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GEVREY CLASS SMOOTHING EFFECT FOR THE PRANDTL EQUATION

WEI-XI LI, DI WU AND CHAO-JIANG XU

ABSTRACT. It is well known that the Prandtl boundary layer equation is unstable, and the well-posedness in Sobolev space for the Cauchy problem is an open problem. Recently, under the Oleinik's monotonicity assumption for the initial datum, [1] have proved the local well-posedness of Cauchy problem in Sobolev space (see also [21]). In this work, we study the Gevrey smoothing effects of the local solution obtained in [1]. We prove that the Sobolev's class solution belongs to some Gevrey class with respect to tangential variables at any positive time.

1. INTRODUCTION

In this work, we study the regularity of solutions to the Prandtl equation which is the foundation of the boundary layer theory introduced by Prandtl in 1904, [24]. The inviscid limit of an incompressible viscous flow with the non-slip boundary condition is still a challenging problem of mathematical analysis due to the appearance of a boundary layer, where the tangential velocity adjusts rapidly from nonzero away from the boundary to zero on the boundary. Prandtl equation describes the behavior of the flow near the boundary in the small viscosity limit, and it reads

$$\begin{cases} u_t + uu_x + vu_y + p_x = u_{yy}, & t > 0, \quad x \in \mathbb{R}, \quad y > 0, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = U(t, x), \\ u|_{t=0} = u_0(x, y), \end{cases}$$

where $u(t, x, y)$ and $v(t, x, y)$ represent the tangential and normal velocities of the boundary layer, with y being the scaled normal variable to the boundary, while $U(t, x)$ and $p(t, x)$ are the values on the boundary of the tangential velocity and pressure of the outflow satisfying the Bernoulli law

$$\partial_t U + U \partial_x U + \partial_x q = 0.$$

Because of the degeneracy in tangential variable, the well-posedness theories and the justification of the Prandtl's boundary layer theory remain as the challenging problems in the mathematical theory of fluid mechanics. Up to now, there are only a few rigorous mathematical results (see [4, 13, 14, 15, 22] and referencesin). Under a monotonic assumption on the tangential velocity of the outflow, Oleinik was the first to obtain the local existence of classical solutions for the initial-boundary value problems, and this result together with some of her works with collaborators were well presented in the monograph [23]. In addition to Oleinik's monotonicity assumption on the velocity field, by imposing a so called favorable condition on the pressure, Xin-Zhang [26] obtained the existence of global weak solutions to the Prandtl equation. All these well-posedness results were based on the Crocco transformation to overcome the main difficulty caused by degeneracy and mixed type of the equation. Very recently the well-posedness in the Sobolev space was explored by virtue of energy method instead of the Crocco transformation; see Alexandre et. all [1] and Masmoudi-Wong [21]. There is very few work concerned with the Prandtl equation without the monotonicity assumption; we refer [2, 3, 20, 9, 25, 30] for the works in the analytic frame, and [12, 17] for the recent works concerned with the existence in Gevrey class. Recall Gevrey class, denoted by G^s , $s \geq 1$, is an intermediate space between analytic functions and C^∞ space. Given a domain Ω , the (global) Gevrey space $G^s(\Omega)$ is consist of such functions that $f \in C^\infty(\Omega)$ and that

$$\|\partial^\alpha f\|_{L^2(\Omega)} \leq L^{|\alpha|+1} (\alpha!)^s$$

for some constant L independent of α . The significant difference between Gevrey ($s > 1$) and analytic ($s = 1$) classes is that there exist nontrivial Gevrey functions admitting compact support.

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We mention that due to the degeneracy in x , it is natural to expect Gevrey regularity rather than analyticity for a subelliptic equation. We refer [5, 6, 7, 8] for the link between subellipticity and Gevrey regularity. In this paper we first study the intrinsic subelliptic structure due to the monotonicity condition, and then deduce, basing on the subelliptic estimate, the Gevrey smoothing effect; that is, given a monotonic initial data belonging to some Sobolev space, the solution will lie in some Gevrey class at positive time, just as like heat equation. It is different from the Gevrey propagation property obtained in the aforementioned works, where the initial data is supposed to be of some Gevrey class, for instance $G^{7/4}$ in [12], and the well-posedness is obtained in the same Gevrey space.

Now we state our main result. Without loss of generality, we only consider here the case of an uniform outflow $U = 1$, and the conclusion will still hold for Gevrey class outflow U . We mention that the Gevrey regularity for outflow U is well developed (see [18] for instance). For the uniform outflow, we get the constant pressure p due to the Bernoulli law. Then the Prandtl equation can be rewritten as

$$(1.1) \quad \begin{cases} u_t + uu_x + vv_y - u_{yy} = 0, & (t, x, y) \in]0, T[\times \mathbb{R}_+^2, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = 1, \\ u|_{t=0} = u_0(x, y), \end{cases}$$

The main result concerned with the Gevrey class regularity can be stated as follows.

Theorem 1.1. *Let $u(t, x, y)$ be a classical local in time solution to Prandtl equation (1.1) on $[0, T]$ with the properties subsequently listed below:*

(i) *There exist two constants $C_* > 1, \sigma > 1/2$ such that for any $(t, x, y) \in [0, T] \times \mathbb{R}_+^2$,*

$$(1.2) \quad \begin{aligned} C_*^{-1} \langle y \rangle^{-\sigma} &\leq \partial_y u(t, x, y) \leq C_* \langle y \rangle^{-\sigma}, \\ |\partial_y^2 u(t, x, y)| + |\partial_y^3 u(t, x, y)| &\leq C_* \langle y \rangle^{-\sigma-1}, \end{aligned}$$

where $\langle y \rangle = (1 + |y|^2)^{1/2}$.

(ii) *There exists $c > 0, C_0 > 0$ and integer $N_0 \geq 7$ such that*

$$(1.3) \quad \|e^{2cy} \partial_x u\|_{L^\infty([0, T]; H^{N_0}(\mathbb{R}_+^2))} + \|e^{2cy} \partial_x \partial_y u\|_{L^2([0, T]; H^{N_0}(\mathbb{R}_+^2))} \leq C_0.$$

Then for any $0 < T_1 < T$, there exists a constant L , such that for any $0 < t \leq T_1$,

$$(1.4) \quad \forall m > 1 + N_0, \quad \|e^{\tilde{c}y} \partial_x^m u(t)\|_{L^2(\mathbb{R}_+^2)} \leq t^{-3(m-N_0-1)} L^m (m!)^{3(1+\sigma)},$$

where $0 < \tilde{c} < c$. The constants L depends only on $C_0, T_1, C_*, c, \tilde{c}$ and σ . Therefore, the solution u belongs to the Gevrey class of index $3(1 + \sigma)$ with respect to $x \in \mathbb{R}$ for any $0 < t \leq T_1$.

Remark 1.2.

1). Such a solution in the above theorem exists, for instance, suppose that the initial data u_0 can be written as

$$u_0(x, y) = u_0^s(y) + \tilde{u}_0(x, y),$$

where u_0^s is a function of y but independent of x such that $C^{-1} \langle y \rangle^{-\sigma} \leq \partial_y u_0^s(y) \leq C \langle y \rangle^{-\sigma}$ for some constant $C \geq 1$, and \tilde{u}_0 is a small perturbation such that its weighted Sobolev norm $\|e^{2cy} \tilde{u}_0\|_{H^{2N_0+7}(\mathbb{R}_+^2)}$ is suitably small. Then using the arguments in [1], we can obtain the desired solution with the properties listed in Theorem 1.1 fulfilled. Precisely, the solution $u(t, x, y)$ is a perturbation of a shear flow $u^s(t, y)$ such that property (i) in the above theorem holds for u , and moreover $e^{2cy} (u - u^s) \in L^\infty([0, T]; H^{N_0+1}(\mathbb{R}_+^2))$.

2). The well-posedness problem of Prandtl's equation depends crucially on the choice of the underlying function spaces, especially on the regularity in the tangential variable x . If the initial datum is analytic in x , then the local in time solution exists (c.f. [20, 25, 30]), but the Cauchy problem is ill-posedness in Sobolev space for linear and non linear Prandtl equation (cf. see [10, 11]). Indeed, the main mathematical difficulty is the lack of control on the x derivatives. For example, v in (1.1) may be written as $-\int_0^y u_x(y') dy'$ by the divergence condition, and here we lose one derivatives in x -regularity. The degeneracy can't be balanced directly by any horizontal diffusion term, so that the standard energy estimates do not apply to establish the existence of local solution. But the results in our main Theorem 1.1 shows that *the loss of derivative in tangential variable x can be partially compensated via the monotonic condition.*

- 3). Under the hypothesis (1.2), the equation (1.1) is a non linear hypoelliptical equation of Hörmander type with a gain of regularity of order $\frac{1}{3}$ in x variable (see Proposition 2.4), so that any C^2 solution is locally C^∞ , see [27, 28, 29]; for the corresponding linear operator, [8] obtained the regularity in the local Gevrey space G^3 . However, in this paper we study the equation (1.1) as a boundary layer equation, so that the local property of solution is not of interest to the physics application, and our goal is then to study the global estimates in Gevrey class. In view of (1.2) we see u_y decays polynomially at infinite, so we only have a weighted subelliptic estimate (see Proposition 2.4). This explains why the Gevrey index, which is $3(1 + \sigma)$, depends also on the decay index σ in (1.2).
- 4). Finally, the estimate (1.4) gives an explicit Gevrey norm of solutions for the Cauchy problem with respect to $t > 0$ when the initial datum is only in some finite order Sobolev space. Since the Gevrey class is an intermediate space between analytic space and Sobolev space, the qualitative study of solutions in Gevrey class can help us to understand the Prandtl boundary layer theory which has been justified in analytic frame.

The paper is organized as follows. In Section 2 we prove Theorem 1.1, and state some preliminaries lemmas used in the proof. The other sections are occupied by the proof of the preliminaries lemmas. Precisely, we prove in Section 3 a subelliptic estimate for the linearized Prandtl operator. Section 4 and Section 5 are devoted to presenting a crucial estimate for an auxilliary function and non linear terms. The last section is an appendix, where the equation fulfilled by the auxilliary function is deduced.

2. PROOF FOR THE MAIN THEOREM

We will prove in this section the Gevrey estimate (1.4) by induction on m . As in [21], we consider the following auxilliary function

$$(2.1) \quad f_m = \partial_x^m \omega - \frac{\partial_y \omega}{\omega} \partial_x^m u = \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right), \quad m \geq 1,$$

where $\omega = \partial_y u > 0$ and u is a solution of equation (1.1) which satisfy the hypotheis (1.2). We also introduce the following inductive weight ,

$$(2.2) \quad W_m^\ell = e^{2cy} \left(1 + \frac{2cy}{(3m + \ell)\sigma} \right)^{-\frac{(3m + \ell)\sigma}{2}} (1 + cy)^{-1} \Lambda^{\frac{\ell}{3}}, \quad 0 \leq \ell \leq 3, \quad m \in \mathbb{N}, \quad y > 0,$$

where $\Lambda^d = \Lambda_x^d$ is the Fourier multiplier of symbol $\langle \xi \rangle^d$ with respect to $x \in \mathbb{R}$. Notting

$$(2.3) \quad W_m^0 \geq e^{cy} (1 + cy)^{-1} \geq c_0 e^{\tilde{c}y},$$

for $0 < \tilde{c} < c$.

Since

$$\left| \frac{\partial_y \omega}{\omega} \right| \leq C_*^2 \langle y \rangle^{-1},$$

we have that , if u is smooth,

$$\|W_m^0 f_m\|_{L^2(\mathbb{R}_+^2)} \leq \|W_m^0 \partial_x^m \omega\|_{L^2(\mathbb{R}_+^2)} + C_*^2 \|W_m^0 \langle y \rangle^{-1} \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}.$$

On the other hand, we have the following Poincaré type inequality.

Lemma 2.1. *There exist $C_1, \tilde{C}_1 > 0$ independents of $m \geq 1, 0 \leq \ell \leq 3$, such that*

$$(2.4) \quad \|\langle y \rangle^{-1} W_m^\ell \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} + \|\langle y \rangle^{-1} W_m^\ell \partial_x^m \omega\|_{L^2(\mathbb{R}_+^2)} \leq C_1 \|W_m^\ell f_m\|_{L^2(\mathbb{R}_+^2)}.$$

As a result,

$$(2.5) \quad \|\Lambda^{-1} W_m^0 f_{m+1}\|_{L^2(\mathbb{R}_+^2)} \leq \tilde{C}_1 \|W_m^0 f_m\|_{L^2(\mathbb{R}_+^2)},$$

and

$$\|\Lambda^{-1} \partial_y W_m^0 f_{m+1}\|_{L^2(\mathbb{R}_+^2)} \leq \tilde{C}_1 \left(\|\partial_y W_m^0 f_m\|_{L^2(\mathbb{R}_+^2)} + \|W_m^0 f_m\|_{L^2(\mathbb{R}_+^2)} \right).$$

We will prove the above lemma in the section 4 as Lemma 4.2.

Since the initial datum of the equation (1.1) is only in Sobolev space H^{N_0+1} , we have to introduce the following cut-off function, with respect to $0 \leq t \leq T \leq 1$, to study the Gevrey smoothing effect by using the hypelliticity,

$$(2.6) \quad \phi_m^\ell = \phi^{3(m - (N_0+1)) + \ell} = (t(T - t))^{3(m - (N_0+1)) + \ell}, \quad m \geq N_0 + 1, \quad 0 \leq \ell \leq 3.$$

We will prove by induction an energy estimate for the function $\phi_m^0 W_m^0 f_m$. For this purpose we need the following lemma concerned with the link between $\phi_{m+1}^0 W_{m+1}^0 f_{m+1}$ and $\phi_m^3 W_m^3 f_m$, whose proof is postponed to the section 4 as Lemma 4.3 and Lemma 4.4.

Lemma 2.2. *There exists a constant C_2 , depending only on the numbers σ , c and the constant C_* in Theorem 1.1, in particular, independents on m , such that for any $m \geq N_0 + 1$,*

$$\begin{aligned} & \left\| \phi_{m+1}^0 W_{m+1}^0 f_{m+1} \right\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} + \sum_{j=1}^2 \left\| \partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_{m+1}^0 W_{m+1}^0 f_{m+1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq C_2 \left\| \phi_m^3 W_m^3 f_m \right\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} + C_2 \sum_{j=1}^2 \left\| \partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_m^3 W_m^3 f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}, \end{aligned}$$

and

$$\begin{aligned} & \left\| \partial_y^3 \Lambda^{-1} \phi_{m+1}^0 W_{m+1}^0 f_{m+1} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq C_2 \left\| \phi_m^3 W_m^3 f_m \right\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} + C_2 \sum_{j=1}^2 \left\| \partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_m^3 W_m^3 f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \quad + C_2 \left\| \partial_y^3 \Lambda^{-1} \phi_m^3 W_m^3 f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}, \end{aligned}$$

and

$$\left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/3} \Lambda_\delta^{-2} W_m^{\ell-1} f_m \right\|_{L^2(\mathbb{R}_+^2)} \leq C_2 \left\| \partial_y \Lambda_\delta^{-2} W_m^\ell f_m \right\|_{L^2(\mathbb{R}_+^2)} + C_2 \left\| \Lambda_\delta^{-2} W_m^\ell f_m \right\|_{L^2(\mathbb{R}_+^2)}.$$

Now we prove Theorem 1.1 by induction on the estimate of $\phi_m^0 W_m^0 f_m$. The procedure of induction is as follows.

Initial hypothesis of the induction. From the hypothesis (1.2) and (1.3) of Theorem 1.1, we have firstly, in view of (2.1),

$$(2.7) \quad 0 \leq m \leq N_0 + 1, \quad \left\| e^{2cy} f_m \right\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} + \sum_{i=1}^3 \left\| e^{2cy} \partial_y^i f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} < C_0.$$

Hypothesis of the induction. Suppose that there exists $A > C_0 + 1$ such that, for some $m \geq N_0 + 1$ and for any $N_0 + 1 \leq k \leq m$, we have

$$(2.8) \quad \partial_y^3 \Lambda^{-1} \phi_k^0 W_k^0 f_k \in L^2([0, T] \times \mathbb{R}_+^2),$$

$$(2.9) \quad \left\| \phi_k^0 W_k^0 f_k \right\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} + \sum_{j=1}^2 \left\| \partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_k^0 W_k^0 f_k \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \leq A^{k-5} ((k-5)!)^{3(1+\sigma)}.$$

Claim I_{m+1} : we claim that (2.8) and (2.9) are also true for $m+1$. As a result, (2.8) and (2.9) hold for all $k \geq N_0 + 1$ by induction.

Completeness of the proof for Theorem 1.1. .

Before proving the above **Claim I_{m+1}** , we remark that Theorem 1.1 is just its immediate consequence. Indeed, induction processus imply that for any $m > 1 + N_0$, we have for any $0 < t < T$,

$$\left\| \phi_m^0 W_m^0 f_m(t) \right\|_{L^2(\mathbb{R}_+^2)} \leq A^{m-5} ((m-5)!)^{3(1+\sigma)} \leq A^m (m!)^{3(1+\sigma)},$$

then with (2.2), (2.3), (2.4) and (2.6), we get

$$\forall 0 < t \leq T_1 < T \leq 1, \quad t^{3(m-N_0-1)} \left\| e^{\tilde{c}y} \partial_x^m u \right\|_{L^2(\mathbb{R}_+^2)} \leq (T - T_1)^{-3(m-N_0-1)} \left\| \phi_m^0 W_m^0 f_m \right\|_{L^2(\mathbb{R}_+^2)},$$

yields, for any $m > N_0 + 1$,

$$\begin{aligned} \forall 0 < t \leq T_1 < T \leq 1, \quad t^{3(m-N_0-1)} \left\| e^{\tilde{c}y} \partial_x^m u \right\|_{L^2(\mathbb{R}_+^2)} & \leq (T - T_1)^{-3(m-N_0-1)} A^m (m!)^{3(1+\sigma)} \\ & \leq (T - T_1)^{-3m} A^m (m!)^{3(1+\sigma)}. \end{aligned}$$

As a result, Theorem 1.1 follows if we take $L = (T - T_1)^{-3} A$. \square

Now we begin to prove **Claim I_{m+1}** , and to do so it is sufficient to prove that the following:
Claim $E_{m,\ell}$, $0 \leq \ell \leq 3$: The following property hold for $0 \leq \ell \leq 3$,

$$(2.10) \quad \begin{aligned} & \partial_y^3 \Lambda^{-1} \phi_m^\ell W_m^\ell f_m \in L^2([0, T] \times \mathbb{R}_+^2), \\ & \|\phi_m^\ell W_m^\ell f_m\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} + \sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_m^\ell W_m^\ell f_m\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq A^{m-5+\frac{\ell}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{\ell(1+\sigma)}. \end{aligned}$$

In fact, **Claim $E_{m,3}$** yields $\partial_y^3 \Lambda^{-1} \phi_m^3 W_m^3 f_m \in L^2([0, T] \times \mathbb{R}_+^2)$ and

$$\begin{aligned} & \|\phi_m^3 W_m^3 f_m\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} + \sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_m^3 W_m^3 f_m\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq A^{m-5+\frac{1}{2}} ((m-5)!)^{3(1+\sigma)} (m-4)^{3(1+\sigma)} \\ & = A^{m-5+\frac{1}{2}} [(m+1)-5!]^{3(1+\sigma)}, \end{aligned}$$

which, along with Lemma 2.2, yields $\partial_y^3 \Lambda^{-1} \phi_{m+1}^0 W_{m+1}^0 f_{m+1} \in L^2([0, T] \times \mathbb{R}_+^2)$ and

$$\begin{aligned} & \|\phi_{m+1}^0 W_{m+1}^0 f_{m+1}\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} + \sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_{m+1}^0 W_{m+1}^0 f_{m+1}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq C_2 A^{m-5+\frac{1}{2}} [(m+1)-5!]^{3(1+\sigma)}, \end{aligned}$$

recalling C_2 is a constant depending only on the numbers σ, c and the constants C_0, C_* in Theorem 1.1. As a result, if we choose A in such a way that

$$A^{1/2} \geq C_2,$$

then we see (2.9) is also valid for $k = m + 1$. Thus the desired **Claim I_{m+1}** follows.

