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PARTIAL NULL CONTROLLABILITY OF PARABOLIC LINEAR SYSTEMS

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Abstract. This paper is devoted to the partial null controllability issue of parabolic linear systems with n equations. Given a bounded domain Ω in $\mathbb{R}^N$ ($N \in \mathbb{N}^*$), we study the effect of m localized controls in a nonempty open subset ω only controlling p components of the solution ($p, m \leqslant n$). The first main result of this paper is a necessary and sufficient condition when the coupling and control matrices are constant. The second result provides, in a first step, a sufficient condition of partial null controllability when the matrices only depend on time. In a second step, through an example of partially controlled 2×2 parabolic system, we will provide positive and negative results on partial null controllability when the coefficients are space dependent.

1. **Introduction and main results.** Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}^*$) with a $\mathcal{C}^2$-class boundary $\partial\Omega$, ω be a nonempty open subset of Ω and $T > 0$. Let p, m, $n \in \mathbb{N}^*$ such that $p, m \leqslant n$. We consider in this paper the following system of n parabolic linear equations

$$\begin{cases} \partial_t y = \Delta y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T := \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $y_0 \in L^2(\Omega)^n$ is the initial data, $u \in L^2(Q_T)^m$ is the control and

$$A \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^n)) \text{ and } B \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)).$$

In many fields such as chemistry, physics or biology it appeared relevant to study the controllability of such a system (see [4]). For example, in [10], the authors study a system of three semilinear heat equations which is a model coming from a mathematical description of the growth of brain tumors. The unknowns are the drug concentration, the density of tumors cells and the density of wealthy cells and the aim is to control only two of them with one control. This practical issue motivates the introduction of the partial null controllability.

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For an initial condition $y(0) = y_0 \in L^2(\Omega)^n$ and a control $u \in L^2(Q_T)^m$, it is well-known that System (1) admits a unique solution in $W(0,T)^n$, where

$$W(0,T) := \{y \in L^2(0,T;H_0^1(\Omega)), \partial_t y \in L^2(0,T;H^{-1}(\Omega))\},$$

with $H^{-1}(\Omega) := H_0^1(\Omega)'$ and the following estimate holds (see [21])

$$\|y\|_{L^2(0,T;H_0^1(\Omega)^n)} + \|y\|_{\mathcal{C}^0([0,T];L^2(\Omega)^n)} \leqslant C(\|y_0\|_{L^2(\Omega)^n} + \|u\|_{L^2(Q_T)^m}), \tag{2}$$

where C does not depend on time. We denote by $y(\cdot; y_0, u)$ the solution to System (1) determined by the couple (y_0, u).

Let us consider Π_p the *projection matrix* of $\mathcal{L}(\mathbb{R}^n)$ given by $\Pi_p := (I_p \quad 0_{p,n-p})$ (I_p is the identity matrix of $\mathcal{L}(\mathbb{R}^p)$ and $0_{p,n-p}$ the null matrix of $\mathcal{L}(\mathbb{R}^{n-p}, \mathbb{R}^p)$), that is,

$$\begin{array}{rccc} \Pi_p : & \mathbb{R}^n & \to & \mathbb{R}^p, \\ & (y_1, ..., y_n) & \mapsto & (y_1, ..., y_p). \end{array}$$

System (1) is said to be

- **$\mathbf{\Pi_p}$-approximately controllable** on the time interval $(0,T)$, if for all real number $\varepsilon > 0$ and $y_0,\ y_T \in L^2(\Omega)^n$ there exists a control $u \in L^2(Q_T)^m$ such that
$$\|\Pi_p y(T; y_0, u) - \Pi_p y_T\|_{L^2(\Omega)^p} \leqslant \varepsilon.$$
- **$\mathbf{\Pi_p}$-null controllable** on the time interval $(0,T)$, if for all initial condition $y_0 \in L^2(\Omega)^n$, there exists a control $u \in L^2(Q_T)^m$ such that
$$\Pi_p y(T; y_0, u) \equiv 0 \text{ in } \Omega.$$

Before stating our main results, let us recall the few known results about the (full) null controllability of System (1). The first of them is about cascade systems (see [19]). The authors prove the null controllability of System (1) with the control matrix $B := e_1$ (the first vector of the canonical basis of $\mathbb{R}^n$) and a coupling matrix A of the form

$$A := \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \cdots & \alpha_{2,n} \\ 0 & \alpha_{3,2} & \alpha_{3,3} & \cdots & \alpha_{3,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{n,n-1} & \alpha_{n,n} \end{pmatrix}, \tag{3}$$

where the coefficients $\alpha_{i,j}$ are elements of $L^\infty(Q_T)$ for all $i, j \in \{1, ..., n\}$ and satisfy for a positive constant C and a nonempty open set ω_0 of ω

$$\alpha_{i+1,i} \geqslant C \text{ in } \omega_0 \quad \text{or} \quad -\alpha_{i+1,i} \geqslant C \text{ in } \omega_0 \quad \text{for all } i \in \{1, ..., n-1\}.$$

A similar result on parabolic systems with cascade coupling matrices can be found in [1].

The null controllability of parabolic 3×3 linear systems with space/time dependent coefficients and non cascade structure is studied in [7] and [22] (see also [19]).

If $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ (the constant case), it has been proved in [3] that System (1) is null controllable on the time interval $(0,T)$ if and only if the following condition holds

$$\text{rank } [A|B] = n, \tag{4}$$

where $[A|B]$, the so-called *Kalman matrix*, is defined as

$$[A|B] := (B|AB|...|A^{n-1}B). \tag{5}$$

For time dependent coupling and control matrices, we need some additional regularity. More precisely, we need to suppose that $A \in \mathcal{C}^{n-1}([0,T];\mathcal{L}(\mathbb{R}^n))$ and $B \in \mathcal{C}^n([0,T];\mathcal{L}(\mathbb{R}^m;\mathbb{R}^n))$. In this case, the associated Kalman matrix is defined as follows. Let us define

$$\begin{cases} B_0(t) := B(t), \\ B_i(t) := A(t)B_{i-1}(t) - \partial_t B_{i-1}(t) \quad \text{for all } i \in \{1,...,n-1\} \end{cases}$$

and denote by $[A|B](\cdot) \in \mathcal{C}^1([0,T];\mathcal{L}(\mathbb{R}^{nm};\mathbb{R}^n))$ the matrix function given by

$$[A|B](\cdot) := (B_0(\cdot)|B_1(\cdot)|...|B_{n-1}(\cdot)). \tag{6}$$

In [2] the authors prove first that, if there exists $t_0 \in [0,T]$ such that

$$\text{rank } [A|B](t_0) = n, \tag{7}$$

then System (1) is null controllable on the time interval $(0,T)$. Secondly that System (1) is null controllable on every interval (T_0,T_1) with $0 \leqslant T_0 < T_1 \leqslant T$ if and only if there exists a dense subset E of $(0,T)$ such that

$$\text{rank } [A|B](t) = n \text{ for every } t \in E. \tag{8}$$

In the present paper, the controls are acting on several equations but on one subset ω of Ω. Concerning the case where the control domains are not identical, we refer to [24].

Our first result is the following:

Theorem 1.1. *Assume that the coupling and control matrices are constant in space and time, i. e., $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$. The condition*

$$\text{rank } \Pi_p[A|B] = p \tag{9}$$

is equivalent to the Π_p-null/approximate controllability on the time interval $(0,T)$ of System (1).

The Condition (9) for Π_p-null controllability reduces to Condition (4) whenever $p = n$. A second result concerns the non-autonomous case:

Theorem 1.2. *Assume that*

$$A \in \mathcal{C}^{n-1}([0,T];\mathcal{L}(\mathbb{R}^n)) \text{ and } B \in \mathcal{C}^n([0,T];\mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)).$$

If

$$\text{rank } \Pi_p[A|B](T) = p, \tag{10}$$

then System (1) is Π_p-null/approximately controllable on the time interval $(0,T)$.

In Theorems 1.1 and 1.2, we control the p first components of the solution y. If we want to control some other components a permutation of lines leads to the same situation.

Remark 1. 1. When the components of the matrices A and B are analytic functions on the time interval $[0,T]$, Condition (7) is necessary for the null controllability of System (1) (see Th. 1.3 in [2]). Under the same assumption, the proof of this result can be adapted to show that the following condition

$$\begin{cases} \text{there exists } t_0 \in [0,T] \text{ such that :} \\ \text{rank } \Pi_p[A|B](t_0) = p, \end{cases}$$

is necessary to the Π_p-null controllability of System (1).

2. As told before, under Condition (7), System (1) is null controllable. But unlike the case where all the components are controlled, the Π_p-null controllability at a time t_0 smaller than T does not imply this property on the time interval $(0,T)$. This roughly explains Condition (10). Furthermore this condition can not be necessary under the assumptions of Theorem 1.2 (for a counterexample we refer to [2]).

Remark 2. In the proofs of Theorems 1.1 and 1.2, we will use a result of null controllability for cascade systems (see Section 2) proved in [2, 19] where the authors consider a time-dependent second order elliptic operator $L(t)$ given by

$$L(t)y(x,t) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\alpha_{i,j}(x,t)\frac{\partial y}{\partial x_j}(x,t)\right) + \sum_{i=1}^{N} b_i(x,t)\frac{\partial y}{\partial x_i}(x,t) + c(x,t)y(x,t), \tag{11}$$

with coefficients $\alpha_{i,j}$, b_i, c satisfying

$$\begin{cases} \alpha_{i,j} \in W^1_\infty(Q_T),\ b_i, c \in L^\infty(Q_T)\ 1 \leqslant i,j \leqslant N, \\ \alpha_{i,j}(x,t) = \alpha_{j,i}(x,t)\ \forall (x,t) \in Q_T,\ 1 \leqslant i,j \leqslant N \end{cases}$$

and the uniform elliptic condition: there exists $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} \alpha_{i,j}(x,t)\xi_i\xi_j \geqslant a_0|\xi|^2,\ \forall (x,t) \in Q_T.$$

Theorems 1.1 and 1.2 remain true if we replace $-\Delta$ by an operator $L(t)$ in System (1).

Now the following question arises: what happens in the case of space and time dependent coefficients ? As it will be shown in the following example, the answer seems to be much more tricky. Let us now consider the following parabolic system of two equations

$$\begin{cases} \partial_t y = \Delta y + \alpha z + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t z = \Delta z & \text{in } Q_T, \\ y = z = 0 & \text{on } \Sigma_T, \\ y(0) = y_0,\ z(0) = z_0 & \text{in } \Omega, \end{cases} \tag{12}$$

for given initial data $y_0,\ z_0 \in L^2(\Omega)$, a control $u \in L^2(Q_T)$ and where the coefficient $\alpha \in L^\infty(\Omega)$.

Theorem 1.3. *(1) Assume that $\alpha \in C^1([0,T])$. Then System (12) is Π_1-null controllable for any open set $\omega \subset \Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}^*$), that is for all initial conditions $y_0, z_0 \in L^2(\Omega)$, there exists a control $u \in L^2(Q_T)$ such that the solution (y,z) to System (12) satisfies $y(T) \equiv 0$ in Ω.*

(2) Let $\Omega := (a,b) \subset \mathbb{R}$ ($a,b \in \mathbb{R}$), $\alpha \in L^\infty(\Omega)$, $(w_k)_{k\geqslant 1}$ be the L^2-normalized eigenfunctions of $-\Delta$ in Ω with Dirichlet boundary conditions and for all $k,l \in \mathbb{N}^*$,

$$\alpha_{kl} := \int_\Omega \alpha(x)w_k(x)w_l(x)\,dx.$$

If the function α satisfies

$$|\alpha_{kl}| \leqslant C_1 e^{-C_2|k-l|} \text{ for all } k,l \in \mathbb{N}^*, \tag{13}$$

for two positive constants $C_1 > 0$ and $C_2 > b-a$, then System (12) is Π_1-null controllable for any open set $\omega \subset \Omega$.

(3) Assume that $\Omega := (0, 2\pi)$ and $\omega \subset (\pi, 2\pi)$. Let us consider $\alpha \in L^\infty(0, 2\pi)$ defined by

$$\alpha(x) := \sum_{j=1}^{\infty} \frac{1}{j^2} \cos(15jx) \text{ for all } x \in (0, 2\pi).$$

Then System (12) is not Π_1-null controllable. More precisely, there exists $k_1 \in \{1, ..., 7\}$ such that for the initial condition $(y_0, z_0) = (0, \sin(k_1 x))$ and any control $u \in L^2(Q_T)$ the solution y to System (12) is not identically equal to zero at time T.

We will not prove item (1) in Theorem 1.3, because it is a direct consequence of Theorem 1.2.

Remark 3. Suppose that $\Omega := (0, \pi)$. Consider $\alpha \in L^\infty(0, \pi)$ and the real sequence $(\alpha_p)_{p\in\mathbb{N}}$ such that for all $x \in (0, \pi)$

$$\alpha(x) := \sum_{p=0}^{\infty} \alpha_p \cos(px).$$

Concerning item (2), we remark that Condition (13) is equivalent to the existence of two constants $C_1 > 0,\ C_2 > \pi$ such that, for all $p \in \mathbb{N}$,

$$|\alpha_p| \leqslant C_1 e^{-C_2 p}.$$

As it will be shown, the proof of item (3) in Theorem 1.3 can be adapted in order to get the same conclusion for any $\alpha \in H^k(0, 2\pi)$ $(k \in \mathbb{N}^*)$ defined by

$$\alpha(x) := \sum_{j=1}^{\infty} \frac{1}{j^{k+1}} \cos((2k + 13)jx) \text{ for all } x \in (0, 2\pi). \tag{14}$$

These given functions α belong to $H^k(0, \pi)$ but not to $D((-\Delta)^{k/2})$. Indeed, in the proof of the third item in Theorem 1.3, we use the fact that the matrix $(\alpha_{kl})_{k,l\in\mathbb{N}^*}$ is sparse (see (103)), what seems true only for coupling terms α of the form (14). Thus α is not zero on the boundary.

