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# Partial null controllability of parabolic linear systems by $m$ forces

Farid AMMAR KHODJA\*    Franz CHOULY\*    Michel DUPREZ\*<sup>†</sup>

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## Abstract

This paper is devoted to the partial null controllability issue of parabolic linear systems with  $n$  equations. Given a bounded domain  $\Omega$  in  $\mathbb{R}^N$  ( $N \in \mathbb{N}^*$ ), we study the effect of  $m$  localized controls in a nonempty open subset  $\omega$  only controlling  $p$  components of the solution ( $p, m \leq n$ ). The first main result of this paper is a necessary and sufficient condition when the coupling and control matrices are constant. The second result provides, in a first step, a sufficient condition of partial null controllability when the matrices only depend on time. In a second step, through an example of partially controlled  $2 \times 2$  parabolic system, we will give positive and negative results on partial null controllability when the coefficients are space dependent.

## 1 Introduction and main results

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ ,  $N \in \mathbb{N}^*$  with a  $\mathcal{C}^2$ -class boundary  $\partial\Omega$ ,  $\omega$  be a nonempty open subset of  $\Omega$  and  $T > 0$ . Let  $p, m, n \in \mathbb{N}^*$  such that  $p, m \leq n$ .

We consider in this paper the following system of  $n$  parabolic linear equations

$$\begin{cases} \partial_t y = \Delta y + Ay + B\mathbf{1}_\omega u & \text{in } Q_T := \Omega \times (0, T), \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0, T), \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where  $y_0 \in L^2(\Omega)^n$  is the initial data,  $u \in L^2(Q_T)^m$  is the control and for all  $(x, t) \in Q_T$

$$A(x, t) \in \mathcal{L}(\mathbb{R}^n) \text{ and } B(x, t) \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n).$$

In many fields such as chemistry, physics or biology it appeared relevant to study the controllability of such a system (see [4]). For example, in [8], the authors study a system of three semilinear heat equations which is a model coming from a mathematical description of the growth of brain tumors. The unknowns are the drug concentration, the density of tumors cells and the density of wealthy cells and the aim is to control only two of them with one control. This practical issue motivates the introduction of the partial null controllability.

For an initial condition  $y(0) = y_0 \in L^2(\Omega)^n$  and a control  $u \in L^2(Q_T)^m$ , it is well-known that System (1.1) admits a unique solution in  $W(0, T)^n$ , where

$$W(0, T) := \{y \in L^2(0, T; H_0^1(\Omega)), \partial_t y \in L^2(0, T; H^{-1}(\Omega))\},$$

and the following estimate holds (see [4])

$$\|y\|_{L^2(0, T; H_0^1(\Omega)^n)} + \|y\|_{C^0([0, T]; L^2(\Omega)^n)} \leq C(\|y_0\|_{L^2(\Omega)^n} + \|u\|_{L^2(Q_T)^m}), \quad (1.2)$$

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where  $C$  does not depend on time. We denote by  $y(\cdot; y_0, u)$  the solution to System (1.1) determined by the couple  $(y_0, u)$ .

Let us consider  $\Pi_p$  the *projection matrix* defined as follows

$$\begin{aligned} \Pi_p : \quad \mathbb{R}^n &\rightarrow \mathbb{R}^p, \\ (y_1, \dots, y_n) &\mapsto (y_1, \dots, y_p). \end{aligned}$$

System (1.1) is said to be

- **$\Pi_p$ -approximately controllable** on the time interval  $(0, T)$ , if for all real number  $\varepsilon > 0$  and  $y_0, y_T \in L^2(\Omega)^n$  there exists a control  $u \in L^2(Q_T)^m$  such that

$$\|\Pi_p y(T; y_0, u) - \Pi_p y_T\|_{L^2(\Omega)^p} \leq \varepsilon.$$

- **$\Pi_p$ -null controllable** on the time interval  $(0, T)$ , if for all initial condition  $y_0 \in L^2(\Omega)^n$ , there exists a control  $u \in L^2(Q_T)^m$  such that

$$\Pi_p y(T; y_0, u) \equiv 0 \text{ in } \Omega.$$

Before stating our main results, let us recall the few known results about the (full) null controllability of System (1.1). The first of them is about cascade systems (see [16]). The authors prove the null controllability of System (1.1) with the control matrix  $B := e_1$  (the first vector of the canonical basis of  $\mathbb{R}^n$ ) and a coupling matrix  $A$  of the form

$$A := \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \cdots & \alpha_{2,n} \\ 0 & \alpha_{3,2} & \alpha_{3,3} & \cdots & \alpha_{3,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{n,n-1} & \alpha_{n,n} \end{pmatrix}, \quad (1.3)$$

where the coefficients  $\alpha_{i,j}$  are elements of  $L^\infty(Q_T)$  for all  $i, j \in \{1, \dots, n\}$  and satisfy for a positive constant  $C$

$$\alpha_{i+1,i} > C \text{ for all } i \in \{1, \dots, n-1\}.$$

A similar result on parabolic systems with cascade coupling matrices can be found in [1].

The null controllability of parabolic  $3 \times 3$  linear systems with space/time dependent coefficients and non cascade structure is studied in [6] and [18] (see also [16]).

If  $A \in \mathcal{L}(\mathbb{R}^n)$  and  $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$  (the constant case), it has been proved in [3] that System (1.1) is null controllable on the time interval  $(0, T)$  if and only if the following condition holds

$$\text{rank } [A|B] = n, \quad (1.4)$$

where  $[A|B]$ , the so-called *Kalman matrix*, is defined as

$$[A|B] := (B|AB|\dots|A^{n-1}B).$$

For time dependent coupling and control matrices, we need some additional regularity. More precisely, we need to suppose that  $A \in \mathcal{C}^{n-1}([0, T]; \mathcal{L}(\mathbb{R}^n))$  and  $B \in \mathcal{C}^n([0, T]; \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n))$ . In this case, the associated Kalman matrix is defined as follows. Let us define

$$\begin{cases} B_0(t) := B(t), \\ B_i(t) := A(t)B_{i-1}(t) - \partial_t B_{i-1}(t), \quad (1 \leq i \leq n-1) \end{cases}$$

and denote by  $[A|B](\cdot) \in \mathcal{C}^1([0, T]; \mathcal{L}(\mathbb{R}^{nm}; \mathbb{R}^n))$  the matrix function given by

$$[A|B](\cdot) := (B_0(\cdot)|B_1(\cdot)|\dots|B_{n-1}(\cdot)). \quad (1.5)$$

In [2] the authors prove first that, if there exists  $t_0 \in [0, T]$  such that

$$\text{rank } [A|B](t_0) = n, \quad (1.6)$$

then System (1.1) is null controllable on the time interval  $(0, T)$ . Secondly that System (1.1) is null controllable on every interval  $(T_0, T_1)$  with  $0 \leq T_0 < T_1 \leq T$  if and only if there exists a dense subset  $E$  of  $(0, T)$  such that

$$\text{rank } [A|B](t) = n \text{ for every } t \in E. \quad (1.7)$$

In the present paper, the controls are acting on several equations but on one subset  $\omega$  of  $\Omega$ . Concerning the case where the control domains are not identical, we refer to [20].

Our first result is the following:

**THEOREM 1.1.** *Assume that the coupling and control matrices are constant in space and time, i. e.,  $A \in \mathcal{L}(\mathbb{R}^n)$  and  $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ . The condition*

$$\text{rank } \Pi_p[A|B] = p \quad (1.8)$$

*is equivalent to the  $\Pi_p$ -null/approximate controllability on the time interval  $(0, T)$  of System (1.1).*

The Condition (1.8) for  $\Pi_p$ -null controllability reduces to Condition (1.4) whenever  $p = n$ . A second result concerns the non-autonomous case:

**THEOREM 1.2.** *Assume that  $A \in \mathcal{C}^{n-1}([0, T]; \mathcal{L}(\mathbb{R}^n))$  and  $B \in \mathcal{C}^n([0, T]; \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n))$ . If*

$$\text{rank } \Pi_p[A|B](T) = p, \quad (1.9)$$

*then System (1.1) is  $\Pi_p$ -null/approximately controllable on the time interval  $(0, T)$ .*

As told before, under Condition (1.6), System (1.1) is (fully) null controllable. But unlike the case where all the components are controlled, the  $\Pi_p$ -null controllability at a time  $t_0$  smaller than  $T$  does not imply this property on the time interval  $(0, T)$ . This roughly explains Condition (1.9). Furthermore this condition can not be necessary (for a counterexample we refer to [2]).

In the proofs of Theorems 1.1 and 1.2, we will use a result of null controllability for cascade systems (see Section 2) proved in [2, 16] where the authors consider a time-dependent second order elliptic operator  $L(t)$  given by

$$L(t)y(x, t) = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( \alpha_{i,j}(x, t) \frac{\partial y}{\partial x_j}(x, t) \right) + \sum_{i=1}^N b_i(x, t) \frac{\partial y}{\partial x_i}(x, t) + c(x, t)y(x, t), \quad (1.10)$$

with coefficients  $\alpha_{i,j}$  satisfying

$$\begin{cases} \alpha_{i,j} \in W_\infty^1(Q_T), \quad b_i, c \in L^\infty(Q_T) \quad 1 \leq i, j \leq N, \\ \alpha_{i,j}(x, t) = \alpha_{j,i}(x, t) \quad \forall (x, t) \in Q_T, \quad 1 \leq i, j \leq N \end{cases}$$

and the uniform elliptic condition: there exists  $a_0 > 0$  such that

$$\sum_{i,j=1}^N \alpha_{i,j}(x, t) \xi_i \xi_j \geq a_0 |\xi|^2, \quad \forall (x, t) \in Q_T.$$

Theorems 1.1 and 1.2 remain true if we replace  $-\Delta$  by an operator  $L(t)$  in System (1.1).

Now the following question arises: what happens in the case of space and time dependent coefficients? As it will be shown in the following example, the answer seems to be much more tricky. Let us now consider the following parabolic system of two equations

$$\begin{cases} \partial_t y = \Delta y + \alpha z + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t z = \Delta z & \text{in } Q_T, \\ y = z = 0 & \text{on } \Sigma_T, \\ y(0) = y_0, \quad z(0) = z_0 & \text{in } \Omega, \end{cases} \quad (1.11)$$

for given initial data  $y_0, z_0 \in L^2(\Omega)$ , a control  $u \in L^2(Q_T)$  and where the coefficient  $\alpha \in L^\infty(\Omega)$ .

**THEOREM 1.3.** (1) Assume that  $\alpha \in C^1([0, T])$ . Then System (1.11) is  $\Pi_1$ -null controllable for any open set  $\omega \subset \Omega \subset \mathbb{R}^N$  ( $N \in \mathbb{N}^*$ ), that is for all initial conditions  $y_0, z_0 \in L^2(\Omega)$ , there exists a control  $u \in L^2(Q_T)$  such that the solution  $(y, z)$  to System (1.11) satisfies  $y(T) \equiv 0$  in  $\Omega$ .

(2) Let us suppose that  $\alpha \in L^\infty(\Omega)$  with  $\Omega := (a, b) \subset \mathbb{R}$  ( $a, b \in \mathbb{R}$ ) and satisfies

$$\left| \int_{\Omega} \alpha w_k w_l dx \right| \leq C_1 e^{-C_2 |k^2 - l^2|} \text{ for all } k, l \in \mathbb{N}^*, \quad (1.12)$$

where  $C_1, C_2$  are two positive constants and  $(w_k)_{k \geq 1}$  are the  $L^2$ -normalized eigenfunctions of  $-\Delta$  in  $\Omega$  with Dirichlet boundary conditions. Then System (1.11) is  $\Pi_1$ -null controllable for any open set  $\omega \subset \Omega$ .

(3) Assume that  $\Omega := (0, 2\pi)$  and  $\omega \subset (\pi, 2\pi)$ . Let us consider  $\alpha \in L^\infty(0, 2\pi)$  defined by

$$\alpha(x) := \sum_{j=1}^{\infty} \frac{1}{j^2} \cos(15jx) \text{ for all } x \in (0, 2\pi).$$

Then System (1.11) is not  $\Pi_1$ -null controllable. More precisely, there exists  $k_1 \in \{1, \dots, 7\}$  such that for the initial condition  $(y_0, z_0) = (0, \sin(k_1 x))$  and any control  $u \in L^2(Q_T)$  the solution  $y$  to System (1.11) is not identically equal to zero at time  $T$ .