Proof of the Claim $E_{m,\ell}$.

The rest of this section is devoted to proving Claim $E_{m,\ell}$ holds for all $0 \leq \ell \leq 3$, supposing the inductive hypothesis (2.8) and (2.8) hold.

We will prove **Claim $E_{m,\ell}$** by iteration on $0 \leq \ell \leq 3$. Obviously **Claim $E_{m,0}$** holds, due to the hypothesis of induction (2.8) and (2.9) with $k = m$. Now supposing **Claim $E_{m,i}$** holds for all $0 \leq i \leq \ell - 1$, i.e., for all $0 \leq i \leq \ell - 1$ we have

$$(2.11) \quad \begin{aligned} & \partial_y^3 \Lambda^{-1} \phi_m^i W_m^i f_m \in L^2([0, T] \times \mathbb{R}_+^2), \\ & \|\phi_m^i W_m^i f_m\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} + \sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_m^i W_m^i f_m\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \leq A^{m-5+\frac{i}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{i(1+\sigma)}, \end{aligned}$$

we will prove in the remaining part **Claim $E_{m,\ell}$** also holds. To do so, we first introduce the mollifier $\Lambda_\delta^{-2} = \Lambda_{\delta,x}^{-2}$ which is the Fourier multiplier with the symbol $\langle \delta \xi \rangle^{-2}$, $0 < \delta < 1$, and then consider the function $F = \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m$. Under the inductive assumption (2.11), we see F is a classical solution to the following problem (See the detail computation in Section 6 and the equation (6.1) fulfilled by f_m):

$$(2.12) \quad \begin{cases} (\partial_t + u \partial_x + v \partial_y - \partial_y^2) F = \mathcal{Z}_{m,\ell,\delta}, \\ \partial_y F|_{y=0} = 0, \\ F|_{t=0} = 0, \end{cases}$$

where

$$(2.13) \quad \mathcal{Z}_{m,\ell,\delta} = \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell \mathcal{Z}_m + \Lambda_\delta^{-2} (\partial_t \phi_m^\ell) W_m^\ell f_m + [u \partial_x + v \partial_y - \partial_y^2, \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell] f_m,$$

with \mathcal{Z}_m given in the appendix (seeing Section 6), that is,

$$\begin{aligned} \mathcal{Z}_m &= - \sum_{j=1}^m \binom{m}{j} (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_y f_{m-1}) \\ &\quad - \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) - 2 \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] f_m. \end{aligned}$$

The initial value and boundary value in (2.12) is take in the sense of trace in Sobolev space, due to the induction hypothesis (2.9) and the facts that $\partial_y \Lambda_\delta^{-2} \phi_m^\ell f_m|_{y=0} = 0$ (seeing (6.5) in the appendix) and

$$\partial_y \left(e^{2cy} \left(1 + \frac{2cy}{(3m+i)\sigma} \right)^{-(3m+i)\sigma/2} (1+cy)^{-1} \right) \Big|_{y=0} = 0.$$

We will prove an energy estimate for the equation (2.12). For this purpose, let $t \in [0, T]$, and take $L^2([0, t] \times \mathbb{R}_+^2)$ inner product with F on both sides of the first equation in (2.12); this gives

$$\operatorname{Re} \left((\partial_t + u\partial_x + v\partial_y - \partial_y^2) F, F \right)_{L^2([0, t] \times \mathbb{R}_+^2)} = \operatorname{Re} \left(\mathcal{Z}_{m, \ell, \delta}, F \right)_{L^2([0, t] \times \mathbb{R}_+^2)}.$$

Moreover observing the initial-boundary conditions in (2.12) and the facts that $u|_{y=0} = v|_{y=0} = 0$ and $\partial_x u + \partial_y v = 0$, we integrate by parts to obtain,

$$\operatorname{Re} \left((\partial_t + u\partial_x + v\partial_y - \partial_y^2) F, F \right)_{L^2([0, t] \times \mathbb{R}_+^2)} = \frac{1}{2} \|F(t)\|_{L^2(\mathbb{R}_+^2)}^2 + \int_0^t \|\partial_y F(t)\|_{L^2(\mathbb{R}_+^2)}^2 dt.$$

Thus we infer

$$\|F\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))}^2 + \|\partial_y F\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \leq 2 \left| (\mathcal{Z}_{m, \ell, \delta}, F)_{L^2([0, T] \times \mathbb{R}_+^2)} \right|,$$

and thus

$$\begin{aligned} & \|F\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))}^2 + \|\partial_y F\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\partial_y^2 \Lambda^{-2/3} F\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ (2.14) \quad & \leq 2 \left| (\mathcal{Z}_{m, \ell, \delta}, F)_{L^2([0, T] \times \mathbb{R}_+^2)} \right| + \|\partial_y^2 \Lambda^{-2/3} F\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \leq 2 \|\phi^{1/2} \mathcal{Z}_{m, \ell, \delta}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \|\phi^{-1/2} F\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \|\partial_y^2 \Lambda^{-2/3} F\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2. \end{aligned}$$

In order to treat the first term on the right hand side, we need the following proposition, whose proof is postponed to Section 5.

Proposition 2.3. *Under the induction hypothesis (2.7)–(2.9) and (2.11), there exists a constant C_3 , such that, using the notation $F = \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m$ and $\tilde{f} = \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m$ with ϕ defined in (2.6),*

$$\begin{aligned} & \|\phi^{1/2} \mathcal{Z}_{m, \ell, \delta}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ (2.15) \quad & \leq mC_3 \|\phi^{-1/2} F\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C_3 \|\partial_y F\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C_3 A^{m-6} ((m-5)!)^{3(1+\sigma)}, \end{aligned}$$

and

$$\begin{aligned} & \|\Lambda^{-1/3} \phi^{1/2} \mathcal{Z}_{m, \ell-1, \delta}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ (2.16) \quad & \leq mC_3 \|\Lambda^{-1/3} \phi^{-1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & + C_3 \|\partial_y \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C_3 A^{m-6} ((m-5)!)^{3(1+\sigma)}, \end{aligned}$$

and

$$\begin{aligned} & \|\Lambda^{-2/3} \partial_y \phi^{1/2} \mathcal{Z}_{m, \ell-1, \delta}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ (2.17) \quad & \leq C_3 \|\langle y \rangle^{-\sigma} \Lambda^{1/3} \tilde{f}\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C_3 \|\partial_y^2 \Lambda^{-2/3} \tilde{f}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & + mC_3 \left(\|\Lambda^{-2/3} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \|\Lambda^{-2/3} \phi^{-1/2} \partial_y \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0, T] \times \mathbb{R}_+^2)} \right) \\ & + C_3 A^{m-6} ((m-5)!)^{3(1+\sigma)}. \end{aligned}$$

The constant C_3 depends only on σ, c , and the constant C_* , but is independent of m and δ .

Now combining (2.15) in the above proposition and (2.14), we have

$$\begin{aligned} & \|F\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))}^2 + \sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} F\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \leq 2mC_3 \|\phi^{-1/2} F\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + (2C_3)^2 \|\phi^{-1/2} F\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \frac{1}{2} \|\partial_y F\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \quad \left(A^{m-6} ((m-5)!)^{3(1+\sigma)} \right)^2 + \|\partial_y^2 \Lambda^{-2/3} F\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2, \end{aligned}$$

which yields, denoting by $C_4 = 4C_3 + 10C_3^2 + 2$,

$$\begin{aligned} & \|F\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))}^2 + \sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} F\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \leq mC_4 \|\phi^{-1/2} F\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + 2 \left(A^{m-6} ((m-5)!)^{3(1+\sigma)} \right)^2 + 2 \|\partial_y^2 \Lambda^{-2/3} F\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2, \end{aligned}$$

or equivalently,

$$(2.18) \quad \begin{aligned} & \|F\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} F\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C_4 \left(m^{1/2} \|\phi^{-1/2} F\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y^2 \Lambda^{-2/3} F\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) + 2A^{m-6} ((m-5)!)^{3(1+\sigma)}. \end{aligned}$$

It remains to treat the right terms on the right hand side. To do so we need to study the subellipticity of the linearized Prandtl equation :

$$(2.19) \quad \mathcal{P}f = \partial_t f + u \partial_x f + v \partial_y f - \partial_y^2 f = h, \quad (t, x, y) \in]0, T[\times \mathbb{R}_+^2,$$

where u, v is solution of Prandtl's equation (1.1) satisfying the condition (1.2) and (1.3). Then we have

Proposition 2.4. *Let $h, g \in L^2([0, T] \times \mathbb{R}_+^2)$ be given such that $\partial_y h, \partial_y g \in L^2([0, T] \times \mathbb{R}_+^2)$. Suppose that $f \in L^2([0, T]; H^2(\mathbb{R}_+^2))$ with $\partial_y^3 f \in L^2([0, T] \times \mathbb{R}_+^2)$, is a classical solution to the equation (2.19) with the following initial and boundary conditions :*

$$(2.20) \quad f(0, x, y) = f(T, x, y) = 0, \quad (x, y) \in \mathbb{R}_+^2,$$

and

$$(2.21) \quad \partial_y f(t, x, 0) = 0, \quad \partial_t f(t, x, 0) = (\partial_y^2 f)(t, x, 0) + g(t, x, 0), \quad (t, x) \in]0, T[\times \mathbb{R}.$$

Then for any $\varepsilon > 0$ there exists a constant C_ε , depending only on ε, σ and the constants C_* , such that

$$(2.22) \quad \begin{aligned} & \|\langle y \rangle^{-\sigma/2} \Lambda^{1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \varepsilon \|\Lambda^{-2/3} \partial_y h\|_{L^2(\mathbb{R}^2)} + C_\varepsilon \left(\|\Lambda^{-1/3} h\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\ & \quad + C_\varepsilon \left(\|\langle y \rangle^{-\frac{\sigma}{2}} \partial_y \Lambda^{1/6} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\langle y \rangle^\sigma \Lambda^{-1/3} \partial_y g\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right). \end{aligned}$$

Moreover

$$\begin{aligned} & \|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \tilde{C} \left(\|\langle y \rangle^{-\sigma/2} \Lambda^{1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda^{-2/3} \partial_y h\|_{L^2(\mathbb{R}^2)} + \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right), \end{aligned}$$

where \tilde{C} is a constant depending only on σ, c , and C_*, C_0 in Theorem 1.1.

We will prove this proposition in next section 3. This subelliptic estimate gives a gain of regularity of order $\frac{1}{3}$ with respect to x variable, so it is sufficient to repeat the same procedure for 3 times to get 1 order of regularity.

Continuation of the proof of the Claim $E_{m,\ell}$.

We now use the above subellipticity for the function $f = \tilde{f}$, with \tilde{f} defined in Proposition 2.3, i.e.,

$$f = \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m = \Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} W_m^{\ell-1} f_m.$$

Similar to (2.12), we see f is a classical solution to the following problem:

$$\begin{cases} (\partial_t + u \partial_x + v \partial_y - \partial_y^2) f = \phi^{1/2} \mathcal{Z}_{m,\ell-1,\delta} + (\partial_t \phi^{1/2}) \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m, \\ \partial_y f|_{y=0} = 0, \\ f|_{t=0} = f|_{t=T} = 0, \end{cases}$$

where $\mathcal{Z}_{m,\ell-1,\delta}$ is defined in (2.13). The initial value and boundary value in (2.12) is take in the sense of trace in Sobolev space. The validity of **Claim $\mathbf{E}_{m,\ell-1}$** due to the inductive assumption (2.11) yields that $\partial_y^3 f \in L^2([0, T] \times \mathbb{R}_+^2)$. Next we calculate $(\partial_t f - \partial_y^2 f)|_{y=0}$. Firstly we have, seeing (6.6) in the appendix,

$$(\partial_t f_m - \partial_y^2 f_m)|_{y=0} = -2 \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] f_m|_{y=0}.$$

Then

$$\begin{aligned} \partial_t f|_{y=0} &= \Lambda_\delta^{-2} \left(\partial_t \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} \right) W_m^{\ell-1} f_m|_{y=0} + \Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} W_m^{\ell-1} \partial_t f_m|_{y=0} \\ &= \Lambda_\delta^{-2} \left(\partial_t \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} \right) W_m^{\ell-1} f_m|_{y=0} + \Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} W_m^{\ell-1} \partial_y^2 f_m|_{y=0} \\ &\quad - 2\Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} W_m^{\ell-1} \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] f_m|_{y=0} \\ &= \Lambda_\delta^{-2} \left(\partial_t \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} \right) \Lambda^{(\ell-1)/3} f_m|_{y=0} + \Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} W_m^{\ell-1} \partial_y^2 f_m|_{y=0} \\ &\quad - 2\Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} \Lambda^{(\ell-1)/3} \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] f_m|_{y=0}. \end{aligned}$$

This, along with the fact that

$$\begin{aligned} &\Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} W_m^{\ell-1} \partial_y^2 f_m|_{y=0} \\ &= \partial_y^2 \Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} W_m^{\ell-1} f_m|_{y=0} - [\partial_y^2, W_m^{\ell-1}] \Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} f_m|_{y=0} \\ &= \partial_y^2 f|_{y=0} - \left(\frac{2c^2}{(3m+\ell-1)\sigma} + 3c^2 \right) \Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} \Lambda^{(\ell-1)/3} f_m|_{y=0} \end{aligned}$$

due to the fact that $\partial_y \Lambda_\delta^{-2} f_m|_{y=0} = 0$ (seeing (6.5) in the appendix), gives

$$\begin{aligned} &(\partial_t f - \partial_y^2 f)|_{y=0} \\ &= \Lambda_\delta^{-2} \left(\partial_t \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} \right) \Lambda^{(\ell-1)/3} f_m|_{y=0} \\ &\quad - \left(\frac{2c^2}{(3m+\ell-1)\sigma} + 3c^2 \right) \Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} \Lambda^{(\ell-1)/3} f_m|_{y=0} \\ &\quad - 2\Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} \Lambda^{(\ell-1)/3} \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] f_m|_{y=0} \\ &\stackrel{\text{def}}{=} g|_{y=0} \end{aligned}$$

with

$$\begin{aligned} (2.23) \quad g &= \Lambda_\delta^{-2} \left(\partial_t \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} \right) \Lambda^{(\ell-1)/3} f_m \\ &\quad - \left(\frac{2c^2}{(3m+\ell-1)\sigma} + 3c^2 \right) \Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} \Lambda^{(\ell-1)/3} f_m \\ &\quad - 2\Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} \Lambda^{(\ell-1)/3} \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] f_m. \end{aligned}$$

Then using Proposition 2.4 for $h = \phi^{1/2} \mathcal{Z}_{m,\ell-1,\delta} + (\partial_t \phi^{1/2}) \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m$ and the above g , we have

$$\begin{aligned} &\| \langle y \rangle^{-\sigma/2} \Lambda^{1/3} f \|_{L^2([0,T] \times \mathbb{R}_+^2)} + \| \partial_y^2 \Lambda^{-1/3} f \|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq \varepsilon \| \Lambda^{-2/3} \partial_y h \|_{L^2(\mathbb{R}^2)} + C_\varepsilon \left(\| \Lambda^{-1/3} h \|_{L^2([0,T] \times \mathbb{R}_+^2)} + \| \partial_y f \|_{L^2([0,T] \times \mathbb{R}_+^2)} + \| f \|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\ &\quad + C_\varepsilon \left(\| \langle y \rangle^{-\frac{\sigma}{2}} \partial_y \Lambda^{1/6} f \|_{L^2([0,T] \times \mathbb{R}_+^2)} + \| \langle y \rangle^\sigma \Lambda^{-1/3} \partial_y g \|_{L^2([0,T] \times \mathbb{R}_+^2)} \right). \end{aligned}$$

We claim, for any $\tilde{\varepsilon} > 0$,

$$\begin{aligned}
 & \varepsilon \left\| \Lambda^{-2/3} \partial_y h \right\|_{L^2(\mathbb{R}^2)} + C_\varepsilon \left(\left\| \Lambda^{-1/3} h \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
 & \quad + C_\varepsilon \left(\left\| \langle y \rangle^{-\frac{\sigma}{2}} \partial_y \Lambda^{1/6} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \langle y \rangle^\sigma \Lambda^{-1/3} \partial_y g \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
 (2.24) \quad & \leq \varepsilon C_5 \left(\left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
 & \quad + \tilde{\varepsilon} m^{-(1+\sigma)/2} \left(\left\| F \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y F \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
 & \quad + C_{\varepsilon, \tilde{\varepsilon}} m^{(1+\sigma)/2} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^s m^{(\ell-1)(1+\sigma)},
 \end{aligned}$$

where C_5 is a constant depending only on σ , c , and the constant C_* , but independent of m and δ , and $C_{\varepsilon, \tilde{\varepsilon}}$ is a constant depending only on $\varepsilon, \tilde{\varepsilon}, \sigma, c$, and the constant C_* , but independent of m and δ . Recall $F = \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m$. The proof of (2.24) is postponed to the end of this section. Now combining the above inequalities and letting ε be small enough, we infer for any $\tilde{\varepsilon} > 0$,

$$\begin{aligned}
 & \left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 (2.25) \quad & \leq \tilde{\varepsilon} m^{-(1+\sigma)/2} \left(\left\| F \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y F \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
 & \quad + C_{\tilde{\varepsilon}} m^{(1+\sigma)/2} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^s m^{(\ell-1)(1+\sigma)}.
 \end{aligned}$$

Now we come back to estimate the terms on the right side of (2.18). To do so we need the following technic Lemma, whose proof is presented at the end of Section 4.

Lemma 2.5. *Recall $F = \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m$ and $f = \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m$. There exists a constant C_6 , depending only on σ, c , and the constant C_* , but independent of m and δ , such that*

$$\begin{aligned}
 & \left\| \phi^{-1/2} F \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y^2 \Lambda^{-2/3} F \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 & \leq C_6 \left(m^{\sigma/2} \left\| \langle y \rangle^{-\frac{\sigma}{2}} \Lambda^{1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
 & \quad + C_6 \left(\left\| \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right),
 \end{aligned}$$

and

$$\begin{aligned}
 & \left\| \partial_y^3 \Lambda^{-1} F \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 (2.26) \quad & \leq C_6 \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C_6 \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 & \quad + C_6 \left(\left\| \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right).
 \end{aligned}$$

End of the proof of the Claim $E_{m, \ell}$.