Remark 4. From Theorem 1.3, one can deduce some new results concerning the null controllability of the heat equation with a right-hand side. Consider the system

$$\begin{cases} \partial_t y = \Delta y + f + \mathbb{1}_\omega u & \text{in } (0, \pi) \times (0, T), \\ y(0) = y(\pi) = 0 & \text{on } (0, T), \\ y(0) = y_0 & \text{in } (0, \pi), \end{cases} \tag{15}$$

where $y_0 \in L^2(0, \pi)$ is the initial data and $f, u \in L^2(Q_T)$ are the right-hand side and the control, respectively. Using the Carleman inequality (see [16]), one can prove that System (15) is null controllable when f satisfies

$$e^{\frac{C}{T-t}} f \in L^2(Q_T), \tag{16}$$

for a positive constant C. For more general right-hand sides it was rather open. The second and third points of Theorem 1.3 provide some positive and negative null controllability results for System (15) with right-hand side f which does not fulfil Condition (16).

Remark 5. Consider the same system as System (12) except that the control is now on the boundary, that is

$$\begin{cases} \partial_t y = \Delta y + \alpha z & \text{in } (0,\pi)\times(0,T), \\ \partial_t z = \Delta z & \text{in } (0,\pi)\times(0,T), \\ y(0,t) = v(t),\ y(\pi,t) = z(0,t) = z(\pi,t) = 0 & \text{on } (0,T), \\ y(x,0) = y_0(x),\ z(x,0) = z_0(x) & \text{in } (0,\pi), \end{cases} \tag{17}$$

where $y_0,\ z_0 \in H^{-1}(0,\pi)$. In Theorem 5.4, we provide an explicit coupling function α for which the Π_1-null controllability of System (17) does not hold. Moreover one can adapt the proof of the second point in Theorem 1.3 to prove the Π_1-null controllability of System (17) under Condition (13).

If the coupling matrix depends on space, the notions of Π_1-null and approximate controllability are not necessarily equivalent. Indeed, according to the choice of the coupling function $\alpha \in L^\infty(\Omega)$, System (12) can be Π_1-null controllable or not. But this system is Π_1-approximately controllable for all $\alpha \in L^\infty(\Omega)$:

Theorem 1.4. *Let $\alpha \in L^\infty(Q_T)$. Then System (12) is Π_1-approximately controllable for any open set $\omega \subset \Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}^*$), that is for all $y_0, y_T, z_0 \in L^2(\Omega)$ and all $\varepsilon > 0$, there exists a control $u \in L^2(Q_T)$ such that the solution (y,z) to System (12) satisfies*

$$\|y(T) - y_T\|_{L^2(\Omega)} \leqslant \varepsilon.$$

This result is a direct consequence of the unique continuation property and existence/unicity of solutions for a single heat equation. Indeed System (12) is Π_1-approximately controllable (see Proposition 1) if and only if for all $\phi_0 \in L^2(\Omega)$ the solution to the adjoint system

$$\begin{cases} -\partial_t \phi = \Delta \phi & \text{in } Q_T, \\ -\partial_t \psi = \Delta \psi + \alpha \phi & \text{in } Q_T, \\ \phi = \psi = 0 & \text{on } \Sigma_T, \\ \phi(T) = \phi_0,\ \psi(T) = 0 & \text{in } \Omega \end{cases} \tag{18}$$

satisfies

$$\phi \equiv 0 \text{ in } \omega \times (0,T) \ \Rightarrow (\phi,\psi) \equiv 0 \text{ in } Q_T.$$

If we assume that, for an initial data $\phi_0 \in L^2(\Omega)$, the solution to System (18) satisfies $\phi \equiv 0$ in $\omega \times (0,T)$, then using Mizohata uniqueness Theorem in [23], $\phi \equiv 0$ in Q_T and consequently $\psi \equiv 0$ in Q_T. For another example of parabolic systems for which these notions are not equivalent we refer for instance to [5].

Remark 6. The quantity α_{kl}, which appears in the second item of Theorem 1.3, has already been considered in some controllability studies for parabolic systems. Let us define for all $k \in \mathbb{N}^*$

$$\begin{cases} I_{1,k}(\alpha) := \displaystyle\int_0^a \alpha(x) w_k(x)^2 dx, \\ \\ I_k(\alpha) := \alpha_{kk}. \end{cases}$$

In [6], the authors have proved that the system

$$\begin{cases} \partial_t y = \Delta y + \alpha z & \text{in } (0,\pi)\times(0,T), \\ \partial_t z = \Delta z + \mathbb{1}_\omega u & \text{in } (0,\pi)\times(0,T), \\ y(0,t) = y(\pi,t) = z(0,t) = z(\pi,t) = 0 & \text{on } (0,T), \\ y(x,0) = y_0(x),\ z(x,0) = z_0(x) & \text{in } (0,\pi), \end{cases} \tag{19}$$

is approximately controllable if and only if

$$|I_k(\alpha)| + |I_{1,k}(\alpha)| \neq 0 \text{ for all } k \in \mathbb{N}^*.$$

A similar result has been obtained for the boundary approximate controllability in [9]. Consider now

$$T_0(\alpha) := \limsup_{k\to\infty} \frac{-\log(\min\{|I_k|, |I_{1,k}|\})}{k^2}.$$

It is also proved in [6] that: If $T > T_0(\alpha)$, then System (19) is null controllable at time T and if $T < T_0(\alpha)$, then System (19) is not null controllable at time T. As in the present paper, we observe a difference between the approximate and null controllability, in contrast with the scalar case (see [4]).

In this paper, the sections are organized as follows. We start with some preliminary results on the null controllability for the cascade systems and on the dual concept associated to the Π_p-null controllability. Theorem 1.1 is proved in a first step with one force i.e. $B \in \mathbb{R}^n$ in Section 3.1 and in a second step with m forces in Section 3.2. Section 4 is devoted to proving Theorem 1.2. We consider the situations of the second and third items of Theorem 1.3 in Section 5.1 and 5.2 respectively. This paper ends with some numerical illustrations of Π_1-null controllability and non Π_1-null controllability of System (12) in Section 5.3.

2. **Preliminaries.**

In this section, we recall a known result about cascade systems and provide a characterization of the Π_p-controllability through the corresponding adjoint system.

2.1. **Cascade systems.**

Some theorems of this paper use the following result of null controllability for the following cascade system of n equations controlled by r distributed functions

$$\begin{cases} \partial_t w = \Delta w + Cw + D\mathbb{1}_\omega u & \text{in } Q_T, \\ w = 0 & \text{on } \Sigma_T, \\ w(0) = w_0 & \text{in } \Omega, \end{cases} \tag{20}$$

where $w_0 \in L^2(\Omega)^n$, $u = (u_1, ..., u_r) \in L^2(Q_T)^r$, with $r \in \{1, ..., n\}$, and the coupling and control matrices $C \in \mathcal{C}^0([0,T]; \mathcal{L}(\mathbb{R}^n))$ and $D \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^n)$ are given by

$$C(t) := \begin{pmatrix} C_{11}(t) & C_{12}(t) & \cdots & C_{1r}(t) \\ 0 & C_{22}(t) & \cdots & C_{2r}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{rr}(t) \end{pmatrix} \tag{21}$$

with

$$C_{ii}(t) := \begin{pmatrix} \alpha^i_{11}(t) & \alpha^i_{12}(t) & \alpha^i_{13}(t) & \cdots & \alpha^i_{1,s_i}(t) \\ 1 & \alpha^i_{22}(t) & \alpha^i_{23}(t) & \cdots & \alpha^i_{2,s_i}(t) \\ 0 & 1 & \alpha^i_{33}(t) & \cdots & \alpha^i_{3,s_i}(t) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha^i_{s_i,s_i}(t) \end{pmatrix},$$

$s_i \in \mathbb{N}$, $\sum_{i=1}^r s_i = n$ and $D := (e_{S_1}|...|e_{S_r})$ with $S_1 = 1$ and $S_i = 1 + \sum_{j=1}^{i-1} s_j$, $i \in \{2, ..., r\}$ (e_j is the j-th element of the canonical basis of $\mathbb{R}^n$).

Theorem 2.1. *System (20) is null controllable on the time interval $(0, T)$, i.e. for all $w_0 \in L^2(\Omega)^n$ there exists $u \in L^2(\Omega)^r$ such that the solution w in $W(0, T)^n$ to System (20) satisfies $w(T) \equiv 0$ in Ω.*

The proof of this result uses a Carleman estimate (see [16]) and can be found in [2] or [19].

2.2. **Partial null controllability of a parabolic linear system by m forces and adjoint system.** It is nowadays well-known that the controllability has a dual concept called *observability* (see for instance [4]). We detail below the observability for the Π_p-controllability.

Proposition 1. *1. System (1) is Π_p-null controllable on the time interval $(0, T)$ if and only if there exists a constant $C_{obs} > 0$ such that for all initial data $\varphi_0 = (\varphi^0_1, ..., \varphi^0_p) \in L^2(\Omega)^p$ the solution $\varphi \in W(0, T)^n$ to the adjoint system*

$$\begin{cases} -\partial_t \varphi = \Delta\varphi + A^*\varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot, T) = \Pi^*_p \varphi_0 = (\varphi^0_1, ..., \varphi^0_p, 0, ..., 0) & \text{in } \Omega \end{cases} \tag{22}$$

satisfies the observability inequality

$$\|\varphi(0)\|^2_{L^2(\Omega)^n} \leqslant C_{obs} \int_0^T \|B^*\varphi\|^2_{L^2(\omega)^m}\, dt. \tag{23}$$

2. System (1) is Π_p-approximately controllable on the time interval $(0, T)$ if and only if for all $\varphi_0 \in L^2(\Omega)^p$ the solution φ to System (22) satisfies

$$B^*\varphi \equiv 0 \text{ in } (0, T) \times \omega \ \Rightarrow \varphi \equiv 0 \text{ in } Q_T.$$

Proof. For all $y_0 \in L^2(\Omega)^n$, and $u \in L^2(Q_T)^m$, we denote by $y(t; y_0, u)$ the solution to System (1) at time $t \in [0, T]$. For all $t \in [0, T]$, let us consider the operators S_t and L_t defined as follows

$$\begin{array}{rccc} S_t: & L^2(\Omega)^n & \to & L^2(\Omega)^n \\ & y_0 & \mapsto & y(t; y_0, 0) \end{array} \quad \text{and} \quad \begin{array}{rccc} L_t: & L^2(Q_T)^m & \to & L^2(\Omega)^n \\ & u & \mapsto & y(t; 0, u). \end{array} \tag{24}$$

1. System (1) is Π_p-null controllable on the time interval $(0, T)$ if and only if

$$\begin{gathered} \forall y_0 \in L^2(\Omega)^n,\ \exists u \in L^2(Q_T)^m \text{ such that} \\ \Pi_p L_T u = -\Pi_p S_T y_0. \end{gathered} \tag{25}$$

Problem (25) admits a solution if and only if

$$\text{Im } \Pi_p S_T \subset \text{Im } \Pi_p L_T. \tag{26}$$

The inclusion (26) is equivalent to (see [11], Lemma 2.48 p. 58)

$$\begin{gathered}\exists C>0 \text{ such that } \forall \varphi_0 \in L^2(\Omega)^p,\\ \|S_T^*\Pi_p^*\varphi_0\|^2_{L^2(\Omega)^n} \leqslant C\|L_T^*\Pi_p^*\varphi_0\|^2_{L^2(Q_T)^m}.\end{gathered} \tag{27}$$

We note that

$$S_T^*\Pi_p^*: \begin{array}{ccc} L^2(\Omega)^p & \to & L^2(\Omega)^n \\ \varphi_0 & \mapsto & \varphi(0)\end{array} \quad \text{and} \quad L_T^*\Pi_p^*: \begin{array}{ccc} L^2(\Omega)^p & \to & L^2(Q_T)^m \\ \varphi_0 & \mapsto & \mathbb{1}_\omega B^*\varphi,\end{array}$$

where $\varphi \in W(0,T)^n$ is the solution to System (22). Indeed, for all $y_0 \in L^2(\Omega)^n$, $u \in L^2(Q_T)^m$ and $\varphi_0 \in L^2(\Omega)^p$

$$\begin{aligned}\langle \Pi_p S_T y_0, \varphi_0\rangle_{L^2(\Omega)^p} &= \langle y(T;y_0,0), \varphi(T)\rangle_{L^2(\Omega)^n}\\ &= \int_0^T \langle \partial_t y(s;y_0,0), \varphi(s)\rangle_{L^2(\Omega)^n}\mathrm{d}s\\ &\quad + \int_0^T \langle y(s;y_0,0), \partial_t\varphi(s)\rangle_{L^2(\Omega)^n}\mathrm{d}s + \langle y_0, \varphi(0)\rangle_{L^2(\Omega)^n}\\ &= \langle y_0, \varphi(0)\rangle_{L^2(\Omega)^n}\end{aligned} \tag{28}$$

and

$$\begin{aligned}\langle \Pi_p L_T u, \varphi_0\rangle_{L^2(\Omega)^p} &= \langle y(T;0,u), \varphi(T)\rangle_{L^2(\Omega)^n}\\ &= \int_0^T \langle \partial_t y(s;0,u), \varphi(s)\rangle_{L^2(\Omega)^n}\mathrm{d}s\\ &\qquad + \int_0^T \langle y(s;0,u), \partial_t\varphi(s)\rangle_{L^2(\Omega)^n}\mathrm{d}s\\ &= \langle \mathbb{1}_\omega B u, \varphi\rangle_{L^2(Q_T)^n} = \langle u, \mathbb{1}_\omega B^*\varphi\rangle_{L^2(Q_T)^m}.\end{aligned} \tag{29}$$

The inequality (27) combined with (28)-(29) lead to the conclusion.