In Theorem 1.3, (1) can be proved with the help of Theorem 1.2. Concerning (2), Condition (1.12) can be rewritten as follows. We suppose that  $\Omega := (0, \pi)$  and consider  $\alpha \in L^\infty(0, \pi)$  and the real sequence  $(\alpha_p)_{p \in \mathbb{N}}$  such that for all  $x \in (0, \pi)$

$$\alpha(x) := \sum_{p=0}^{\infty} \alpha_p \cos(px).$$

Then Condition (1.12) is equivalent to find two positive constants  $C_1, C_2$  such that, for all  $p \in \mathbb{N}$ ,

$$|\alpha_p| \leq C_1 e^{-C_2 p^2}.$$

And, as it will be shown, the proof of (3) in Theorem 1.3 can be adapted in order to get the same conclusion for any  $\alpha \in H^k(0, 2\pi)$  ( $k \in \mathbb{N}^*$ ) defined by

$$\alpha(x) := \sum_{j=1}^{\infty} \frac{1}{j^{k+1}} \cos((2k+13)jx) \text{ for all } x \in (0, 2\pi).$$

Unlike the case with a time dependent coupling matrix, if the coupling matrix depends on space, the notions of  $\Pi_1$ -null and approximate controllability are not necessarily equivalent. Indeed, according to the choice of the coupling function  $\alpha \in L^\infty(\Omega)$ , System (1.1) can be  $\Pi_p$ -null controllable or not. But this system is  $\Pi_1$ -approximately controllable for all  $\alpha \in L^\infty(\Omega)$ :

**THEOREM 1.4.** Let  $\alpha \in L^\infty(Q_T)$ . Then System (1.11) is  $\Pi_1$ -approximately controllable for any open set  $\omega \subset \Omega \subset \mathbb{R}^N$  ( $N \in \mathbb{N}^*$ ), that is for all  $y_0, y_T, z_0 \in L^2(\Omega)$  and all  $\varepsilon > 0$ , there exists a control  $u \in L^2(Q_T)$  such that the solution  $(y, z)$  to System (1.11) satisfies

$$\|y(T) - y_T\|_{L^2(\Omega)} \leq \varepsilon.$$

This result is a direct consequence of the approximate controllability for the heat equation. Indeed System (1.11) is  $\Pi_1$ -approximately controllable (see Proposition 2.1) if and only if for all  $\phi_0 \in L^2(\Omega)$  the solution to the adjoint system

$$\begin{cases} -\partial_t \phi = \Delta \phi & \text{in } Q_T, \\ -\partial_t \psi = \Delta \psi + \alpha \phi & \text{in } Q_T, \\ \phi = \psi = 0 & \text{on } \Sigma_T, \\ \phi(T) = \phi_0, \psi(T) = 0 & \text{in } \Omega \end{cases}$$

satisfies

$$\phi \equiv 0 \text{ in } (0, T) \times \omega \Rightarrow (\phi, \psi) \equiv 0 \text{ in } Q_T.$$

If we assume that, for an initial data  $\phi_0 \in L^2(\Omega)$ , the solution to System (5.2) satisfies  $\phi \equiv 0$  in  $(0, T) \times \omega$ , then using Mizohata uniqueness Theorem in [19],  $\phi \equiv 0$  in  $Q_T$  and consequently  $\psi \equiv 0$  in  $Q_T$ . For another example of parabolic systems for which those notions are not equivalent we refer to [5].

In this paper, the sections are organized as follows. We start with some preliminary results on the null controllability for the cascade systems and on the dual concept associated to the  $\Pi_p$ -null controllability. Theorem 1.1 is proved in a first step with one force i.e.  $B \in \mathbb{R}^n$  in Section 3.1 and in a second step with  $m$  forces in Section 3.2. Section 4 is devoted to proving Theorem 1.2. We consider the situations of the second and third items of Theorem 1.3 in Section 5.1 and 5.2 respectively. This paper ends with some numerical illustrations of  $\Pi_1$ -null controllability and non  $\Pi_1$ -null controllability of System (1.11) in Section 5.3.

## 2 Preliminaries

In this section, we recall a known result about cascade systems and provide a characterization of the  $\Pi_p$ -controllability through the corresponding dual system.

### 2.1 Cascade systems

Some theorems of this paper use the following result of null controllability for the following cascade system of  $n$  equations controlled by  $r$  distributed functions

$$\begin{cases} \partial_t w = \Delta w + Cw + D\mathbf{1}_\omega u & \text{in } Q_T, \\ w = 0 & \text{on } \Sigma_T, \\ w(0) = w_0 & \text{in } \Omega, \end{cases} \quad (2.1)$$

where  $w_0 \in L^2(\Omega)^n$ ,  $u = (u_1, \dots, u_r) \in L^2(Q_T)^r$ , with  $r \in \{1, \dots, n\}$ , and the coupling and control matrices  $C \in \mathcal{C}^0([0, T]; \mathcal{L}(\mathbb{R}^n))$  and  $D \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^n)$  are given by

$$C(t) := \begin{pmatrix} C_{11}(t) & C_{12}(t) & \cdots & C_{1r}(t) \\ 0 & C_{22}(t) & \cdots & C_{2r}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{rr}(t) \end{pmatrix} \quad (2.2)$$

with

$$C_{ii}(t) := \begin{pmatrix} \alpha_{11}^i(t) & \alpha_{12}^i(t) & \alpha_{13}^i(t) & \cdots & \alpha_{1, s_i}^i(t) \\ 1 & \alpha_{22}^i(t) & \alpha_{23}^i(t) & \cdots & \alpha_{2, s_i}^i(t) \\ 0 & 1 & \alpha_{33}^i(t) & \cdots & \alpha_{3, s_i}^i(t) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{s_i, s_i}^i(t) \end{pmatrix},$$

$s_i \in \mathbb{N}$ ,  $\sum_{i=1}^r s_i = n$  and  $D := (e_{S_1} | \dots | e_{S_r})$  with  $S_1 = 1$  and  $S_i = 1 + \sum_{j=1}^{i-1} s_j$ ,  $i \in \{2, \dots, r\}$  ( $e_j$  is the  $j$ -th element of the canonical basis of  $\mathbb{R}^n$ ).

**THEOREM 2.1.** *System (2.1) is null controllable on the time interval  $(0, T)$ , i.e. for all  $w_0 \in L^2(\Omega)^n$  there exists  $u \in L^2(\Omega)^r$  such that the solution  $w$  in  $W(0, T)^n$  to System (2.1) satisfies  $w(T) \equiv 0$  in  $\Omega$ .*

This result can be found in [2] or [16].

## 2.2 Partial null controllability of a parabolic linear system by $m$ forces and adjoint system

It is nowadays well-known that the controllability has a dual concept called *observability* (see for instance [4]). We detail below the observability for the  $\Pi_p$ -controllability.

**PROPOSITION 2.1.** *1. System (1.1) is  $\Pi_p$ -null controllable on the time interval  $(0, T)$  if and only if there exists a constant  $C_{obs} > 0$  such that for all  $\varphi_0 = (\varphi_1^0, \dots, \varphi_p^0) \in L^2(\Omega)^p$  the solution  $\varphi \in W(0, T)^n$  to the adjoint system*

$$\begin{cases} -\partial_t \varphi = \Delta \varphi + A^* \varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot, T) = \Pi_p^* \varphi_0 = (\varphi_1^0, \dots, \varphi_p^0, 0, \dots, 0) & \text{in } \Omega \end{cases} \quad (2.3)$$

*satisfies the observability inequality*

$$\|\varphi(0)\|_{L^2(\Omega)^n}^2 \leq C_{obs} \int_0^T \|B^* \varphi\|_{L^2(\omega)^m}^2 dt. \quad (2.4)$$

*2. System (1.1) is  $\Pi_p$ -approximately controllable on the time interval  $(0, T)$  if and only if for all  $\varphi_0 \in L^2(\Omega)^p$  the solution  $\varphi$  to System (2.3) satisfies*

$$B^* \varphi \equiv 0 \text{ in } (0, T) \times \omega \Rightarrow \varphi \equiv 0 \text{ in } Q_T.$$

*Proof.* For all  $y_0 \in L^2(\Omega)^n$ , and  $u \in L^2(Q_T)^m$ , we denote by  $y(t; y_0, u)$  the solution to System (1.1) at time  $t \in [0, T]$ . For all  $t \in [0, T]$ , let us consider the operators  $S_t$  and  $L_t$  defined as follows

$$\begin{array}{ccc} S_t : L^2(\Omega)^n & \rightarrow & L^2(\Omega)^n \\ y_0 & \mapsto & y(t; y_0, 0) \end{array} \quad \text{and} \quad \begin{array}{ccc} L_t : L^2(Q_T)^m & \rightarrow & L^2(\Omega)^n \\ u & \mapsto & y(t; 0, u). \end{array} \quad (2.5)$$

1. System (1.1) is  $\Pi_p$ -null controllable on the time interval  $(0, T)$  if and only if

$$\begin{aligned} \forall y_0 \in L^2(\Omega)^n, \exists u \in L^2(Q_T)^m \text{ such that} \\ \Pi_p L_T u = -\Pi_p S_T y_0. \end{aligned} \quad (2.6)$$

Problem (2.6) admits a solution if and only if

$$\text{Im } \Pi_p S_T \subset \text{Im } \Pi_p L_T. \quad (2.7)$$

The inclusion (2.7) is equivalent to (see [9], Lemma 2.48 p. 58)

$$\begin{aligned} \exists C > 0 \text{ such that } \forall \varphi_0 \in L^2(\Omega)^p, \\ \|S_T^* \Pi_p^* \varphi_0\|_{L^2(\Omega)^n}^2 \leq C \|L_T^* \Pi_p^* \varphi_0\|_{L^2(Q_T)^m}^2. \end{aligned} \quad (2.8)$$

We note that

$$\begin{array}{ccc} S_T^* \Pi_p^* : L^2(\Omega)^p & \rightarrow & L^2(\Omega)^n \\ \varphi_0 & \mapsto & \varphi(0) \end{array} \quad \text{and} \quad \begin{array}{ccc} L_T^* \Pi_p^* : L^2(\Omega)^p & \rightarrow & L^2(Q_T)^m \\ \varphi_0 & \mapsto & \mathbf{1}_\omega B^* \varphi, \end{array}$$

where  $\varphi \in W(0, T)^n$  is the solution to System (2.3). Indeed, for all  $y_0 \in L^2(\Omega)^n$ ,  $u \in L^2(Q_T)^m$  and  $\varphi_0 \in L^2(\Omega)^p$

$$\begin{aligned} \langle \Pi_p S_T y_0, \varphi_0 \rangle_{L^2(\Omega)^p} &= \langle y(T; y_0, 0), \varphi(T) \rangle_{L^2(\Omega)^n} \\ &= \int_0^T \langle \partial_t y(s; y_0, 0), \varphi(s) \rangle_{L^2(\Omega)^n} ds \\ &\quad + \int_0^T \langle y(s; y_0, 0), \partial_t \varphi(s) \rangle_{L^2(\Omega)^n} ds + \langle y_0, \varphi(0) \rangle_{L^2(\Omega)^n} \\ &= \langle y_0, \varphi(0) \rangle_{L^2(\Omega)^n} \end{aligned} \quad (2.9)$$

and

$$\begin{aligned}
\langle \Pi_p L_T u, \varphi_0 \rangle_{L^2(\Omega)^p} &= \langle y(T; 0, u), \varphi(T) \rangle_{L^2(\Omega)^n} \\
&= \int_0^T \langle \partial_t y(s; 0, u), \varphi(s) \rangle_{L^2(\Omega)^n} ds + \int_0^T \langle y(s; 0, u), \partial_t \varphi(s) \rangle_{L^2(\Omega)^n} ds \\
&= \langle \mathbb{1}_\omega B u, \varphi \rangle_{L^2(Q_T)^n} = \langle u, \mathbb{1}_\omega B^* \varphi \rangle_{L^2(Q_T)^m}.
\end{aligned} \tag{2.10}$$

The inequality (2.8) combined with (2.9)-(2.10) lead to the conclusion.

2. In view of the definition in (2.5) of  $S_T$  and  $L_T$ , System (1.1) is  $\Pi_p$ -approximately controllable on the time interval  $(0, T)$  if and only if

$$\forall (y_0, y_T) \in L^2(\Omega)^n \times L^2(\Omega)^p, \forall \varepsilon > 0, \exists u \in L^2(Q_T)^m \text{ such that } \|\Pi_p L_T u + \Pi_p S_T y_0 - y_T\|_{L^2(\Omega)^p} \leq \varepsilon.$$

This is equivalent to

$$\forall \varepsilon > 0, \forall z_T \in L^2(\Omega)^p, \exists u \in L^2(Q_T)^m \text{ such that } \|\Pi_p L_T u - z_T\|_{L^2(\Omega)^p} \leq \varepsilon.$$

That means

$$\overline{\Pi_p L_T(L^2(Q_T)^m)} = L^2(\Omega)^p.$$

In other words

$$\ker L_T^* \Pi_p^* = \{0\}.$$

Thus System (1.1) is  $\Pi_p$ -approximately controllable on the time interval  $(0, T)$  if and only if for all  $\varphi_0 \in L^2(\Omega)^p$

$$L_T^* \Pi_p^* \varphi_0 = \mathbb{1}_\omega B^* \varphi \equiv 0 \text{ in } Q_T \Rightarrow \varphi \equiv 0 \text{ in } Q_T.$$

□

**Corollary 2.1.** *Let us suppose that for all  $\varphi_0 \in L^2(\Omega)^p$ , the solution  $\varphi$  to the adjoint System (2.3) satisfies the observability inequality (2.4). Then for all initial condition  $y_0 \in L^2(\Omega)^n$ , there exists a control  $u \in L^2(Q_T)^m$  ( $q_T := \omega \times (0, T)$ ) such that the solution  $y$  to System (1.1) satisfies  $\Pi_p y(T) \equiv 0$  in  $\Omega$  and*

$$\|u\|_{L^2(q_T)^m} \leq \sqrt{C_{obs}} \|y_0\|_{L^2(\Omega)^n}. \tag{2.11}$$

The proof is classical and will be omitted (to get (2.11) the method developed by Fursikov and Immanuvilov in [13] could be used).