We combine (2.25) and the first estimate in Lemma 2.5, to conclude

$$\begin{aligned}
 & \left\| \phi^{-1/2} F \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y^2 \Lambda^{-2/3} F \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 & \leq \tilde{\varepsilon} C_6 m^{-1/2} \left(\left\| F \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y F \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
 & \quad + C_6 C_{\tilde{\varepsilon}} m^{\frac{1}{2}+\sigma} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^s m^{(\ell-1)(1+\sigma)} \\
 & \quad + C_6 \left(\left\| \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
 & \leq \tilde{\varepsilon} C_6 m^{-1/2} \left(\left\| F \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \partial_y F \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\
 & \quad + (C_6 C_{\tilde{\varepsilon}} + C_6) m^{\frac{1}{2}+\sigma} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)}
 \end{aligned}$$

the last inequality using (2.11). This along with (2.18) yields

$$\begin{aligned} & \|F\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} F\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \tilde{\varepsilon} C_4 C_6 \left(\|F\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y F\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\ & \quad + C_4 (C_6 C_{\tilde{\varepsilon}} + C_6) m^{1+\sigma} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)} + 2A^{m-6} ((m-5)!)^{3(1+\sigma)}. \end{aligned}$$

Consequently, letting $\tilde{\varepsilon} > 0$ be small sufficiently,

$$\begin{aligned} & \|F\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} F\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C_7 m^{1+\sigma} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)} + C_7 A^{m-6} ((m-5)!)^{3(1+\sigma)} \\ & \leq C_8 (m-4)^{1+\sigma} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)} + C_7 A^{m-6} ((m-5)!)^{3(1+\sigma)}, \end{aligned}$$

where C_7, C_8 are two constants depending only on σ, c , and the constants C_0, C_* in Theorem 1.1, but is independent of m and δ . Now we choose A such that

$$A \geq (2C_8 + 2C_7 + 1)^6.$$

It then follows that, observing $\ell \geq 1$,

$$\|F\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} F\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq A^{m-5+\frac{\ell}{6}} ((m-5)!)^s (m-4)^{\ell(1+\sigma)}.$$

Observe the above constant A is independent of δ , and thus letting $\delta \rightarrow 0$, we see (2.11) holds for $i = \ell$. It remains to prove that $\partial_y^3 \Lambda^{-1} \phi_m^\ell W_m^\ell f_m$. The above estimate together with (2.25) gives

$$\| \langle y \rangle^{-\sigma/2} \Lambda^{1/3} f \|_{L^2([0,T] \times \mathbb{R}_+^2)} + \| \partial_y^2 \Lambda^{-1/3} f \|_{L^2([0,T] \times \mathbb{R}_+^2)} < C_{m,1}$$

with $C_{m,1}$ a constant depending on m but independent of δ , and thus, using the last estimate in Proposition 2.4 and (2.24),

$$\| \partial_y^3 \Lambda^{-2/3} f \|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C_{m,2},$$

with $C_{m,2}$ a constant depending on m but independent of δ . As a result, combining (2.26), we conclude

$$\| \partial_y^3 \Lambda^{-1} F \|_{L^2([0,T] \times \mathbb{R}_+^2)} < C_{m,3}$$

with $C_{m,3}$ a constant depending on m but independent of δ . Thus letting $\delta \rightarrow 0$, we see $\partial_y^3 \Lambda^{-1} \phi_m^\ell W_m^\ell f_m \in L^2([0, T] \times \mathbb{R}_+^2)$. Thus **Claim $E_{m,\ell}$** holds. This completes the proof of **Claim I_{m+1}** , and thus the proof of Theorem 1.1.

We end up this section by the following

Proof of the estimate (2.24). In the proof we use C to denote different constants depending only on σ, c , and the constants C_0, C_* in Theorem 1.1, but is independent of m and δ .

(a) We first estimate $\| \Lambda^{-1/3} h \|_{L^2([0,T] \times \mathbb{R}_+^2)}$, recalling

$$h = \phi^{1/2} \mathcal{Z}_{m,\ell-1,\delta} + \left(\partial_t \phi^{1/2} \right) \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m.$$

Using interpolation inequality gives, observing $|\partial_t \phi^{1/2}| \leq \phi^{-1/2}$,

$$\begin{aligned} & \| \Lambda^{-1/3} \left(\partial_t \phi^{1/2} \right) \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \|_{L^2(\mathbb{R}_x)} \\ & \leq m^{-1/2} \phi^{1/2} \| \left(\partial_t \phi^{1/2} \right) \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \|_{L^2(\mathbb{R}_x)} \\ & \quad + m^{(\ell+1)/2} \phi^{-(\ell+1)/2} \| \Lambda^{-1-\frac{\ell-1}{3}} \left(\partial_t \phi^{1/2} \right) \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \|_{L^2(\mathbb{R}_x)} \\ & \leq m^{-1/2} \| \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \|_{L^2(\mathbb{R}_x)} + m^{(\ell+1)/2} \| \Lambda^{-1-\frac{\ell-1}{3}} \phi^{-(\ell+2)/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \|_{L^2(\mathbb{R}_x)} \\ & \leq m^{-1/2} \| \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \|_{L^2(\mathbb{R}_x)} + m^{(\ell+1)/2} \| \Lambda^{-1} \Lambda_\delta^{-2} \phi_{m-1}^0 W_{m-1}^0 f_m \|_{L^2(\mathbb{R}_x)}, \end{aligned}$$

the last inequality following from (2.2) which shows $W_i^0, i \geq 1$, is a decreasing sequence of functions as i varies in \mathbb{N} , and the fact that

$$\phi^{-(\ell+2)/2} \phi_m^{\ell-1} \leq \phi_{m-1}^0.$$

Moreover, using (2.5) and the inductive assumptions (2.11) and (2.9), we compute, observing $\ell/2 + 1 \leq 3(1 + \sigma)$,

$$\begin{aligned} & m^{-1/2} \|\Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2(\mathbb{R}_x)} + m^{(\ell+1)/2} \|\Lambda^{-1} \Lambda_\delta^{-2} \phi_{m-1}^0 W_{m-1}^0 f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq m^{-1/2} \|\phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2(\mathbb{R}_x)} + \tilde{C}_1 m^{(\ell+1)/2} \|\phi_{m-1}^0 W_{m-1}^0 f_{m-1}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq m^{-1/2} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)} + \tilde{C}_1 m^{(\ell+1)/2} A^{m-6} ((m-6)!)^{3(1+\sigma)} \\ & \leq C m^{-1/2} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)}. \end{aligned}$$

Thus we have, combining the above inequalities,

$$(2.27) \quad \begin{aligned} & \|\Lambda^{-1/3} (\partial_t \phi^{1/2}) \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C m^{-1/2} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)}. \end{aligned}$$

Similarly, we can show that

$$(2.28) \quad \begin{aligned} & \|\partial_y \Lambda^{-1/3} (\partial_t \phi^{1/2}) \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C m^{-1/2} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)}. \end{aligned}$$

Using (2.16) in Proposition 2.3, we have

$$\begin{aligned} & \|\Lambda^{-1/3} \phi^{1/2} \mathcal{Z}_{m,\ell-1,\delta}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq m C_3 \|\Lambda^{-1/3} \phi^{-1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C_3 \|\partial_y \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + C_3 A^{m-6} ((m-5)!)^s, \end{aligned}$$

and moreover repeating the arguments as in (2.27) and (2.28), with $\partial_t \phi^{1/2}$ there replaced by $\phi^{-1/2}$,

$$\begin{aligned} & m C_3 \|\Lambda^{-1/3} \phi^{-1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C_3 \|\partial_y \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C m^{1/2} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)}, \end{aligned}$$

and thus

$$\|\Lambda^{-1/3} \phi^{1/2} \mathcal{Z}_{m,\ell-1,\delta}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C m^{1/2} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)}.$$

This along with (2.27) yields

$$\|\Lambda^{-1/3} h\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C m^{1/2} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)}.$$

(b) In this step we treat $\|\Lambda^{-2/3} \partial_y h\|_{L^2([0,T] \times \mathbb{R}_+^2)}$. It follows from (2.28) that

$$\begin{aligned} & \|\Lambda^{-2/3} \partial_y (\partial_t \phi^{1/2}) \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C m^{-1/2} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)}. \end{aligned}$$

On the other hand, by (2.17) we have, recalling $\tilde{f} = f = \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m$,

$$\begin{aligned} & \|\Lambda^{-2/3} \partial_y \phi^{1/2} \mathcal{Z}_{m,\ell-1,\delta}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C_3 \|\langle y \rangle^{-\sigma} \Lambda^{1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C_3 \|\partial_y^2 \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + m C_3 \left(\|\Lambda^{-2/3} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda^{-2/3} \phi^{-1/2} \partial_y \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\ & \quad + C_3 A^{m-6} ((m-5)!)^s \end{aligned}$$

and moreover similar to (2.27) and (2.28), we have

$$\begin{aligned} & m C_3 \left(\|\Lambda^{-2/3} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda^{-2/3} \phi^{-1/2} \partial_y \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\ & \leq C m^{1/2} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)}, \end{aligned}$$

since $|\partial_t \phi^{1/2}| \geq 1$. Combining the above three inequalities gives

$$\begin{aligned} \|\Lambda^{-2/3} \partial_y h\|_{L^2([0,T] \times \mathbb{R}_+^2)} & \leq C \left(\|\langle y \rangle^{-\sigma} \Lambda^{1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y^2 \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\ & \quad + C m^{1/2} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)}. \end{aligned}$$

(c) It follows from the inductive assumption (2.11) that, observing $\phi^{1/2} \leq 1$,

$$\begin{aligned} \sum_{j=0}^1 \|\partial_y^j f\|_{L^2([0,T] \times \mathbb{R}_+^2)} & \leq \|\phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)}. \end{aligned}$$

Now we estimate $\|\langle y \rangle^\sigma \Lambda^{-1/3} \partial_y g\|_{L^2([0,T] \times \mathbb{R}_+^2)}$, with g is defined in (2.23). It is quite similar as in step (a). For instance,

$$\begin{aligned} & \|\langle y \rangle^\sigma \Lambda^{-1/3} \partial_y \Lambda_\delta^{-2} \left(\partial_t \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} \right) \Lambda^{(\ell-1)/3} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \|\Lambda^{-1/3} \phi^{-1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} \partial_y f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C \|\Lambda^{-1/3} \phi^{-1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \|\partial_y \Lambda^{-1/3} \phi^{-1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

Then similar to (2.27) and (2.28), we conclude

$$\begin{aligned} & \|\langle y \rangle^\sigma \Lambda^{-1/3} \partial_y \Lambda_\delta^{-2} \left(\partial_t \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} \right) \Lambda^{(\ell-1)/3} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C m^{-1/2} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)}. \end{aligned}$$

The other terms in (2.23) can be estimated similarly, and a classical commutator estimate (see Lemma 3.1 in the following section) will be used for treatment of the third term in (2.23). Thus we conclude

$$\|\langle y \rangle^\sigma \Lambda^{-1/3} \partial_y g\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)}.$$

(d) It remains to estimate $\|\langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}$, and we have

$$\begin{aligned} & \|\langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 = \|\langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} \Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-\frac{1}{2}} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & = \left(\langle y \rangle^{-\sigma} \partial_y \Lambda^{1/3} \Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell} W_m^{\ell-1} f_m, \partial_y \Lambda_\delta^{-2} \phi^{3(m-N_0-1)+\ell-1} W_m^{\ell-1} f_m \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \|\langle y \rangle^{-\sigma} \partial_y \Lambda^{1/3} \Lambda_\delta^{-2} \phi_m^\ell W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \|\partial_y \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C \left(\|\partial_y \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \|\partial_y \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)}, \end{aligned}$$

the last inequality following from the third estimate in Lemma 2.2. This, along with the inductive assumption (2.11) implies, for any $\tilde{\varepsilon} > 0$,

$$\begin{aligned} & \|\langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \tilde{\varepsilon} m^{-(1+\sigma)/2} \left(\|F\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y F\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) + C_{\tilde{\varepsilon}} m^{(1+\sigma)/2} A^{m-5+\frac{\ell-1}{6}} ((m-5)!)^s m^{(\ell-1)(1+\sigma)}, \end{aligned}$$

recalling $F = \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m$.

Now combining the estimates in the above steps (a)-(d), we obtain the desired (2.24). \square

3. SUBELLIPTIC ESTIMATE

In this section we prove the Proposition 2.4. We need the following commutators estimates. Throughout the paper we use $[Q_1, Q_2]$ to denote the commutator between two operators Q_1 and Q_2 , which is defined by

$$[Q_1, Q_2] = Q_1Q_2 - Q_2Q_1 = -[Q_2, Q_1].$$

We have

$$(3.1) \quad [Q_1, Q_2Q_3] = Q_2[Q_1, Q_3] + [Q_1, Q_2]Q_3.$$

Lemma 3.1. *Denote by $[\alpha]$ the largest integer less than or equal to $\alpha \geq 0$. For any $\tau \in \mathbb{R}$ and $a \in C_b^{[[\tau]]+1}(\mathbb{R}_+^2)$, the space of functions such that all their derivatives up to the order of $[[\tau]] + 1$ are continuous and bounded, there exists $C > 0$ such that for suitable function f and any $0 < \delta < 1$,*

$$\| [a, \Lambda^\tau \Lambda_\delta^{-2}] f \|_{L^2(\mathbb{R}_+^2)} \leq C \| \Lambda^{\tau-1} \Lambda_\delta^{-2} f \|_{L^2(\mathbb{R}_+^2)},$$

and

$$\| [a \partial_x, \Lambda^\tau \Lambda_\delta^{-2}] f \|_{L^2(\mathbb{R}_+^2)} \leq C \| \Lambda^\tau \Lambda_\delta^{-2} f \|_{L^2(\mathbb{R}_+^2)}.$$

The constant C depends on only on τ and $\|a\|_{C_b^{[[\tau]]+1}(\mathbb{R}_+^2)}$.

Since $\Lambda^\tau \Lambda_\delta^{-2}$ is only a Fourier multiplier of x variable, so we can prove the above Lemma by direct calculus or pseudo-differential computation, cf. [16, 19]. In this section, we use above Lemma with $a = u$ or $a = v$ and $\tau = -1/3, -2/3$. So that with hypothesis (1.3), the constant in Lemma 3.1 depends only on the constant C_0 in Theorem 1.1.

Proof of the Proposition 2.4. Taking the operator $\Lambda^{-2/3}$ on both sides of (2.19), we see the function $\Lambda^{-2/3} f$ satisfies the following equation in $]0, T[\times \mathbb{R}_+^2$:

$$(3.2) \quad \begin{aligned} & \partial_t \Lambda^{-2/3} f + u \partial_x \Lambda^{-2/3} f + v \partial_y \Lambda^{-2/3} f - \partial_y^2 \Lambda^{-2/3} f \\ & = \Lambda^{-2/3} h + [u \partial_x + v \partial_y, \Lambda^{-2/3}] f, \end{aligned}$$

and that

$$(3.3) \quad \Lambda^{-2/3} f|_{t=0} = \Lambda^{-2/3} f|_{t=T} = 0, \quad \partial_y \Lambda^{-2/3} f|_{y=0} = 0$$

due to (2.20) and (2.21), since $\Lambda^{-2/3}$ is an operator acting only on x variable. Recall $[u \partial_x + v \partial_y, \Lambda^{-2/3}]$ stands for the commutator between $u \partial_x + v \partial_y$ and $\Lambda^{-2/3}$.

Step 1). We will show in this step that

$$(3.4) \quad \begin{aligned} & \| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \leq 2 \left| \operatorname{Re} \left(\partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \right| + \| \partial_y^2 \Lambda^{-1/3} f \|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \quad + C \left(\| \Lambda^{-1/3} h \|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \| \partial_y f \|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \| f \|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right). \end{aligned}$$

To do so, we take $L^2([0, T] \times \mathbb{R}_+^2)$ inner product with the function $\partial_y \partial_x \Lambda^{-2/3} f \in L^2([0, T] \times \mathbb{R}_+^2)$ on the both sides of equation (3.2), and then consider the real parts; this gives

$$(3.5) \quad \begin{aligned} & - \operatorname{Re} \left(u \partial_x \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & = \operatorname{Re} \left(\partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} - \operatorname{Re} \left(\partial_y^2 \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \quad + \operatorname{Re} \left(v \partial_y \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} - \operatorname{Re} \left(\Lambda^{-2/3} h, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & \quad - \operatorname{Re} \left([u \partial_x + v \partial_y, \Lambda^{-2/3}] f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)}. \end{aligned}$$

We will treat the terms on both sides. For the term on left hand side we integrate by parts to obtain, here we use $u|_{y=0} = 0$,

$$\begin{aligned} & -\operatorname{Re} \left(u \partial_x \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= -\frac{1}{2} \left\{ \left(u \partial_x \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} + \left(\partial_y \partial_x \Lambda^{-2/3} f, u \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right\} \\ &= \frac{1}{2} \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2. \end{aligned}$$

Next we estimate the terms on the right hand side and have, by Cauchy-Schwarz's inequality,

$$\begin{aligned} & \left| -\operatorname{Re} \left(\partial_y^2 \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \leq \frac{1}{2} \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \frac{1}{2} \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2, \\ & \left| \operatorname{Re} \left(\Lambda^{-2/3} h, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \leq \left\| \Lambda^{-1/3} h \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \end{aligned}$$

and

$$\begin{aligned} & \left| -\operatorname{Re} \left(v \partial_y \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ & \leq \left| \left(\partial_y f, [\Lambda^{-2/3}, v] \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| + \left| \left(v \partial_y f, \Lambda^{-2/3} \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ & \leq C \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2, \end{aligned}$$

the last inequality using Lemma 3.1. Finally

$$\begin{aligned} & \left| -\operatorname{Re} \left([u \partial_x + v \partial_y, \Lambda^{-2/3}] f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ & \leq \left\| \Lambda^{1/3} [u \partial_x + v \partial_y, \Lambda^{-2/3}] f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \leq 2 \left(\left\| [u \partial_x + v \partial_y, \Lambda^{1/3} \Lambda^{-2/3}] f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| [u \partial_x + v \partial_y, \Lambda^{1/3}] \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right) \\ & \quad + \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \leq C \left(\left\| f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right), \end{aligned}$$

These inequalities, together with (3.5), yields the desired (3.4).