2. In view of the definition in (24) of S_T and L_T, System (1) is Π_p-approximately controllable on the time interval $(0,T)$ if and only if

$$\begin{gathered}\forall (y_0,y_T) \in L^2(\Omega)^n \times L^2(\Omega)^p,\ \forall \varepsilon>0,\ \exists u \in L^2(Q_T)^m \text{ such that}\\ \|\Pi_p L_T u + \Pi_p S_T y_0 - y_T\|_{L^2(\Omega)^p} \leqslant \varepsilon.\end{gathered}$$

This is equivalent to

$$\begin{gathered}\forall \varepsilon>0,\ \forall z_T \in L^2(\Omega)^p, \exists u \in L^2(Q_T)^m \text{ such that}\\ \|\Pi_p L_T u - z_T\|_{L^2(\Omega)^p} \leqslant \varepsilon.\end{gathered}$$

That means

$$\overline{\Pi_p L_T(L^2(Q_T)^m)} = L^2(\Omega)^p.$$

In other words

$$\ker L_T^*\Pi_p^* = \{0\}.$$

Thus System (1) is Π_p-approximately controllable on the time interval $(0,T)$ if and only if for all $\varphi_0 \in L^2(\Omega)^p$

$$L_T^*\Pi_p^*\varphi_0 = \mathbb{1}_\omega B^*\varphi \equiv 0 \text{ in } Q_T \ \Rightarrow \varphi \equiv 0 \text{ in } Q_T.$$

□

Corollary 1. *Let us suppose that for all $\varphi_0 \in L^2(\Omega)^p$, the solution φ to the adjoint System (22) satisfies the observability inequality (23). Then for all initial condition*

$y_0 \in L^2(\Omega)^n$, there exists a control $u \in L^2(q_T)^m$ ($q_T := \omega \times (0,T)$) such that the solution y to System (1) satisfies $\Pi_p y(T) \equiv 0$ in Ω and

$$\|u\|_{L^2(q_T)^m} \leqslant \sqrt{C_{obs}} \|y_0\|_{L^2(\Omega)^n}. \tag{30}$$

The proof is classical and will be omitted (estimate (30) can be obtained directly following the method developed in [15]).

3. **Partial null controllability with constant coupling matrices.** Let us consider the system

$$\begin{cases} \partial_t y = \Delta y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{31}$$

where $y_0 \in L^2(\Omega)^n$, $u \in L^2(Q_T)^m$, $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$. Let the natural number s be defined by

$$s := \text{rank } [A|B] \tag{32}$$

and $X \subset \mathbb{R}^n$ be the linear space spanned by the columns of $[A|B]$.

In this section, we prove Theorem 1.1 in two steps. In subsection 3.1, we begin by studying the case where $B \in \mathbb{R}^n$ and the general case is considered in subsection 3.2.

All along this section, we will use the lemma below which proof is straightforward.

Lemma 3.1. *Let be $y_0 \in L^2(\Omega)^n$, $u \in L^2(Q_T)^m$ and $P \in \mathcal{C}^1([0,T], \mathcal{L}(\mathbb{R}^n))$ such that $P(t)$ is invertible for all $t \in [0,T]$. Then the change of variable $w = P^{-1}(t)y$ transforms System (31) into the equivalent system*

$$\begin{cases} \partial_t w = \Delta w + C(t)w + D(t)\mathbb{1}_\omega u & \text{in } Q_T, \\ w = 0 & \text{on } \Sigma_T, \\ w(0) = w_0 & \text{in } \Omega, \end{cases} \tag{33}$$

with $w_0 := P^{-1}(0)y_0$, $C(t) := -P^{-1}(t)\partial_t P(t) + P^{-1}(t)AP(t)$ and $D(t) := P^{-1}(t)B$. Moreover

$$\Pi_p y(T) \equiv 0 \text{ in } \Omega \;\Leftrightarrow\; \Pi_p P(T) w(T) \equiv 0 \text{ in } \Omega.$$

If P is constant, we have

$$[C|D] = P^{-1}[A|B].$$

3.1. **One control force.** In this subsection, we suppose that $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathbb{R}^n$ and denote by $[A|B] =: (k_{ij})_{1 \leqslant i,j \leqslant n}$ and $s := \text{rank } [A|B]$. We begin with the following observation.

Lemma 3.2. *$\{B, ..., A^{s-1}B\}$ is a basis of X.*

Proof. If $s = \text{rank } [A|B] = 1$, then the conclusion of the lemma is clearly true, since $B \neq 0$. Let $s \geqslant 2$. Suppose to the contrary that $\{B, ..., A^{s-1}B\}$ is not a basis of X, that is for some $i \in \{0, ..., s-2\}$ the family $\{B, ..., A^iB\}$ is linearly independent and $A^{i+1}B \in \text{span}(B, ..., A^iB)$, that is $A^{i+1}B = \sum_{k=0}^{i} \alpha_k A^k B$ with $\alpha_0, ..., \alpha_i \in \mathbb{R}$. Multiplying by A this expression, we deduce that $A^{i+2}B \in \text{span}(AB, ..., A^{i+1}B) = \text{span}(B, ..., A^iB)$. Thus, by induction, $A^lB \in \text{span}(B, ..., A^iB)$ for all $l \in \{i+1, ..., n-1\}$. Then rank $(B|AB|...|A^{n-1}B) = \text{rank } (B|AB|...|A^iB) = i+1 < s$, contradicting with (32). □

Proof of Theorem 1.1. Let us remark that

$$\text{rank } \Pi_p[A|B] = \dim \Pi_p[A|B](\mathbb{R}^n) \leqslant \text{rank } [A|B] = s. \tag{34}$$

Lemma 3.2 yields

$$\text{rank } (B|AB|...|A^{s-1}B) = \text{rank } [A|B] = s. \tag{35}$$

Thus, for all $l \in \{s, s+1, ..., n\}$ and $i \in \{0, ..., s-1\}$, there exist $\alpha_{l,i}$ such that

$$A^l B = \sum_{i=0}^{s-1} \alpha_{l,i} A^i B. \tag{36}$$

Since, for all $l \in \{s, ..., n\}$, $\Pi_p A^l B = \sum_{i=0}^{s-1} \alpha_{l,i} \Pi_p A^i B$, then

$$\text{rank } \Pi_p(B|AB|...|A^{s-1}B) = \text{rank } \Pi_p[A|B]. \tag{37}$$

We first prove in (a) that condition (9) is sufficient, and then in (b) that this condition is necessary.

(a) Sufficiency part: Let us assume first that condition (9) holds. Then, using (37), we have

$$\text{rank } \Pi_p(B|AB|...|A^{s-1}B) = p. \tag{38}$$

Let be $y_0 \in L^2(\Omega)^n$. We will study the Π_p-null controllability of System (31) according to the values of p and s.

Case 1 : $p = s$. The idea is to find an appropriate change of variable P to the solution y to System (31). More precisely, we would like the new variable $w := P^{-1}y$ to be the solution to a cascade system and then, apply Theorem 2.1. So let us define, for all $t \in [0, T]$,

$$P(t) := (B|AB|...|A^{s-1}B|P_{s+1}(t)|...|P_n(t)), \tag{39}$$

where, for all $l \in \{s+1, ..., n\}$, $P_l(t)$ is the solution in $\mathcal{C}^1([0, T])^n$ to the system of ordinary differential equations

$$\begin{cases} \partial_t P_l(t) = AP_l(t) \text{ in } [0, T], \\ P_l(T) = e_l. \end{cases} \tag{40}$$

Using (39) and (40), we can write

$$P(T) = \begin{pmatrix} P_{11} & 0 \\ P_{21} & I_{n-s} \end{pmatrix}, \tag{41}$$

where $P_{11} := \Pi_p(B|AB|...|A^{s-1}B) \in \mathcal{L}(\mathbb{R}^s)$, $P_{21} \in \mathcal{L}(\mathbb{R}^s, \mathbb{R}^{n-s})$ and I_{n-s} is the identity matrix of size $n-s$. Using (38), P_{11} is invertible and thus $P(T)$ also. Furthermore, since $P(t) \in \mathcal{C}^1([0, T], \mathcal{L}(\mathbb{R}^n))$ continuous in time on the time interval $[0, T]$, there exists $T^* \in [0, T)$ such that $P(t)$ is invertible for all $t \in [T^*, T]$.

Let us suppose first that $T^* = 0$. Since $P(t) \in \mathcal{C}^1([0, T], \mathcal{L}(\mathbb{R}^n))$ and invertible, in view of Lemma 3.1: for a fixed control $u \in L^2(Q_T)$, y is the solution to System (31) if and only if $w := P(t)^{-1}y$ is the solution to System (33) where C, D are given by

$$C(t) := -P^{-1}(t)\partial_t P(t) + P^{-1}(t)AP(t) \quad \text{and} \quad D(t) := P^{-1}(t)B,$$

for all $t \in [0, T]$. Using (36) and (40), we obtain

$$\begin{cases} -\partial_t P(t) + AP(t) = (AB|...|A^sB|0|...|0) = P(t) \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} & \text{in } [0, T], \\ P(t)e_1 = B & \text{in } [0, T], \end{cases} \tag{42}$$

where

$$C_{11} := \begin{pmatrix} 0 & 0 & 0 & \dots & \alpha_{s,0} \\ 1 & 0 & 0 & \dots & \alpha_{s,1} \\ 0 & 1 & 0 & \dots & \alpha_{s,2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \alpha_{s,s-1} \end{pmatrix} \in \mathcal{L}(\mathbb{R}^s). \tag{43}$$

Then

$$C(t) = \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } D(t) = e_1. \tag{44}$$

Using Theorem 2.1, there exists $u \in L^2(Q_T)$ such that the solution to System (33) satisfies $w_1(T) \equiv ... \equiv w_s(T) \equiv 0$ in Ω. Moreover, using (41), we have

$$\Pi_s y(T) = (y_1(T), ..., y_s(T)) = P_{11}(w_1(T), ..., w_s(T)) \equiv 0 \text{ in } \Omega.$$

If now $T^* \neq 0$, let $\overline{y}$ be the solution in $W(0, T^*)^n$ to System (31) with the initial condition $\overline{y}(0) = y_0$ in Ω and the control $u \equiv 0$ in $\Omega \times (0, T^*)$. We use the same argument as above to prove that System (31) is Π_s-null controllable on the time interval $[T^*, T]$. Let v be a control in $L^2(\Omega \times (T^*, T))$ such that the solution z in $W(T^*, T)^n$ to System (31) with the initial condition $z(T^*) = \overline{y}(T^*)$ in Ω and the control v satisfies $\Pi_s z(T) \equiv 0$ in Ω. Thus if we define y and u as follows

$$(y, u) := \begin{cases} (\overline{y}, 0) \text{ if } t \in [0, T^*], \\ (z, v) \text{ if } t \in [T^*, T], \end{cases}$$

then, for this control u, y is the solution in $W(0, T)^n$ to System (31). Moreover y satisfies

$$\Pi_s y(T) \equiv 0 \text{ in } \Omega.$$

Case 2 : $p < s$. In order to use Case 1, we would like to apply an appropriate change of variable Q to the solution y to System (31). If we denote by $[A|B] =: (k_{ij})_{ij}$, equalities (35) and (38) can be rewritten

$$\text{rank} \begin{pmatrix} k_{11} & \cdots & k_{1s} \\ \vdots & & \vdots \\ k_{n1} & \cdots & k_{ns} \end{pmatrix} = s \quad \text{and} \quad \text{rank} \begin{pmatrix} k_{11} & \cdots & k_{1s} \\ \vdots & & \vdots \\ k_{p1} & \cdots & k_{ps} \end{pmatrix} = p.$$

Then there exist distinct natural numbers $\lambda_{p+1}, ..., \lambda_s$ such that we have $\{\lambda_{p+1}, ..., \lambda_s\} \subset \{p+1, ..., n\}$ and

$$\text{rank} \begin{pmatrix} k_{11} & \cdots & k_{1s} \\ \vdots & & \vdots \\ k_{p1} & \cdots & k_{ps} \\ k_{\lambda_{p+1}1} & \cdots & k_{\lambda_{p+1}s} \\ \vdots & & \vdots \\ k_{\lambda_s 1} & \cdots & k_{\lambda_s s} \end{pmatrix} = s. \tag{45}$$

Let Q be the matrix defined by

$$Q := (e_1|...|e_p|e_{\lambda_{p+1}}|...|e_{\lambda_n})^{\mathrm{t}},$$

where $\{\lambda_{s+1}, ..., \lambda_n\} := \{p+1, ..., n\} \backslash \{\lambda_{p+1}, ..., \lambda_s\}$. Q is invertible, so taking $w := P^{-1} y$ with $P := Q^{-1}$, for a fixed control u in $L^2(Q_T)$, y is

solution to System (31) if and only if w is solution to System (33) where $w_0 := Qy_0$, $C := QAQ^{-1} \in \mathcal{L}(\mathbb{R}^n)$ and $D := QB \in \mathcal{L}(\mathbb{R};\mathbb{R}^n)$. Moreover there holds

$$[C|D] = Q[A|B].$$

Thus, equation (45) yields

$$\text{rank } \Pi_s[C|D] = \text{rank } \Pi_s Q[A|B] = \text{rank } \begin{pmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & & \vdots \\ k_{p1} & \cdots & k_{pn} \\ k_{\lambda_{p+1}1} & \cdots & k_{\lambda_{p+1}n} \\ \vdots & & \vdots \\ k_{\lambda_s 1} & \cdots & k_{\lambda_s n} \end{pmatrix} = s.$$

Since rank $[C|D]$ = rank $[A|B] = s$, we proceed as in Case 1 forward deduce that System (33) is Π_s-null controllable, that is there exists a control $u \in L^2(Q_T)$ such that the solution w to System (33) satisfies

$$\Pi_s w(T) \equiv 0 \text{ in } \Omega.$$

Moreover the matrix Q can be rewritten

$$Q = \begin{pmatrix} I_p & 0 \\ 0 & Q_{22} \end{pmatrix},$$

where $Q_{22} \in \mathcal{L}(\mathbb{R}^{n-p})$. Thus

$$\Pi_p y(T) = \Pi_p Q y(T) = \Pi_p w(T) \equiv 0 \text{ in } \Omega.$$

(b) Necessary part: Let us denote by $[A|B] =: (k_{ij})_{ij}$. We suppose now that (9) is not satisfied: there exist $\overline{p} \in \{1, ..., p\}$ and β_i for all $i \in \{1, ..., p\} \backslash \{\overline{p}\}$ such that $k_{\overline{p}j} = \sum_{i=1, i\neq \overline{p}}^{p} \beta_i k_{ij}$ for all $j \in \{1, ..., s\}$. The idea is to find a change of variable $w := Qy$ that allows to handle more easily our system. We will achieve this in three steps starting from the simplest situation.