### 3 Partial null controllability with constant coupling matrices

In this section, we prove Theorem 1.1 in two steps. In subsection 3.1, we begin by studying the case where  $B \in \mathbb{R}^n$  and the general case is considered in subsection 3.2.

Let us consider the system

$$\begin{cases} \partial_t y = \Delta y + Ay + B \mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where  $y_0 \in L^2(\Omega)^n$ ,  $u \in L^2(Q_T)^m$ ,  $A \in \mathcal{L}(\mathbb{R}^n)$  and  $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ .

Let the natural number  $s$  be defined by

$$s := \text{rank } [A|B] \tag{3.2}$$

and  $X \subset \mathbb{R}^n$  be the linear space spanned by the columns of  $[A|B]$ .

All along this section, we will use the lemma below which proof is straightforward.



**Lemma 3.1.** *Let be  $y_0 \in L^2(\Omega)^n$ ,  $u \in L^2(Q_T)^m$  and  $P \in C^1([0, T], \mathcal{L}(\mathbb{R}^n))$  such that  $P(t)$  is invertible for all  $t \in [0, T]$ . Then the change of variable  $w = P^{-1}(t)y$  transforms System (3.1) into the equivalent system*

$$\begin{cases} \partial_t w = \Delta w + C(t)w + D(t)\mathbb{1}_\omega u & \text{in } Q_T, \\ w = 0 & \text{on } \Sigma_T, \\ w(0) = w_0 & \text{in } \Omega, \end{cases} \quad (3.3)$$

with  $w_0 := P^{-1}(0)y_0$ ,  $C(t) := -P^{-1}(t)\partial_t P(t) + P^{-1}(t)AP(t)$  and  $D(t) := P^{-1}(t)B$ . Moreover

$$\Pi_p y(T) \equiv 0 \text{ in } \Omega \Leftrightarrow \Pi_p P(T)w(T) \equiv 0 \text{ in } \Omega.$$

If  $P$  is constant, we have

$$[C|D] = P^{-1}[A|B].$$

### 3.1 One control force

In this subsection, we suppose that  $A \in \mathcal{L}(\mathbb{R}^n)$ ,  $B \in \mathbb{R}^n$  and denote by  $[A|B] := (k_{ij})_{1 \leq i, j \leq n}$  and  $s := \text{rank } [A|B]$ . We begin with the following observation.

**Lemma 3.2.**  $\{B, \dots, A^{s-1}B\}$  is a basis of  $X$ .

*Proof.* If  $s = \text{rank } [A|B] = 1$ , since  $B \neq 0$ , the conclusion of the lemma is clearly true. Let  $s \geq 2$ . Suppose to the contrary that  $\{B, \dots, A^{s-1}B\}$  is not a basis of  $X$ , that is for some  $i \in \{0, \dots, s-2\}$  the family  $\{B, \dots, A^i B\}$  is linearly independent and  $A^{i+1}B \in \text{span}(B, \dots, A^i B)$ . Hence, by induction,  $A^l B \in \text{span}(B, \dots, A^i B)$  for all  $l \in \{i+1, \dots, n-1\}$ . Then  $\text{rank } (B|AB|\dots|A^{n-1}B) = \text{rank } (B|AB|\dots|A^i B) = i+1 < s$ , contradicting with (3.2).  $\square$

*Proof of Theorem 1.1.* Let us remark that

$$\text{rank } \Pi_p [A|B] = \dim \Pi_p [A|B](\mathbb{R}^n) \leq \text{rank } [A|B] = s. \quad (3.4)$$

Lemma 3.2 yields

$$\text{rank } (B|AB|\dots|A^{s-1}B) = \text{rank } [A|B] = s. \quad (3.5)$$

Thus, for all  $l \in \{s, s+1, \dots, n\}$  and  $i \in \{0, \dots, s-1\}$ , there exist  $\alpha_{l,i}$  such that

$$A^l B = \sum_{i=0}^{s-1} \alpha_{l,i} A^i B. \quad (3.6)$$

Since, for all  $l \in \{s, \dots, n\}$ ,  $\Pi_p A^l B = \sum_{i=0}^{s-1} \alpha_{l,i} \Pi_p A^i B$ , then

$$\text{rank } \Pi_p (B|AB|\dots|A^{s-1}B) = \text{rank } \Pi_p [A|B]. \quad (3.7)$$

Let us assume first that condition (1.8) holds. Then, using (3.7), we have

$$\text{rank } \Pi_p (B|AB|\dots|A^{s-1}B) = p. \quad (3.8)$$

Let be  $y_0 \in L^2(\Omega)^n$ . We will study the  $\Pi_p$ -null controllability of System (3.1) according to the values of  $p$  and  $s$ .

*Case 1 :  $p = s$ .* The idea is to find an appropriate change of variable  $P$  to the solution  $y$  to System (3.1). More precisely, we would like the new variable  $w := P^{-1}y$  to be the solution to a cascade system apply Theorem 2.1. So let us define, for all  $t \in [0, T]$ ,

$$P(t) := (B|AB|\dots|A^{s-1}B|P_{s+1}(t)|\dots|P_n(t)), \quad (3.9)$$

where, for all  $l \in \{s+1, \dots, n\}$ ,  $P_l(t)$  is the solution in  $\mathcal{C}^1([0, T])^n$  to the system of ordinary differential equations

$$\begin{cases} \partial_t P_l(t) = AP_l(t) \text{ in } [0, T], \\ P_l(T) = e_l. \end{cases} \quad (3.10)$$

Using (3.9) and (3.10), we can write

$$P(T) = \begin{pmatrix} P_{11} & 0 \\ P_{21} & I_{n-s} \end{pmatrix}, \quad (3.11)$$

where  $P_{11} := \Pi_p(B|AB|\dots|A^{s-1}B) \in \mathcal{L}(\mathbb{R}^s)$ ,  $P_{21} \in \mathcal{L}(\mathbb{R}^s, \mathbb{R}^{n-s})$  and  $I_{n-s}$  is the identity matrix of size  $n-s$ . Using (3.8),  $P_{11}$  is invertible and thus  $P(T)$  also. Furthermore, since  $P(t)$  is an element of  $\mathcal{C}^1([0, T], \mathcal{L}(\mathbb{R}^n))$  continuous in time on the time interval  $[0, T]$ , there exists  $T^* \in [0, T)$  such that  $P(t)$  is invertible for all  $t \in [T^*, T]$ .

Let us suppose first that  $T^* = 0$ . Since  $P(t)$  is an element of  $\mathcal{C}^1([0, T], \mathcal{L}(\mathbb{R}^n))$  and invertible, in view of Lemma 3.1: for a fixed control  $u \in L^2(Q_T)$ ,  $y$  is the solution to System (3.1) if and only if  $w := P(t)^{-1}y$  is the solution to System (3.3) where  $C, D$  are given by

$$C(t) := -P^{-1}(t)\partial_t P(t) + P^{-1}(t)AP(t) \quad \text{and} \quad D(t) := P^{-1}(t)B,$$

for all  $t \in [0, T]$ . Using (3.6) and (3.10), we obtain

$$\begin{cases} -\partial_t P(t) + AP(t) = (AB|\dots|A^s B|0|\dots|0) = P(t) \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} & \text{in } [0, T], \\ P(t)e_1 = B & \text{in } [0, T], \end{cases} \quad (3.12)$$

where

$$C_{11} := \begin{pmatrix} 0 & 0 & 0 & \dots & \alpha_{s,0} \\ 1 & 0 & 0 & \dots & \alpha_{s,1} \\ 0 & 1 & 0 & \dots & \alpha_{s,2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \alpha_{s,s-1} \end{pmatrix} \in \mathcal{L}(\mathbb{R}^s). \quad (3.13)$$

Then

$$C(t) = \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad D(t) = e_1. \quad (3.14)$$

Using Theorem 2.1, there exists  $u \in L^2(Q_T)$  such that the solution to System (3.3) satisfies  $w_1(T) \equiv \dots \equiv w_s(T) \equiv 0$  in  $\Omega$ . Moreover, using (3.11), we have

$$\Pi_s y(T) = (y_1(T), \dots, y_s(T)) = P_{11}(w_1(T), \dots, w_s(T)) \equiv 0 \text{ in } \Omega.$$

If now  $T^* \neq 0$ , let  $\bar{y}$  be the solution in  $W(0, T^*)^n$  to System (3.1) with the initial condition  $\bar{y}(0) = y_0$  in  $\Omega$  and the control  $u \equiv 0$  in  $\Omega \times (0, T^*)$ . We use the same argument as above to prove that System (3.1) is  $\Pi_s$ -null controllable on the time interval  $[T^*, T]$ . Let  $v$  be a control in  $L^2(\Omega \times (T^*, T))$  such that the solution  $z$  in  $W(T^*, T)^n$  to System (3.1) with the initial condition  $z(T^*) = \bar{y}(T^*)$  in  $\Omega$  and the control  $v$  satisfies  $\Pi_s z(T) \equiv 0$  in  $\Omega$ . Thus if we define  $y$  and  $u$  as follows

$$(y, u) := \begin{cases} (\bar{y}, 0) & \text{if } t \in [0, T^*], \\ (z, v) & \text{if } t \in [T^*, T], \end{cases}$$

then, for this control  $u$ ,  $y$  is the solution in  $W(0, T)^n$  to System (3.1). Moreover  $y$  satisfies

$$\Pi_s y(T) \equiv 0 \text{ in } \Omega.$$

*Case 2* :  $p < s$ . In order to use the previous item (i), we would like to apply an appropriate change of variable  $Q$  to the solution  $y$  to System (3.1). If we denote by  $[A|B] =: (k_{ij})_{ij}$ , equalities (3.5) and (3.8) can be rewritten

$$\text{rank} \begin{pmatrix} k_{11} & \cdots & k_{1s} \\ \vdots & & \vdots \\ k_{n1} & \cdots & k_{ns} \end{pmatrix} = s \quad \text{and} \quad \text{rank} \begin{pmatrix} k_{11} & \cdots & k_{1s} \\ \vdots & & \vdots \\ k_{p1} & \cdots & k_{ps} \end{pmatrix} = p.$$

Then there exist distinct natural numbers  $\lambda_{p+1}, \dots, \lambda_s$  such that  $\{\lambda_{p+1}, \dots, \lambda_s\} \subset \{p+1, \dots, n\}$  and

$$\text{rank} \begin{pmatrix} k_{11} & \cdots & k_{1s} \\ \vdots & & \vdots \\ k_{p1} & \cdots & k_{ps} \\ k_{\lambda_{p+1}1} & \cdots & k_{\lambda_{p+1}s} \\ \vdots & & \vdots \\ k_{\lambda_s1} & \cdots & k_{\lambda_s s} \end{pmatrix} = s. \quad (3.15)$$

Let  $Q$  be the matrix defined by

$$Q := (e_1 | \dots | e_p | e_{\lambda_{p+1}} | \dots | e_{\lambda_s})^t,$$

where  $\{\lambda_{s+1}, \dots, \lambda_n\} := \{p+1, \dots, n\} \setminus \{\lambda_{p+1}, \dots, \lambda_s\}$ .  $Q$  is invertible, so taking  $w := P^{-1}y$  with  $P := Q^{-1}$ , for a fixed control  $u$  in  $L^2(Q_T)$ ,  $y$  is solution to System (3.1) if and only if  $w$  is solution to System (3.3) where  $w_0 := Qy_0$ ,  $C := QAQ^{-1} \in \mathcal{L}(\mathbb{R}^n)$  and  $D := QB \in \mathcal{L}(\mathbb{R}; \mathbb{R}^n)$ . Moreover there holds

$$[C|D] = Q[A|B].$$

Thus, equation (3.15) yields

$$\text{rank} \Pi_s [C|D] = \text{rank} \Pi_s Q[A|B] = \text{rank} \begin{pmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & & \vdots \\ k_{p1} & \cdots & k_{pn} \\ k_{\lambda_{p+1}1} & \cdots & k_{\lambda_{p+1}n} \\ \vdots & & \vdots \\ k_{\lambda_s1} & \cdots & k_{\lambda_s n} \end{pmatrix} = s.$$

Since  $\text{rank} [C|D] = \text{rank} [A|B] = s$ , we proceed as in Case 1 forward deduce that System (3.3) is  $\Pi_s$ -null controllable, that is there exists a control  $u \in L^2(Q_T)$  such that the solution  $w$  to System (3.3) satisfies

$$\Pi_s w(T) \equiv 0 \text{ in } \Omega.$$

Moreover the matrix  $Q$  can be rewritten

$$Q = \begin{pmatrix} I_p & 0 \\ 0 & Q_{22} \end{pmatrix},$$

where  $Q_{22} \in \mathcal{L}(\mathbb{R}^{n-p})$ . Thus

$$\Pi_p y(T) = \Pi_p Q y(T) = \Pi_p w(T) \equiv 0 \text{ in } \Omega.$$

Let us denote by  $[A|B] =: (k_{ij})_{ij}$ . We suppose now that (1.8) is not satisfied: there exist  $\bar{p} \in \{1, \dots, p\}$  and  $\beta_i$  for all  $i \in \{1, \dots, p\} \setminus \{\bar{p}\}$  such that  $k_{\bar{p}j} = \sum_{i=1, i \neq \bar{p}}^p \beta_i k_{ij}$  for all  $j \in \{1, \dots, s\}$ . The idea is to find a change of variable  $w := Qy$  that allows to handle more easily our system. We will achieve this in two steps starting from the simplest situation.