Step 2). In this step we will estimate the second term on the right hand side of (3.4) and show that for any $\varepsilon > 0$,

$$\begin{aligned} & \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ (3.6) \quad & \leq \varepsilon \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \quad + C_\varepsilon \left(\left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \Lambda^{-1/3} h \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right), \end{aligned}$$

with C_ε a constant depending on ε . We see that the function $\Lambda^{-1/3} f$ satisfies the equation in $]0, T[\times \mathbb{R}_+^2$,

$$\begin{aligned} (3.7) \quad & \partial_t \Lambda^{-1/3} f + (u \partial_x + v \partial_y) \Lambda^{-1/3} f - \partial_y^2 \Lambda^{-1/3} f \\ & = \Lambda^{-1/3} h + [u \partial_x + v \partial_y, \Lambda^{-1/3}] f, \end{aligned}$$

with the boundary condition

$$(3.8) \quad \Lambda^{-1/3} f|_{t=0} = \Lambda^{-1/3} f|_{t=T} = 0, \quad \partial_y \Lambda^{-1/3} f|_{y=0} = 0.$$

Now we take $L^2([0, T] \times \mathbb{R}_+^2)$ inner product with the function $-\partial_y^2 \Lambda^{-1/3} f \in L^2([0, T] \times \mathbb{R}_+^2)$ on both sides of (3.7), and then consider the real parts; this gives

$$(3.9) \quad \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2(\mathbb{R}_+^3)}^2 \leq \sum_{p=1}^4 J_p,$$

where

$$\begin{aligned} J_1 &= \left| \operatorname{Re} \left(\partial_t \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right|, \\ J_2 &= \left| \operatorname{Re} \left((u \partial_x + v \partial_y) \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{[0,T] \times L^2([0,T] \times \mathbb{R}_+^2)} \right|, \\ J_3 &= \left| \operatorname{Re} \left(\Lambda^{-1/3} h, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right|, \\ J_4 &= \left| \operatorname{Re} \left([u \partial_x + v \partial_y, \Lambda^{-1/3}] f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right|. \end{aligned}$$

Integrating by parts and observing the condition (3.8), we see

$$\left(\partial_t \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} = - \left(\partial_t \partial_y \Lambda^{-1/3} f, \partial_y \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)},$$

which along with the fact

$$\operatorname{Re} \left(\partial_t \partial_y \Lambda^{-1/3} f, \partial_y \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} = 0$$

due to (3.8), implies

$$J_1 = \left| \operatorname{Re} \left(\partial_t \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| = 0.$$

About J_2 we integrate by parts again and observe the boundary condition (3.8), to compute

$$\begin{aligned} & \operatorname{Re} \left(u \partial_x \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= - \operatorname{Re} \left(u \partial_x \Lambda^{-1/3} \partial_y f, \Lambda^{-1/3} \partial_y f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} - \operatorname{Re} \left((\partial_y u) \partial_x \Lambda^{-1/3} f, \Lambda^{-1/3} \partial_y f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= \frac{1}{2} \left((\partial_x u) \Lambda^{-1/3} \partial_y f, \Lambda^{-1/3} \partial_y f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} - \operatorname{Re} \left((\partial_y u) \partial_x \Lambda^{-1/3} f, \Lambda^{-1/3} \partial_y f \right)_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

This gives

$$\begin{aligned} & \left| \operatorname{Re} \left(u \partial_x \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ & \leq \|\Lambda^{-1/3} (\partial_y u) \partial_x \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \leq \left(\|(\partial_y u) \Lambda^{-1/3} \partial_x \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|[\Lambda^{-1/3}, \partial_y u] \partial_x \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + C \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \leq C \left(\|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \leq \varepsilon \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C_\varepsilon \left(\|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right). \end{aligned}$$

Moreover integrating by part, we obtain

$$\begin{aligned} \left| \operatorname{Re} \left(v \partial_y \Lambda^{-1/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| &= \frac{1}{2} \left| \left((\partial_y v) \partial_y \Lambda^{-1/3} f, \partial_y \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ &\leq C \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2. \end{aligned}$$

Thus

$$(3.10) \quad J_2 \leq \varepsilon \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C_\varepsilon \left(\|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right).$$

It remains to estimate J_3 and J_4 . Let $\tilde{\varepsilon} > 0$ be an arbitrarily small number. Cauchy-Schwarz's inequality gives

$$\begin{aligned} J_3 &= \left| \operatorname{Re} \left(\Lambda^{-1/3} h, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ &\leq \tilde{\varepsilon} \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \left\| \Lambda^{-1/3} h \right\|_{L^2(\mathbb{R}_+^3)}^2, \end{aligned}$$

and for J_4 , Lemma 3.1 implies

$$\begin{aligned} J_4 &= \left| \operatorname{Re} \left([u \partial_x + v \partial_y, \Lambda^{-1/3}] f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2(\mathbb{R}_+^2)} \right| \\ &\leq \tilde{\varepsilon} \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \left(\left\| f \right\|_{L^2(\mathbb{R}_+^3)}^2 + \left\| \partial_y f \right\|_{L^2(\mathbb{R}_+^3)}^2 \right), \end{aligned}$$

where \tilde{C}_ε is constant depending on $\tilde{\varepsilon}$. Now the above two estimates for J_3 and J_4 , along with (3.9) - (3.10), gives

$$\begin{aligned} \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 &\leq \tilde{\varepsilon} \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \varepsilon \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ &\quad + C_\varepsilon \left(\left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right) \\ &\quad + C_{\tilde{\varepsilon}} \left(\left\| \Lambda^{-1/3} h \right\|_{L^2(\mathbb{R}_+^3)}^2 + \left\| f \right\|_{L^2(\mathbb{R}_+^3)}^2 + \left\| \partial_y f \right\|_{L^2(\mathbb{R}_+^3)}^2 \right), \end{aligned}$$

and thus, letting $\tilde{\varepsilon}$ small sufficiently,

$$\begin{aligned} &\left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ &\leq \varepsilon \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C_\varepsilon \left(\left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \Lambda^{-1/3} h \right\|_{L^2(\mathbb{R}_+^3)}^2 \right). \end{aligned}$$

This is just the desired estimate (3.6).

Combining the estimates (3.4) and (3.6), we obtain, choosing ε sufficiently small,

$$\begin{aligned} &\left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ (3.11) \quad &\leq C \left| \operatorname{Re} \left(\partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ &\quad + C \left(\left\| \Lambda^{-1/3} h \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right). \end{aligned}$$

Step 3) It remains to treat the first term on the right hand side of (3.11). In this step we will prove that, for any $\varepsilon_1 > 0$,

$$\begin{aligned} &\left| \operatorname{Re} \left(\partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ (3.12) \quad &\leq \varepsilon_1 \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dx dt + C_{\varepsilon_1} \left\| \langle y \rangle \Lambda^{-1/3} \partial_y g \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ &\quad + \varepsilon_1^{-1} C \left(\left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right). \end{aligned}$$

For this purpose we integrate by parts again and observe the boundary condition (3.3), to compute

$$\begin{aligned} &\left(\partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= - \left(\Lambda^{-2/3} f, \partial_t \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= \left(\partial_x \Lambda^{-2/3} f, \partial_t \partial_y \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= - \left(\partial_y \partial_x \Lambda^{-2/3} f, \partial_t \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} + \int_0^T \int_{\mathbb{R}} \left(\partial_t \Lambda^{-2/3} f(t, x, 0) \right) \left(\partial_x \Lambda^{-2/3} f(t, x, 0) \right) dx dt, \end{aligned}$$

which, along with the fact that

$$\begin{aligned} & 2 \operatorname{Re} \left(\partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= \left(\partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} + \left(\partial_y \partial_x \Lambda^{-2/3} f, \partial_t \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)}, \end{aligned}$$

yields, for any $\varepsilon_1 > 0$,

$$\begin{aligned} & \left| \operatorname{Re} \left(\partial_t \Lambda^{-2/3} f, \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ &= \frac{1}{2} \left| \int_0^T \int_{\mathbb{R}} \left(\partial_t \Lambda^{-2/3} f(t, x, 0) \right) \left(\partial_x \Lambda^{-2/3} f(t, x, 0) \right) dx dt \right| \\ (3.13) \quad &= \frac{1}{2} \left| \int_0^T \int_{\mathbb{R}} \left(\Lambda^{1/6} \partial_t \Lambda^{-2/3} f(t, x, 0) \right) \left(\Lambda^{-1/6} \partial_x \Lambda^{-2/3} f(t, x, 0) \right) dx dt \right| \\ &\leq \varepsilon_1 \int_0^T \int_{\mathbb{R}} \left(\partial_t \Lambda^{-1/2} f(t, x, 0) \right)^2 dx dt + \varepsilon_1^{-1} \int_0^T \int_{\mathbb{R}} \left(\Lambda^{1/6} f(t, x, 0) \right)^2 dx dt. \end{aligned}$$

Moreover observing

$$\Lambda^{1/6} f(t, x, 0) = \left(\langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right) (t, x, 0),$$

it then follows from Sobolev inequality that

$$\begin{aligned} \left| \Lambda^{1/6} f(t, x, 0) \right|^2 &\leq C \left(\left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|_{L^2(\mathbb{R}_+)}^2 + \left\| \partial_y \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|_{L^2(\mathbb{R}_+)}^2 \right) \\ &\leq C \left(\left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|_{L^2(\mathbb{R}_+)}^2 + \left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|_{L^2(\mathbb{R}_+)}^2 \right) \end{aligned}$$

with C a constant independent of t, x . And thus

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \left(\Lambda^{1/6} f(t, x, 0) \right)^2 dx dt \\ (3.14) \quad &\leq C \left(\left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right) \\ &\leq C \left(\left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} \partial_y f \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right). \end{aligned}$$

Using the fact that

$$\partial_t \Lambda^{-1/2} f(t, x, 0) = \left(\partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) + \Lambda^{-1/2} g(t, x, 0)$$

due to assumption (2.21), we conclude

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \left(\partial_t \Lambda^{-1/2} f(t, x, 0) \right)^2 dx dt \\ &\leq \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dx dt + \int_0^T \int_{\mathbb{R}} \left| \Lambda^{-1/2} g(t, x, 0) \right|^2 dx dt. \end{aligned}$$

Moreover observe

$$\begin{aligned} \left| \Lambda^{-1/2} g(t, x, 0) \right| &= \left| - \int_0^{+\infty} \partial_{\tilde{y}} \Lambda^{-1/2} g(t, x, \tilde{y}) d\tilde{y} \right| \\ &\leq \left(\int_0^{+\infty} \langle \tilde{y} \rangle^{-2\sigma} d\tilde{y} \right)^{1/2} \left(\int_0^{+\infty} \langle \tilde{y} \rangle^{2\sigma} \left| \Lambda^{-1/2} \partial_{\tilde{y}} g(t, x, \tilde{y}) \right|^2 d\tilde{y} \right)^{1/2}, \end{aligned}$$

which implies

$$\int_0^T \int_{\mathbb{R}} \left| \Lambda^{-1/2} g(t, x, 0) \right|^2 dx dt \leq C \left\| \langle y \rangle^\sigma \Lambda^{-1/2} \partial_y g \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \leq C \left\| \langle y \rangle^\sigma \Lambda^{-1/3} \partial_y g \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2,$$

and thus

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \left(\partial_t \Lambda^{-1/2} f(t, x, 0) \right)^2 dx dt \\ & \leq \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dx dt + C \|\langle y \rangle^\sigma \Lambda^{-1/3} \partial_y g\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2. \end{aligned}$$

This along with (3.13) and (3.14) yields the desired (3.12).

Step 4) Combining (3.11) and (3.12), we have, for any $\varepsilon_1 > 0$,

$$\begin{aligned} & \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \leq \varepsilon_1 \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dx dt + C_{\varepsilon_1} \|\langle y \rangle^\sigma \Lambda^{-1/3} \partial_y g\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \quad + \varepsilon_1^{-1} C \left(\|\langle y \rangle^{-\sigma/2} \Lambda^{1/6} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\ & \quad + C \left(\|\partial_y f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\Lambda^{-1/3} h\|_{L^2(\mathbb{R}_+^3)}^2 \right). \end{aligned}$$

Moreover we use the monotonicity condition and interpolation inequality to get, for any $\varepsilon_2 > 0$

$$\begin{aligned} \|\langle y \rangle^{-\sigma/2} \Lambda^{1/6} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 & \leq \varepsilon_2 \|\langle y \rangle^{-\sigma/2} \Lambda^{1/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \varepsilon_2^{-1} \|\langle y \rangle^{-\sigma/2} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \leq \varepsilon_2 \|\langle y \rangle^{-\sigma/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C_{\varepsilon_2} \|\langle y \rangle^{-\sigma/2} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \leq \varepsilon_2 \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C_{\varepsilon_2} \|f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2. \end{aligned}$$

From the above inequalities, we infer that, choosing ε_2 small enough,

$$\begin{aligned} & \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ (3.15) \quad & \leq \varepsilon_1 \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dx dt \\ & \quad + C_{\varepsilon_1} \left(\|\langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\langle y \rangle^\sigma \Lambda^{-1/3} \partial_y g\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\ & \quad + C_{\varepsilon_1} \left(\|f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\partial_y f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\Lambda^{-1/3} h\|_{L^2(\mathbb{R}_+^3)}^2 \right). \end{aligned}$$

Step 5) In this step we treat the first term on the right side of (3.15), and show that, for any $0 < \varepsilon < 1$,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dx dt \\ (3.16) \quad & \leq C \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \varepsilon C \|\Lambda^{-2/3} \partial_y h\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \quad + C_\varepsilon \left(\|f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \|\partial_y f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right). \end{aligned}$$

To do so, we integrate by parts to get

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dx dt & = 2\operatorname{Re} \left(\partial_y^3 \Lambda^{-1/2} f, \partial_y^2 \Lambda^{-1/2} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\ & = 2\operatorname{Re} \left(\partial_y^3 \Lambda^{-2/3} f, \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)}. \end{aligned}$$

This yields

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right) (t, x, 0) \right|^2 dx dt \\ (3.17) \quad & \leq \frac{\varepsilon}{2} \|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + 2\varepsilon^{-1} \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\ & \leq \varepsilon \|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C_\varepsilon \|\partial_y f\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2, \end{aligned}$$

the last inequality holding because we can use (2.21) to integrate by parts and then obtain

$$(3.18) \quad \begin{aligned} \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 &= \left(\partial_y^2 \Lambda^{-2/3} f, \partial_y^2 f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \leq \left| \left(\partial_y^3 \Lambda^{-2/3} f, \partial_y f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \right| \\ &\leq \|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

Thus in order to prove (3.16) it suffices to estimate $\|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}$. We study the equation

$$\begin{aligned} &\partial_t \Lambda^{-2/3} \partial_y f + u \partial_x \Lambda^{-2/3} \partial_y f + v \partial_y \Lambda^{-2/3} \partial_y f - \partial_y^3 \Lambda^{-2/3} f \\ &= \Lambda^{-2/3} \partial_y h + [u \partial_x + v \partial_y, \Lambda^{-2/3}] \partial_y f - \Lambda^{-2/3} (\partial_y u) \partial_x f - \Lambda^{-2/3} (\partial_y v) \partial_y f, \end{aligned}$$

which implies, by taking L^2 inner product with $-\partial_y^3 \Lambda^{-2/3} f$,

$$(3.19) \quad \begin{aligned} \|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 &= -\operatorname{Re} \left(\partial_t \Lambda^{-2/3} \partial_y f, \partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad - \operatorname{Re} \left(u \partial_x \Lambda^{-2/3} \partial_y f + v \partial_y \Lambda^{-2/3} \partial_y f, \partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad + \operatorname{Re} \left(\Lambda^{-2/3} \partial_y h, \partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad + \operatorname{Re} \left([u \partial_x + v \partial_y, \Lambda^{-2/3}] \partial_y f, \partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad - \operatorname{Re} \left(\Lambda^{-2/3} (\partial_y u) \partial_x f, \partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad - \operatorname{Re} \left(\Lambda^{-2/3} (\partial_y v) \partial_y f, \partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

Next we will treat the terms on the right hand side. Observing

$$\partial_t \Lambda^{-2/3} \partial_y f|_{y=0} = 0$$

due to (2.21), we integrate by part to compute

$$\begin{aligned} &-\operatorname{Re} \left(\partial_t \Lambda^{-2/3} \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= -\operatorname{Re} \left(\partial_t \partial_y^2 \Lambda^{-2/3} f, \partial_y^2 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= 0, \end{aligned}$$

the last equality holding because

$$\partial_y^2 \Lambda^{-2/3} f|_{t=0} = \partial_y^2 \Lambda^{-2/3} f|_{t=T} = 0$$

due to (2.20). Since $u|_{y=0}$ then integrating by parts gives

$$\begin{aligned} &-\operatorname{Re} \left(u \partial_x \Lambda^{-2/3} \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= -\operatorname{Re} \left(u \partial_x \Lambda^{-2/3} \partial_y^2 f, \partial_y^2 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad - \operatorname{Re} \left((\partial_y u) \partial_x \Lambda^{-2/3} \partial_y f, \partial_y^2 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= \frac{1}{2} \left((\partial_x u) \Lambda^{-2/3} \partial_y^2 f, \Lambda^{-2/3} \partial_y^2 f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad - \operatorname{Re} \left((\partial_y u) \partial_x \Lambda^{-2/3} \partial_y f, \partial_y^2 \Lambda^{-2/3} f \right)_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq \frac{1}{2} \|\partial_x u\|_{L^\infty} \|\Lambda^{-2/3} \partial_y^2 f\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad + \|\Lambda^{-1/3} (\partial_y u) \partial_x \Lambda^{-2/3} \partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2. \end{aligned}$$

On the other hand, using Lemma 3.1 gives

$$\begin{aligned}
 & \left\| \Lambda^{-1/3} (\partial_y u) \partial_x \Lambda^{-2/3} \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
 \leq & 2 \left\| \Lambda^{-1/3} \partial_x \Lambda^{-2/3} (\partial_y u) \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + 2 \left\| \Lambda^{-1/3} [\partial_y u, \partial_x \Lambda^{-2/3}] \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
 \leq & C \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2.
 \end{aligned}$$

Thus

$$\begin{aligned}
 & - \operatorname{Re} \left(u \partial_x \Lambda^{-2/3} \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 \leq & C \left(\left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \right) \\
 \leq & \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2,
 \end{aligned}$$

where the last inequality using (3.18). Using (3.18) we conclude

$$\begin{aligned}
 & - \operatorname{Re} \left(v \partial_y \Lambda^{-2/3} \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 \leq & \frac{\tilde{\varepsilon}}{2} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
 \leq & \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2.
 \end{aligned}$$

Cauchy-Schwarz inequality gives, for any $\tilde{\varepsilon} > 0$,

$$\begin{aligned}
 & \operatorname{Re} \left(\Lambda^{-2/3} \partial_y h, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 \leq & \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \tilde{\varepsilon}^{-1} \left\| \Lambda^{-2/3} \partial_y h \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2,
 \end{aligned}$$

and

$$\begin{aligned}
 & - \operatorname{Re} \left(\Lambda^{-2/3} (\partial_y v) \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 \leq & \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \tilde{\varepsilon}^{-1} \left\| \partial_y v \right\|_{L^\infty}^2 \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2
 \end{aligned}$$

and

$$\begin{aligned}
 & \operatorname{Re} \left([u \partial_x + v \partial_y, \Lambda^{-2/3}] \partial_y f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 (3.20) \quad & \leq \frac{\tilde{\varepsilon}}{2} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + 2\tilde{\varepsilon}^{-1} \left\| [u \partial_x + v \partial_y, \Lambda^{-2/3}] \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
 & \leq \frac{\tilde{\varepsilon}}{2} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
 & \leq \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \left\| \partial_y f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2,
 \end{aligned}$$

the second inequality using Lemma 3.1, while the last inequality following from (3.18). Finally,

$$\begin{aligned}
 & - \operatorname{Re} \left(\Lambda^{-2/3} (\partial_y u) \partial_x f, -\partial_y^3 \Lambda^{-2/3} f \right)_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 \leq & \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \tilde{\varepsilon}^{-1} \left\| \Lambda^{-2/3} (\partial_y u) \partial_x f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
 \leq & \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + \tilde{\varepsilon}^{-1} \left\| (\partial_y u) \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
 & + \tilde{\varepsilon}^{-1} \left\| [\partial_y u, \Lambda^{-2/3}] \partial_x f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
 \leq & \tilde{\varepsilon} \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2 \\
 & + C_{\tilde{\varepsilon}} \left\| f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}^2.
 \end{aligned}$$

This, along with (3.19) -(3.20), yields, for any $\tilde{\varepsilon} > 0$,

$$\begin{aligned} & \|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \leq \tilde{\varepsilon} \|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C_{\tilde{\varepsilon}} \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \quad + C_{\tilde{\varepsilon}} \left(\|\Lambda^{-2/3} \partial_y h\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right). \end{aligned}$$

Thus letting $\tilde{\varepsilon}$ be small enough, we have

$$\begin{aligned} & \|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ (3.21) \quad & \leq C \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \quad + C \left(\|\Lambda^{-2/3} \partial_y h\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right). \end{aligned}$$

This along with (3.17) yields the desired estimate (3.16).