Step 1. Let us suppose first that

$$k_{11} = ... = k_{1s} = 0 \quad \text{and} \quad \text{rank } \begin{pmatrix} k_{21} & \cdots & k_{2s} \\ \vdots & & \vdots \\ k_{s+1,1} & \cdots & k_{s+1,s} \end{pmatrix} = s. \tag{46}$$

We want to prove that, for some initial condition $y_0 \in L^2(\Omega)^n$, a control $u \in L^2(Q_T)$ cannot be found such that the solution to System (31) satisfies $y_1(T) \equiv 0$ in Ω. Let us consider the matrix $P \in \mathcal{L}(\mathbb{R}^n)$ defined by

$$P := (B|...|A^{s-1}B|e_1|e_{s+2}|...|e_n). \tag{47}$$

Using the assumption (46), P is invertible. Thus, in view of Lemma 3.1, for a fixed control $u \in L^2(Q_T)$, y is a solution to System (31) if and only if $w := P^{-1}y$ is a solution to System (33) where C, D are given by $C := P^{-1}AP$ and $D := P^{-1}B$. Using (36) we remark that

$$A(B|AB|...|BA^{s-1}) = (B|AB|...|BA^{s-1}) \begin{pmatrix} C_{11} \\ 0 \end{pmatrix},$$

with C_{11} defined in (43). Then C can be rewritten as

$$C = \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix}, \tag{48}$$

where $C_{12} \in \mathcal{L}(\mathbb{R}^{n-s}, \mathbb{R}^s)$ and $C_{22} \in \mathcal{L}(\mathbb{R}^{n-s})$. Furthermore

$$D = P^{-1}B = P^{-1}Pe_1 = e_1.$$

and with the Definition (47) of P we get

$$y_1(T) = w_{s+1}(T) \text{ in } \Omega.$$

Thus we need only to prove that there exists $w_0 \in L^2(\Omega)^n$ such that we cannot find a control $u \in L^2(Q_T)$ with the corresponding solution w to System (33) satisfying $w_{s+1}(T) \equiv 0$ in Ω. Therefore we apply Proposition 1 and prove that the observability inequality (23) can not be satisfied. More precisely, for all $w_0 \in L^2(\Omega)^n$, there exists a control $u \in L^2(Q_T)$ such that the solution to System (33) satisfies $w_{s+1}(T) \equiv 0$ in Ω, if and only if there exists $C_{obs} > 0$ such that for all $\varphi^0_{s+1} \in L^2(\Omega)$ the solution to the adjoint system

$$\begin{cases} -\partial_t \varphi &= \Delta\varphi + \begin{pmatrix} C^*_{11} & 0 \\ C^*_{12} & C^*_{22} \end{pmatrix} \varphi & \text{in } Q_T, \\ \varphi &= 0 & \text{on } \Sigma_T, \\ \varphi(T) &= (0, ..., 0, \varphi^0_{s+1}, 0, ..., 0)^{\mathrm{t}} = e_{s+1}\varphi^0_{s+1} & \text{in } \Omega \end{cases} \tag{49}$$

satisfies the observability inequality

$$\int_\Omega \varphi(0)^2 \, dx \leqslant C_{obs} \int_{\omega\times(0,T)} \varphi_1^2 \, dx \, dt. \tag{50}$$

But for all $\varphi^0_{s+1} \not\equiv 0$ in Ω, the inequality (50) is not satisfied. Indeed, we remark first that, since $\varphi_1(T) = ... = \varphi_s(T) = 0$ in Ω, we have $\varphi_1 = ... = \varphi_s = 0$ in Q_T, so that $\int_{\omega\times(0,T)} \varphi_1^2 \, dx = 0$, while, if we choose $\varphi^0_{s+1} \not\equiv 0$ in Ω, using the results on backward uniqueness for this type of parabolic system (see [17]), we have clearly $(\varphi_{s+1}(0), ..., \varphi_n(0)) \not\equiv 0$ in Ω.

Step 2. Let us suppose only that $k_{11} = ... = k_{1s} = 0$. Since rank $(B|...|A^{s-1}B) = s$, there exists distinct $\lambda_1, ..., \lambda_s \in \{2, ..., n\}$ such that

$$\text{rank} \begin{pmatrix} k_{\lambda_1,1} & \cdots & k_{\lambda_1,s} \\ \vdots & & \vdots \\ k_{\lambda_s,1} & \cdots & k_{\lambda_s,s} \end{pmatrix} = s.$$

Let us consider the following matrix

$$Q := (e_1|e_{\lambda_1}|...|e_{\lambda_{n-1}})^{\mathrm{t}},$$

where $\{\lambda_{s+1}, ..., \lambda_{n-1}\} = \{2, ..., n\} \backslash \{\lambda_1, ..., \lambda_s\}$. Thus, for $P := Q^{-1}$, again, for a fixed control $u \in L^2(Q_T)$, y is a solution to System (31) if and only if $w := P^{-1}y$ is a solution to System (33) where C, D are given by $C := QAQ^{-1}$ and $D := QB$. Moreover, we have

$$[C|D] = Q[A|B].$$

If we note $(\tilde{k}_{ij})_{ij} := [C|D]$, this implies $\tilde{k}_{11} = ... = \tilde{k}_{1s} = 0$ and

$$\text{rank}\begin{pmatrix} \tilde{k}_{21} & \cdots & \tilde{k}_{2s} \\ \vdots & & \vdots \\ \tilde{k}_{s+1,1} & \cdots & \tilde{k}_{s+1,s} \end{pmatrix} = \text{rank}\begin{pmatrix} k_{\lambda_1 1} & \cdots & k_{\lambda_1 s} \\ \vdots & & \vdots \\ k_{\lambda_s,1} & \cdots & k_{\lambda_s,s} \end{pmatrix} = s.$$

Proceeding as in Step 1 for w, there exists an initial condition w_0 such that for all control u in $L^2(Q_T)$ the solution w to System (33) satisfies $w_1(T) \not\equiv 0$ in Ω. Thus, for the initial condition $y_0 := Q^{-1}w_0$, for all control u in $L^2(Q_T)$, the solution y to System (31) satisfies

$$y_1(T) = w_1(T) \not\equiv 0 \text{ in } \Omega.$$

Step 3. Without loss of generality, we can suppose that there exists β_i for all $i \in \{2, ..., p\}$ such that $k_{1j} = \sum_{i=2}^{p} \beta_i k_{ij}$ for all $j \in \{1, ..., s\}$ (otherwise a permutation of lines leads to this case). Let us define the following matrix

$$Q := \left((e_1 - \sum_{i=2}^{p} \beta_i e_i)|e_2|...|e_n \right)^{\text{t}}.$$

Thus, for $P := Q^{-1}$, again, for a fixed initial condition $y_0 \in L^2(\Omega)^n$ and a control $u \in L^2(Q_T)$, consider System (33) with $w := P^{-1}y$, y being a solution to System (31). We remark that if we denote by $(\tilde{k}_{ij}) := [C|D]$, we have $\tilde{k}_{11} = ... = \tilde{k}_{1s} = 0$. Applying step 2 to w, there exists an initial condition w_0 such that for all control u in $L^2(Q_T)$ the solution w to System (33) satisfies

$$w_1(T) \not\equiv 0 \text{ in } \Omega. \tag{51}$$

Thus, with the definition of Q, for all control u in $L^2(Q_T)$ the solution y to System (31) satisfies

$$w_1(T) = y_1(T) - \sum_{i=2}^{p} \beta_i y_i(T) \text{ in } \Omega.$$

Suppose $\Pi_p y(T) \equiv 0$ in Ω, then $w_1(T) \equiv 0$ in Ω and this contradicts (51).

As a consequence of Proposition 1, the Π_p-null controllability implies the Π_p-approximate controllability of System (33). If now Condition (9) is not satisfied, as for the Π_p-null controllability, we can find a solution to System (49) such that $\phi_1 \equiv 0$ in $\omega \times (0, T)$ and $\phi \not\equiv 0$ in Q_T and we conclude again with Proposition 1.

□

3.2. ***m*-control forces.** In this subsection, we will suppose that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. We denote by $B =: (b^1|...|b^m)$. To prove Theorem 1.1, we will use the following lemma which can be found in [2].

Lemma 3.3. *There exist $r \in \{1, ..., s\}$ and sequences $\{l_j\}_{1 \leqslant j \leqslant r} \subset \{1, ..., m\}$ and $\{s_j\}_{1 \leqslant j \leqslant r} \subset \{1, ..., n\}$ with $\sum_{j=1}^{r} s_j = s$, such that*

$$\mathcal{B} := \bigcup_{j=1}^{r} \{b^{l_j}, Ab^{l_j}, ..., A^{s_j - 1}b^{l_j}\}$$

is a basis of X. Moreover, for every $1 \leqslant j \leqslant r$*, there exist* $\alpha^i_{k,s_j} \in \mathbb{R}$ *for* $1 \leqslant i \leqslant j$ *and* $1 \leqslant k \leqslant s_j$ *such that*

$$A^{s_j} b^{l_j} = \sum_{i=1}^{j} \left(\alpha^i_{1,s_j} b^{l_i} + \alpha^i_{2,s_j} A b^{l_i} + ... + \alpha^i_{s_i,s_j} A^{s_i-1} b^{l_i} \right). \tag{52}$$

Proof of Theorem 1.1. Consider the basis $\mathcal{B}$ of X given by Lemma 3.3. Note that

$$\text{rank } \Pi_p[A|B] = \dim \Pi_p[A|B](\mathbb{R}^n) \leqslant \text{rank } [A|B] = s.$$

If M is the matrix whose columns are the elements of $\mathcal{B}$, i.e.

$$M = (m_{ij})_{ij} := \left(b^{l_1} | A b^{l_1} | ... | A^{s_1-1} b^{l_1} | ... | b^{l_r} | A b^{l_r} | ... | A^{s_r-1} b^{l_r} \right),$$

we can remark that

$$\text{rank } \Pi_p M = \text{rank } \Pi_p[A|B]. \tag{53}$$

Indeed, relationship (52) yields

$$\Pi_p A^{s_j} b^{l_j} = \sum_{i=1}^{j} \left(\alpha^i_{1,s_j} \Pi_p b^{l_i} + \alpha^i_{2,s_j} \Pi_p A b^{l_i} + ... + \alpha^i_{s_i,s_j} \Pi_p A^{s_i-1} b^{l_i} \right).$$

We first prove in (a) that condition (9) is sufficient, and then in (b) that this condition is necessary.

(a) Sufficiency part: Let us suppose first that (9) is satisfied. Let be $y_0 \in L^2(\Omega)^n$. We will prove that we need only r forces to control System (31). More precisely, we will study the Π_p-null controllability of the system

$$\begin{cases} \partial_t y = \Delta y + Ay + \tilde{B} 1_\omega v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{54}$$

where $\tilde{B} = (b^{l_1} | b^{l_2} | \cdots | b^{l_r}) \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^n)$. Using (9) and (53), we have

$$\text{rank } \Pi_p (b^{l_1} | A b^{l_1} | ... | A^{s_1-1} b^{l_1} | ... | b^{l_r} | A b^{l_r} | ... | A^{s_r-1} b^{l_r}) = p. \tag{55}$$

Case 1 : $p = s$. As in the case of one control force, we want to apply a change of variable P to the solution y to System (54). Let us define for all $t \in [0, T]$ the following matrix

$$P(t) := (b^{l_1} | A b^{l_1} | ... | A^{s_1-1} b^{l_1} | ... | b^{l_r} | A b^{l_r} | ... | A^{s_r-1} b^{l_r} | P_{s+1}(t) | ... | P_n(t)) \in \mathcal{L}(\mathbb{R}^n), \tag{56}$$

where for all $l \in \{s+1, ..., n\}$, P_l is solution in $\mathcal{C}^1([0,T])^n$ to the system of ordinary differential equations

$$\begin{cases} \partial_t P_l(t) = A P_l(t) \text{ in } [0, T], \\ P_l(T) = e_l. \end{cases} \tag{57}$$

Using (56) and (57) we have

$$P(T) = \begin{pmatrix} P_{11} & 0 \\ P_{21} & I_{n-s} \end{pmatrix}, \tag{58}$$

where $P_{11} := \Pi_s (b^{l_1} | A b^{l_1} | ... | A^{s_1-1} b^{l_1} | ... | b^{l_r} | A b^{l_r} | ... | A^{s_r-1} b^{l_r}) \in \mathcal{L}(\mathbb{R}^s)$ and $P_{21} \in \mathcal{L}(\mathbb{R}^{n-s}, \mathbb{R}^s)$. From (55), P_{11} and thus $P(T)$ are invertible. Furthermore, since P is continuous on $[0, T]$, there exists a $T^* \in [0, T)$ such that $P(t)$ is invertible for all $t \in [T^*, T]$.