Step 1. Let us suppose first that

$$k_{11} = \dots = k_{1s} = 0 \quad \text{and} \quad \text{rank} \begin{pmatrix} k_{21} & \cdots & k_{2s} \\ \vdots & & \vdots \\ k_{s+1,1} & \cdots & k_{s+1,s} \end{pmatrix} = s. \quad (3.16)$$

We want to prove that, for some initial condition  $y_0 \in L^2(\Omega)^n$ , a control  $u \in L^2(Q_T)$  cannot be found such that the solution to System (3.1) satisfies  $y_1(T) \equiv 0$  in  $\Omega$ . Let us consider the matrix  $P \in \mathcal{L}(\mathbb{R}^n)$  defined by

$$P := (B|\dots|A^{s-1}B|e_1|e_{s+2}|\dots|e_n). \quad (3.17)$$

Using the assumption (3.16),  $P$  is invertible. Thus, in view of Lemma 3.1, for a fixed control  $u \in L^2(Q_T)$ ,  $y$  is a solution to System (3.1) if and only if  $w := P^{-1}y$  is a solution to System (3.3) where  $C, D$  are given by  $C := P^{-1}AP$  and  $D := P^{-1}B$ . Using (3.6) we remark that

$$A(B|AB|\dots|BA^{s-1}) = (B|AB|\dots|BA^{s-1}) \begin{pmatrix} C_{11} \\ 0 \end{pmatrix},$$

with  $C_{11}$  defined in (3.13). Then  $C$  can be rewritten as

$$C = \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix}, \quad (3.18)$$

where  $C_{12} \in \mathcal{L}(\mathbb{R}^{n-s}, \mathbb{R}^s)$  and  $C_{22} \in \mathcal{L}(\mathbb{R}^{n-s})$ . Furthermore

$$D = P^{-1}B = P^{-1}Pe_1 = e_1.$$

and with the Definition (3.17) of  $P$  we get

$$y_1(T) = w_{s+1}(T) \text{ in } \Omega.$$

Thus we need only to prove that there exists  $w_0 \in L^2(\Omega)^n$  such that we cannot find a control  $u \in L^2(Q_T)$  with the corresponding solution  $w$  to System (3.3) satisfying  $w_{s+1}(T) \equiv 0$  in  $\Omega$ . Therefore we apply Proposition 2.1 and prove that the observability inequality (2.4) can not be satisfied. More precisely, for all  $w_0 \in L^2(\Omega)^n$ , there exists a control  $u \in L^2(Q_T)$  such that the solution to System (3.3) satisfies  $w_{s+1}(T) \equiv 0$  in  $\Omega$ , if and only if there exists  $C_{obs} > 0$  such that for all  $\varphi_{s+1}^0 \in L^2(\Omega)$  the solution to the adjoint system

$$\begin{cases} -\partial_t \varphi &= \Delta \varphi + \begin{pmatrix} C_{11}^* & 0 \\ C_{12}^* & C_{22}^* \end{pmatrix} \varphi & \text{in } Q_T, \\ \varphi &= 0 & \text{on } \Sigma_T, \\ \varphi(T) &= (0, \dots, 0, \varphi_{s+1}^0, 0, \dots, 0)^t = e_{s+1} \varphi_{s+1}^0 & \text{in } \Omega \end{cases} \quad (3.19)$$

satisfies the observability inequality

$$\int_{\Omega} \varphi(0)^2 dx \leq C_{obs} \int_{\omega \times (0, T)} \varphi_1^2 dx dt. \quad (3.20)$$

But for all  $\varphi_{s+1}^0 \neq 0$  in  $\Omega$ , the inequality (3.20) is not satisfied. Indeed, we remark first that, since  $\varphi_1(T) = \dots = \varphi_s(T) = 0$  in  $\Omega$ , we have  $\varphi_1 = \dots = \varphi_s = 0$  in  $Q_T$ , so that  $\int_{\omega \times (0, T)} \varphi_1^2 dx = 0$ , while, if we choose  $\varphi_{s+1}^0 \neq 0$  in  $\Omega$ , using the results on backward uniqueness for this type of parabolic system (see [14]), we have clearly  $(\varphi_{s+1}(0), \dots, \varphi_n(0)) \neq 0$  in  $\Omega$ .

Step 2. Let us suppose only that  $k_{11} = \dots = k_{1s} = 0$ . Since  $\text{rank}(B|\dots|A^{s-1}B) = s$ , there exists distinct  $\lambda_1, \dots, \lambda_s \in \{2, \dots, n\}$  such that

$$\text{rank} \begin{pmatrix} k_{\lambda_1,1} & \cdots & k_{\lambda_1,s} \\ \vdots & & \vdots \\ k_{\lambda_s,1} & \cdots & k_{\lambda_s,s} \end{pmatrix} = s.$$

Let us consider the following matrix

$$Q := (e_1|e_{\lambda_1}|\dots|e_{\lambda_{n-1}})^t,$$

where  $\{\lambda_{s+1}, \dots, \lambda_{n-1}\} = \{2, \dots, n\} \setminus \{\lambda_1, \dots, \lambda_s\}$ . Thus, for  $P := Q^{-1}$ , again, for a fixed control  $u \in L^2(Q_T)$ ,  $y$  is a solution to System (3.1) if and only if  $w := P^{-1}y$  is a solution to System (3.3) where  $C, D$  are given by  $C := QAQ^{-1}$  and  $D := QB$ . Moreover, we have

$$[C|D] = Q[A|B].$$

If we note  $(\tilde{k}_{ij})_{ij} := [C|D]$ , this implies  $\tilde{k}_{11} = \dots = \tilde{k}_{1s} = 0$  and

$$\text{rank} \begin{pmatrix} \tilde{k}_{21} & \cdots & \tilde{k}_{2s} \\ \vdots & & \vdots \\ \tilde{k}_{s+1,1} & \cdots & \tilde{k}_{s+1,s} \end{pmatrix} = \text{rank} \begin{pmatrix} k_{\lambda_1,1} & \cdots & k_{\lambda_1,s} \\ \vdots & & \vdots \\ k_{\lambda_s,1} & \cdots & k_{\lambda_s,s} \end{pmatrix} = s.$$

Proceeding as in Step 1 for  $w$ , there exists an initial condition  $w_0$  such that for all control  $u$  in  $L^2(Q_T)$  the solution  $w$  to System (3.3) satisfies  $w_1(T) \not\equiv 0$  in  $\Omega$ . Thus, for the initial condition  $y_0 := Q^{-1}w_0$ , for all control  $u$  in  $L^2(Q_T)$ , the solution  $y$  to System (3.1) satisfies

$$y_1(T) = w_1(T) \not\equiv 0 \text{ in } \Omega.$$

Step 3. Without loss of generality, we can suppose that there exists  $\beta_i$  for all  $i \in \{2, \dots, p\}$  such that  $k_{1j} = \sum_{i=2}^p \beta_i k_{ij}$  for all  $j \in \{1, \dots, s\}$  (otherwise a permutation of lines leads to this case). Let us define the following matrix

$$Q := \left( (e_1 - \sum_{i=2}^p \beta_i e_i) | e_2 | \dots | e_n \right)^t.$$

Thus, for  $P := Q^{-1}$ , again, for a fixed initial condition  $y_0 \in L^2(\Omega)^n$  and a control  $u \in L^2(Q_T)$ , consider System (3.3) with  $w := P^{-1}y$ ,  $y$  being a solution to System (3.1). We remark that if we denote by  $(\tilde{k}_{ij}) := [C|D]$ , we have  $\tilde{k}_{11} = \dots = \tilde{k}_{1s} = 0$ . Applying step 2 to  $w$ , there exists an initial condition  $w_0$  such that for all control  $u$  in  $L^2(Q_T)$  the solution  $w$  to System (3.3) satisfies

$$w_1(T) \not\equiv 0 \text{ in } \Omega. \tag{3.21}$$

Thus, with the definition of  $Q$ , for all control  $u$  in  $L^2(Q_T)$  the solution  $y$  to System (3.1) satisfies

$$w_1(T) = y_1(T) - \sum_{i=2}^p \beta_i y_i(T) \text{ in } \Omega.$$

Suppose  $\Pi_p y(T) \equiv 0$  in  $\Omega$ , then  $w_1(T) \equiv 0$  in  $\Omega$  and this contradicts (3.21).

As a consequence of Proposition 2.1, the  $\Pi_p$ -null controllability implies the  $\Pi_p$ -approximate controllability of System (3.3). If now Condition (1.8) is not satisfied, as for the  $\Pi_p$ -null controllability, we can find a solution to System (3.19) such that  $\phi_1 \equiv 0$  in  $\omega \times (0, T)$  and  $\phi \not\equiv 0$  in  $Q_T$  and we conclude again with Proposition 2.1.  $\square$

### 3.2 $m$ -control forces

In this subsection, we will suppose that  $A \in \mathcal{L}(\mathbb{R}^n)$  and  $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ . We denote by  $B =: (b^1 | \dots | b^m)$ . To prove Theorem 1.1, we will use the following lemma which can be found in [2].

**Lemma 3.3.** *There exist  $r \in \{1, \dots, s\}$  and sequences  $\{l_j\}_{1 \leq j \leq r} \subset \{1, \dots, m\}$  and  $\{s_j\}_{1 \leq j \leq r} \subset \{1, \dots, n\}$  with  $\sum_{j=1}^r s_j = s$ , such that*

$$\mathcal{B} := \bigcup_{j=1}^r \{b^{l_j}, Ab^{l_j}, \dots, A^{s_j-1}b^{l_j}\}$$

is a basis of  $X$ . Moreover, for every  $1 \leq j \leq r$ , there exist  $\alpha_{k,s_j}^i \in \mathbb{R}$  for  $1 \leq i \leq j$  and  $1 \leq k \leq s_j$  such that

$$A^{s_j}b^{l_j} = \sum_{i=1}^j \left( \alpha_{1,s_j}^i b^{l_i} + \alpha_{2,s_j}^i Ab^{l_i} + \dots + \alpha_{s_i,s_j}^i A^{s_i-1}b^{l_i} \right). \quad (3.22)$$

*Proof of Theorem 1.1.* Consider the basis  $\mathcal{B}$  of  $X$  given by Lemma 3.3. Note that

$$\text{rank } \Pi_p[A|B] = \dim \Pi_p[A|B](\mathbb{R}^n) \leq \text{rank } [A|B] = s.$$

If  $M$  is the matrix whose columns are the elements of  $\mathcal{B}$ , i.e.

$$M = (m_{ij})_{ij} := (b^{l_1} | Ab^{l_1} | \dots | A^{s_1-1}b^{l_1} | \dots | b^{l_r} | Ab^{l_r} | \dots | A^{s_r-1}b^{l_r}),$$

we can remark that

$$\text{rank } \Pi_p M = \text{rank } \Pi_p[A|B]. \quad (3.23)$$

Indeed, relationship (3.22) yields

$$\Pi_p A^{s_j} b^{l_j} = \sum_{i=1}^j \left( \alpha_{1,s_j}^i \Pi_p b^{l_i} + \alpha_{2,s_j}^i \Pi_p Ab^{l_i} + \dots + \alpha_{s_i,s_j}^i \Pi_p A^{s_i-1} b^{l_i} \right).$$

Let us suppose first that (1.8) is satisfied. Let be  $y_0 \in L^2(\Omega)^n$ . We will prove that we need only  $r$  forces to control System (3.1). More precisely, we will study the  $\Pi_p$ -null controllability of the system

$$\begin{cases} \partial_t y = \Delta y + Ay + \tilde{B}1_\omega v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \quad (3.24)$$

where  $\tilde{B} = (b^{l_1} | b^{l_2} | \dots | b^{l_r}) \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^n)$ . Using (1.8) and (3.23), we have

$$\text{rank } \Pi_p (b^{l_1} | Ab^{l_1} | \dots | A^{s_1-1}b^{l_1} | \dots | b^{l_r} | Ab^{l_r} | \dots | A^{s_r-1}b^{l_r}) = p. \quad (3.25)$$

*Case 1 :  $p = s$ .* As in the case of one control force, we want to apply a change of variable  $P$  to the solution  $y$  to System (3.24). Let us define for all  $t \in [0, T]$  the following matrix

$$P(t) := (b^{l_1} | Ab^{l_1} | \dots | A^{s_1-1}b^{l_1} | \dots | b^{l_r} | Ab^{l_r} | \dots | A^{s_r-1}b^{l_r} | P_{s+1}(t) | \dots | P_n(t)) \in \mathcal{L}(\mathbb{R}^n), \quad (3.26)$$

where for all  $l \in \{s+1, \dots, n\}$ ,  $P_l$  is solution in  $\mathcal{C}^1([0, T])^n$  to the system of ordinary differential equations

$$\begin{cases} \partial_t P_l(t) = AP_l(t) & \text{in } [0, T], \\ P_l(T) = e_l. \end{cases} \quad (3.27)$$

Using (3.26) and (3.27) we have

$$P(T) = \begin{pmatrix} P_{11} & 0 \\ P_{21} & I_{n-s} \end{pmatrix}, \quad (3.28)$$

where  $P_{11} := \Pi_s(b^{l_1}|Ab^{l_1}|...|A^{s_1-1}b^{l_1}|...|b^{l_r}|Ab^{l_r}|...|A^{s_r-1}b^{l_r}) \in \mathcal{L}(\mathbb{R}^s)$  and  $P_{21} \in \mathcal{L}(\mathbb{R}^{n-s}, \mathbb{R}^s)$ . From (3.25),  $P_{11}$  and thus  $P(T)$  are invertible. Furthermore, since  $P$  is continuous on  $[0, T]$ , there exists a  $T^* \in [0, T)$  such that  $P(t)$  is invertible for all  $t \in [T^*, T]$ .