Step 6) Now we combine (3.15) and (3.16) to conclude for any $0 < \varepsilon, \varepsilon_1 < 1$,

$$\begin{aligned} & \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \leq \varepsilon_1 C \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \varepsilon_1 \varepsilon C \|\Lambda^{-2/3} \partial_y h\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \quad + C_{\varepsilon_1, \varepsilon} \left(\|\langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\langle y \rangle^\sigma \Lambda^{-1/3} \partial_y g\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right) \\ & \quad + C_{\varepsilon_1, \varepsilon} \left(\|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\Lambda^{-1/3} h\|_{L^2(\mathbb{R}_+^3)}^2 \right), \end{aligned}$$

which implies, choosing $\varepsilon_1 > 0$ sufficiently small,

$$\begin{aligned} & \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \leq \varepsilon \|\Lambda^{-2/3} \partial_y h\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \quad + C_\varepsilon \left(\|\langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\langle y \rangle^\sigma \Lambda^{-1/3} \partial_y g\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right) \\ & \quad + C_\varepsilon \left(\|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\Lambda^{-1/3} h\|_{L^2(\mathbb{R}_+^3)}^2 \right), \end{aligned}$$

with $\varepsilon > 0$ arbitrarily small. This, along with

$$\begin{aligned} \|\langle y \rangle^{-\sigma/2} \Lambda^{1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 & \leq C \|(\partial_y u)^{1/2} \Lambda^{1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \leq C \|\langle y \rangle^{-\sigma/2} \partial_x \Lambda^{-2/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + C \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \end{aligned}$$

due to (1.2), implies, for any $\varepsilon > 0$,

$$\begin{aligned} & \|\langle y \rangle^{-\sigma/2} \Lambda^{1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \leq \varepsilon \|\Lambda^{-2/3} \partial_y h\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \\ & \quad + C_\varepsilon \left(\|\langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\langle y \rangle^\sigma \Lambda^{-1/3} \partial_y g\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 \right) \\ & \quad + C_\varepsilon \left(\|\partial_y f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|f\|_{L^2([0,T] \times \mathbb{R}_+^2)}^2 + \|\Lambda^{-1/3} h\|_{L^2(\mathbb{R}_+^3)}^2 \right). \end{aligned}$$

This is just the first estimate in Proposition 2.4. And the second estimate follows from (3.21) since $|\partial_y u|$ is bounded from above by $\langle y \rangle^{-\sigma}$. Thus the proof of Proposition 2.4 is complete. \square

4. PROPERTY OF INDUCATIVE WEIGHT FUNCTIONS

This section is devoted to proving the Lemma 2.1, Lemma 2.2 and Lemma 2.5, used in Section 2.

Recall, for $m \geq N_0 + 1$ and $0 \leq \ell \leq 3, y > 0, 0 \leq t \leq T < 1$,

$$W_m^\ell = e^{2cy} \left(1 + \frac{2cy}{(3m + \ell)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} (1 + cy)^{-1} \Lambda^{\frac{\ell}{3}}, \quad \phi_m^\ell = \phi^{3(m-N_0-1)+\ell}.$$

thus

$$(4.1) \quad \phi_{m_1}^{\ell_1} \leq \phi_{m_2}^{\ell_2}$$

provided $N_0 + 1 \leq m_2 \leq m_1$ and $0 \leq \ell_2 \leq \ell_1 \leq 3$.

Next we list some inequalities for the weight W_m^ℓ . Observe the function

$$\gamma \longrightarrow \left(1 + \frac{cy}{\gamma}\right)^{-\gamma}$$

is a monotonically decreasing function as γ varies in the interval $[1, +\infty[$ for $y \geq 0$. Thus

$$(4.2) \quad 0 \leq \ell \leq 3, \quad \|W_{m_1}^\ell f\|_{L^2(\mathbb{R}_x)} \leq \|W_{m_2}^\ell f\|_{L^2(\mathbb{R}_x)}$$

and

$$(4.3) \quad \forall 0 \leq \ell \leq i \leq 3, \quad \|W_{m_1}^i f\|_{L^2(\mathbb{R}_x)} \leq \|W_{m_2}^{i-\ell} \Lambda^{\ell/3} f\|_{L^2(\mathbb{R}_x)} \leq \|W_{m_3}^i f\|_{L^2(\mathbb{R}_x)},$$

provided that $m_1 \geq m_2 \geq 1$, and that $3m_2 + i - \ell \geq 3m_3 + i$. Moreover, since

$$\forall 0 \leq \alpha \leq 3, \forall \gamma \geq 1, \quad \left| \partial_y^\alpha e^{2cy} \left(1 + \frac{cy}{\gamma}\right)^{-\gamma} (1 + cy)^{-1} \right| \leq C_\alpha e^{2cy} \left(1 + \frac{cy}{\gamma}\right)^{-\gamma} (1 + cy)^{-1},$$

with C_α a constant independent of γ , then the following estimates:

$$(4.4) \quad \|[\partial_y, W_m^i] f\|_{L^2(\mathbb{R}_+^2)} \leq C \|W_m^i f\|_{L^2(\mathbb{R}_+^2)},$$

$$(4.5) \quad \begin{aligned} \|[\partial_y^2, W_m^i] f\|_{L^2(\mathbb{R}_+^2)} &\leq C \left(\|W_m^i f\|_{L^2(\mathbb{R}_+^2)} + \|W_m^i \partial_y f\|_{L^2(\mathbb{R}_+^2)} \right) \\ &\leq \tilde{C} \left(\|W_m^i f\|_{L^2(\mathbb{R}_+^2)} + \|\partial_y W_m^i f\|_{L^2(\mathbb{R}_+^2)} \right) \end{aligned}$$

$$(4.6) \quad \begin{aligned} \|[\partial_y^3, W_m^i] f\|_{L^2(\mathbb{R}_+^2)} &\leq C \left(\|W_m^i f\|_{L^2(\mathbb{R}_+^2)} + \|W_m^i \partial_y f\|_{L^2(\mathbb{R}_+^2)} + \|W_m^i \partial_y^2 f\|_{L^2(\mathbb{R}_+^2)} \right) \\ &\leq \tilde{C} \left(\|W_m^i f\|_{L^2(\mathbb{R}_+^2)} + \|\partial_y W_m^i f\|_{L^2(\mathbb{R}_+^2)} + \|\partial_y^2 W_m^i f\|_{L^2(\mathbb{R}_+^2)} \right) \end{aligned}$$

hold for all integers m, i with $m \geq 1$ and $0 \leq i \leq 3$, where C, \tilde{C} are two constants independent of m .

Lemma 4.1. *Under the assumption (1.2) and (1.3). Let c be the constant given in (2.2), and $\Lambda^{\tau_1}, \Lambda_\delta^{\tau_2}$ be the Fourier multiplier associate with the symbols $\langle \xi \rangle^{\tau_1}$ and $\langle \delta \xi \rangle^{\tau_2}$, respectively. Then there exists a constant C , such that for any $m, n \geq 1, 0 \leq \ell \leq 3$, and for any $0 < \tilde{c} < c$, we have*

$$(4.7) \quad \|e^{\tilde{c}y} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \leq C \|\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} W_n^\ell f_m\|_{L^2(\mathbb{R}_+^2)},$$

and

$$(4.8) \quad \|\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_x^m v\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{R}_x))} \leq C \|\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} W_n^\ell f_{m+1}\|_{L^2(\mathbb{R}_+^2)}.$$

Proof. In the proof we use C to denote different constants which are independent of m . Observe $\omega \in L^\infty$ and $\omega > 0$ then

$$\|e^{\tilde{c}y} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \leq C \|e^{\tilde{c}y} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}$$

On the other hand, integrating by parts we have

$$\begin{aligned} \|e^{\tilde{c}y} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}^2 &= \int_{\mathbb{R}} \int_0^\infty e^{2\tilde{c}y} \left(\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \frac{\partial_x^m u}{\omega} \right) \overline{\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \frac{\partial_x^m u}{\omega}} dy dx \\ &= \frac{1}{2\tilde{c}} \int_{\mathbb{R}} \int_0^\infty (\partial_y e^{2\tilde{c}y}) \left(\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \frac{\partial_x^m u}{\omega} \right) \overline{\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \frac{\partial_x^m u}{\omega}} dy dx \\ &= -\frac{1}{2\tilde{c}} \int_{\mathbb{R}} \int_0^\infty e^{2\tilde{c}y} \left[\partial_y \left(\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \frac{\partial_x^m u}{\omega} \right) \right] \overline{\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \frac{\partial_x^m u}{\omega}} dy dx \\ &\quad - \frac{1}{2\tilde{c}} \int_{\mathbb{R}} \int_0^\infty e^{2\tilde{c}y} \left(\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \frac{\partial_x^m u}{\omega} \right) \overline{\partial_y \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \frac{\partial_x^m u}{\omega}} dy dx \\ &\leq \frac{1}{\tilde{c}} \|e^{\tilde{c}y} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)} \|e^{\tilde{c}y} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_y \left(\frac{\partial_x^m u}{\omega} \right)\|_{L^2(\mathbb{R}_+^2)}, \end{aligned}$$

which implies

$$\begin{aligned} \left\| e^{\tilde{c}y} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \frac{\partial_x^m u}{\omega} \right\|_{L^2(\mathbb{R}_+^2)} &\leq \left\| e^{\tilde{c}y} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_y \left(\frac{\partial_x^m u}{\omega} \right) \right\|_{L^2(\mathbb{R}_+^2)} \\ &= \left\| \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} e^{\tilde{c}y} \omega^{-1} \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right) \right\|_{L^2(\mathbb{R}_+^2)} \\ &\leq \left\| e^{\tilde{c}y} \omega^{-1} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right) \right\|_{L^2(\mathbb{R}_+^2)} + \left\| [e^{\tilde{c}y} \omega^{-1}, \Lambda^{\tau_1} \Lambda_\delta^{\tau_2}] \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right) \right\|_{L^2(\mathbb{R}_+^2)}. \end{aligned}$$

Thus we have, by the above inequalities,

$$\left\| e^{\tilde{c}y} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_x^m u \right\|_{L^2(\mathbb{R}_+^2)} \leq C \left\| e^{\tilde{c}y} \omega^{-1} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right) \right\|_{L^2(\mathbb{R}_+^2)} + C \left\| [e^{\tilde{c}y} \omega^{-1}, \Lambda^{\tau_1} \Lambda_\delta^{\tau_2}] \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right) \right\|_{L^2(\mathbb{R}_+^2)}.$$

On the other hand, (1.2) and (1.3) enables us to use Lemma 3.1 to obtain

$$\begin{aligned} \left\| [e^{\tilde{c}y} \omega^{-1}, \Lambda^{\tau_1} \Lambda_\delta^{\tau_2}] \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right) \right\|_{L^2(\mathbb{R}_+^2)} &\leq C \left\| \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right) \right\|_{L^2(\mathbb{R}_+^2)} \\ &\leq C \left\| e^{\tilde{c}y} \omega^{-1} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right) \right\|_{L^2(\mathbb{R}_+^2)}. \end{aligned}$$

As a result,

$$\left\| e^{\tilde{c}y} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_x^m u \right\|_{L^2(\mathbb{R}_+^2)} \leq C \left\| e^{\tilde{c}y} \omega^{-1} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right) \right\|_{L^2(\mathbb{R}_+^2)} \leq C \left\| \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} W_n^\ell f_m \right\|_{L^2(\mathbb{R}_+^2)},$$

the last inequality using the fact that $f_m = \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right)$, and that

$$e^{\tilde{c}y} \omega^{-1} \leq C e^{\tilde{c}y} (1+y)^\sigma \leq C e^{2cy} \left(1 + \frac{2cy}{\gamma} \right)^{-\gamma/2}$$

for any $\gamma \geq 1$. This is just the desired (4.7). Now we prove (4.8). Recall $v(t, x, y) = -\int_0^y \partial_x u(t, x, y') dy'$. Then we have

$$\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_x^m v = -\int_0^y \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_x^{m+1} u(x, y') dy'$$

Therefore

$$\begin{aligned} \left\| \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_x^m v \right\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{R}_x))} &\leq \left\| e^{-\tilde{c}y} \right\|_{L^2(\mathbb{R}_+)} \left\| e^{\tilde{c}y} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial_x^{m+1} u \right\|_{L^2(\mathbb{R}_+^2)} \\ &\leq C \left\| \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} W_n^\ell f_{m+1} \right\|_{L^2(\mathbb{R}_+^2)}, \end{aligned}$$

the last inequality using (4.7). Thus the desired (4.8) follows and the proof of Lemma 4.1 is complete. \square

We prove now Lemma 2.1, recall

$$f_m = \partial_x^m \omega - \frac{\partial_y \omega}{\omega} \partial_x^m u = \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right).$$

Lemma 4.2. *There exists a constant C , such that*

$$(4.9) \quad \left\| \langle y \rangle^{-1} W_m^\ell \partial_x^m u \right\|_{L^2(\mathbb{R}_+^2)} + \left\| \langle y \rangle^{-1} W_m^\ell \partial_x^m \omega \right\|_{L^2(\mathbb{R}_+^2)} \leq C \left\| W_m^\ell f_m \right\|_{L^2(\mathbb{R}_+^2)}.$$

As a result, for some constant \tilde{C} ,

$$\left\| \Lambda^{-1} W_m^0 f_{m+1} \right\|_{L^2(\mathbb{R}_+^2)} \leq \tilde{C} \left\| W_m^0 f_m \right\|_{L^2(\mathbb{R}_+^2)},$$

and

$$\left\| \Lambda^{-1} \partial_y W_m^0 f_{m+1} \right\|_{L^2(\mathbb{R}_+^2)} \leq \tilde{C} \left(\left\| \partial_y W_m^0 f_m \right\|_{L^2(\mathbb{R}_+^2)} + \left\| W_m^0 f_m \right\|_{L^2(\mathbb{R}_+^2)} \right).$$

Proof. In the proof we use C to denote different constants which depend only on σ , c , and C_* and are independent of m . We first prove (4.9). Observe

$$\begin{aligned} \omega \langle y \rangle^{-1} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-(3m+\ell)\sigma/2} (1+cy)^{-1} &\leq C(1+y)^{-\sigma-1} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-(3m+\ell)\sigma/2} \\ &\leq CR^{\sigma+1}(R+y)^{-\sigma-1} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-(3m+\ell)\sigma/2}, \end{aligned}$$

where $R \geq 1$ is a large number to be determined later. Thus using the notation

$$b_{m,\ell}^R(y) = \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-(3m+\ell)\sigma/2} (R+y)^{-\sigma-1},$$

we have

$$\begin{aligned} \|\langle y \rangle^{-1} W_m^\ell \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} &= \|\langle y \rangle^{-1} W_m^\ell \left(\omega \frac{\partial_x^m u}{\omega} \right)\|_{L^2(\mathbb{R}_+^2)} \\ &\leq \|\omega \langle y \rangle^{-1} W_m^\ell \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)} + \|\langle y \rangle^{-1} [W_m^\ell, \omega] \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)} \\ &\leq CR^{\sigma+1} \|e^{2cy} b_{m,\ell}^R \frac{\Lambda^{\ell/3} \partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)} + \|\langle y \rangle^{-1} [W_m^\ell, \omega] \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}. \end{aligned}$$

On the other hand, using Lemma 3.1

$$\|\langle y \rangle^{-1} [W_m^\ell, \omega] \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)} \leq R \|\Lambda^{\frac{1}{3}}, \omega\| e^{2cy} b_{m,\ell}^R \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)} \leq CR \|e^{2cy} b_{m,\ell}^R \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}$$

Combining these inequalities we conclude

$$(4.10) \quad \|\langle y \rangle^{-1} W_m^\ell \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \leq CR^{\sigma+1} \|e^{2cy} b_{m,\ell}^R \Lambda^{\ell/3} \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}.$$

Moreover, observe $u|_{y=0} = 0$ and thus we have, by integrating by parts,

$$\begin{aligned} \|e^{2cy} b_{m,\ell}^R \Lambda^{\ell/3} \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}^2 &= \int_{\mathbb{R}} \int_0^\infty e^{4cy} (b_{m,\ell}^R(y))^2 \left(\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right) \overline{\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega}} dy dx \\ &= \frac{1}{4c} \int_{\mathbb{R}} \int_0^\infty (\partial_y e^{4cy}) (b_{m,\ell}^R(y))^2 \left(\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right) \overline{\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega}} dy dx \\ &= -\frac{1}{2c} \int_{\mathbb{R}} \int_0^\infty e^{4cy} b_{m,\ell}^R(y) (\partial_y b_{m,\ell}^R(y)) \left(\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right) \overline{\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega}} dy dx \\ &\quad - \frac{1}{4c} \int_{\mathbb{R}} \int_0^\infty e^{4cy} (b_{m,\ell}^R(y))^2 \left[\partial_y \left(\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right) \right] \overline{\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega}} dy dx \\ &\quad - \frac{1}{4c} \int_{\mathbb{R}} \int_0^\infty e^{4cy} (b_{m,\ell}^R(y))^2 \left(\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right) \overline{\partial_y \Lambda^{\ell/3} \left(\frac{\partial_x^m u}{\omega} \right)} dy dx, \end{aligned}$$

which, along with the estimate

$$|\partial_y b_{m,\ell}^R| \leq (c + (\sigma + 1)R^{-1}) b_{m,\ell}^R,$$

gives

$$\begin{aligned} \|e^{2cy} b_{m,\ell}^R \Lambda^{\ell/3} \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}^2 &\leq \frac{c + (\sigma + 1)R^{-1}}{2c} \|e^{2cy} b_{m,\ell}^R \Lambda^{\ell/3} \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}^2 \\ &\quad + \frac{1}{2c} \|e^{2cy} b_{m,\ell}^R \Lambda^{\ell/3} \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)} \|e^{2cy} b_{m,\ell}^R \partial_y \left(\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right)\|_{L^2(\mathbb{R}_+^2)}. \end{aligned}$$

Now we choose $R = 1 + 2(\sigma + 1)c^{-1}$, which gives $R \geq 1$ and

$$(\sigma + 1)R^{-1} \leq \frac{c}{2}.$$

Then we deduce, from the above inequalities,

$$\|e^{2cy} b_{m,\ell}^R \Lambda^{\ell/3} \frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)} \leq \frac{2}{c} \|e^{2cy} b_{m,\ell}^R \partial_y \Lambda^{\ell/3} \left(\frac{\partial_x^m u}{\omega} \right)\|_{L^2(\mathbb{R}_+^2)}.$$