We suppose first that $T^* = 0$. Since P is invertible and continuous on $[0, T]$, for a fixed control $v \in L^2(Q_T)^r$, y is the solution to System (54)

if and only if $w := P(t)^{-1}y$ is the solution to System (33) where C, D are given by

$$C(t) := -P^{-1}(t)\partial_t P(t) + P^{-1}(t)AP(t) \quad \text{and} \quad D(t) := P^{-1}(t)\tilde{B},$$

for all $t \in [0,T]$. Using (52) and (57), we obtain

$$\left\{\begin{array}{rcl} -\partial_t P(t) + AP(t) & = & (Ab^{l_1}|A^2b^{l_1}|...|A^{s_1}b^{l_1}|...|Ab^{l_r}|A^2b^{l_r}|...|A^{s_r}b^{l_r}|0|...|0), \\ & = & P(t)\begin{pmatrix} \tilde{C}_{11} & 0 \\ 0 & 0 \end{pmatrix} \text{ in } [0,T], \\ P(t)e_{S_i} & = & b^{l_i} \text{ in } [0,T], \end{array}\right. \tag{59}$$

where $S_i = 1 + \sum_{j=1}^{i-1} s_j$ for $i \in \{1,...,r\}$,

$$\tilde{C}_{11} := \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1r} \\ 0 & C_{22} & \cdots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{rr} \end{pmatrix} \in \mathcal{L}(\mathbb{R}^s) \tag{60}$$

and for $1 \leqslant i \leqslant j \leqslant r$ the matrices $C_{ij} \in \mathcal{L}(\mathbb{R}^{s_j}, \mathbb{R}^{s_i})$ are given by

$$C_{ii} := \begin{pmatrix} 0 & 0 & 0 & \dots & \alpha^i_{1,s_i} \\ 1 & 0 & 0 & \dots & \alpha^i_{2,s_i} \\ 0 & 1 & 0 & \dots & \alpha^i_{3,s_i} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \alpha^i_{s_i,s_i} \end{pmatrix}$$

and

$$C_{ij} := \begin{pmatrix} 0 & 0 & 0 & \dots & \alpha^i_{1,s_j} \\ 0 & 0 & 0 & \dots & \alpha^i_{2,s_j} \\ 0 & 0 & 0 & \dots & \alpha^i_{3,s_j} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \alpha^i_{s_i,s_j} \end{pmatrix} \text{ for } j > i.$$

Then

$$C(t) = \begin{pmatrix} \tilde{C}_{11} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } D(t) = (e_{S_1}|...|e_{S_r}). \tag{61}$$

Using Theorem 2.1, there exists $v \in L^2(Q_T)^r$ such that the solution to System (33) satisfies $w_1(T) = ... = w_s(T) \equiv 0$ in Ω. Moreover, using (58), we have

$$\Pi_s y(T) = (y_1(T), ..., y_s(T)) = P_{11}(w_1(T), ..., w_s(T)) \equiv 0 \text{ in } \Omega.$$

If now $T^* \neq 0$, we conclude as in the proof of Theorem 1.1 with one force (see § 3.1).

Case 2 : $p < s$. The proof is a direct adaptation of the proof of Theorem 1.1 with one force, it is possible to find a change of variable in order to get back to the situation of Case 1 (see § 3.1).

(b) Necessary part: If (9) is not satisfied, there exist $\overline{p} \in \{1,...,p\}$ and, for all $i \in \{1,...,p\}\backslash\{\overline{p}\}$, scalars β_i such that $m_{\overline{p}j} = \sum_{i=1, i\neq\overline{p}}^{p} \beta_i m_{ij}$ for all $j \in \{1,...,s\}$. As

previously, without loss of generality, we can suppose that

$$m_{11} = ... = m_{1s} = 0 \quad \text{and} \quad \text{rank} \begin{pmatrix} m_{21} & \cdots & m_{2s} \\ \vdots & & \vdots \\ m_{s+1,1} & \cdots & m_{s+1,s} \end{pmatrix} = s \tag{62}$$

(otherwise a permutation of lines leads to this case). Let us consider the matrix P defined by

$$P := (b^{l_1}|Ab^{l_1}|...|A^{s_1-1}b^{l_1}|...|b^{l_r}|Ab^{l_r}|...|A^{s_r-1}b^{l_r}|e_1|e_{s+2}|...|e_n). \tag{63}$$

Relationship ensures (62) that P is invertible. Thus, again, for a fixed control $u \in L^2(Q_T)^m$, y is the solution to System (31) if and only if $w := P^{-1}y$ is the solution to System (33) where C, D are given by $C := P^{-1}AP$ and $D := P^{-1}B$. Using (52), we remark that

$$\begin{aligned} &A(b^{l_1}|Ab^{l_1}|...|A^{s_1-1}b^{l_1}|...|b^{l_r}|Ab^{l_r}|...|A^{s_r-1}b^{l_r}) \\ &= (Ab^{l_1}|A^2b^{l_1}|...|A^{s_1}b^{l_1}|...|Ab^{l_r}|A^2b^{l_r}|...|A^{s_r}b^{l_r}) = P \begin{pmatrix} \tilde{C}_{11} \\ 0 \end{pmatrix}, \end{aligned}$$

where $\tilde{C}_{11}$ is defined in (60). Then C can be written as

$$C = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ 0 & \tilde{C}_{22} \end{pmatrix}, \tag{64}$$

where $\tilde{C}_{12} \in \mathcal{L}(\mathbb{R}^s, \mathbb{R}^{n-s})$ and $\tilde{C}_{22} \in \mathcal{L}(\mathbb{R}^{n-s})$. Furthermore, the matrix D can be written

$$D = \begin{pmatrix} D_1 \\ 0 \end{pmatrix},$$

where $D_1 \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^s)$. Using (63), we get

$$y_1(T) = w_{s+1}(T) \text{ in } \Omega.$$

Thus, we need only to prove that there exists $w_0 \in L^2(\Omega)^n$ such that we cannot find a control $u \in L^2(Q_T)^m$ with the corresponding solution w to System (33) satisfying $w_{s+1}(T) \equiv 0$ in Ω. Therefore we apply Proposition 1 and prove that the observability inequality (23) can not be satisfied. More precisely, for all $w_0 \in L^2(\Omega)^n$, there exists a control $u \in L^2(Q_T)^m$ such that the solution w to System (33) satisfies $w_{s+1}(T) \equiv 0$ in Ω, if and only if there exists $C_{obs} > 0$ such that for all $\varphi^0_{s+1} \in L^2(\Omega)$ the solution to the adjoint system

$$\left\{ \begin{array}{rcll} -\partial_t \varphi &=& \Delta\varphi + \begin{pmatrix} \tilde{C}^*_{11} & 0 \\ \tilde{C}^*_{12} & \tilde{C}^*_{22} \end{pmatrix} \varphi & \text{in } Q_T, \\ \varphi &=& 0 & \text{on } \Sigma_T, \\ \varphi(T) &=& (0, ..., 0, \varphi^0_{s+1}, 0, ..., 0)^{\mathrm{t}} = e_{s+1}\varphi^0_{s+1} & \text{in } \Omega \end{array} \right. \tag{65}$$

satisfies the observability inequality

$$\int_\Omega \varphi(0)^2 \, dx \leqslant C_{obs} \int_{\omega\times(0,T)} (D^*_1(\varphi_1, ..., \varphi_s)^{\mathrm{t}})^2 \, dx \, dt. \tag{66}$$

But for all $\varphi^0_{s+1} \not\equiv 0$ in Ω, the inequality (66) is not satisfied. Indeed, we remark first that, since $\varphi_1(T) = ... = \varphi_s(T) = 0$ in Ω, we have $\varphi_1 = ... = \varphi_s = 0$ in Q_T. Furthermore, if we choose $\varphi^0_{s+1} \not\equiv 0$ in Ω, as previously, we get $(\varphi_{s+1}(0), ..., \varphi_n(0)) \not\equiv 0$ in Ω.

We recall that, as a consequence of Proposition 1, the Π_p-null controllability implies the Π_p-approximate controllability of System (54). If Condition (9) is not

satisfied, as for the Π_p-null controllability, we can find a solution to System (65) such that $D_1^*(\phi_1, ..., \phi_s)^{\mathrm{t}} \equiv 0$ in $\omega \times (0,T)$ and $\phi \not\equiv 0$ in Q_T and we conclude again with Proposition 1.

□

4. **Partial null controllability with time dependent matrices.** We recall that $[A|B](\cdot) = (B_0(\cdot)|...|B_{n-1}(\cdot))$ (see (6)). Since $A(t) \in \mathcal{C}^{n-1}([0,T]; \mathcal{L}(\mathbb{R}^n))$ and $B(t) \in \mathcal{C}^n([0,T]; \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n))$, we remark that the matrix $[A|B]$ is well defined and is an element of $\mathcal{C}^1([0,T], \mathcal{L}(\mathbb{R}^{mn}, \mathbb{R}^n)$. We will use the notation $B_i =: (b_1^i|...|b_m^i)$ for all $i \in \{0, ..., n-1\}$. To prove Theorem 1.2, we will use the following lemma of [19]

Lemma 4.1. *Assume that* $\max\{\text{rank } [A|B](t) : t \in [0,T]\} = s \leqslant n$. *Then there exist* $T_0,\ T_1 \in [0,T]$, *with* $T_0 < T_1$, $r \in \{1, ..., m\}$ *and sequences* $(s_j)_{1 \leqslant j \leqslant r} \subset \{1, ..., n\}$, *with* $\sum_{i=1}^r s_j = s$, *and* $(l_j)_{1 \leqslant j \leqslant r} \subset \{1, ..., m\}$ *such that, for every* $t \in [T_0, T_1]$, *the set*

$$\mathcal{B}(t) = \bigcup_{j=1}^{r} \{b_0^{l_j}(t), b_1^{l_j}(t), ..., b_{s_j-1}^{l_j}(t)\}, \tag{67}$$

is linearly independent, spans the columns of $[A|B](t)$ *and satisfies*

$$b_{s_j}^{l_j}(t) = \sum_{k=1}^{j} \left(\theta_{s_j,0}^{l_j,l_k}(t) b_0^{l_k}(t) + \theta_{s_j,1}^{l_j,l_k}(t) b_1^{l_k}(t) + ... + \theta_{s_j,s_k-1}^{l_j,l_k}(t) b_{s_k-1}^{l_k}(t) \right), \tag{68}$$

for every $t \in [T_0, T_1]$ *and* $j \in \{1, ..., r\}$, *where*

$$\theta_{s_j,0}^{l_j,l_k}(t),\ \theta_{s_j,1}^{l_j,l_k}(t),\ ..., \theta_{s_j,s_k-1}^{l_j,l_k}(t) \in \mathcal{C}^1([T_0, T_1]).$$

With exactly the same argument for the proof of the previous lemma, we can obtain the

Lemma 4.2. *If* rank $[A|B](T) = s$, *then the conclusions of Lemma 4.1 hold true with* $T_1 = T$.

Proof of Theorem 1.2. Let $y_0 \in L^2(\Omega)^n$ and s be the rank of the matrix $[A|B](T)$. As in the proof of the controllability by one force with constant matrices, let X being the linear space spanned by the columns of the matrix $[A|B](T)$. We consider $\mathcal{B} = \mathcal{B}(t)$ the basis of X defined in (67).

As in the constant case, we will prove that we need only r forces to control System (1) that is we study the partial null controllability of System (54) with the coupling matrix $A(t) \in \mathcal{C}^{n-1}([0,T]; \mathcal{L}(\mathbb{R}^n))$ and the control matrix $\tilde{B}(t) = (B_{l_1}(t)|B_{l_2}(t)|\cdots|B_{l_r}(t)) \in \mathcal{C}^n([0,T]; \mathcal{L}(\mathbb{R}^r, \mathbb{R}^n))$. If we define M as the matrix whose columns are the elements of $\mathcal{B}(t)$, i.e. for all $t \in [0,T]$

$$M(t) = (m_{ij}(t))_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant s} := \left(b_0^{l_1}(t)|b_1^{l_1}(t)|...|b_{s_1-1}^{l_1}(t)|...|b_0^{l_r}(t)|b_1^{l_r}(t)|...|b_{s_r-1}^{l_r}(t) \right),$$

we can remark that

$$\text{rank } \Pi_p M(T) = \text{rank } \Pi_p [A|B](T) = p. \tag{69}$$

Indeed, using (68),

$$\Pi_p b_{s_j}^{l_j}(t) = \sum_{k=1}^{j} \left(\theta_{s_j,0}^{l_j,l_k}(t) \Pi_p b_0^{l_k}(t) + \theta_{s_j,1}^{l_j,l_k}(t) \Pi_p b_1^{l_k}(t) + ... + \theta_{s_j,s_k-1}^{l_j,l_k}(t) \Pi_p b_{s_k-1}^{l_k}(t) \right).$$

Case 1 : $p = s$. As in the constant case, we want to apply a change of variable P to the solution y to System (54). Let us define for all $t \in [0,T]$ the following matrix

$$\begin{aligned} P(t) := (&b_0^{l_1}(t)|b_1^{l_1}(t)|...|b_{s_1-1}^{l_1}(t)| \\ &...|b_0^{l_r}(t)|b_1^{l_r}(t)|...|b_{s_r-1}^{l_r}(t)|P_{s+1}(t)|...|P_n(t)) \in \mathcal{L}(\mathbb{R}^n), \end{aligned} \tag{70}$$

where for all $i \in \{s+1,...,n\}$, P_l is solution in $\mathcal{C}^1([0,T])^n$ to the system of ordinary differential equations

$$\begin{cases} \partial_t P_l(t) = AP_l(t) \text{ in } [0,T], \\ P_l(T) = e_l. \end{cases} \tag{71}$$

Using (70) and (71), $P(T)$ can be rewritten

$$P(T) = \begin{pmatrix} P_{11} & 0 \\ P_{21} & I_{n-s} \end{pmatrix}, \tag{72}$$

where $P_{11} := \Pi_p(b_0^{l_1}(T)|b_1^{l_1}(T)|...|b_{s_1-1}^{l_1}(T)|...|b_0^{l_r}(T)|b_1^{l_r}(T)|...|b_{s_r-1}^{l_r}(T)) \in \mathcal{L}(\mathbb{R}^s)$ and $P_{21} \in \mathcal{L}(\mathbb{R}^{n-s}, \mathbb{R}^s)$. Using (69), P_{11}, and thus $P(T)$, are invertible. Furthermore, since P is continuous on $[0,T]$, there exists a $T^* \in [0,T)$ such that $P(t)$ is invertible for all $t \in [T^*, T]$.