We suppose first that  $T^* = 0$ . Since  $P$  is invertible and continuous on  $[0, T]$ , for a fixed control  $v \in L^2(Q_T)^r$ ,  $y$  is the solution to System (3.24) if and only if  $w := P(t)^{-1}y$  is the solution to System (3.3) where  $C, D$  are given by

$$C(t) := -P^{-1}(t)\partial_t P(t) + P^{-1}(t)AP(t) \quad \text{and} \quad D(t) := P^{-1}(t)\tilde{B},$$

for all  $t \in [0, T]$ . Using (3.22) and (3.27), we obtain

$$\begin{cases} -\partial_t P(t) + AP(t) &= (Ab^{l_1}|A^2b^{l_1}|...|A^{s_1}b^{l_1}|...|Ab^{l_r}|A^2b^{l_r}|...|A^{s_r}b^{l_r}|0|...|0), \\ &= P(t) \begin{pmatrix} \tilde{C}_{11} & 0 \\ 0 & 0 \end{pmatrix} & \text{in } [0, T], \\ P(t)e_{S_i} &= b^{l_i} & \text{in } [0, T], \end{cases} \quad (3.29)$$

where  $S_i = 1 + \sum_{j=1}^{i-1} s_j$  for  $i \in \{1, \dots, r\}$ ,

$$\tilde{C}_{11} := \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1r} \\ 0 & C_{22} & \cdots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{rr} \end{pmatrix} \in \mathcal{L}(\mathbb{R}^s) \quad (3.30)$$

and for  $1 \leq i \leq j \leq r$  the matrices  $C_{ij} \in \mathcal{L}(\mathbb{R}^{s_j}, \mathbb{R}^{s_i})$  are given by

$$C_{ii} := \begin{pmatrix} 0 & 0 & 0 & \cdots & \alpha_{1,s_i}^i \\ 1 & 0 & 0 & \cdots & \alpha_{2,s_i}^i \\ 0 & 1 & 0 & \cdots & \alpha_{3,s_i}^i \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{s_i,s_i}^i \end{pmatrix} \quad \text{and} \quad C_{ij} := \begin{pmatrix} 0 & 0 & 0 & \cdots & \alpha_{1,s_j}^i \\ 0 & 0 & 0 & \cdots & \alpha_{2,s_j}^i \\ 0 & 0 & 0 & \cdots & \alpha_{3,s_j}^i \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \alpha_{s_i,s_j}^i \end{pmatrix} \quad \text{for } j > i. \quad (3.31)$$

Then

$$C(t) = \begin{pmatrix} \tilde{C}_{11} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad D(t) = (e_{S_1}|...|e_{S_r}). \quad (3.32)$$

Using Theorem 2.1, there exists  $v \in L^2(Q_T)^r$  such that the solution to System (3.3) satisfies  $w_1(T) = \dots = w_s(T) \equiv 0$  in  $\Omega$ . Moreover, using (3.28), we have

$$\Pi_s y(T) = (y_1(T), \dots, y_s(T)) = P_{11}(w_1(T), \dots, w_s(T)) \equiv 0 \text{ in } \Omega.$$

If now  $T^* \neq 0$ , we conclude as in the proof of Theorem 1.1 with one force (see § 3.1).

*Case 2* :  $p < s$ . The proof is a direct adaptation of the proof of Theorem 1.1 with one force, it is possible to find a change of variable in order to get back to the situation of Case 1 (see § 3.1).

If (1.8) is not satisfied, there exist  $\bar{p} \in \{1, \dots, p\}$  and, for all  $i \in \{1, \dots, p\} \setminus \{\bar{p}\}$ , scalars  $\beta_i$  such that  $m_{\bar{p}j} = \sum_{i=1, i \neq \bar{p}}^p \beta_i m_{ij}$  for all  $j \in \{1, \dots, s\}$ . As previously, without loss of generality, we can suppose that

$$m_{11} = \dots = m_{1s} = 0 \quad \text{and} \quad \text{rank} \begin{pmatrix} m_{21} & \cdots & m_{2s} \\ \vdots & & \vdots \\ m_{s+1,1} & \cdots & m_{s+1,s} \end{pmatrix} = s \quad (3.33)$$

(otherwise a permutation of lines leads to this case). Let us consider the matrix  $P$  defined by

$$P := (b^{l_1}|Ab^{l_1}|\dots|A^{s_1-1}b^{l_1}|\dots|b^{l_r}|Ab^{l_r}|\dots|A^{s_r-1}b^{l_r}|e_1|e_{s+2}|\dots|e_n). \quad (3.34)$$

Relationship ensures (3.33) that  $P$  is invertible. Thus, again, for a fixed control  $u \in L^2(Q_T)^m$ ,  $y$  is the solution to System (3.1) if and only if  $w := P^{-1}y$  is the solution to System (3.3) where  $C, D$  are given by  $C := P^{-1}AP$  and  $D := P^{-1}B$ . Using (3.22), we remark that

$$\begin{aligned} & A(b^{l_1}|Ab^{l_1}|\dots|A^{s_1-1}b^{l_1}|\dots|b^{l_r}|Ab^{l_r}|\dots|A^{s_r-1}b^{l_r}) \\ &= (Ab^{l_1}|A^2b^{l_1}|\dots|A^{s_1}b^{l_1}|\dots|Ab^{l_r}|A^2b^{l_r}|\dots|A^{s_r}b^{l_r}) = P \begin{pmatrix} \tilde{C}_{11} \\ 0 \end{pmatrix}, \end{aligned}$$

where  $\tilde{C}_{11}$  is defined in (3.30). Then  $C$  can be written as

$$C = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ 0 & \tilde{C}_{22} \end{pmatrix}, \quad (3.35)$$

where  $\tilde{C}_{12} \in \mathcal{L}(\mathbb{R}^s, \mathbb{R}^{n-s})$  and  $\tilde{C}_{22} \in \mathcal{L}(\mathbb{R}^{n-s})$ . Furthermore, the matrix  $D$  can be written

$$D = \begin{pmatrix} D_1 \\ 0 \end{pmatrix},$$

where  $D_1 \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^s)$ . Using (3.34), we get

$$y_1(T) = w_{s+1}(T) \text{ in } \Omega.$$

Thus, we need only to prove that there exists  $w_0 \in L^2(\Omega)^n$  such that we cannot find a control  $u \in L^2(Q_T)^m$  with the corresponding solution  $w$  to System (3.3) satisfying  $w_{s+1}(T) \equiv 0$  in  $\Omega$ . Therefore we apply Proposition 2.1 and prove that the observability inequality (2.4) can not be satisfied. More precisely, for all  $w_0 \in L^2(\Omega)^n$ , there exists a control  $u \in L^2(Q_T)^m$  such that the solution  $w$  to System (3.3) satisfies  $w_{s+1}(T) \equiv 0$  in  $\Omega$ , if and only if there exists  $C_{obs} > 0$  such that for all  $\varphi_{s+1}^0 \in L^2(\Omega)$  the solution to the adjoint system

$$\begin{cases} -\partial_t \varphi &= \Delta \varphi + \begin{pmatrix} \tilde{C}_{11}^* & 0 \\ \tilde{C}_{12}^* & \tilde{C}_{22}^* \end{pmatrix} \varphi & \text{in } Q_T, \\ \varphi &= 0 & \text{on } \Sigma_T, \\ \varphi(T) &= (0, \dots, 0, \varphi_{s+1}^0, 0, \dots, 0)^t = e_{s+1} \varphi_{s+1}^0 & \text{in } \Omega \end{cases} \quad (3.36)$$

satisfies the observability inequality

$$\int_{\Omega} \varphi(0)^2 dx \leq C_{obs} \int_{\omega \times (0, T)} (D_1^*(\varphi_1, \dots, \varphi_s)^t)^2 dx dt. \quad (3.37)$$

But for all  $\varphi_{s+1}^0 \neq 0$  in  $\Omega$ , the inequality (3.37) is not satisfied. Indeed, we remark first that, since  $\varphi_1(T) = \dots = \varphi_s(T) = 0$  in  $\Omega$ , we have  $\varphi_1 = \dots = \varphi_s = 0$  in  $Q_T$ . Furthermore, if we choose  $\varphi_{s+1}^0 \neq 0$  in  $\Omega$ , as previously, we get  $(\varphi_{s+1}(0), \dots, \varphi_n(0)) \neq 0$  in  $\Omega$ .

We recall that, as a consequence of Proposition 2.1, the  $\Pi_p$ -null controllability implies the  $\Pi_p$ -approximate controllability of System (3.24). If Condition (1.8) is not satisfied, as for the  $\Pi_p$ -null controllability, we can find a solution to System (3.36) such that  $D_1^*(\phi_1, \dots, \phi_s)^t \equiv 0$  in  $\omega \times (0, T)$  and  $\phi \neq 0$  in  $Q_T$  and we conclude again with Proposition 2.1.  $\square$



## 4 Partial null controllability with time dependent matrices

We recall that  $[A|B](\cdot) = (B_0(\cdot)|\dots|B_{n-1}(\cdot))$  (see (1.5)). Since  $A(t) \in \mathcal{C}^{n-1}([0, T]; \mathcal{L}(\mathbb{R}^n))$  and  $B(t) \in \mathcal{C}^n([0, T]; \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n))$ , we remark that the matrix  $[A|B]$  is well defined and is an element of  $\mathcal{C}^1([0, T], \mathcal{L}(\mathbb{R}^{mn}, \mathbb{R}^n))$ . We will use the notation  $B_i =: (b_0^i|\dots|b_m^i)$  for all  $i \in \{0, \dots, n-1\}$ . To prove Theorem 1.2, we will use the following lemma of [16]

**Lemma 4.1.** *Assume that  $\max\{\text{rank } [A|B](t) : t \in [0, T]\} = s \leq n$ . Then there exist  $T_0, T_1 \in [0, T]$ , with  $T_0 < T_1$ ,  $r \in \{1, \dots, m\}$  and sequences  $(s_j)_{1 \leq j \leq r} \subset \{1, \dots, n\}$ , with  $\sum_{i=1}^r s_j = s$ , and  $(l_j)_{1 \leq j \leq r} \subset \{1, \dots, m\}$  such that, for every  $t \in [T_0, T_1]$ , the set*

$$\mathcal{B}(t) = \bigcup_{j=1}^r \{b_0^{l_j}(t), b_1^{l_j}(t), \dots, b_{s_j-1}^{l_j}(t)\}, \quad (4.1)$$

is linearly independent, spans the columns of  $[A|B](t)$  and satisfies

$$b_{s_j}^{l_j}(t) = \sum_{k=1}^j \left( \theta_{s_j, 0}^{l_j, l_k}(t) b_0^{l_k}(t) + \theta_{s_j, 1}^{l_j, l_k}(t) b_1^{l_k}(t) + \dots + \theta_{s_j, s_k-1}^{l_j, l_k}(t) b_{s_k-1}^{l_k}(t) \right), \quad (4.2)$$

for every  $t \in [T_0, T_1]$  and  $j \in \{1, \dots, r\}$ , where

$$\theta_{s_j, 0}^{l_j, l_k}(t), \theta_{s_j, 1}^{l_j, l_k}(t), \dots, \theta_{s_j, s_k-1}^{l_j, l_k}(t) \in \mathcal{C}^1([T_0, T_1]).$$

With exactly the same argument for the proof of the previous lemma, we can obtain the

**Lemma 4.2.** *If  $\text{rank } [A|B](T) = s$ , then the conclusions of Lemma 4.1 hold true with  $T_1 = T$ .*

*Proof of Theorem 1.2.* Let  $y_0 \in L^2(\Omega)^n$  and  $s$  be the rank of the matrix  $[A|B](T)$ . As in the proof of the controllability by one force with constant matrices, let  $X$  being the linear space spanned by the columns of the matrix  $[A|B](T)$ . We consider  $\mathcal{B} = \mathcal{B}(t)$  the basis of  $X$  defined in (4.1).