Moreover, observe $R \geq c^{-1} + 1$ and the monotonicity assumption $\omega \geq C_*^{-1}(1+y)^{-\sigma}$, and thus

$$\begin{aligned} b_{m,\ell}^R &\leq c(1+y)^{-\sigma}(1+cy)^{-1} \left(1 + \frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2} \\ &\leq cC_*\omega(1+cy)^{-1} \left(1 + \frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2}. \end{aligned}$$

As a result, we obtain

$$\|e^{2cy}b_{m,\ell}^R\Lambda^{\ell/3}\frac{\partial_x^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)} \leq C_*\|\omega W_m^\ell \partial_y \left(\frac{\partial_x^m u}{\omega}\right)\|_{L^2(\mathbb{R}_+^2)},$$

which along with (4.10) gives

$$\begin{aligned} &\|\langle y \rangle^{-1} W_m^\ell \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \\ &\leq C\|\omega W_m^\ell \partial_y \left(\frac{\partial_x^m u}{\omega}\right)\|_{L^2(\mathbb{R}_+^2)} \\ &\leq C\|W_m^\ell \omega \partial_y \left(\frac{\partial_x^m u}{\omega}\right)\|_{L^2(\mathbb{R}_+^2)} + C\|[\omega, W_m^\ell] \partial_y \left(\frac{\partial_x^m u}{\omega}\right)\|_{L^2(\mathbb{R}_+^2)}. \end{aligned}$$

Using the notation $\rho_{m,\ell}(y) = e^{2cy} \left(1 + \frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2} (1+cy)^{-1}$,

$$\begin{aligned} \|[\omega, W_m^\ell] \partial_y \left(\frac{\partial_x^m u}{\omega}\right)\|_{L^2(\mathbb{R}_+^2)} &= \|[\omega, \Lambda^{\ell/3}] \rho_{m,\ell}(y) \partial_y \left(\frac{\partial_x^m u}{\omega}\right)\|_{L^2(\mathbb{R}_+^2)} \\ &= \|[\omega \langle y \rangle^\sigma, \Lambda^{\ell/3}] \langle y \rangle^{-\sigma} \rho_{m,\ell}(y) \partial_y \left(\frac{\partial_x^m u}{\omega}\right)\|_{L^2(\mathbb{R}_+^2)} \\ &\leq C\|\langle y \rangle^{-\sigma} \rho_{m,\ell}(y) \partial_y \left(\frac{\partial_x^m u}{\omega}\right)\|_{L^2(\mathbb{R}_+^2)} \\ &\leq C\|\rho_{m,\ell}(y) \omega \partial_y \left(\frac{\partial_x^m u}{\omega}\right)\|_{L^2(\mathbb{R}_+^2)} \\ &\leq C\|W_m^\ell \omega \partial_y \left(\frac{\partial_x^m u}{\omega}\right)\|_{L^2(\mathbb{R}_+^2)}. \end{aligned}$$

Then, combining these inequalities we conclude,

$$\|\langle y \rangle^{-1} W_m^\ell \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \leq C\|W_m^\ell \omega \partial_y \left(\frac{\partial_x^m u}{\omega}\right)\|_{L^2(\mathbb{R}_+^2)} = C\|W_m^\ell f_m\|_{L^2(\mathbb{R}_+^2)}.$$

For the other terms in (4.9), we have

$$\begin{aligned} &\|\langle y \rangle^{-1} W_m^\ell \partial_x^m \omega\|_{L^2(\mathbb{R}_+^2)} \\ &\leq \|\langle y \rangle^{-1} W_m^\ell f_m\|_{L^2(\mathbb{R}_+^2)} + \|\langle y \rangle^{-1} W_m^\ell ((\partial_y \omega)/\omega) \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \\ &\leq \|\langle y \rangle^{-1} W_m^\ell f_m\|_{L^2(\mathbb{R}_+^2)} + \|((\partial_y \omega)/\omega) \langle y \rangle^{-1} W_m^\ell \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \\ &\quad + \|[(\partial_y \omega)/\omega, W_m^\ell] \langle y \rangle^{-1} \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \\ &\leq \|\langle y \rangle^{-1} W_m^\ell f_m\|_{L^2(\mathbb{R}_+^2)} + C\|\langle y \rangle^{-1} W_m^\ell \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \\ &\leq C\|W_m^\ell f_m\|_{L^2(\mathbb{R}_+^2)}, \end{aligned}$$

Thus the desired estimate (4.9) follows. As a result, we have

$$\begin{aligned} \|\Lambda^{-1} W_m^0 f_{m+1}\|_{L^2(\mathbb{R}_+^2)} &\leq \|\Lambda^{-1} W_m^0 \partial_x f_m\|_{L^2(\mathbb{R}_+^2)} + \|\Lambda^{-1} W_m^0 [\partial_x ((\partial_y \omega)/\omega)] \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \\ &\leq \|W_m^0 f_m\|_{L^2(\mathbb{R}_+^2)} + \|\langle y \rangle^{-1} W_m^0 \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \leq C\|W_m^0 f_m\|_{L^2(\mathbb{R}_+^2)}. \end{aligned}$$

Similarly, we can deduce that, using (4.4),

$$\|\Lambda^{-1} \partial_y W_m^0 f_{m+1}\|_{L^2(\mathbb{R}_+^2)} \leq C \left(\|\partial_y W_m^0 f_m\|_{L^2(\mathbb{R}_+^2)} + \|W_m^0 f_m\|_{L^2(\mathbb{R}_+^2)} \right).$$

Thus the proof of Lemma 4.2 is complete. \square

We prove now the Lemma 2.2 by the following 2 lemmas .

Lemma 4.3. *There exists a constant C such that, for any $m \geq 1$ and $1 \leq \ell \leq 3$,*

$$\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/3} \Lambda_\delta^{-2} W_m^{\ell-1} f_m \|_{L^2(\mathbb{R}_+^2)} \leq C \| \partial_y \Lambda_\delta^{-2} W_m^\ell f_m \|_{L^2(\mathbb{R}_+^2)} + C \| \Lambda_\delta^{-2} W_m^\ell f_m \|_{L^2(\mathbb{R}_+^2)}.$$

Proof. We can write

$$\Lambda^{1/3} \Lambda_\delta^{-2} W_m^{\ell-1} = e^{2cy} \left(1 + \frac{2cy}{(3m+\ell-1)\sigma} \right)^{-\frac{(3m+\ell-1)\sigma}{2}} (1+cy)^{-1} \Lambda^{\frac{\ell}{3}} \Lambda_\delta^{-2} = a_{m,\ell}(y) \Lambda_\delta^{-2} W_m^\ell,$$

where

$$a_{m,\ell}(y) = \left(1 + \frac{2cy}{(3m+\ell-1)\sigma} \right)^{-\frac{(3m+\ell-1)\sigma}{2}} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{\frac{(3m+\ell)\sigma}{2}}.$$

Direct computation gives

$$\begin{aligned} |a_{m,\ell}(y)| &= \left(1 + \frac{2cy}{(3m+\ell-1)\sigma} \right)^{\sigma/2} \left(1 + \frac{2cy}{(3m+\ell-1)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{\frac{(3m+\ell)\sigma}{2}} \\ &\leq \left(1 + \frac{2cy}{(3m+\ell-1)\sigma} \right)^{\sigma/2} \leq C \langle y \rangle^{\sigma/2}. \end{aligned}$$

Moreover observe $|\partial_y a_{m,\ell}(y)| \leq 2c |a_{m,\ell}(y)|$, and thus

$$|\partial_y a_{m,\ell}(y)| \leq C \langle y \rangle^{\sigma/2}.$$

As a result,

$$\begin{aligned} &\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/3} \Lambda_\delta^{-2} W_m^{\ell-1} f_m \|_{L^2(\mathbb{R}_+^2)} = \| \langle y \rangle^{-\sigma/2} \partial_y (a_{m,\ell} \Lambda_\delta^{-2} W_m^\ell f_m) \|_{L^2(\mathbb{R}_+^2)} \\ &\leq \| \langle y \rangle^{-\sigma/2} a_{m,\ell} \partial_y \Lambda_\delta^{-2} W_m^\ell f_m \|_{L^2(\mathbb{R}_+^2)} + \| \langle y \rangle^{-\sigma/2} (\partial_y a_{m,\ell}) \Lambda_\delta^{-2} W_m^\ell f_m \|_{L^2(\mathbb{R}_+^2)} \\ &\leq C \left(\| \partial_y \Lambda_\delta^{-2} W_m^\ell f_m \|_{L^2(\mathbb{R}_+^2)} + \| \Lambda_\delta^{-2} W_m^\ell f_m \|_{L^2(\mathbb{R}_+^2)} \right). \end{aligned}$$

The proof of Lemma 4.3 is thus complete. \square

Lemma 4.4. *There exists a constant C , depending only on σ , c , and C_* , such that for any integers $m \geq N_0 + 1$, we have*

$$\begin{aligned} &\| \phi_{m+1}^0 W_{m+1}^0 f_{m+1} \|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \sum_{j=1}^2 \| \partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_{m+1}^0 W_{m+1}^0 f_{m+1} \|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq C \| \phi_m^3 W_m^3 f_m \|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + C \sum_{j=1}^3 \| \partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_m^3 W_m^3 f_m \|_{L^2([0,T] \times \mathbb{R}_+^2)}, \end{aligned}$$

and

$$\begin{aligned} &\| \partial_y^3 \Lambda^{-1} \phi_{m+1}^0 W_{m+1}^0 f_{m+1} \|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq C \| \phi_m^3 W_m^3 f_m \|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + C \sum_{j=1}^2 \| \partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_m^3 W_m^3 f_m \|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad + C \| \partial_y^3 \Lambda^{-1} \phi_m^3 W_m^3 f_m \|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

Proof. In the proof we use C to denote different constants which are independent of m . In view of the definition (2.1) of f_m , we have, observing (4.1),

$$\begin{aligned} &\| \phi_{m+1}^0 W_{m+1}^0 f_{m+1} \|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \\ &\leq \| \phi_{m+1}^0 W_{m+1}^0 \partial_x f_m \|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + \| \phi_{m+1}^0 W_{m+1}^0 [\partial_x ((\partial_y \omega)/\omega)] \partial_x^m u \|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \\ &\leq \| \phi_m^3 W_{m+1}^0 \Lambda^1 f_m \|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + C \| \langle y \rangle^{-1} \phi_m^3 W_m^3 \partial_x^m u \|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \\ &\leq C \| \phi_m^3 W_m^3 f_m \|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))}, \end{aligned}$$

the last inequality using (4.9) and (4.3). Similarly, using (4.4), we can deduce that

$$\|\partial_y \phi_{m+1}^0 W_{m+1}^0 f_{m+1}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \leq C \|\partial_y \phi_m^3 W_m^3 f_m\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + C \|\phi_m^3 W_m^3 f_m\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))}.$$

The other terms

$$\|\partial_y^2 \Lambda^{-2/3} \phi_{m+1}^0 W_{m+1}^0 f_{m+1}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))}, \quad \|\partial_y^3 \Lambda^{-1} \phi_{m+1}^0 W_{m+1}^0 f_{m+1}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))}$$

can be treated in the same way, thanks to (4.5) and (4.6). So we omit it here. Thus the proof of Lemma 4.4 is complete. \square

Proof of Lemma 2.5. Observe

$$\begin{aligned} (1+y)^{-\frac{\sigma}{2}} &= \left(\frac{(3m+\ell-1)\sigma}{2c} \right)^{-\frac{\sigma}{2}} \left(\frac{2c}{(3m+\ell-1)\sigma} + \frac{2cy}{(3m+\ell-1)\sigma} \right)^{-\frac{\sigma}{2}} \\ &\geq C \left(\frac{(3m+\ell-1)\sigma}{2c} \right)^{-\frac{\sigma}{2}} \left(1 + \frac{2cy}{(3m+\ell-1)\sigma} \right)^{-\frac{\sigma}{2}} \\ &\geq Cm^{-\frac{\sigma}{2}} \left(1 + \frac{2cy}{(3m+\ell-1)\sigma} \right)^{-\frac{\sigma}{2}}. \end{aligned}$$

Then

$$(4.11) \quad (1+y)^{-\frac{\sigma}{2}} \left(1 + \frac{2cy}{(3m+\ell-1)\sigma} \right)^{-\frac{(3m+\ell-1)\sigma}{2}} \geq Cm^{-\frac{\sigma}{2}} \left(1 + \frac{2cy}{(3m+\ell-1)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}}.$$

Moreover we find

$$\begin{aligned} \left(1 + \frac{2cy}{(3m+\ell-1)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} &= \left(\frac{1}{(3m+\ell-1)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} ((3m+\ell-1)\sigma + 2cy)^{-\frac{(3m+\ell)\sigma}{2}} \\ &= \left(\frac{(3m+\ell)\sigma}{(3m+\ell-1)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} \left(\frac{(3m+\ell-1)}{(3m+\ell)} + \frac{2cy}{(3m+\ell)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} \\ &\geq \left(\frac{3m+\ell}{3m+\ell-1} \right)^{-\frac{(3m+\ell)\sigma}{2}} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} \\ &\geq C \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}}, \end{aligned}$$

which along with (4.11) gives

$$\left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} \leq Cm^{\frac{\sigma}{2}} (1+y)^{-\frac{\sigma}{2}} \left(1 + \frac{2cy}{(3m+\ell-1)\sigma} \right)^{-\frac{(3m+\ell-1)\sigma}{2}}.$$

As a result, recalling

$$(1+y)^{-\frac{\sigma}{2}} \Lambda^{1/3} W_m^{\ell-1} = (1+y)^{-\frac{\sigma}{2}} e^{2cy} \left(1 + \frac{2cy}{(3m+\ell-1)\sigma} \right)^{-\frac{(3m+\ell-1)\sigma}{2}} (1+cy)^{-1} \Lambda^{\frac{\ell}{3}},$$

we have, observing $\phi^{-\frac{1}{2}} \phi_m^\ell = \phi^{\frac{1}{2}} \phi_m^{\ell-1}$

$$\|\phi^{-\frac{1}{2}} \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq Cm^{\sigma/2} \|(1+y)^{-\frac{\sigma}{2}} \Lambda^{1/3} \phi^{\frac{1}{2}} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)},$$

that is, recalling $F = \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m$ and $f = \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m$,

$$\|\phi^{-1/2} F\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq Cm^{\sigma/2} \|\langle y \rangle^{-\frac{\sigma}{2}} \Lambda^{1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)}.$$

Moreover, using (4.3) and (4.5) we have, observing $\phi_m^\ell \leq \phi^{1/2} \phi_m^{\ell-1}$,

$$\begin{aligned} &\|\partial_y^2 \Lambda^{-2/3} F\|_{L^2([0,T] \times \mathbb{R}_+^2)} = \|\partial_y^2 \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq C \|\partial_y^2 \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_m^\ell W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \|\partial_y \Lambda^{-1/3} \Lambda_\delta^{-2} \phi_m^\ell W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad + C \|\Lambda^{-1/3} \Lambda_\delta^{-2} \phi_m^\ell W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\leq C \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \left(\|\phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\partial_y \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right). \end{aligned}$$

Then combining the above inequalities, the first estimate in Lemma 2.5 follows. The second one can be deduced similarly. In fact using (4.3) and (4.6) gives

$$\begin{aligned}
 & \left\| \partial_y^3 \Lambda^{-1} F \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} = \left\| \partial_y^3 \Lambda^{-1} \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 & \leq C \left\| \partial_y^3 \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_m^\ell W_m^{\ell-1} f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \left\| \partial_y^2 \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_m^\ell W_m^{\ell-1} f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 & \quad + C \left\| \partial_y \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \left\| \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 & \leq C \left\| \partial_y^3 \Lambda^{-2/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \left\| \partial_y^2 \Lambda^{-1/3} f \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 & \quad + C \left(\left\| \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \left\| \partial_y \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \right).
 \end{aligned}$$

This is just the second estimate in Lemma 2.5. The proof is thus complete. \square

5. ESTIMATES OF THE NONLINEAR TERMS

In this section we estimate the nonlinear terms $\mathcal{Z}_{m, \ell, \delta}$ defined in (2.13), and prove the Proposition 2.3. Recall

$$\begin{aligned}
 \mathcal{Z}_{m, \ell, \delta} &= - \sum_{j=1}^m \binom{m}{j} \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j v) \partial_y f_{m-j} \\
 & \quad - \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) - 2 \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] f_m \\
 & \quad + \Lambda_\delta^{-2} (\partial_t \phi_m^\ell) W_m^\ell f_m + [u \partial_x + v \partial_y - \partial_y^2, \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell] f_m \\
 &= \mathcal{J}_{m, \ell, \delta} + \Lambda_\delta^{-2} (\partial_t \phi_m^\ell) W_m^\ell f_m + [u \partial_x + v \partial_y - \partial_y^2, \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell] f_m,
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{J}_{m, \ell, \delta} &= - \sum_{j=1}^m \binom{m}{j} \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j v) \partial_y f_{m-j} \\
 & \quad - \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) - 2 \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] f_m.
 \end{aligned}$$

We remark it suffices to prove the estimates (2.15) and (2.17) in Proposition 2.3, since the estimate (2.16) can be treated exactly similar as (2.15). Next we will proceed to prove (2.15) and (2.17) through the following Proposition 5.1 and Proposition 5.2. Proposition 5.2 is devoted to treating the term $\mathcal{J}_{m, \ell, \delta}$ in the definition of $\mathcal{Z}_{m, \ell, \delta}$, while the other two terms are estimated in Proposition 5.1.

To simplify the notations, we will use C to denote different constants depending only on σ , c , and the constants C_0, C_* in Theorem 1.1, but independent of m and δ .

