m,$$

where

$$\begin{aligned} \mathcal{Z}_m &= - \sum_{j=1}^m \binom{m}{j} (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_y f_{m-1}) \\ &\quad - \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) + (\partial_x^m u) f_1 \\ &\quad + \left( \partial_t \left( \frac{\partial_y \omega}{\omega} \right) + u \partial_x \left( \frac{\partial_y \omega}{\omega} \right) + v \partial_y \left( \frac{\partial_y \omega}{\omega} \right) - \partial_y^2 \left( \frac{\partial_y \omega}{\omega} \right) \right) \partial_x^m u \\ &\quad - 2 \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \partial_y \partial_x^m u. \end{aligned}$$

On the other hand we notice that

$$\begin{aligned}
& \partial_t \left( \frac{\partial_y \omega}{\omega} \right) + u \partial_x \left( \frac{\partial_y \omega}{\omega} \right) + v \partial_y \left( \frac{\partial_y \omega}{\omega} \right) - \partial_y^2 \left( \frac{\partial_y \omega}{\omega} \right) \\
&= \frac{1}{\omega} (\partial_t \partial_y \omega + u \partial_x \partial_y \omega + v \partial_y \partial_y \omega - \partial_y^2 \partial_y \omega) \\
&\quad - \frac{\partial_y \omega}{\omega^2} (\partial_t \omega + u \partial_x \omega + v \partial_y \omega - \partial_y^2 \omega) + 2 \frac{\partial_y \omega}{\omega} \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \\
&= -\partial_x \omega + \frac{(\partial_x u)(\partial_y \omega)}{\omega} + 2 \frac{\partial_y \omega}{\omega} \partial_y \left( \frac{\partial_y \omega}{\omega} \right)
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\mathcal{Z}_m &= -\sum_{j=1}^m \binom{m}{j} (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_y f_{m-1}) \\
&\quad - \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) + (\partial_x^m u) f_1 \\
&\quad + \left( \partial_x \omega - \frac{(\partial_x u)(\partial_y \omega)}{\omega} \right) \partial_x^m u + 2 \frac{\partial_y \omega}{\omega} \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \partial_x^m u - 2 \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \partial_y \partial_x^m u \\
&= -\sum_{j=1}^m \binom{m}{j} (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_y f_{m-1}) \\
&\quad - \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) + \partial_y \left( \left( \frac{\partial_y \omega}{\omega} \right)^2 \right) \partial_x^m u \\
&\quad - 2 \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \partial_x^m \omega \\
&= -\sum_{j=1}^m \binom{m}{j} (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_y f_{m-1}) \\
&\quad - \partial_y \left( \frac{\partial_y \omega}{\omega} \right) \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) - 2 \partial_y \left( \frac{\partial_y \omega}{\omega} \right) f_m
\end{aligned}$$

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WEI-XI LI, SCHOOL OF MATHEMATICS, WUHAN UNIVERSITY 430072, WUHAN, P.R. CHINA  
*E-mail address:* wei-xi.li@whu.edu.cn

DI WU, SCHOOL OF MATHEMATICS, WUHAN UNIVERSITY 430072, WUHAN, P.R. CHINA

C.-J. XU, UNIVERSITÉ DE ROUEN, CNRS UMR 6085-MATHÉMATIQUES, 76801 SAINT-ETIENNE DU ROUVRAY, FRANCE, AND,  
 SCHOOL OF MATHEMATICS, WUHAN UNIVERSITY 430072, WUHAN, P.R. CHINA  
*E-mail address:* Chao-Jiang.Xu@univ-rouen.fr