As previously it is sufficient to prove the result for $T^* = 0$. Since $P(t) \in \mathcal{C}^1([0,T], \mathcal{L}(\mathbb{R}^n))$ and is invertible on the time interval $[0,T]$, again, for a fixed control $v \in L^2(Q_T)^r$, y is the solution to System (54) if and only if $w := P(t)^{-1}y$ is the solution to System (33) where C, D are given by

$$C(t) := -P^{-1}(t)\partial_t P(t) + P^{-1}(t)AP(t) \quad \text{and} \quad D(t) := P^{-1}(t)\tilde{B},$$

for all $t \in [0,T]$. Using (68) and (71), we obtain

$$\begin{cases} -\partial_t P(t) + AP(t) &= (b_1^{l_1}(t)|b_2^{l_1}(t)|...|b_{s_1}^{l_1}(t)|...|b_1^{l_r}(t)|b_2^{l_r}(t)|...|b_{s_r}^{l_r}(t)|0|...|0), \\ &= P(t)\begin{pmatrix} \tilde{C}_{11} & 0 \\ 0 & 0 \end{pmatrix} \text{ in } [0,T], \\ P(t)e_{S_i} &= b_0^{l_i} \text{ in } [0,T], \end{cases} \tag{73}$$

where $S_i = 1 + \sum_{j=1}^{i-1} s_j$ for $1 \leqslant i \leqslant r$,

$$\tilde{C}_{11} := \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1r} \\ 0 & C_{22} & \cdots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{rr} \end{pmatrix} \in \mathcal{L}(\mathbb{R}^s), \tag{74}$$

and for $1 \leqslant i \leqslant j \leqslant r$, the matrices $C_{ij} \in \mathcal{C}^0([0,T]; \in \mathcal{L}(\mathbb{R}^{s_j}, \mathbb{R}^{s_i}))$ are given here by

$$C_{ii} = \begin{pmatrix} 0 & 0 & 0 & \dots & \theta_{s_i,0}^{l_i,l_i} \\ 1 & 0 & 0 & \dots & \theta_{s_i,1}^{l_i,l_i} \\ 0 & 1 & 0 & \dots & \theta_{s_i,2}^{l_i,l_i} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \theta_{s_i,s_i-1}^{l_i,l_i} \end{pmatrix}$$

and

$$C_{ij} = \begin{pmatrix} 0 & 0 & 0 & \dots & \theta^{l_j,l_i}_{s_j,0} \\ 0 & 0 & 0 & \dots & \theta^{l_j,l_i}_{s_j,1} \\ 0 & 0 & 0 & \dots & \theta^{l_j,l_i}_{s_j,2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \theta^{l_j,l_i}_{s_j,s_i-1} \end{pmatrix} \text{ for j > i.}$$

Then

$$C = \begin{pmatrix} \tilde{C}_{11} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } D = (e_{S_1}|...|e_{S_r}). \tag{75}$$

Using Theorem 2.1, there exists $v \in L^2(Q_T)^r$ such that the solution to System (33) satisfies $w_1(T) = ... = w_s(T) \equiv 0$ in Ω. Moreover, the equality (72) leads to

$$\Pi_s y(T) = (y_1(T), ..., y_s(T))^{\mathrm{t}} = P_{11}(w_1(T), ..., w_s(T))^{\mathrm{t}} \equiv 0 \text{ in } \Omega.$$

Case 2 : $p < s$. The same method as in the constant case leads to the conclusion (see § 3.1).

The π_p-approximate controllability can proved also as in the constant case.

□

5. **Partial null controllability for a space dependent coupling matrix.** All along this section, the dimension N will be equal to 1, more precisely $\Omega := (0, \pi)$ with the exception of the proof of the third point in Theorem 1.3 and the numerical illustration in Section 5.3 where $\Omega := (0, 2\pi)$. We recall that the eigenvalues of $-\Delta$ in Ω with Dirichlet boundary conditions are given by $\mu_k := k^2$ for all $k \geqslant 1$ and we will denote by $(w_k)_{k\geqslant 1}$ the associated L^2-normalized eigenfunctions. Let us consider the following parabolic system of two equations

$$\begin{cases} \partial_t y = \Delta y + \alpha z + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t z = \Delta z & \text{in } Q_T, \\ y = z = 0 & \text{on } \Sigma_T, \\ y(0) = y_0,\ z(0) = z_0 & \text{in } \Omega, \end{cases} \tag{76}$$

where $y_0,\ z_0 \in L^2(\Omega)$ are the initial data, $u \in L^2(Q_T)$ is the control and the coupling coefficient α is in $L^\infty(\Omega)$. We recall that System (76) is Π_1-*null controllable* if for all $y^0, z^0 \in L^2(\Omega)$, we can find a control $u \in L^2(Q_T)$ such that the solution $(y, z) \in W(0, T)^2$ to System (76) satisfies $y(T) \equiv 0$ in Ω.

5.1. **Example of controllability.** In this subsection, we will provide an example of Π_1-null controllability for System (76) with the help of the method of moments initially developed in [13]. As already mentioned, we suppose that $\Omega := (0, \pi)$, but the argument of Section 5.1 can be adapted for any open bounded interval of $\mathbb{R}$. Let us introduce the adjoint system associated to our control problem

$$\begin{cases} -\partial_t \phi = \Delta \phi & \text{in } (0, \pi) \times (0, T), \\ -\partial_t \psi = \Delta \psi + \alpha \phi & \text{in } (0, \pi) \times (0, T), \\ \phi(0) = \phi(\pi) = \psi(0) = \psi(\pi) = 0 & \text{on } (0, T), \\ \phi(T) = \phi_0,\ \psi(T) = 0 & \text{in } (0, \pi), \end{cases} \tag{77}$$

where $\phi_0 \in L^2(0,\pi)$. For an initial data $\phi_0 \in L^2(0,\pi)$ in adjoint System (77), we get

$$\int_0^\pi \phi_0 y(T)\,dx - \int_0^\pi \phi(0) y_0\,dx - \int_0^\pi \psi(0) z_0\,dx = \iint_{q_T} \phi u\,dx\,dt, \tag{78}$$

with the notation $q_T := \omega \times (0,T)$. Since $(w_k)_{k\geqslant 1}$ spans $L^2(0,\pi)$, System (76) is Π_1-*null controllable* if and only if there exists $u \in L^2(q_T)$ such that, for all $k \in \mathbb{N}^*$, the solution to System (77) satisfies the following equality

$$-\int_0^\pi \phi_k(0) y_0\,dx - \int_0^\pi \psi_k(0) z_0\,dx = \iint_{q_T} \phi_k u\,dx\,dt, \tag{79}$$

where (ϕ_k, ψ_k) is the solution to adjoint System (77) for the initial data $\phi_0 := w_k$.

Let $k \in \mathbb{N}^*$. With the initial condition $\phi_0 := w_k$ is associated the solution (ϕ_k, ψ_k) to adjoint System (77):

$$\phi_k(t) = e^{-k^2(T-t)} w_k \text{ in } (0,\pi)$$

for all $t \in [0,T]$. If we write:

$$\psi_k(x,t) := \sum_{l\geqslant 1} \psi_{kl}(t) w_l(x) \text{ for all } (x,t) \in (0,\pi)\times(0,T),$$

then a simple computation leads to the formula

$$\psi_{kl}(t) \quad = \quad \frac{e^{-k^2(T-t)} - e^{-l^2(T-t)}}{-k^2+l^2}\alpha_{kl} \text{ for all } l \geqslant 1,\ t \in (0,T), \tag{80}$$

where, for all $k, l \in \mathbb{N}^*$, α_{kl} is defined in (2). In (80) we implicitly used the convention: if $l = k$ the term $(e^{-k^2(T-t)} - e^{-l^2(T-t)})/(-k^2+l^2)$ is replaced by $(T-t)e^{-k^2(T-t)}$. With these expressions of ϕ_k and ψ_k, the equality (79) reads for all $k \geqslant 1$

$$-e^{-k^2T} y_k^0 - \sum_{l\geqslant 1} \frac{e^{-k^2T} - e^{-l^2T}}{-k^2+l^2}\alpha_{kl} z_l^0 = \iint_{q_T} e^{-k^2(T-t)} w_k(x) u(t,x)\,dx\,dt. \tag{81}$$

In the proof of Theorem 1.3, we will look for a control u expressed as $u(x,t) = f(x)\gamma(t)$ with $\gamma(t) = \sum_{k\geqslant 1} \gamma_k q_k(t)$ and $(q_k)_{k\geqslant 1}$ a family biorthogonal to $(e^{-k^2 t})_{k\geqslant 1}$. Thus, we will need the two following lemma

Lemma 5.1. *(see Lemma 5.1, [6]) There exists $f \in L^2(0,\pi)$ such that* Supp $f \subset \omega$ *and for a constant β, one has*

$$\inf_{k\geqslant 1} f_k k^3 = \beta > 0,$$

where, for all $k \in \mathbb{N}^*$, $f_k := \int_0^\pi f w_k\,dx$.

Lemma 5.2. *(see Corollary 3.2, [13]) There exists a sequence $(q_k)_{k\geqslant 1} \subset L^2(0,T)$ biorthogonal to $(e^{-k^2 t})_{k\geqslant 1}$, that is*

$$\langle q_k, e^{-l^2 t}\rangle_{L^2(0,T)} = \delta_{kl}.$$

Moreover, for all $\varepsilon > 0$, there exists $C_{T,\varepsilon} > 0$, independent of k, such that

$$\|q_k\|_{L^2(0,T)} \leqslant C_{T,\varepsilon} e^{(\pi+\varepsilon)k},\ \forall k \geqslant 1. \tag{82}$$

Remark 7. When $\Omega := (a,b)$ with $a, b \in \mathbb{R}$, the inequality (82) of Lemma 5.2 is replaced by

$$\|q_k\|_{L^2(0,T)} \leqslant C_{T,\varepsilon} e^{(b-a+\varepsilon)k},\ \forall k \geqslant 1.$$

Proof of the second point in Theorem 1.3. As mentioned above, let us look for the control u of the form $u(x,t) = f(x)\gamma(t)$, where f is as in Lemma 5.1. Since $f_k \neq 0$ for all $k \in \mathbb{N}^*$, using (81), the Π_1-null controllability of System (76) is reduced to find a solution $\gamma \in L^2(0,T)$ to the following problem of moments:

$$\int_0^T \gamma(T-t)e^{-k^2 t}\,dt = f_k^{-1}\left(-e^{-k^2T}y_k^0 - \sum_{l\geqslant 1}\frac{e^{-k^2T}-e^{-l^2T}}{-k^2+l^2}\alpha_{kl}z_l^0\right) := M_k \ \forall k \geqslant 0. \tag{83}$$

The function $\gamma(t) := \sum_{k\geqslant 1} M_k q_k(T-t)$ is a solution to this problem of moments. We need only to prove that $\gamma \in L^2(0,T)$. Using the convexity of the exponential function, we get for all $k \in \mathbb{N}^*$,

$$\begin{aligned}\sum_{l\geqslant 1}\left|\frac{e^{-k^2T}-e^{-l^2T}}{-k^2+l^2}\right||\alpha_{kl}| &= \sum_{l=1}^{k}\left|\frac{e^{-k^2T}-e^{-l^2T}}{-k^2+l^2}\right||\alpha_{kl}| \\ &\quad + \sum_{l=k+1}^{\infty}\left|\frac{e^{-k^2T}-e^{-l^2T}}{-k^2+l^2}\right||\alpha_{kl}| \\ &\leqslant \sum_{l=1}^{k} Te^{-l^2T}|\alpha_{kl}| + \sum_{l=k+1}^{\infty} Te^{-k^2T}|\alpha_{kl}| \\ &=: A_{1,k} + A_{2,k}.\end{aligned} \tag{84}$$

With the Condition (13) on α, there exists a positive constant C_T which do not depend on k such that for all $k \in \mathbb{N}^*$

$$\begin{aligned}A_{1,k} &\leqslant C_1 T\sum_{l=1}^{k} e^{-l^2T}e^{-C_2(k-l)} \\ &\leqslant C_1 Te^{-C_2k}\sum_{l=1}^{\infty} e^{-l^2T+C_2l} \\ &\leqslant C_T e^{-C_2k}\end{aligned} \tag{85}$$

and

$$\begin{aligned}A_{2,k} &\leqslant C_1 Te^{-k^2T}\sum_{l=k+1}^{\infty} e^{-C_2(l-k)} \\ &\leqslant C_1 Te^{-k^2T}\sum_{j=0}^{\infty}(e^{-C_2})^j \\ &\leqslant C_1 Te^{-k^2T}\frac{1}{1-e^{-C_2}}.\end{aligned} \tag{86}$$

Combining the three last inequalities (84)-(86), for all $k \in \mathbb{N}^*$

$$\sum_{l\geqslant 1}\left|\frac{e^{-k^2T}-e^{-l^2T}}{-k^2+l^2}\right||\alpha_{kl}| \leqslant C_T e^{-C_2k}, \tag{87}$$

where C_T is a positive constant independent of k. Let $\varepsilon \in (0,1)$. Then, with Lemma 5.1, (83) and (87), there exists a positive constant $C_{T,\varepsilon}$ independent of k such that for all $k \in \mathbb{N}^*$

$$\begin{aligned}|M_k| &\leqslant \beta^{-1}k^3\left(e^{-k^2T}\|y_0\|_{L^2(0,\pi)} + C_T e^{-C_2k}\|z_0\|_{L^2(0,\pi)}\right) \\ &\leqslant C_{T,\varepsilon}e^{-C_2(1-\varepsilon)k}(\|y_0\|_{L^2(0,\pi)} + \|z_0\|_{L^2(0,\pi)}).\end{aligned}$$

Thus, using Lemma 5.2, for ε small enough and a positive constant $C_{T,\varepsilon}$

$$\|\gamma\|_{L^2(0,T)} \leqslant C_{T,\varepsilon}\Big(\sum_{k\in\mathbb{N}^*} e^{-[C_2(1-\varepsilon)-\pi+\varepsilon]k}\Big)(\|y_0\|_{L^2(0,\pi)} + \|z_0\|_{L^2(0,\pi)}) < \infty.$$

□

5.2. **Example of non controllability.** In this subsection, to provide an example of non Π_1-null controllability of System (76), we will first study the boundary controllability of the following parabolic system of two equations

$$\begin{cases} \partial_t y = \Delta y + \alpha z & \text{in } Q_T := (0,\pi)\times(0,T), \\ \partial_t z = \Delta z & \text{in } Q_T, \\ y(0,t) = v(t),\ y(\pi,t) = z(0,t) = z(\pi,t) = 0 & \text{on } (0,T), \\ y(x,0) = y_0(x),\ z(x,0) = z_0(x) & \text{in } \Omega := (0,\pi), \end{cases} \tag{88}$$

where $y_0,\ z_0 \in H^{-1}(0,\pi)$ are the initial data, $v \in L^2(0,T)$ is the boundary control and $\alpha \in L^\infty(0,\pi)$. For any given $y_0, z_0 \in H^{-1}(0,\pi)$ and $v \in L^2(0,T)$, System (88) has a unique solution in $L^2(Q_T)^2 \cap \mathcal{C}^0([0,T]; H^{-1}(\Omega)^2)$ (defined by transposition; see [14]).