Proposition 5.1. *We have, denoting $F = \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m$ and $\tilde{f} = \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m$,*

$$\begin{aligned}
 & \left\| \phi^{1/2} \Lambda_\delta^{-2} (\partial_t \phi_m^\ell) W_m^\ell f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \left\| \phi^{1/2} [u \partial_x + v \partial_y - \partial_y^2, \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell] f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 & \leq m C \left\| \phi^{-1/2} F \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \left\| \partial_y F \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}
 \end{aligned}$$

and

$$\begin{aligned}
 & \left\| \Lambda^{-\frac{2}{3}} \partial_y \phi^{1/2} \Lambda_\delta^{-2} (\partial_t \phi_m^{\ell-1}) W_m^{\ell-1} f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 & \quad + \left\| \Lambda^{-\frac{2}{3}} \partial_y \phi^{1/2} [u \partial_x + v \partial_y - \partial_y^2, \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1}] f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 & \leq C \left\| \langle y \rangle^{-\sigma} \Lambda^{\frac{1}{3}} \tilde{f} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \left\| \partial_y^2 \Lambda^{-\frac{2}{3}} \tilde{f} \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 & \quad + m C \left\| \Lambda^{-\frac{2}{3}} \phi^{-1/2} \partial_y \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \left\| \Lambda^{-\frac{2}{3}} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0, T] \times \mathbb{R}_+^2)}.
 \end{aligned}$$

Proof. It is sufficient to prove the second estimate in Proposition 5.1, since the treatment of the first one is similar and easier and we omit it here for brevity. Observe

$$\left| \partial_t \phi_m^{\ell-1} \right| \leq 3m \phi_m^{\ell-2} \leq 3m \phi_m^{\ell-1} \phi^{-1},$$

and thus

$$(5.1) \quad \|\Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\Lambda_\delta^{-2}(\partial_t\phi_m^{\ell-1})W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \leq 3m\|\Lambda^{-\frac{2}{3}}\phi^{-1/2}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}.$$

We write, using (3.1),

$$\begin{aligned} & \|\Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}[u\partial_x + v\partial_y - \partial_y^2, \Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}]f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \leq \| [u\partial_x + v\partial_y - \partial_y^2, \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}]f_m \|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \quad + \| [u\partial_x + v\partial_y - \partial_y^2, \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}]\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m \|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \stackrel{\text{def}}{=} Q_{5,1} + Q_{5,2}. \end{aligned}$$

We first estimate $Q_{5,1}$. Observe

$$\begin{aligned} & \| [u\partial_x, \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}]f_m \|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \leq \| [u\partial_x, \Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\partial_y]\phi^{1/2}f_m \|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \quad + \| [u\partial_x, \Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}[\partial_y, W_m^{\ell-1}]]\phi^{1/2}f_m \|_{L^2([0,T]\times\mathbb{R}_+^2)}. \end{aligned}$$

On the other hand, we compute, using Lemma 3.1 and (4.4),

$$\begin{aligned} & \| [u\partial_x, \Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\partial_y]\phi^{1/2}f_m \|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \leq \| [u\partial_x, \Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}]\partial_y\phi^{1/2}f_m \|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \quad + \| \Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}[u\partial_x, \partial_y]\phi^{1/2}f_m \|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \leq C\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\partial_y\phi^{1/2}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} + \|\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}(\partial_y u)\partial_x\phi^{1/2}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \leq C\|\partial_y\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\phi^{1/2}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} + C\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\phi^{1/2}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \quad + \|(\partial_y u)\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\partial_x\phi^{1/2}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \quad + \|[\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}, (\partial_y u)]\partial_x\phi^{1/2}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \leq C\|\partial_y\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} + C\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \quad + C\|\langle y \rangle^{-\sigma}\Lambda^{\frac{1}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}. \end{aligned}$$

Similarly we also have, using again Lemma 3.1,

$$\begin{aligned} & \| [u\partial_x, \Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}[\partial_y, W_m^{\ell-1}]]\phi^{1/2}f_m \|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \leq C\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}. \end{aligned}$$

As a result, combining these inequalities, we have

$$\begin{aligned} & \| [u\partial_x, \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}]f_m \|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \leq C\|\partial_y\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} + C\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \quad + C\|\langle y \rangle^{-\sigma}\Lambda^{\frac{1}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}. \end{aligned}$$

Similarly, repeating the above arguments with $u\partial_x$ replaced by $v\partial_y$ and ∂_y^2 respectively, one has

$$\begin{aligned} & \| [v\partial_y, \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}]f_m \|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \leq C\|\partial_y^2\Lambda^{-\frac{2}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} + C\|\Lambda^{-\frac{2}{3}}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \quad + C\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}, \end{aligned}$$

and

$$\begin{aligned} & \left\| [\partial_y^2, \Lambda^{-\frac{2}{3}} \partial_y \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1}] f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C \left\| \partial_y^2 \Lambda^{-\frac{2}{3}} \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \left\| \Lambda^{-\frac{2}{3}} \partial_y \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + C \left\| \Lambda^{-\frac{2}{3}} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

As a result, we conclude, combining these inequalities,

$$\begin{aligned} Q_{5,1} & = \left\| [u \partial_x + v \partial_y - \partial_y^2, \Lambda^{-\frac{2}{3}} \partial_y \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1}] f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C \left\| \langle y \rangle^{-\sigma} \Lambda^{\frac{1}{3}} \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \left\| \partial_y^2 \Lambda^{-\frac{2}{3}} \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + C \left\| \partial_y \Lambda^{-\frac{2}{3}} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \left\| \Lambda^{-\frac{2}{3}} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

The term $Q_{5,2}$ can be treated similarly and easily, and we have

$$\begin{aligned} Q_{5,2} & = \left\| [u \partial_x + v \partial_y - \partial_y^2, \Lambda^{-\frac{2}{3}} \partial_y \phi^{1/2}] \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C \left\| \langle y \rangle^{-\sigma} \Lambda^{\frac{1}{3}} \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \left\| \partial_y^2 \Lambda^{-\frac{2}{3}} \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + C \left\| \partial_y \Lambda^{-\frac{2}{3}} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \left\| \Lambda^{-\frac{2}{3}} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

Thus

$$\begin{aligned} & \left\| \Lambda^{-\frac{2}{3}} \partial_y \phi^{1/2} [u \partial_x + v \partial_y - \partial_y^2, \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1}] f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C \left\| \langle y \rangle^{-\sigma} \Lambda^{\frac{1}{3}} \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \left\| \partial_y^2 \Lambda^{-\frac{2}{3}} \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + C \left\| \partial_y \Lambda^{-\frac{2}{3}} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \left\| \Lambda^{-\frac{2}{3}} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

This along with (5.1) gives the second estimate in Proposition 5.1. The proof is thus complete. \square

Proposition 5.2. *Under the induction hypothesis (2.9), (2.10), we have, denoting $F = \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m$,*

$$\left\| \phi^{1/2} \mathcal{J}_{m,\ell,\delta} \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq mC \|F\|_{L^2([0,T] \times \mathbb{R}_+^2)} + CA^{m-6} ((m-5)!)^{3(1+\sigma)},$$

and

$$\begin{aligned} & \left\| \Lambda^{-2/3} \partial_y \phi^{1/2} \mathcal{J}_{m,\ell-1,\delta} \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq mC \left(\left\| \Lambda^{-2/3} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \left\| \Lambda^{-2/3} \partial_y \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \right) \\ & \quad + CA^{m-6} ((m-5)!)^{3(1+\sigma)}, \end{aligned}$$

where the constant $C > 0$ is independent on m and $\delta > 0$.

We first prove the first estimate in Proposition 5.2. In view of the definition given at the beginning of this section, we see,

$$\begin{aligned} & \left\| \phi^{1/2} \mathcal{J}_{m,\ell,\delta} \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq \left\| \mathcal{J}_{m,\ell,\delta} \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \sum_{j=1}^m \binom{m}{j} \left\| \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j u) f_{m+1-j} \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ (5.2) \quad & + \sum_{j=1}^{m-1} \binom{m}{j} \left\| \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j v) \partial_y f_{m-j} \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & + \sum_{j=1}^{m-1} \binom{m}{j} \left\| \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell [\partial_y (\partial_y \omega / \omega)] (\partial_x^j v) (\partial_x^{m-j} u) \right\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & + 2 \left\| \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell [\partial_y (\partial_y \omega / \omega)] f_m \right\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

And we will proceed to estimate the each term on the right hand side of (5.2), and state as the following three Lemmas.

Lemma 5.3. *Under the same assumption as in Proposition 2.3, we have*

$$\begin{aligned} & \sum_{j=1}^{m-1} \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq mC \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + CA^{m-6} ((m-5)!)^{3(1+\sigma)}. \end{aligned}$$

Proof. We first split the summation as follows:

$$\begin{aligned} & \sum_{j=1}^{m-1} \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & = \sum_{j=m-2}^{m-1} \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \sum_{j=1}^{m-3} \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

Moreover as for the last term on the right hand side, we use (4.3) to compute,

$$\begin{aligned} & \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \|\phi_m^\ell W_m^0 \Lambda^{\ell/3} (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \|\phi_m^\ell W_m^0 (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\phi_m^\ell W_m^0 \partial_x (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \|\phi_m^\ell W_m^0 (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \|\phi_m^\ell W_m^0 (\partial_x^j v) (\partial_y \partial_x f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

Thus we have

$$\begin{aligned} & \sum_{j=1}^{m-1} \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \sum_{j=m-2}^{m-1} \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \sum_{j=1}^{m-3} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \sum_{j=1}^{m-3} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \quad + \sum_{j=1}^{m-3} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j v) (\partial_y \partial_x f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned} \tag{5.3}$$

Next we estimate step by step the terms on the right side of (5.3).

(a) We treat in this step the first term on the right hand side of (5.3), and prove that

$$\begin{aligned} & \sum_{j=m-2}^{m-1} \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq mC \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + CA^{m-6} ((m-5)!)^{3(1+\sigma)}. \end{aligned} \tag{5.4}$$

To do so, direct computation gives

$$\begin{aligned}
 & \sum_{j=m-2}^{m-1} \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 & \leq \sum_{j=m-2}^{m-1} \binom{m}{j} \|\Lambda_\delta^{-2} \Lambda^{\ell/3} e^{2cy} (1+cy)^{-1} \phi_m^\ell (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 & \leq \sum_{j=m-2}^{m-1} \binom{m}{j} \|e^{2cy} (\partial_y f_{m-j}) \Lambda_\delta^{-2} \Lambda^{\ell/3} (1+cy)^{-1} \phi_m^\ell \partial_x^j v\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 & \quad + \sum_{j=m-2}^{m-1} \binom{m}{j} \|[e^{2cy} (\partial_y f_{m-j}), \Lambda_\delta^{-2} \Lambda^{\ell/3}] (1+cy)^{-1} \phi_m^\ell \partial_x^j v\|_{L^2([0,T] \times \mathbb{R}_+^2)}.
 \end{aligned}$$

On the other hand, by (2.7),

$$\begin{aligned}
 & \sum_{j=m-2}^{m-1} \binom{m}{j} \|e^{2cy} (\partial_y f_{m-j}) \Lambda_\delta^{-2} \Lambda^{\ell/3} (1+cy)^{-1} \phi_m^\ell \partial_x^j v\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 & \leq C \sum_{j=m-2}^{m-1} \binom{m}{j} \|(1+cy)^{-1}\|_{L^2(\mathbb{R}_+; L^\infty([0,T] \times \mathbb{R}_x))} \|\Lambda_\delta^{-2} \phi_m^\ell \Lambda^{\ell/3} \partial_x^j v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))} \\
 & \leq C \sum_{j=m-2}^{m-1} \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell \Lambda^{\ell/3} \partial_x^j v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))}.
 \end{aligned}$$

Similarly, we have, by virtue of Lemma 3.1,

$$\begin{aligned}
 & \sum_{j=m-2}^{m-1} \binom{m}{j} \|[e^{2cy} (\partial_y f_{m-j}), \Lambda_\delta^{-2} \Lambda^{\ell/3}] (1+cy)^{-1} \phi_m^\ell \partial_x^j v\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 & \leq C \sum_{j=m-2}^{m-1} \binom{m}{j} \|(1+cy)^{-1} \phi_m^\ell \partial_x^j v\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 & \leq C \sum_{j=m-2}^{m-1} \binom{m}{j} \|\phi_m^\ell \Lambda^{\ell/3} \partial_x^j v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))}.
 \end{aligned}$$

Thus combining these inequalities, we obtain

$$\begin{aligned}
 & \sum_{j=m-2}^{m-1} \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 & \leq C \sum_{j=m-2}^{m-1} \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell \Lambda^{\ell/3} \partial_x^j v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))} \\
 & \leq Cm \|\Lambda_\delta^{-2} \phi_m^\ell \Lambda^{\ell/3} \partial_x^{m-1} v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))} + Cm^2 \|\Lambda_\delta^{-2} \phi_m^\ell \Lambda^{\ell/3} \partial_x^{m-2} v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))}.
 \end{aligned}$$

Moreover, observe

$$\begin{aligned}
 & \|\Lambda_\delta^{-2} \phi_m^\ell \Lambda^{\ell/3} \partial_x^{m-2} v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))}^2 \\
 & \leq \|\Lambda_\delta^{-2} \phi_m^\ell \Lambda^{\ell/3} \partial_x^{m-1} v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))} \|\Lambda_\delta^{-2} \phi_m^\ell \Lambda^{\ell/3} \partial_x^{m-3} v\|_{L^\infty(\mathbb{R}_+; L^2([0,T] \times \mathbb{R}_x))},
 \end{aligned}$$

and thus

$$\begin{aligned}
 & m^2 \|\Lambda_\delta^{-2} \phi_m^\ell \Lambda^{\ell/3} \partial_x^{m-2} v\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} \\
 \leq & m \|\Lambda_\delta^{-2} \phi_m^\ell \Lambda^{\ell/3} \partial_x^{m-1} v\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} + m^3 \|\Lambda_\delta^{-2} \phi_m^\ell \Lambda^{\ell/3} \partial_x^{m-3} v\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} \\
 \leq & m \|\Lambda_\delta^{-2} \phi_m^\ell \Lambda^{\ell/3} \partial_x^{m-1} v\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} + m^3 \|\Lambda_\delta^{-2} \phi_m^\ell \partial_x^{m-3} v\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} \\
 & + m^3 \|\Lambda_\delta^{-2} \phi_m^\ell \partial_x^{m-2} v\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} \\
 \leq & m \|\Lambda_\delta^{-2} \phi_m^\ell \Lambda^{\ell/3} \partial_x^{m-1} v\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} + m^3 \|\phi_{m-2}^0 \partial_x^{m-3} v\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} \\
 & + m^3 \|\phi_{m-1}^0 \partial_x^{m-2} v\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))}.
 \end{aligned}$$

Then we have, combining the above inequalities,

$$\begin{aligned}
 & \sum_{j=m-2}^{m-1} \binom{m}{j} \|e^{2cy} (\partial_y f_{m-j}) \Lambda_\delta^{-2} \Lambda^{\ell/3} (1+cy)^{-1} \phi_m^\ell \partial_x^j v\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 \leq & Cm \|\Lambda_\delta^{-2} \phi_m^\ell \Lambda^{\ell/3} \partial_x^{m-1} v\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} + Cm^3 \|\phi_{m-2}^0 \partial_x^{m-3} v\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} \\
 & + Cm^3 \|\phi_{m-1}^0 \partial_x^{m-2} v\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} \\
 \leq & Cm \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} + Cm^3 \|\phi_{m-2}^0 W_{m-2}^0 f_{m-2}\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} \\
 & + Cm^3 \|\phi_{m-1}^0 W_{m-1}^0 f_{m-1}\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))},
 \end{aligned}$$

the last inequality following from (4.8). This, along with the estimate

$$\begin{aligned}
 & m^3 \|\phi_{m-2}^0 W_{m-2}^0 f_{m-2}\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} + m^3 \|\phi_{m-1}^0 W_{m-1}^0 f_{m-1}\|_{L^\infty(\mathbb{R}_+; L^2([0, T] \times \mathbb{R}_x))} \\
 \leq & CA^{m-6} ((m-5)!)^{3(1+\sigma)}
 \end{aligned}$$

due to the inductive assumption (2.9), gives the desired estimate (5.4).

(b) We will estimate in this step the second and the third terms on the right hand side of (5.3), and prove that

$$\begin{aligned}
 (5.5) \quad & \sum_{j=1}^{m-3} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 & + \sum_{j=1}^{m-3} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v) \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 \leq & CA^{m-6} ((m-5)!)^{3(1+\sigma)}.
 \end{aligned}$$

For this purpose we write, denoting by $[m/2]$ the largest integer less than or equal to $m/2$,

$$\begin{aligned}
 (5.6) \quad & \sum_{j=1}^{m-3} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v) \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 \leq & \sum_{j=1}^{[m/2]} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v) (\partial_y f_{m-j})\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 & + \sum_{j=[m/2]+1}^{m-3} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v) \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 = & S_1 + S_2.
 \end{aligned}$$

We first treat S_1 . Using the inequality

$$\phi_m^\ell \leq \phi_m^0 \leq \phi_{j+3}^0 \phi_{m-j}^0, \quad W_m^0 \leq W_{m-j}^0 \text{ for } j \geq 1,$$

gives

$$\begin{aligned}
 (5.7) \quad S_1 &= \sum_{j=1}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 &\leq \sum_{j=1}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_{j+3}^0 \partial_x^{j+1} v\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \|\phi_{m-j}^0 W_{m-j}^0 \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)}.
 \end{aligned}$$

By Sobolev inequality, we have

$$\begin{aligned}
 &\|\phi_{j+3}^0 \partial_x^{j+1} v\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \\
 &\leq C \|\phi_{j+3}^0 \partial_x^{j+1} v\|_{L^\infty([0,T] \times \mathbb{R}_+; L^2(\mathbb{R}_x))} + C \|\phi_{j+3}^0 \partial_x^{j+2} v\|_{L^\infty([0,T] \times \mathbb{R}_+; L^2(\mathbb{R}_x))} \\
 &\leq C \|\phi_{j+2}^0 \partial_x^{j+1} v\|_{L^\infty([0,T] \times \mathbb{R}_+; L^2(\mathbb{R}_x))} + C \|\phi_{j+3}^0 \partial_x^{j+2} v\|_{L^\infty([0,T] \times \mathbb{R}_+; L^2(\mathbb{R}_x))} \\
 &\leq C \|\phi_{j+2}^0 W_{j+2}^0 f_{j+2}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} + C \|\phi_{j+3}^0 W_{j+3}^0 f_{j+3}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))},
 \end{aligned}$$

the second inequality using (4.1) and the last inequality following from (4.8). As a result, we use the hypothesis of induction (2.9) and the initial hypothesis of induction (2.7) to conclude that if $4 \leq j \leq \lfloor m/2 \rfloor$ then

$$\|\phi_{j+3}^0 \partial_x^{j+1} v\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \leq C \left(A^{j-3} ((j-3)!)^{3(1+\sigma)} + A^{j-2} ((j-2)!)^{3(1+\sigma)} \right) \leq C A^{j-2} ((j-2)!)^{3(1+\sigma)},$$

and if $1 \leq j \leq 3$

$$\|\phi_{j+3}^0 \partial_x^{j+1} v\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \leq C.$$

Moreover, using (4.4) and also the inductive assumption (2.9), we calculate, for any $1 \leq j \leq \lfloor m/2 \rfloor$,

$$\begin{aligned}
 &\|\phi_{m-j}^0 W_{m-j}^0 \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 &\leq \|\partial_y \phi_{m-j}^0 W_{m-j}^0 f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|\phi_{m-j}^0 [\partial_y, W_{m-j}^0] f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\
 &\leq \|\partial_y \phi_{m-j}^0 W_{m-j}^0 f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} + C \|\phi_{m-j}^0 W_{m-j}^0 f_{m-j}\|_{L^\infty([0,T], L^2(\mathbb{R}_+^2))} \\
 &\leq C A^{m-j-5} ((m-j-5)!)^{3(1+\sigma)}.
 \end{aligned}$$