As in the constant case, we will prove that we need only  $r$  forces to control System (1.1) that is we study the partial null controllability of System (3.24) with the coupling matrix  $A(t) \in \mathcal{C}^{n-1}([0, T]; \mathcal{L}(\mathbb{R}^n))$  and the control matrix  $\tilde{B}(t) = (B_{l_1}(t)|B_{l_2}(t)|\dots|B_{l_r}(t)) \in \mathcal{C}^n([0, T]; \mathcal{L}(\mathbb{R}^r, \mathbb{R}^n))$ . If we define  $M$  as the matrix whose columns are the elements of  $\mathcal{B}(t)$ , i.e. for all  $t \in [0, T]$

$$M(t) = (m_{ij}(t))_{1 \leq i \leq n, 1 \leq j \leq s} := \left( b_0^{l_1}(t)|b_1^{l_1}(t)|\dots|b_{s_1-1}^{l_1}(t)|\dots|b_0^{l_r}(t)|b_1^{l_r}(t)|\dots|b_{s_r-1}^{l_r}(t) \right),$$

we can remark that

$$\text{rank } \Pi_p M(T) = \text{rank } \Pi_p [A|B](T) = p. \quad (4.3)$$

Indeed, using (4.2),

$$\Pi_p b_{s_j}^{l_j}(t) = \sum_{k=1}^j \left( \theta_{s_j, 0}^{l_j, l_k}(t) \Pi_p b_0^{l_k}(t) + \theta_{s_j, 1}^{l_j, l_k}(t) \Pi_p b_1^{l_k}(t) + \dots + \theta_{s_j, s_k-1}^{l_j, l_k}(t) \Pi_p b_{s_k-1}^{l_k}(t) \right).$$

*Case 1 :  $p = s$ .* As in the constant case, we want to apply a change of variable  $P$  to the solution  $y$  to System (3.24). Let us define for all  $t \in [0, T]$  the following matrix

$$P(t) := (b_0^{l_1}(t)|b_1^{l_1}(t)|\dots|b_{s_1-1}^{l_1}(t)|\dots|b_0^{l_r}(t)|b_1^{l_r}(t)|\dots|b_{s_r-1}^{l_r}(t)|P_{s+1}(t)|\dots|P_n(t)) \in \mathcal{L}(\mathbb{R}^n), \quad (4.4)$$

where for all  $i \in \{s+1, \dots, n\}$ ,  $P_i$  is solution in  $\mathcal{C}^1([0, T])^n$  to the system of ordinary differential equations

$$\begin{cases} \partial_t P_l(t) = A P_l(t) \text{ in } [0, T], \\ P_l(T) = e_l. \end{cases} \quad (4.5)$$

Using (4.4) and (4.5),  $P(T)$  can be rewritten

$$P(T) = \begin{pmatrix} P_{11} & 0 \\ P_{21} & I_{n-s} \end{pmatrix}, \quad (4.6)$$

where  $P_{11} := \Pi_p(b_0^{l_1}(T)|b_1^{l_1}(T)|\dots|b_{s_1-1}^{l_1}(T)|\dots|b_0^{l_r}(T)|b_1^{l_r}(T)|\dots|b_{s_r-1}^{l_r}(T)) \in \mathcal{L}(\mathbb{R}^s)$  and  $P_{21} \in \mathcal{L}(\mathbb{R}^{n-s}, \mathbb{R}^s)$ . Using (4.3),  $P_{11}$ , and thus  $P(T)$ , are invertible. Furthermore, since  $P$  is continuous on  $[0, T]$ , there exists a  $T^* \in [0, T)$  such that  $P(t)$  is invertible for all  $t \in [T^*, T]$ .

As previously it is sufficient to prove the result for  $T^* = 0$ . Since  $P(t) \in \mathcal{C}^1([0, T], \mathcal{L}(\mathbb{R}^n))$  and is invertible on the time interval  $[0, T]$ , again, for a fixed control  $v \in L^2(Q_T)^r$ ,  $y$  is the solution to System (3.24) if and only if  $w := P(t)^{-1}y$  is the solution to System (3.3) where  $C, D$  are given by

$$C(t) := -P^{-1}(t)\partial_t P(t) + P^{-1}(t)AP(t) \quad \text{and} \quad D(t) := P^{-1}(t)\tilde{B},$$

for all  $t \in [0, T]$ . Using (4.2) and (4.5), we obtain

$$\begin{cases} -\partial_t P(t) + AP(t) &= (b_1^{l_1}(t)|b_2^{l_1}(t)|\dots|b_{s_1}^{l_1}(t)|\dots|b_1^{l_r}(t)|b_2^{l_r}(t)|\dots|b_{s_r}^{l_r}(t)|0|\dots|0), \\ &= P(t) \begin{pmatrix} \tilde{C}_{11} & 0 \\ 0 & 0 \end{pmatrix} & \text{in } [0, T], \\ P(t)e_{S_i} &= b_0^{l_i} & \text{in } [0, T], \end{cases} \quad (4.7)$$

where  $S_i = 1 + \sum_{j=1}^{i-1} s_j$  for  $1 \leq i \leq r$ ,

$$\tilde{C}_{11} := \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1r} \\ 0 & C_{22} & \dots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_{rr} \end{pmatrix} \in \mathcal{L}(\mathbb{R}^s), \quad (4.8)$$

and for  $1 \leq i \leq j \leq r$ , the matrices  $C_{ij} \in \mathcal{C}^0([0, T]; \mathcal{L}(\mathbb{R}^{s_j}, \mathbb{R}^{s_i}))$  are given here by

$$C_{ii} = \begin{pmatrix} 0 & 0 & 0 & \dots & \theta_{s_j,0}^{l_i,l_i} \\ 1 & 0 & 0 & \dots & \theta_{s_j,1}^{l_i,l_i} \\ 0 & 1 & 0 & \dots & \theta_{s_j,2}^{l_i,l_i} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \theta_{s_i,s_i-1}^{l_i,l_i} \end{pmatrix} \quad \text{and} \quad C_{ij} = \begin{pmatrix} 0 & 0 & 0 & \dots & \theta_{s_j,0}^{l_j,l_i} \\ 0 & 0 & 0 & \dots & \theta_{s_j,1}^{l_j,l_i} \\ 0 & 0 & 0 & \dots & \theta_{s_j,2}^{l_j,l_i} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \theta_{s_j,s_i-1}^{l_j,l_i} \end{pmatrix} \quad \text{for } j > i. \quad (4.9)$$

Then

$$C = \begin{pmatrix} \tilde{C}_{11} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad D = (e_{S_1}|\dots|e_{S_r}). \quad (4.10)$$

Using Theorem 2.1, there exists  $v \in L^2(Q_T)^r$  such that the solution to System (3.3) satisfies  $w_1(T) = \dots = w_s(T) \equiv 0$  in  $\Omega$ . Moreover, the equality (4.6) leads to

$$\Pi_s y(T) = (y_1(T), \dots, y_s(T))^t = P_{11}(w_1(T), \dots, w_s(T))^t \equiv 0 \text{ in } \Omega.$$

*Case 2* :  $p < s$ . The same method as in the constant case leads to the conclusion (see § 3.1).

The  $\pi_p$ -approximate controllability can be proved also as in the constant case. □

## 5 Partial null controllability for a space dependent coupling matrix

All along this section, the dimension  $N$  will be equal to 1, more precisely  $\Omega := (0, \pi)$  with the exception of the proof of the third point in Theorem 1.3 and the numerical illustration in Section 5.3 where  $\Omega := (0, 2\pi)$ . We recall that the eigenvalues of  $-\Delta$  in  $\Omega$  with Dirichlet boundary conditions are given by  $\mu_k := k^2$  for all  $k \geq 1$  and we will denote by  $(w_k)_{k \geq 1}$  the associated  $L^2$ -normalized eigenfunctions. Let us consider the following parabolic system of two equations

$$\begin{cases} \partial_t y = \Delta y + \alpha z + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t z = \Delta z & \text{in } Q_T, \\ y = z = 0 & \text{on } \Sigma_T, \\ y(0) = y_0, z(0) = z_0 & \text{in } \Omega, \end{cases} \quad (5.1)$$

where  $y_0, z_0 \in L^2(\Omega)$  are the initial data,  $u \in L^2(Q_T)$  is the control and the coupling coefficient  $\alpha$  is an element of  $L^\infty(\Omega)$ . We recall that System (5.1) is  $\Pi_1$ -null controllable, if for all  $y^0, z^0 \in L^2(\Omega)$ , we can find a control  $u \in L^2(Q_T)$  such that the solution  $(y, z) \in W(0, T)^2$  to System (5.1) satisfies  $y(T) \equiv 0$  in  $\Omega$ .

### 5.1 Example of controllability

In this subsection, we will give an example of  $\Pi_1$ -null controllability for System (5.1) with the help of the method of moments initially developed in [11]. As already mentioned, we suppose that  $\Omega := (0, \pi)$ , but the argument of Section 5.1 can be adapted for any open bounded interval of  $\mathbb{R}$ . Let us introduce the adjoint system associated to our control problem

$$\begin{cases} -\partial_t \phi = \Delta \phi & \text{in } (0, \pi) \times (0, T), \\ -\partial_t \psi = \Delta \psi + \alpha \phi & \text{in } (0, \pi) \times (0, T), \\ \phi(0) = \phi(\pi) = \psi(0) = \psi(\pi) = 0 & \text{on } (0, T), \\ \phi(T) = \phi_0, \psi(T) = 0 & \text{in } (0, \pi), \end{cases} \quad (5.2)$$

where  $\phi_0 \in L^2(0, \pi)$ . For all  $k \geq 1$ , if we consider  $\phi_0 := w_k$  in dual System (5.2), we get

$$\int_0^\pi w_k y(T) dx - \int_0^\pi \phi(0) y_0 dx - \int_0^\pi \psi(0) z_0 dx = \int_0^T \int_\omega \phi u dx dt, \quad (5.3)$$

Since  $(w_k)_{k \geq 1}$  spans  $L^2(0, \pi)$ , System (5.1) is  $\Pi_1$ -null controllable if and only if there exists  $u \in L^2(Q_T)$  such that for all  $\phi_0 = w_k \in L^2(0, \pi)$ ,  $k \geq 1$ ; the solution to System (5.2) satisfies the following equality

$$-\int_0^\pi \phi(0) y_0 dx - \int_0^\pi \psi(0) z_0 dx = \int_0^T \int_\omega \phi u dx dt, \quad (5.4)$$

where  $(\phi, \psi)$  is the solution to dual System (5.2) for the initial data  $\phi_0 = w_k$ .