As in Section 5.1, for an initial data $(y_0, z_0) \in H^{-1}(0,\pi)^2$ we can find a control $v \in L^2(0,T)$ such that the solution to (88) satisfies $y(T) \equiv 0$ in $(0,\pi)$ if and only if for all $\phi_0 \in H_0^1(0,\pi)$ the solution to System (77) verifies the equality

$$-\langle y_0, \phi(0)\rangle_{H^{-1},H_0^1} - \langle z_0, \psi(0)\rangle_{H^{-1},H_0^1} = \int_0^T v(t)\phi_x(0,t)\,dt, \tag{89}$$

where the duality bracket $\langle\cdot,\cdot\rangle_{H^{-1},H_0^1}$ is defined as $\langle f, g\rangle_{H^{-1},H_0^1} := f(g)$ for all $f \in H^{-1}(0,\pi)$ and all $g \in H_0^1(0,\pi)$.

The used strategy here is inspired from [20]. The idea involves constructing particular initial data for adjoint System (77):

Lemma 5.3. *Let $m, G \in \mathbb{N}^*$. For all $M \in \mathbb{N}\backslash\{0,1\}$, there exists $\phi_{0,M} \in L^2(0,\pi)$ given by*

$$\phi_{0,M} = \sum_{i=1}^{m} \phi_{GM+i}^{0,M} w_{GM+i},$$

with $\phi_{GM+1}^{0,M}, ..., \phi_{GM+m}^{0,M} \in \mathbb{R}$, such that the solution (ϕ_M, ψ_M) to adjoint System (77) with $\phi_0 = \phi_{0,M}$ satisfies

$$\left(\int_0^T (\phi_M)_x(0,t)^2\,dt\right)^{1/2} \leqslant \frac{\gamma_1}{M^{(2m-5)/2}}, \tag{90}$$

where γ_1 does not depend on M. Morover for an increasing sequence $(M_j)_{j\in\mathbb{N}} \subset \mathbb{N}\backslash\{0,1\}$ and a $k_1 \in \{1,...,m\}$, we have $|\phi_{GM_j+k_1}^{0,j}| = 1$ for all $G \in \mathbb{N}^*$ and $j \in \mathbb{N}$.

To study the controllability of System (88) we will use the fact that for fixed $m, G \in \mathbb{N}^*$, the quantity in the left-side hand in (90) converge to zero when M goes to infinity.

Proof. We remark first that

$$A_M := \int_0^T (\phi_M)_x(0,t)^2\,dt = \int_0^T \left|\sum_{k=GM+1}^{GM+m} k e^{-k^2(T-t)}\phi_k^{0,M}\right|^2 dt. \tag{91}$$

We can rewrite A_M as follows:

$$\begin{aligned} A_M &= \int_0^T \left| \sum_{j=1}^m (GM+j) e^{-(G^2M^2+2GMj+j^2)(T-t)} \phi_{GM+j}^{0,M} \right|^2 dt \\ &= \int_0^T e^{-2G^2M^2(T-t)} g_M(t)\, dt, \end{aligned} \tag{92}$$

where, for all $t \in [0,T]$, $g_M(t) := f_M(t)^2$ with

$$f_M(t) := \sum_{j=1}^m (GM+j) e^{-(2GMj+j^2)(T-t)} \phi_{GM+j}^{0,M}.$$

Let $(\phi_{GM+1}^{0,M}, \phi_{GM+2}^{0,M}, ..., \phi_{GM+m}^{0,M})$ be a nontrivial solution of the following homogeneous linear system of $m-1$ equations with m unknowns

$$f_M^{(l)}(T) = \sum_{j=1}^m (GM+j)(2GMj+j^2)^l \phi_{GM+j}^{0,M} = 0, \text{ for all } \ l \in \{0, ..., m-2\}. \tag{93}$$

Using Leibniz formula

$$g_M^{(l)} = \sum_{k=0}^l \left(\begin{array}{c} l \\ k \end{array} \right) f_M^{(k)} f_M^{(l-k)}$$

we deduce that

$$g_M^{(l)}(T) = 0, \text{ for all } \ l \in \{0, ..., 2m-4\}. \tag{94}$$

Using (94), after $2m-3$ integrations by part in (92), we obtain

$$\begin{aligned} A_M &= \frac{-g_M(0)e^{-2G^2M^2T}}{2G^2M^2} + \int_0^T \frac{e^{-2G^2M^2(T-t)}}{(-2G^2M^2)} g_M^{(1)}(t) \mathrm{d}t \\ &= \sum_{l=0}^{2m-4} \frac{g_M^{(l)}(0)e^{-2G^2M^2T}}{(-2G^2M^2)^{l+1}} + \int_0^T \frac{e^{-2G^2M^2(T-t)}}{(-2G^2M^2)^{2m-3}} g_M^{(2m-3)}(t)\, dt. \end{aligned}$$

By linearity, in (93) we can choose $\phi_{GM+1}^{0,M}, \ ..., \phi_{GM+m}^{0,M}$ such that

$$\sup_{i \in \{1,...,m\}} |\phi_{GM+i}^{0,M}| = 1. \tag{95}$$

Thus, for all $l \in \mathbb{N}$ and all $t \in [0,T]$, the following estimate holds

$$\begin{aligned} |g_M^{(l)}(t)| &= \left| \sum_{k=0}^l \left(\begin{array}{c} l \\ k \end{array} \right) f_M^{(k)}(t) f_M^{(l-k)}(t) \right| \\ &\leqslant \sum_{k=0}^l \left(\begin{array}{c} l \\ k \end{array} \right) \left| \sum_{j=1}^m (GM+j)(2GMj+j^2)^k e^{-(2GMj+j^2)(T-t)} \phi_{GM+j}^{0,M} \right| \\ &\qquad \times \left| \sum_{j=1}^m (GM+j)(2GMj+j^2)^{l-k} e^{-(2GMj+j^2)(T-t)} \phi_{GM+j}^{0,M} \right| \\ &\leqslant (GM+m)^2 m^2 \sum_{k=0}^l \left(\begin{array}{c} l \\ k \end{array} \right) (2GMm+m^2)^l \\ &\leqslant CM^{l+2}, \end{aligned}$$

where C does not depend on M. Then, since $\sup_{i\in\{1,\dots,m\}} |\phi^{0,M}_{GM+i}| = 1$, there exist $C,\tau>0$ such that

$$\begin{aligned} A_M &\leqslant e^{-2G^2M^2T}\sum_{l=0}^{2m-4}\frac{\|g_M^{(l)}\|_\infty}{(2G^2M^2)^{l+1}} + \frac{T\|g_M^{(2m-3)}\|_\infty}{(2G^2M^2)^{2m-3}} \\ &\leqslant e^{-\tau M^2}\sum_{l=0}^{\infty}\frac{C}{M^l} + \frac{C}{M^{2m-5}} \\ &\leqslant CM^{-2}e^{-\tau M^2}\frac{1}{1-M^{-2}} + \frac{C}{M^{2m-5}}. \end{aligned}$$

Thus there exists $\gamma_1>0$ such that we have the estimate

$$A_M \leqslant \frac{\gamma_1}{M^{2m-5}},$$

where γ_1 does not depend on M. Using (104), for all $M\geqslant 2$, there exists $k_1(M)\in\{1,...,7\}$, such that $|\phi^{0,M}_{15M+k_1(M)}|=1$. Thus there exists an increasing sequence $(M_j)_{j\in\mathbb{N}^*}$ such that $|\phi^{0,M_j}_{15M_j+k_1}|=1$ for a $k_1\in\{1,...,m\}$ independent of j. □

Theorem 5.4. *Let $T>0$ and α be the function of $L^\infty(0,\pi)$ defined by*

$$\alpha(x) := \sum_{j=1}^{\infty}\frac{1}{j^2}\cos(15jx) \text{ for all } x\in(0,\pi). \tag{96}$$

Then there exists $k_1\in\{1,..,7\}$ such that for $(y_0,z_0):=(0,w_{k_1})$ and all control $v\in L^2(0,T)$, the solution to System (88) verifies $y(T)\not\equiv 0$ in $(0,\pi)$.

Proof. To understand why the number ≪15≫ appears in the definition (96) of the function α, we will consider for all $x\in(0,\pi)$

$$\alpha(x) := \sum_{j=1}^{\infty}\frac{1}{j^2}\cos(Gjx) \text{ for all } x\in(0,\pi), \tag{97}$$

where $G\in\mathbb{N}^*$. We recall that for an initial condition $(y_0,z_0)\in L^2(0,\pi)^2$ and a control $v\in L^2(0,T)$, the solution to System (96) satisfies $y(T)\equiv 0$ in $(0,\pi)$ if and only if for all $\phi_0\in L^2(0,\pi)$, we have the equality

$$-\langle y_0,\phi(0)\rangle_{H^{-1},H_0^1} - \langle z_0,\psi(0)\rangle_{H^{-1},H_0^1} = \int_0^T v(t)\phi_x(0,t)\ dt, \tag{98}$$

where (ϕ,ψ) is the solution to the adjoint System (77). Let us consider the sequences $(M_j)_{j\in\mathbb{N}^*}$ and $(\phi_{0,M_j})_{j\in\mathbb{N}}$, k_1 defined in Lemma 5.3 and (ϕ_{M_j},ψ_{M_j}) the solution to

$$\begin{cases} -\partial_t\phi_{M_j} = \Delta\phi_{M_j} & \text{in } (0,\pi)\times(0,T), \\ -\partial_t\psi_{M_j} = \Delta\psi_{M_j} + \alpha\phi_{M_j} & \text{in } (0,\pi)\times(0,T), \\ \phi_{M_j}(0)=\phi_{M_j}(\pi)=\psi_{M_j}(0)=\psi_{M_j}(\pi)=0 & \text{on } (0,T), \\ \phi_{M_j}(T)=\phi_{0,M_j},\ \psi_{M_j}(T)=0 & \text{in } (0,\pi). \end{cases}$$

The goal is to prove that for the initial data $(y_0,z_0):=(0,w_{k_1})$ and ϕ_{0,M_j} for j large enough, the equality (98) does not holds. Using Lemma 5.3, we have

$$\left|\int_0^T v(t)(\phi_{M_j})_x(0,t)\ dt\right| \leqslant \frac{\gamma_1\|v\|_{L^2(q_T)}}{M_j^{(2m-5)/2}}. \tag{99}$$

Since $y_0=0$, we obtain

$$\langle y_0,\phi_{M_j}(0)\rangle_{H^{-1},H_0^1} = 0. \tag{100}$$

Let us now estimate the term $\langle z_0, \psi_{M_j}(0)\rangle_{H^{-1},H_0^1}$ in the equality (98). We recall that the expression of α is given in (97). Then, the function α is of the form $\alpha(x) = \sum_{p=0}^{\infty} \alpha_p \cos(px)$ for all $x \in (0,\pi)$, with

$$\alpha_p := \begin{cases} \frac{1}{i^2} & \text{if } p = Gi \text{ with } i \in \mathbb{N}^*, \\ 0 & \text{otherwise.} \end{cases} \tag{101}$$

From the definition of α_{kl} in (2), there holds for all $k, l \in \mathbb{N}^*$

$$\alpha_{kl} \quad = \quad \tfrac{1}{\pi}(\alpha_{|k-l|} - \alpha_{k+l}).$$

Let $k \in \{1, ..., m\}$ and $l \in \{GM_j + 1, ..., GM_j + m\}$. We have

$$k + l \in \{GM_j + 2, ..., GM_j + 2m\}.$$

Thus if we choose

$$G \geqslant 2m + 1, \tag{102}$$

using (101), we obtain

$$\alpha_{k+l} = 0$$

and

$$\alpha_{|k-l|} = \begin{cases} \dfrac{1}{{M_j}^2} & \text{if } |k-l| = GM_j, \\ 0 & \text{otherwise.} \end{cases}$$

So that we have the following submatrix of $(\alpha_{kl})_{1\leqslant k,l\leqslant GM+m}$:

$$(\alpha_{kl})_{1\leqslant k\leqslant m, GM_j+1\leqslant l\leqslant GM_j+m} = \frac{1}{\pi {M_j}^2} I_{\mathbb{R}^m}. \tag{103}$$

According to Lemma 5.3, there exists $k_1 \in \{1, ..., m\}$ such that

$$|\phi^{0,M_j}_{GM_j+k_1}| = 1. \tag{104}$$

Furthermore, since $k_1 \in \{1, ..., m\}$,

$$|e^{-k_1^2 T} - e^{-(GM_j+k_1)^2 T}| \geqslant |e^{-m^2 T} - e^{-G^2 {M_j}^2 T}| \tag{105}$$

and

$$(GM_j + k_1)^2 - k_1^2 \leqslant (GM_j + m)^2 - 1. \tag{106}$$

Since $z_0 = w_{k_1}$, the equality (103) leads to

$$\begin{aligned} \left| \int_0^\pi z_0 \psi_{M_j}(0)\, dx \right| &= \left| \sum_{s=1}^{7} \frac{e^{-k_1^2 T} - e^{-(GM_j+s)2T}}{-k_1^2 + (GM_j+s)^2} \alpha_{k_1, GM_j+s} \phi^{0,M_j}_{GM_j+s} \right| \\ &= \left| \frac{e^{-k_1^2 T} - e^{-(GM_j+k_1)^2 T}}{-k_1^2 + (GM_j+k_1)^2} \frac{1}{\pi {M_j}^2} \right|. \end{aligned}$$

Then using (105) and (106) for all $j \in \mathbb{N}^*$

$$\left| \langle z_0, \psi_{M_j}(0)\rangle_{H^{-1},H_0^1} \right| = \left| \int_0^\pi z_0 \psi_{M_j}(0)\, dx \right| \geqslant \frac{\gamma_2}{{M_j}^4}, \tag{107}$$

where γ_2 does not depend on j. Combining (99) and (107), we obtain a contradiction with equality (98). Thus, for this initial condition y_0 and z_0, we can not find a control $v \in L^2(0,T)$ such that the solution (y,z) to system (96) satisfies $y(T) \equiv 0$ in $(0,\pi)$.