Putting these inequalities into (5.7) gives

$$\begin{aligned}
 (5.8) \quad S_1 &\leq C \sum_{j=4}^{\lfloor m/2 \rfloor} \frac{m!}{j!(m-j)!} A^{j-2} ((j-2)!)^{3(1+\sigma)} \left(A^{m-j-5} ((m-j-5)!)^{3(1+\sigma)} \right) \\
 &\quad + C \sum_{j=1}^3 \frac{m!}{j!(m-j)!} \left(A^{m-j-5} ((m-j-5)!)^{3(1+\sigma)} \right) \\
 &\leq C \sum_{j=4}^{\lfloor m/2 \rfloor} \frac{m!}{j^2(m-j)^5} A^{m-7} ((j-2)!)^{3(1+\sigma)-1} ((m-j-5)!)^{3(1+\sigma)-1} \\
 &\quad + C A^{m-6} ((m-5)!)^{3(1+\sigma)} \\
 &\leq C \sum_{j=4}^{\lfloor m/2 \rfloor} \frac{m!}{j^2(m/2)^5} A^{m-7} ((m-7)!)^{3(1+\sigma)-1} + C A^{m-6} ((m-5)!)^{3(1+\sigma)} \\
 &\leq C (m-5)! A^{m-7} ((m-7)!)^{3(1+\sigma)-1} + C A^{m-6} ((m-5)!)^{3(1+\sigma)} \\
 &\leq C A^{m-6} ((m-5)!)^{3(1+\sigma)}.
 \end{aligned}$$

We now treat S_2 . Using the inequality

$$\phi_m^\ell \leq \phi_m^0 \leq \phi_{j+2}^0 \phi_{m-j+1}^0, \quad W_m^0 \leq W_{m-j+1}^0 \text{ for } j \geq 1,$$

and thus

$$\begin{aligned}
 (5.9) \quad S_2 &= \sum_{j=\lfloor m/2 \rfloor + 1}^{m-3} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v) \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 &\leq \sum_{j=\lfloor \frac{m}{2} \rfloor + 1}^{m-3} \binom{m}{j} \|\phi_{j+2}^0 \partial_x^{j+1} v\|_{L^\infty([0, T] \times \mathbb{R}_+; L^2(\mathbb{R}_x))} \\
 &\quad \times \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+; L^\infty(\mathbb{R}_x))} \\
 &\leq \sum_{j=\lfloor m/2 \rfloor + 1}^{m-3} \binom{m}{j} \|\phi_{j+2}^0 W_{j+2}^0 f_{j+2}\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} \\
 &\quad \times \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+; L^\infty(\mathbb{R}_x))},
 \end{aligned}$$

the last inequality using (4.8). As for the last factor in the above inequality, we use Sobolev inequality, (4.1) and (4.2) to compute

$$\begin{aligned}
 &\|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+; L^\infty(\mathbb{R}_x))} \\
 &\leq C \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y \partial_x f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 &\leq C \|\phi_{m-j}^0 W_{m-j}^0 \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y \partial_x f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)}.
 \end{aligned}$$

On the other hand, in view of the definition of f_m , we have

$$\begin{aligned}
 &\|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y \partial_x f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 &\leq \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j+1}\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y (\partial_x^{m-j} u) \partial_x ((\partial_y \omega)/\omega)\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 &\leq \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j+1}\|_{L^2([0, T] \times \mathbb{R}_+^2)} + \|\phi_{m-j+1}^0 W_{m-j+1}^0 (\partial_x^{m-j} \omega) \partial_x ((\partial_y \omega)/\omega)\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 &\quad + \|\phi_{m-j+1}^0 W_{m-j+1}^0 (\partial_x^{m-j} u) \partial_x \partial_y ((\partial_y \omega)/\omega)\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 &\leq C \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j+1}\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \|\langle y \rangle^{-1} \phi_{m-j}^0 W_{m-j}^0 \partial_x^{m-j} u\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 &\quad + C \|\langle y \rangle^{-1} \phi_{m-j}^0 W_{m-j}^0 \partial_x^{m-j} \omega\|_{L^2([0, T] \times \mathbb{R}_+^2)},
 \end{aligned}$$

the last inequality using (4.1) and (4.2). Combining these inequalities, we conclude

$$\begin{aligned}
 &\|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+; L^\infty(\mathbb{R}_x))} \\
 &\leq C \|\phi_{m-j}^0 W_{m-j}^0 \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j+1}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 &\quad + C \|\langle y \rangle^{-1} \phi_{m-j}^0 W_{m-j}^0 \partial_x^{m-j} \omega\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \|\langle y \rangle^{-1} \phi_{m-j}^0 W_{m-j}^0 \partial_x^{m-j} u\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 &\leq C \|\partial_y \phi_{m-j}^0 W_{m-j}^0 f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \|\partial_y \phi_{m-j+1}^0 W_{m-j+1}^0 f_{m-j+1}\|_{L^2([0, T] \times \mathbb{R}_+^2)} \\
 &\quad + C \|\phi_{m-j}^0 W_{m-j}^0 f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+^2)} + C \|\phi_{m-j+1}^0 W_{m-j+1}^0 f_{m-j+1}\|_{L^2([0, T] \times \mathbb{R}_+^2)},
 \end{aligned}$$

where the last inequality follows from (4.9) and (4.4). This, along with the inductive assumptions (2.9), yields, if $\lfloor m/2 \rfloor + 1 \leq j \leq m-4$ then

$$\begin{aligned}
 &\|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+; L^\infty(\mathbb{R}_x))} \\
 &\leq C A^{m-j-5} ((m-j-5)!)^{3(1+\sigma)} + C A^{m-j-4} ((m-j-4)!)^{3(1+\sigma)} \\
 &\leq C A^{m-j-4} ((m-j-4)!)^{3(1+\sigma)},
 \end{aligned}$$

and if $j = m-3$ then

$$\|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j}\|_{L^2([0, T] \times \mathbb{R}_+; L^\infty(\mathbb{R}_x))} \leq C$$

due to the initial hypothesis of induction (2.7). On the other hand, the inductive assumptions (2.9) yields, for any $[m/2] + 1 \leq j \leq m - 3$,

$$\|\phi_{j+2}^0 W_{j+2}^0 f_{j+2}\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))} \leq A^{j-3} ((j-3)!)^{3(1+\sigma)}.$$

Putting these estimates into (5.9), we have

$$\begin{aligned} S_2 &\leq C \sum_{j=[m/2]+1}^{m-4} \frac{m!}{j!(m-j)!} A^{j-3} ((j-3)!)^{3(1+\sigma)} \left(A^{m-j-4} ((m-j-4)!)^{3(1+\sigma)} \right) \\ &\quad + C \sum_{j=m-3}^{m-3} \frac{m!}{j!(m-j)!} A^{j-3} ((j-3)!)^{3(1+\sigma)} \\ &\leq C \sum_{j=[m/2]+1}^{m-4} \frac{m!}{j^3(m-j)^4} A^{m-7} ((j-3)!)^{3(1+\sigma)-1} ((m-j-4)!)^{3(1+\sigma)-1} \\ &\quad + C A^{m-6} ((m-5)!)^{3(1+\sigma)} \\ &\leq C \sum_{j=[m/2]+1}^{m-4} \frac{m!}{(m/2)^3(m-j)^4} A^{m-7} ((m-7)!)^{3(1+\sigma)-1} + C A^{m-6} ((m-5)!)^{3(1+\sigma)} \\ &\leq C(m-3)! A^{m-7} ((m-7)!)^{3(1+\sigma)-1} + C A^{m-6} ((m-5)!)^{3(1+\sigma)} \\ &\leq C A^{m-6} ((m-5)!)^{3(1+\sigma)}. \end{aligned}$$

This along with (5.8) and (5.6) yields

$$\sum_{j=1}^{m-3} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^{j+1} v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C A^{m-6} ((m-5)!)^{3(1+\sigma)}.$$

Similarly, we have

$$\sum_{j=1}^{m-3} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j v) \partial_y f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C A^{m-6} ((m-5)!)^{3(1+\sigma)}.$$

Then the desired estimate (5.5) follows.

(c) It remains to prove that

$$(5.10) \quad \sum_{j=1}^{m-3} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j v) (\partial_y \partial_x f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C A^{m-6} ((m-5)!)^{3(1+\sigma)}.$$

The proof is quite similar as in the previous step. To do so we first write

$$\begin{aligned} &\sum_{j=1}^{m-3} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j v) (\partial_y \partial_x f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= \sum_{j=1}^{[m/2]} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j v) (\partial_y \partial_x f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &\quad + \sum_{j=[m/2]+1}^{m-3} \binom{m}{j} \|\phi_m^\ell W_m^0 (\partial_x^j v) (\partial_y \partial_x f_{m-j})\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ &= \tilde{S}_1 + \tilde{S}_2. \end{aligned}$$

For the term \tilde{S}_1 , we use

$$\phi_m^\ell \leq \phi_m^0 \leq \phi_{j+2}^0 \phi_{m-j+1}^0, \quad W_m^0 \leq W_{m-j+1}^0 \text{ for } j \geq 2,$$

to obtain

$$\tilde{S}_1 \leq \sum_{j=1}^{\lfloor m/2 \rfloor} \binom{m}{j} \|\phi_{j+2}^0 \partial_x^j v\|_{L^\infty([0,T] \times \mathbb{R}_+^2)} \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y \partial_x f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)}.$$

Then repeating the arguments used to estimate S_1 and S_2 in the previous step, we can deduce that

$$\tilde{S}_1 \leq CA^{m-6} ((m-6)!)^{3(1+\sigma)}.$$

As for \tilde{S}_2 , using the inequality

$$\phi_m^\ell \leq \phi_m^0 \leq \phi_{j+1}^0 \phi_{m-j+2}^0, \quad W_m^0 \leq W_{m-j+2}^0 \text{ for } j \geq 2,$$

gives

$$\tilde{S}_2 \leq \sum_{j=\lfloor m/2 \rfloor + 1}^{m-3} \binom{m}{j} \|\phi_{j+1}^0 \partial_x^j v\|_{L^\infty([0,T] \times \mathbb{R}_+; L^2(\mathbb{R}_x))} \|\phi_{m-j+2}^0 W_{m-j+2}^0 \partial_y \partial_x f_{m-j}\|_{L^2([0,T] \times \mathbb{R}_+; L^\infty(\mathbb{R}_x))}.$$

Then repeating the arguments used to estimate S_2 in the previous step, we have

$$\tilde{S}_2 \leq CA^{m-6} ((m-5)!)^{3(1+\sigma)}.$$

This along with the estimate on \tilde{S}_1 yields (5.10). Finally, combining (5.3), (5.4), (5.5) and (5.10) gives the desired estimate in Lemma 5.3, and thus the proof is complete. \square

Lemma 5.4. *Under the same assumption as in Proposition 2.3, we have*

$$\begin{aligned} & \sum_{j=1}^m \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial_x^j u) f_{m+1-j}\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & + \sum_{j=1}^{m-1} \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell [\partial_y ((\partial_y \omega)/\omega)] (\partial_x^j v) (\partial_x^{m-j} u)\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq mC \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + CA^{m-6} ((m-5)!)^{3(1+\sigma)}. \end{aligned}$$

The proof of this Lemma is quite similar as in Lemma 5.3, so we omit it.

Lemma 5.5. *Under the same assumption as in Proposition 2.3, we have*

$$2\|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell [\partial_y ((\partial_y \omega)/\omega)] f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \leq C\|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)}.$$

Proof. This is a just direct verification. Indeed, Lemma 3.1 gives

$$\begin{aligned} & \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell [\partial_y ((\partial_y \omega)/\omega)] f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq \|[\partial_y ((\partial_y \omega)/\omega)] \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} + \|[\partial_y ((\partial_y \omega)/\omega), \Lambda_\delta^{-2} W_m^\ell] \phi_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)} \\ & \leq C\|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T] \times \mathbb{R}_+^2)}. \end{aligned}$$

Then the desired estimate follows and thus the proof of Lemma 5.5 is complete. \square

Proof of Proposition 5.2. In view of (5.2), we combine the estimates in Lemma 5.3-Lemma 5.5, to get the first estimate in Proposition 5.2. The second one can be treated quite similarly and the main difference is that we will use here additionally the inductive estimates on the terms of the following form

$$\|\partial_y^2 \Lambda^{-2/3} \phi_j^0 W_j^0 f_j\|_{L^2([0,T] \times \mathbb{R}_+^2)}, \quad 6 \leq j \leq m,$$

while in the proof of Lemma 5.3, we only used the estimates on the following two forms

$$\|\phi_j^0 W_j^0 f_j\|_{L^\infty([0,T]; L^2(\mathbb{R}_+^2))}, \quad \|\partial_y \phi_j^0 W_j^0 f_j\|_{L^2([0,T] \times \mathbb{R}_+^2)}, \quad 6 \leq j \leq m.$$

So we omit the treatment of the second estimate for brevity, and thus the proof of Proposition 5.2 is complete. \square

Completeness of the proof of Proposition 2.3. The estimates (2.15) follows from the combination of Propostion 5.1 and the first estimate in Proposition 5.2, while the estimate (2.17) in Proposition 2.3 follows from Propostion 5.1 and the second estimate in Proposition 5.2. The treatment of (2.16) is exactly the same as (2.15). The proof of Proposition 2.3 is thus complete. \square

6. APPENDIX

Here we deduce the equation fulfilled by f_m (cf. [21]). Recall that

$$f_m = \partial_x^m \omega - \frac{\partial_y \omega}{\omega} \partial_x^m u, \quad m \geq 1,$$

where u is a smooth solution to Prandtl equation (1.1) and $\omega = \partial_y u$. We will verify that

$$(6.1) \quad \partial_t f_m + u \partial_x f_m + v \partial_y f_m - \partial_y^2 f_m = \mathcal{Z}_m,$$

where

$$\begin{aligned} \mathcal{Z}_m &= - \sum_{j=1}^m \binom{m}{j} (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_y f_{m-j}) \\ &\quad - \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) - 2 \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] f_m. \end{aligned}$$

To do so, we firstly notice that

$$(6.2) \quad u_t + uu_x + vu_y - u_{yy} = 0,$$

and

$$\omega_t + u\omega_x + v\omega_y - \omega_{yy} = 0.$$

Thus by Leibniz's formula, $\partial_x^m u, \partial_x^m \omega$ satisfy, respectively, the following equation

$$\begin{aligned} &\partial_t \partial_x^m u + u \partial_x \partial_x^m u + v \partial_y \partial_x^m u - \partial_y^2 \partial_x^m u \\ (6.3) \quad &= - \sum_{j=1}^m \binom{m}{j} (\partial_x^j u) (\partial_x^{m-j+1} u) - \sum_{j=1}^m (\partial_x^j v) (\partial_y \partial_x^{m-j} u) \\ &= - \sum_{j=1}^m \binom{m}{j} (\partial_x^j u) (\partial_x^{m-j+1} u) - \sum_{j=1}^{m-1} (\partial_x^j v) (\partial_y \partial_x^{m-j} u) - (\partial_x^m v) (\partial_y u) \end{aligned}$$

and

$$\begin{aligned} &\partial_t \partial_x^m \omega + u \partial_x \partial_x^m \omega + v \partial_y \partial_x^m \omega - \partial_y^2 \partial_x^m \omega \\ (6.4) \quad &= - \sum_{j=1}^m \binom{m}{j} (\partial_x^j u) (\partial_x^{m-j+1} \omega) - \sum_{j=1}^m (\partial_x^j v) (\partial_y \partial_x^{m-j} \omega) \\ &= - \sum_{j=1}^m \binom{m}{j} (\partial_x^j u) (\partial_x^{m-j+1} \omega) - \sum_{j=1}^{m-1} (\partial_x^j v) (\partial_y \partial_x^{m-j} \omega) - (\partial_x^m v) (\partial_y \omega). \end{aligned}$$

In order to eliminate the last terms on the right sides of the above two equations, we observe $\partial_y u = \omega > 0$ and thus multiply (6.3) by $-\frac{\partial_y \omega}{\omega}$, and then add the resulting equation to (6.4); this gives

$$\partial_t f_m + u \partial_x f_m + v \partial_y f_m - \partial_y^2 f_m = \mathcal{Z}_m$$

where

$$\begin{aligned} \mathcal{Z}_m &= - \sum_{j=1}^m \binom{m}{j} (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_y f_{m-j}) \\ &\quad - \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) + (\partial_x^m u) f_1 \\ &\quad + \left(\partial_t \left(\frac{\partial_y \omega}{\omega} \right) + u \partial_x \left(\frac{\partial_y \omega}{\omega} \right) + v \partial_y \left(\frac{\partial_y \omega}{\omega} \right) - \partial_y^2 \left(\frac{\partial_y \omega}{\omega} \right) \right) \partial_x^m u \\ &\quad - 2 \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] \partial_y \partial_x^m u. \end{aligned}$$

On the other hand we notice that

$$\begin{aligned}
 & \partial_t \left(\frac{\partial_y \omega}{\omega} \right) + u \partial_x \left(\frac{\partial_y \omega}{\omega} \right) + v \partial_y \left(\frac{\partial_y \omega}{\omega} \right) - \partial_y^2 \left(\frac{\partial_y \omega}{\omega} \right) \\
 &= \frac{1}{\omega} (\partial_t \partial_y \omega + u \partial_x \partial_y \omega + v \partial_y \partial_y \omega - \partial_y^2 \partial_y \omega) \\
 & \quad - \frac{\partial_y \omega}{\omega^2} (\partial_t \omega + u \partial_x \omega + v \partial_y \omega - \partial_y^2 \omega) + 2 \frac{\partial_y \omega}{\omega} \partial_y \left(\frac{\partial_y \omega}{\omega} \right) \\
 &= -\partial_x \omega + \frac{(\partial_x u)(\partial_y \omega)}{\omega} + 2 \frac{\partial_y \omega}{\omega} \partial_y \left(\frac{\partial_y \omega}{\omega} \right)
 \end{aligned}$$

Therefore we have

$$\begin{aligned}
 \mathcal{Z}_m &= -\sum_{j=1}^m \binom{m}{j} (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_y f_{m-1}) \\
 & \quad - \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) + (\partial_x^m u) f_1 \\
 & \quad + \left(\partial_x \omega - \frac{(\partial_x u)(\partial_y \omega)}{\omega} \right) \partial_x^m u + 2 \frac{\partial_y \omega}{\omega} \partial_y \left(\frac{\partial_y \omega}{\omega} \right) \partial_x^m u - 2 \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] \partial_y \partial_x^m u \\
 &= -\sum_{j=1}^m \binom{m}{j} (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_y f_{m-1}) \\
 & \quad - \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) + \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right)^2 \right] \partial_x^m u \\
 & \quad - 2 \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] \partial_x^m \omega \\
 &= -\sum_{j=1}^m \binom{m}{j} (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_y f_{m-1}) \\
 & \quad - \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v) (\partial_x^{m-j} u) - 2 \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] f_m.
 \end{aligned}$$

Next we will give the boundary value of $\partial_y f_m$ and $\partial_t f_m - \partial_y^2 f_m$. In view of (6.2), we infer, recalling $u|_{y=0} = v|_{y=0} = 0$,

$$\partial_y \omega|_{y=0} = \partial_y^2 u|_{y=0} = 0.$$

As a result, observing

$$\partial_y f_m = \partial_y \partial_x^m \omega - \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] \partial_x^m u - \left(\frac{\partial_y \omega}{\omega} \right) \partial_y \partial_x^m u,$$

we have

$$(6.5) \quad \partial_y f_m|_{y=0} = 0.$$

Direct verification shows

$$\mathcal{Z}_m|_{y=0} = -2 \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] f_m|_{y=0},$$

and thus

$$(6.6) \quad (\partial_t f_m - \partial_y^2 f_m)|_{y=0} = \mathcal{Z}_m|_{y=0} = -2 \left[\partial_y \left(\frac{\partial_y \omega}{\omega} \right) \right] f_m|_{y=0},$$

due to the equation fulfilled by f_m .

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