Let us consider  $\phi_0 \in L^2(0, \pi)$  defined by

$$\phi_0(x) = \sum_{k=1}^{\infty} \phi_k^0 w_k(x) \text{ for all } x \in (0, \pi).$$

With this initial condition  $\phi_0$  is associated the solution  $(\phi, \psi)$  to adjoint System (5.2):

$$\phi(t) = \sum_{k=1}^{\infty} e^{-k^2(T-t)} \phi_k^0 w_k \text{ in } (0, \pi)$$

for all  $t \in [0, T]$ . If we write:

$$\psi(x, t) := \sum_{k=1}^{\infty} \psi_k(t) w_k(x) \text{ for all } (x, t) \in (0, \pi) \times (0, T),$$

then a simple computation leads to the formula

$$\psi_k(t) = \sum_{l=1}^{\infty} \frac{e^{-k^2(T-t)} - e^{-l^2(T-t)}}{-k^2 + l^2} \alpha_{kl} \phi_l^0 \text{ for all } k \geq 1, t \in (0, T), \quad (5.5)$$

where, for all  $k, l \in \mathbb{N}^*$ ,

$$\alpha_{kl} := \int_0^{\pi} \alpha w_k w_l dx. \quad (5.6)$$

In (5.5) we implicitly used the convention: if  $l = k$  the term  $(e^{-k^2(T-t)} - e^{-l^2(T-t)})/(-k^2 + l^2)$  is replaced by  $(T-t)e^{-k^2(T-t)}$ . With these expressions of  $\phi$  and  $\psi$ , the equality (5.4) reads for all  $k \geq 1$

$$-e^{-k^2 T} y_k^0 - \sum_{l \geq 1} \frac{e^{-k^2 T} - e^{-l^2 T}}{-k^2 + l^2} \alpha_{kl} z_l^0 = \int_0^T \int_{\omega} e^{-k^2(T-t)} w_k(x) u(t, x) dx dt. \quad (5.7)$$

In the proof of Theorem 1.3, we will search a control  $u$  expressed as  $u(x, t) = f(x)\gamma(t)$  with  $\gamma(t) = \sum_{k \geq 1} \gamma_k q_k(t)$  and  $(q_k)_{k \geq 1}$  a family biorthogonal to  $(e^{-k^2 t})_{k \geq 1}$ . Thus, we will need the two following lemma

**Lemma 5.1.** (see [5]) *There exists  $f \in L^2(0, \pi)$  such that  $\text{Supp } f \subset \omega$  and for all  $\varepsilon > 0$  one has  $\inf_{k \geq 1} f_k e^{\varepsilon k^2} > 0$ , where, for all  $k \in \mathbb{N}^*$ ,  $f_k := \int_0^{\pi} f w_k dx$ .*

**Lemma 5.2.** (see [11]) *There exists a biorthogonal sequence  $(q_k)_{k \geq 1} \subset L^2(0, T)$  to  $(e^{k^2 t})_{k \geq 1}$  that is*

$$\langle q_k, e^{l^2 t} \rangle_{L^2(0, T)} = \delta_{kl}.$$

Moreover, for all  $\varepsilon > 0$ , there exists  $C_{\varepsilon} > 0$ , independent of  $k$ , such that

$$\|q_k\|_{L^2(0, T)} \leq C_{\varepsilon} e^{\varepsilon k^2}, \quad \forall k \geq 1.$$

*Proof of the second point in Theorem 1.3.* As mentioned above, let us search the control  $u$  of the form  $u(x, t) = f(x)\gamma(t)$ , where  $f$  is as in Lemma 5.1. Since  $f_k \neq 0$  for all  $k \in \mathbb{N}^*$ , using (5.7), the  $\Pi_1$ -null controllability of System (5.1) is reduced to find a solution  $\gamma \in L^2(0, T)$  to the following problem of moments:

$$\int_0^T \gamma(T-t) e^{-k^2 t} dt = f_k^{-1} \left( -e^{-k^2 T} y_k^0 - \sum_{l \geq 1} \frac{e^{-k^2 T} - e^{-l^2 T}}{-k^2 + l^2} \alpha_{kl} z_l^0 \right) := M_k \quad \forall k \geq 0. \quad (5.8)$$

The function  $\gamma(t) := \sum_{k \geq 1} M_k q_k(T-t)$  is a solution to this problem of moments. We need only to prove that  $\gamma \in L^2(0, T)$ . Using the convexity of the exponential function, we get for all  $k \in \mathbb{N}^*$ ,

$$\begin{aligned} \sum_{l \geq 1} \left| \frac{e^{-k^2 T} - e^{-l^2 T}}{-k^2 + l^2} \right| |\alpha_{kl}| &= \sum_{l=1}^k \left| \frac{e^{-k^2 T} - e^{-l^2 T}}{-k^2 + l^2} \right| |\alpha_{kl}| + \sum_{l=k+1}^{\infty} \left| \frac{e^{-k^2 T} - e^{-l^2 T}}{-k^2 + l^2} \right| |\alpha_{kl}| \\ &\leq \sum_{l=1}^k T e^{-l^2 T} |\alpha_{kl}| + \sum_{l=k+1}^{\infty} T e^{-k^2 T} |\alpha_{kl}| \\ &=: A_1 + A_2. \end{aligned} \quad (5.9)$$

Let us suppose for the moment that  $T \leq C_2$ . With Condition (1.12) on  $\alpha$  we deduce that

$$A_1 \leq C_1 T \sum_{l=1}^k e^{-l^2 T} e^{-C_2(k^2-l^2)} \leq C_1 T k e^{-k^2 T} \quad (5.10)$$

and

$$\begin{aligned} A_2 &\leq C_1 T e^{-k^2 T} \sum_{l=k+1}^{\infty} e^{-C_2(l^2-k^2)} \\ &= C_1 T e^{-k^2 T} \sum_{j=0}^{\infty} e^{-C_2((j+k+1)^2-k^2)} \\ &\leq C_1 T e^{-k^2 T} \sum_{j=0}^{\infty} (e^{-C_2})^j \\ &\leq C_1 T e^{-k^2 T} \frac{1}{1 - e^{-C_2}}. \end{aligned} \quad (5.11)$$

Let  $\varepsilon \in (0, 1)$ . Combining the three last inequalities (5.9)-(5.11), there exists  $k_1 \in \mathbb{N}^*$  such that for all  $k \geq k_1$ ,

$$\sum_{l \geq 1} \left| \frac{e^{-k^2 T} - e^{-l^2 T}}{-k^2 + l^2} \right| |\alpha_{kl}| \leq C k e^{-k^2 T} \leq C_\varepsilon e^{-(1-\varepsilon)k^2 T}, \quad (5.12)$$

where, here and thereafter,  $C_\varepsilon$  is a constant which does not depend on  $k$ . Then, with Lemma 5.1, (5.8) and (5.12), there exists  $k_2 \geq k_1$  such that for all  $k \geq k_2$

$$\begin{aligned} |M_k| &\leq e^{\varepsilon k^2 T} \left( e^{-k^2 T} \|y_0\|_{L^2(0,\pi)} + C_\varepsilon e^{-(1-\varepsilon)k^2 T} \|z_0\|_{L^2(0,\pi)} \right) \\ &\leq C_\varepsilon e^{-(1-2\varepsilon)k^2 T} (\|y_0\|_{L^2(0,\pi)} + \|z_0\|_{L^2(0,\pi)}). \end{aligned}$$

Thus, using Lemma 5.2, for  $\varepsilon < 1/3$ ,

$$\|\gamma\|_{L^2(0,T)} \leq (C_\varepsilon + C_\varepsilon \sum_{k \geq k_2} e^{-(1-3\varepsilon)k^2 T}) (\|y_0\|_{L^2(0,\pi)} + \|z_0\|_{L^2(0,\pi)}) < \infty.$$

If  $T > C_2$ , we conclude with the same argument that was ending of the proof of Theorem 1.1 with one force (see Section 3.1, end of *Case 1*). More precisely, we consider System (5.1) in  $(0, T - C_2)$  with  $u \equiv 0$  in  $\omega \times (0, T - C_2)$ , and applying the above strategy we construct a control in  $(T - C_2, T)$  such that the solution to System (5.1) in  $(T - C_2, T)$  satisfies  $y(T) \equiv 0$  in  $(0, \pi)$ .  $\square$

## 5.2 Example of non controllability

In this subsection, to give an example of non  $\Pi_1$ -null controllability of System (5.1), we will first study the boundary controllability of the following parabolic system of two equations

$$\begin{cases} \partial_t y = \Delta y + \alpha z & \text{in } Q_T := (0, \pi) \times (0, T), \\ \partial_t z = \Delta z & \text{in } Q_T, \\ y(0, t) = v(t), \quad y(\pi, t) = z(0, t) = z(\pi, t) = 0 & \text{on } (0, T), \\ y(x, 0) = y_0(x), \quad z(x, 0) = z_0(x) & \text{in } \Omega := (0, \pi), \end{cases} \quad (5.13)$$

where  $y_0, z_0 \in L^2(0, \pi)$  are the initial data,  $v \in L^2(0, T)$  is the boundary control and  $\alpha \in L^\infty(0, \pi)$ . For all  $K \geq 1$ , we define the set

$$E_K := \text{span}(w_k : k \leq K). \quad (5.14)$$

For any given  $y_0, z_0 \in L^2(0, \pi)$  and  $v \in L^2(0, T)$ , System (5.13) has a unique solution in  $L^2(Q_T)^2 \cap C^0([0, T]; H^{-1}(\Omega)^2)$  (defined by transposition; see [12]).

System (5.13) is  $\Pi_1$ -null controllable if and only if there exists a constant  $C$  such that for all  $\phi_0 \in L^2(0, \pi)$  the solution to adjoint System (5.2) satisfies the following observability inequality

$$\int_0^\pi \phi(0)^2 dx + \int_0^\pi \psi(0)^2 dx \leq C \int_0^T \phi_x(0, t)^2 dt. \quad (5.15)$$

In the same way as in (5.7), if System (5.13) is  $\Pi_1$ -null controllable, there exists  $C_{obs} > 0$  such that for a  $K > 0$  and all  $\phi_0 := \sum_{k=1}^K \phi_k^0 w_k \in E_K$ , the solution  $(\phi, \psi)$  to the adjoint System (5.2) satisfies

$$\sum_{k=1}^K e^{-2k^2 T} (\phi_k^0)^2 + \sum_{k \geq 1} \left| \sum_{l=1}^K \frac{e^{-k^2 T} - e^{-l^2 T}}{-k^2 + l^2} \alpha_{kl} \phi_l^0 \right|^2 \leq C_{obs} \int_0^T \left| \sum_{k=1}^K k e^{-k^2(T-t)} \phi_k^0 \right|^2 dt. \quad (5.16)$$

**THEOREM 5.1.** *Let  $T > 0$  and let  $\alpha$  be the function of  $L^\infty(0, \pi)$  defined by*

$$\alpha(x) := \sum_{j=1}^{\infty} \frac{1}{j^2} \cos(15jx) \text{ for all } x \in (0, \pi). \quad (5.17)$$

*Then System (5.13) is not  $\Pi_1$ -null controllable.*

*Proof.* This proof is inspired from [17]. To understand why the number «15» appears in the definition (5.17) of the function  $\alpha$ , we will consider for all  $x \in (0, \pi)$

$$\alpha(x) := \sum_{j=1}^{\infty} \frac{1}{j^2} \cos(Gjx) \text{ for all } x \in (0, \pi),$$

where  $G \in \mathbb{N}^*$ . The idea is to prove that there exist a sequence of initial conditions  $(\phi_{0,M})_{M \in \mathbb{N}}$  and two positive constants  $\gamma_1$  and  $\gamma_2$  such that, for all  $M \in \{2, \dots\}$ , the solution  $(\phi_M, \psi_M)$  to the adjoint System (5.2) satisfies

$$\int_0^T (\phi_M)_x(0, t)^2 dt \leq \frac{\gamma_1}{M^9} \quad \text{and} \quad \int_0^\pi \psi_M(0)^2 dx \geq \frac{\gamma_2}{M^8}.$$

So that, for a sufficiently large  $M$ , the observability inequality (5.15) cannot be true. Let us take  $M \in \{2, \dots\}$ . We will search a sequence of initial conditions of the form

$$\phi_{0,M} = (0, \dots, 0, \phi_{GM+1}^{0,M}, \phi_{GM+2}^{0,M}, \dots, \phi_{GM+m}^{0,M}, 0, \dots),$$

for a fixed  $m \in \mathbb{N}$ , independent of  $M$ . The value of  $m$  will be determined in the proof. First, we set

$$A := \int_0^\pi \psi_M(0)^2 dx = \sum_{k \geq 1} \left| \sum_{l=GM+1}^{GM+m} \frac{e^{-k^2 T} - e^{-l^2 T}}{-k^2 + l^2} \alpha_{kl} \phi_l^{0,M} \right|^2 \quad (5.18)$$

and

$$B := \int_0^T (\phi_M)_x(0, t)^2 dt = \int_0^T \left| \sum_{k=GM+1}^{GM+m} k e^{-k^2(T-t)} \phi_k^{0,M} \right|^2 dt. \quad (5.19)$$

**Estimate for (5.19)**

We can rewrite  $B$  as follows:

$$B = \int_0^T \left| \sum_{j=1}^m (GM+j) e^{-(G^2 M^2 + 2GMj + j^2)(T-t)} \phi_{GM+j}^{0,M} \right|^2 dt = \int_0^T e^{-2G^2 M^2(T-t)} g_M(t) dt, \quad (5.20)$$

where, for all  $t \in [0, T]$ ,  $g_M(t) := f_M(t)^2$  with

$$f_M(t) := \sum_{j=1}^m (GM + j) e^{-(2GMj+j^2)(T-t)} \phi_{GM+j}^{0,M}.$$

Let  $(\phi_{GM+1}^{0,M}, \phi_{GM+2}^{0,M}, \dots, \phi_{GM+m}^{0,M})$  be a non nontrivial solution of the following homogeneous linear system of  $m - 1$  equations with  $m$  unknowns

$$f_M^{(l)}(T) = \sum_{j=1}^m (GM + j)(2GMj + j^2)^l \phi_{GM+j}^{0,M} = 0, \text{ for all } l \in \{0, \dots, m - 2\}. \quad (5.21)$$

Using Leibniz formula

$$g_M^{(l)} = \sum_{k=0}^l \binom{l}{k} f_M^{(k)} f_M^{(l-k)}$$

we deduce that

$$g_M^{(l)}(T) = 0, \text{ for all } l \in \{0, \dots, 2m - 4\}. \quad (5.22)$$

Using (5.22), after  $2m - 3$  integrations by part in (5.20), we obtain

$$\begin{aligned} B &= \frac{-g_M(0)e^{-2G^2M^2T}}{2G^2M^2} + \int_0^T \frac{e^{-2G^2M^2(T-t)}}{(-2G^2M^2)} g_M^{(1)}(t) dt \\ &= \sum_{l=0}^{2m-4} \frac{g_M^{(l)}(0)e^{-2G^2M^2T}}{(-2G^2M^2)^{l+1}} + \int_0^T \frac{e^{-2G^2M^2(T-t)}}{(-2G^2M^2)^{2m-3}} g_M^{(2m-3)}(t) dt. \end{aligned}$$