□

Proof of the third point in Theorem 1.3. Using Theorem 5.4, for the initial data $(p_0, q_0) := (0, w_{k_1}) \in L^2(0,\pi)^2$ and all control $v \in L^2(0,T)$, the solution $(p,q) \in W(0,T)^2$ (defined by transposition) to the system

$$\begin{cases} \partial_t p = \Delta p + \alpha q & \text{in } (0,\pi)\times(0,T), \\ \partial_t q = \Delta q & \text{in } (0,\pi)\times(0,T), \\ p(\pi,\cdot) = v,\ p(0,\cdot) = q(0,\cdot) = q(\pi,\cdot) = 0 & \text{on } (0,T), \\ p(\cdot,0) = p_0,\ q(\cdot,0) = q_0 & \text{in } (0,\pi) \end{cases} \tag{108}$$

satisfies $p(T) \not\equiv 0$ in $(0,\pi)$. Consider now $\overline{p}_0$, $\overline{q}_0 \in L^2(0,2\pi)$ defined by

$$\overline{p}_0(x) = 0 \quad \text{and} \quad \overline{q}_0(x) = \sqrt{\frac{2}{\pi}}\sin(k_1 x) \quad \text{for all } x \in (0,2\pi).$$

Remark that $(\overline{p}_{0|(0,\pi)}, \overline{q}_{0|(0,\pi)}) = (p_0, q_0)$. Let $\omega \subset (0,\pi)$. Suppose now that the system

For given $(y_0, z_0) : (0,2\pi) \to \mathbb{R}^2,\ u : (0,2\pi)\times(0,T) \to \mathbb{R}$,
Find $(y,z) : (0,2\pi)\times(0,T) \to \mathbb{R}^2$ such that

$$\begin{cases} \partial_t y = \Delta y + \alpha z + \mathbb{1}_\omega u & \text{in } (0,2\pi)\times(0,T), \\ \partial_t z = \Delta z & \text{in } (0,2\pi)\times(0,T), \\ y(0,\cdot) = y(2\pi,\cdot) = z(0,\cdot) = z(2\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0,\ z(\cdot,0) = z_0 & \text{in } (0,2\pi) \end{cases} \tag{109}$$

is Π_1-null controllable, more particularly for the initial conditions $y(0) = \overline{p}_0$ and $z(0) = \overline{q}_0$ in $(0,2\pi)$, there exists a control u in $L^2((0,2\pi)\times(0,T))$ such that the solution (y,z) to System (109) satisfies $y(T) \equiv 0$ in $(0,2\pi)$. We remark now that $(p,q) := (y|_{(0,\pi)}, z|_{(0,\pi)})$ is a solution of (108) with $(p(0),q(0)) = (p_0,q_0)$ in $(0,\pi)$, $v(t) = y(\pi,t)$ in $(0,T)$ and satisfying $p(T) \equiv 0$ in $(0,\pi)$. This contradicts that for any control $v \in L^2(0,T)$ the solution (p,q) to System (108) can not be identically equal to zero at time T. □

5.3. **Numerical illustration.** In this section, we illustrate numerically the results obtained previously in Sections 5.1 and 5.2. We adapt the HUM method to our control problem. For all penalty parameter $\varepsilon > 0$, we compute the control that minimizes the penalized HUM functional F_ε given by

$$F_\varepsilon(u) := \frac{1}{2}\|u\|^2_{L^2(\omega\times(0,T))} + \frac{1}{2\varepsilon}\|y(T;y_0,u)\|^2_{L^2(\Omega)},$$

where y is the solution to (76). We can find in [8] the argument relating the null/approximate controllability and this kind of functional. Using the Fenchel-Rockafellar theory (see [12] p. 59) we know that the minimum of F_ε is equal to the opposite of the minimum of J_ε, the so-called dual functional, defined for all $\varphi_0 \in L^2(\Omega)$ by

$$J_\varepsilon(\varphi_0) \quad := \quad \frac{1}{2}\|\varphi\|^2_{L^2(q_T)} + \frac{\varepsilon}{2}\|\varphi_0\|^2_{L^2(Q_T)} + \langle y(T;y_0,0),\varphi_0\rangle_{L^2(\Omega)},$$

where φ is the solution to the backward System (110). Moreover the minimizers u_ε and $\varphi_{0,\varepsilon}$ of the functionals F_ε and J_ε respectively, are related through the equality $u_\varepsilon = \mathbb{1}_\omega\varphi_\varepsilon$, where φ_ε is the solution to the backward System (110) with the initial data $\varphi(T) = \varphi_{0,\varepsilon}$. A simple computation leads to

$$\nabla J_\varepsilon(\varphi_0) = \Lambda\varphi_0 + \varepsilon\varphi_0 + y(T;y_0,0),$$

with the Gramiam operator Λ defined as follows

$$\begin{array}{rccc} \Lambda: & L^2(\Omega) & \mapsto & L^2(\Omega), \\ & \varphi_0 & \to & w(T), \end{array}$$

where w is the solution to the following backward and forward systems

$$\begin{cases} -\partial_t \varphi = \Delta \varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(T) = \varphi_0 & \text{in } \Omega \end{cases} \tag{110}$$

and

$$\begin{cases} \partial_t w = \Delta w + \mathbb{1}_\omega \varphi & \text{in } Q_T, \\ w = 0, & \text{on } \Sigma_T, \\ w(0) = 0 & \text{in } \Omega. \end{cases} \tag{111}$$

Then the minimizer u_ε of F_ε will be computed with the help of the minimizer $\varphi_{0,\varepsilon}$ of J_ε which is the solution to the linear problem

$$(\Lambda + \varepsilon)\varphi_{0,\varepsilon} = -y(T; y_0, 0).$$

Remark 8. The proof of Theorem 1.7 in [8] can be adapted to prove that

(i) System (76) is Π_1-null controllable if and only if $\sup_{\varepsilon>0} \left(\inf_{L^2(\omega\times(0,T))} F_\varepsilon \right) < \infty$,

(ii) System (76) is Π_1-approximately controllable if and only if $y_\varepsilon(T) \underset{\varepsilon\to 0}{\longrightarrow} 0$,

where y_ε is the solution to System (76) for the control u_ε.

System (76) with $T = 0.005$, $\Omega := (0, 2\pi)$, $\omega := (0, \pi)$ and $y_0 := 100 \sin(x)$ has been considered. We take the two expressions below for the coupling coefficient α that correspond respectively to Cases (1)-(2) and (3) in Theorem 1.3:

(a) $\alpha(x) = 1$,

(b) $\alpha(x) = \sum_{p\geqslant 0} \frac{1}{p^2} cos(15px)$.

Systems (76) and (110)-(111) are discretized with backward Euler time-marching scheme (time step $\delta t = 1/400$) and standard piecewise linear Lagrange finite elements on a uniform mesh of size h successively equal to $2\pi/50$, $2\pi/100$, $2\pi/200$ and $2\pi/300$. We follow the methodology of F. Boyer (see [8]) that introduces a penalty parameter $\varepsilon = \phi(h) := h^4$. We denote by E_h, U_h and $L^2_{\delta t}(0, T; U_h)$ the fully-discretized spaces associated to $L^2(\Omega)$, $L^2(\omega)$ and $L^2(q_T)$. $F_\varepsilon^{h,\delta t}$ is the discretization of F_ε and $(y_\varepsilon^{h,\delta t}, z_\varepsilon^{h,\delta t}, u_\varepsilon^{h,\delta t})$ the solution to the corresponding fully-discrete problem of minimisation. For more details on the fully-discretization of System (76) and Gramiam Λ (used to the minimisation of F_ϵ), we refer to Section 3 in [8] and in [18, p. 37] respectively. The results are depicted Figure 1 and 2.

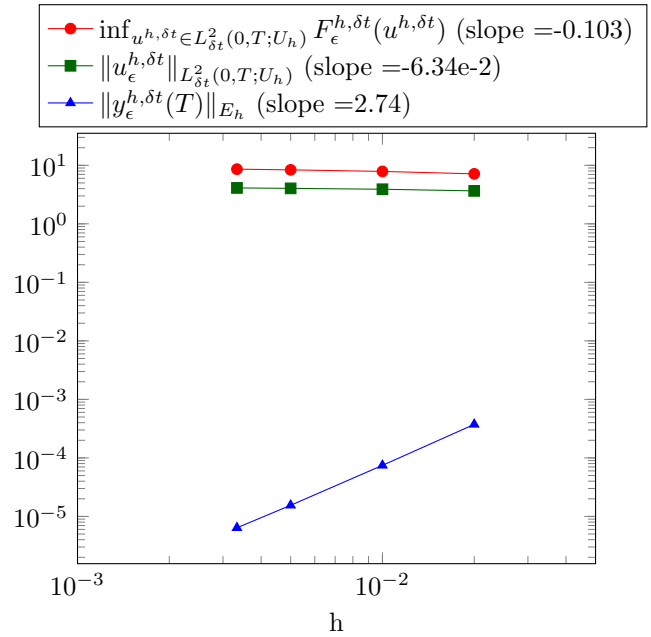

FIGURE 1. Minimal value of the functional $F^{h,\delta t}_\epsilon$ in $L^2_{\delta t}(0,T;U_h)$, norm of the control $\|u^{h,\delta t}_\epsilon\|_{L^2_{\delta t}(0,T;U_h)}$, and distance to the target $\|y^{h,\delta t}_\epsilon(T)\|_{E_h}$ in Case (a).

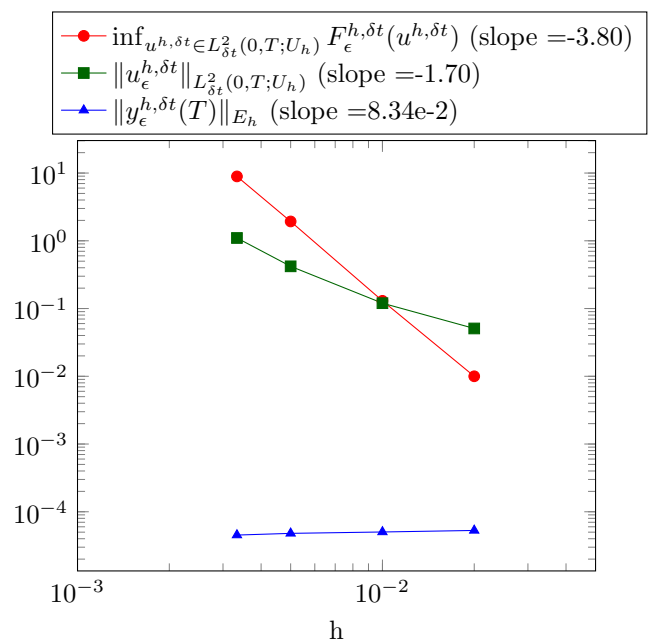

FIGURE 2. Minimal value of the functional $F^{h,\delta t}_\epsilon$ in $L^2_{\delta t}(0,T;U_h)$, norm of the control $\|u^{h,\delta t}_\epsilon\|_{L^2_{\delta t}(0,T;U_h)}$, and distance to the target $\|y^{h,\delta t}_\epsilon(T)\|_{E_h}$ in Case (b).

As mentioned in the introduction of the present article (see Theorem 1.3), in both situations (a) and (b), System (76) is Π_1-approximately controllable and we observe indeed in Figure 1 and 2 that the norm of the numerical solution to System (76) at time T ($-\blacktriangle-$) is decreasing when reducing the penality parameter $\varepsilon = h^4$.

In Figure 1, the minimal value of the functional $F_\varepsilon^{h,\delta t}$ ($-\bullet-$) as well as the L^2-norm of the control $u_\varepsilon^{h,\delta t}$ ($-\blacksquare-$) remain roughly constant whatever is the value of h (and $\varepsilon = h^4$). This appears in agreement with the results (1)-(2) of Theorem 1.3, that state the Π_1-null controllability of System (76) in Case (a) of a constant coupling coefficient α (see Remark 8 (i)). Furthermore the convergence to the null target is approximately of order 2 (slope of 2.27). This is in agreement with the convergence rate established in [8, Proposition 2.2], which should be h^2 for $\varepsilon = h^4$ (this result should be in fact slightly adapted to consider Π_1-null controllability).

At the opposite, in Figure 2, the minimal value of the functional $F_\varepsilon^{h,\delta t}$ as well as the L^2-norm of the control $u_\varepsilon^{h,\delta t}$ are strongly increasing whenever h (and ε) become smaller. This coincides with point (3) of Theorem 1.3: for the chosen value of the coupling coefficient α in Case (b), no Π_1-null controllability of System (76) is expected. Moreover, convergence to the null target is quite slow, with a slope of approximately $8.34e-2$.

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