By linearity, in (5.21) we can choose  $\phi_{GM+1}^{0,M}, \dots, \phi_{GM+m}^{0,M}$  such that

$$\|\phi^{0,M}\|_\infty = 1. \quad (5.23)$$

Thus, for all  $l \in \mathbb{N}$  and all  $t \in [0, T]$ , the following estimate holds

$$\begin{aligned} |g_M^{(l)}(t)| &= \left| \sum_{k=0}^l \binom{l}{k} f_M^{(k)}(t) f_M^{(l-k)}(t) \right| \\ &\leq \sum_{k=0}^l \binom{l}{k} \left| \sum_{j=1}^m (GM + j)(2GMj + j^2)^k e^{-(2GMj+j^2)(T-t)} \phi_{GM+j}^{0,M} \right| \\ &\quad \times \left| \sum_{j=1}^m (GM + j)(2GMj + j^2)^{l-k} e^{-(2GMj+j^2)(T-t)} \phi_{GM+j}^{0,M} \right| \\ &\leq (GM + m)^2 m^2 \sum_{k=0}^l \binom{l}{k} (2GMm + m^2)^l \\ &\leq CM^{l+2}, \end{aligned}$$

where  $C$  does not depend on  $M$ . Then, since  $\|\phi^{0,M}\|_\infty = 1$ , there exist  $C, \tau > 0$  such that

$$\begin{aligned} B &\leq e^{-2G^2M^2T} \sum_{l=0}^{2m-4} \frac{\|g_M^{(l)}\|_\infty}{(2G^2M^2)^{l+1}} + \frac{T \|g_M^{(2m-3)}\|_\infty}{(2G^2M^2)^{2m-3}} \\ &\leq e^{-\tau M^2} \sum_{l=0}^{\infty} \frac{C}{M^l} + \frac{C}{M^{2m-5}} \\ &\leq CM^{-2} e^{-\tau M^2} \frac{1}{1 - M^{-2}} + \frac{C}{M^{2m-5}}. \end{aligned}$$

Thus there exists  $\gamma_1 > 0$  such that we have the estimate

$$B \leq \frac{\gamma_1}{M^{2m-5}}, \quad (5.24)$$

where  $\gamma_1$  does not depend on  $M$ .

**Estimate for (5.18)**

We recall that for all  $x \in (0, \pi)$

$$\alpha(x) := \sum_{j=1}^{\infty} \frac{1}{j^2} \cos(Gjx).$$

The function  $\alpha$  is of the form  $\alpha(x) = \sum_{p=0}^{\infty} \alpha_p \cos(px)$  for all  $x \in (0, \pi)$ , with

$$\alpha_p := \begin{cases} \frac{1}{j^2} & \text{if } p = Gj \text{ with } j \in \mathbb{N}^*, \\ 0 & \text{otherwise.} \end{cases} \quad (5.25)$$

From the definition of  $\alpha_{kl}$  in (5.6), there holds for all  $k, l \in \mathbb{N}^*$

$$\alpha_{kl} = \frac{1}{2}(\alpha_{|k-l|} - \alpha_{k+l}).$$

Let  $k \in \{1, \dots, m\}$  and  $l \in \{GM + 1, \dots, GM + m\}$ . We have  $k + l \in \{GM + 2, \dots, GM + 2m\}$ . Thus if we choose

$$G \geq 2m + 1, \quad (5.26)$$

using (5.25), we obtain

$$\alpha_{k+l} = 0$$

and

$$\alpha_{|k-l|} = \begin{cases} \frac{1}{M^2} & \text{if } |k-l| = GM, \\ 0 & \text{otherwise.} \end{cases}$$

So that we have the following submatrix of  $(\alpha_{kl})_{1 \leq k, l \leq GM+m}$ :

$$(\alpha_{kl})_{1 \leq k \leq m, GM+1 \leq l \leq GM+m} = \frac{1}{2M^2} I_{\mathbb{R}^m}. \quad (5.27)$$

According to (5.23), there exists  $k_1 \in \{1, \dots, m\}$  such that

$$|\phi_{GM+k_1}^{0,M}| = 1. \quad (5.28)$$

Furthermore, since  $m \leq GM$  (see (5.26)),

$$|e^{-k_1^2 T} - e^{-(GM+k_1)^2 T}| \geq |e^{-m^2 T} - e^{-G^2 M^2 T}| \quad (5.29)$$

and

$$(GM + k_1)^2 - k_1^2 \leq (GM + m)^2 - 1. \quad (5.30)$$

Then, with the help of (5.27)-(5.30) in (5.18) we bound  $A$

$$\begin{aligned} A &\geq \left| \frac{e^{-k_1^2 T} - e^{-(GM+k_1)^2 T}}{-k_1^2 + (GM + k_1)^2} \frac{1}{2M^2} \right|^2 \\ &\geq \left| \frac{e^{-m^2 T} - e^{-G^2 M^2 T}}{(GM + m)^2 - 1} \frac{1}{2M^2} \right|^2. \end{aligned}$$



Thus, there exists  $\gamma_2 > 0$  such that,

$$A \geq \frac{\gamma_2}{M^8}, \quad (5.31)$$

where  $\gamma_2$  depends on  $G$  and  $m$  but does not depend on  $M$  and  $k_1$ .

We conclude from (5.24), (5.26) and (5.31), that for all  $m, G \in \mathbb{N}^*$  with  $G \geq 2m + 1$ , there exist  $\gamma_1, \gamma_2 > 0$  such that

$$B \leq \frac{\gamma_1}{M^{2m-5}} \quad \text{and} \quad A \geq \frac{\gamma_2}{M^8}.$$

Thus, if we choose

$$m \geq 7 \quad \text{so that} \quad G \geq 15,$$

the observability inequality (5.16) is not satisfied for  $M$  large enough.

### Construction of an initial condition

Let us fix  $m = 7$  and  $G = 15$ . We want to construct an initial condition  $y_0, z_0$  such that, for  $\alpha$  defined in (5.17) and all control  $v \in L^2(0, T)$  the solution  $y$  to System (5.13) can not be identically equal to zero at  $t = T$ . Let us define  $(\phi_{0,M})_{M \geq 2}$  as

$$\phi_{0,M} = (0, \dots, 0, \phi_{15M+1}^{0,M}, \dots, \phi_{15M+7}^{0,M}, 0, \dots),$$

where  $(\phi_{15M+1}^{0,M}, \dots, \phi_{15M+7}^{0,M})$  is a nontrivial solution to linear System (5.21). Using (5.28), for all  $M \geq 2$ , there exists  $k_1(M) \in \{1, \dots, 7\}$ , such that  $|\phi_{15M+k_1(M)}^{0,M}| = 1$ . We can extract a subsequence  $(M_j)_{j \in \mathbb{N}}$  of  $\{2, \dots\}$  such that  $|\phi_{15M_j+k_1}^{0,M_j}| = 1$  for a  $k_1$  independent of  $M_j$ .

Let  $y_0 := 0$  and  $z_0 := w_{k_1}$ . Let us suppose that, for this initial condition  $y_0$  and  $z_0$ , System (5.17) is  $\Pi_1$ -null controllable to the boundary, that is there exists a control  $v \in L^2(0, T)$  such that  $y(T) \equiv 0$ . Thus, for all  $j \in \mathbb{N}^*$ , we have the equality

$$\int_0^\pi y_0 \phi_j(0) dx + \int_0^\pi z_0 \psi_j(0) dx = \int_0^T v(t) (\phi_j)_x(0, t) dt, \quad (5.32)$$

where  $(\phi_j, \psi_j)$  is the solution to

$$\begin{cases} -\partial_t \phi_j = \Delta \phi_j & \text{in } (0, \pi) \times (0, T), \\ -\partial_t \psi_j = \Delta \psi_j + \alpha \phi_j & \text{in } (0, \pi) \times (0, T), \\ \phi_j(0) = \phi_j(\pi) = \psi_j(0) = \psi_j(\pi) = 0 & \text{on } (0, T), \\ \phi_j(T) = \phi_{0,M_j}, \psi_j(T) = 0 & \text{in } (0, \pi). \end{cases}$$

We remark that, for all  $j \in \mathbb{N}^*$ ,

$$\int_0^\pi y_0 \phi_j(0) dx = 0. \quad (5.33)$$

Since  $z_0 = w_{k_1}$ , the equality (5.27) leads to

$$\begin{aligned} \left| \int_0^\pi z_0 \psi_j(0) dx \right| &= \left| \sum_{s=1}^7 \frac{e^{-k_1^2 T} - e^{-(GM_j+s)2T}}{-k_1^2 + (GM_j+s)^2} \alpha_{k_1, GM_j+s} \phi_{GM_j+s}^{0,M_j} \right| \\ &= \left| \frac{e^{-k_1^2 T} - e^{-(GM_j+k_1)^2 T}}{-k_1^2 + (GM_j+k_1)^2} \frac{1}{M_j^2} \right|. \end{aligned}$$

Then for all  $j \in \mathbb{N}^*$

$$\left| \int_0^\pi z_0 \psi_j(0) dx \right| \geq \frac{\sqrt{\gamma_2}}{M_j^4}, \quad (5.34)$$

with  $\gamma_2$  determined in (5.31). And, using (5.24) we have

$$\left| \int_0^T v(t)(\phi_j)_x(0, t) dt \right| \leq \|v\|_{L^2(0, T)} \left( \int_0^T ((\phi_j)_x(0, t))^2 dt \right)^{1/2} \leq \frac{\sqrt{\gamma_1} \|v\|_{L^2(0, T)}}{M_j^{9/2}}. \quad (5.35)$$

Combining (5.33), (5.34) and (5.35), we obtain a contradiction with (5.32). Thus, for this initial condition  $y_0$  and  $z_0$ , we can not find a control  $v \in L^2(0, T)$  such that the  $(y, z)$  solution to system (5.17) satisfies  $y(T) \equiv 0$  in  $(0, \pi)$ .  $\square$

*Proof of the third point in Theorem 1.3.* Using Theorem 5.1, the following system

$$\begin{aligned} & \text{For given } (p_0, q_0) : (0, \pi) \rightarrow \mathbb{R}^2, \quad v : (0, T) \rightarrow \mathbb{R}, \\ & \text{Find } (p, q) : (0, \pi) \times (0, T) \rightarrow \mathbb{R}^2 \text{ such that} \\ & \begin{cases} \partial_t p = \Delta p + \alpha q & \text{in } (0, \pi) \times (0, T), \\ \partial_t q = \Delta q & \text{in } (0, \pi) \times (0, T), \\ p(\pi, t) = v(t), \quad p(0, t) = q(0, t) = q(\pi, t) = 0 & \text{on } (0, T), \\ p(x, 0) = p_0(x), \quad q(x, 0) = q_0(x) & \text{in } (0, \pi) \end{cases} \end{aligned} \quad (5.36)$$

is not  $\Pi_1$ -null controllable, that is there exist initial conditions  $p_0, q_0 \in L^2(0, \pi)$  such that for any control  $v \in L^2(0, T)$  the solution  $(p, q)$  to System (5.36) is not identically equal to zero at time T. There exist two sequences  $(p_k^0)_k$  and  $(q_k^0)$  such that

$$p_0(x) = \sum_{k \geq 1} p_k^0 w_k(x) \quad \text{and} \quad q_0(x) = \sum_{k \geq 1} q_k^0 w_k(x) \quad \text{for all } x \in (0, \pi).$$

We consider now  $\bar{p}_0, \bar{q}_0 \in L^2(0, 2\pi)$  defined by

$$\bar{p}_0(x) = \sum_{k \geq 1} p_k^0 w_k(x) \quad \text{and} \quad \bar{q}_0(x) = \sum_{k \geq 1} q_k^0 w_k(x) \quad \text{for all } x \in (0, 2\pi).$$

Let  $\omega \subset (0, \pi)$ . Suppose now that the system

$$\begin{aligned} & \text{For given } (y_0, z_0) : (0, 2\pi) \rightarrow \mathbb{R}^2, \quad u : (0, 2\pi) \times (0, T) \rightarrow \mathbb{R}, \\ & \text{Find } (y, z) : (0, 2\pi) \times (0, T) \rightarrow \mathbb{R}^2 \text{ such that} \\ & \begin{cases} \partial_t y = \Delta y + \alpha z + \mathbf{1}_\omega u & \text{in } (0, 2\pi) \times (0, T), \\ \partial_t z = \Delta z & \text{in } (0, 2\pi) \times (0, T), \\ y(0, t) = y(2\pi, t) = z(0, t) = z(2\pi, t) = 0 & \text{on } (0, T), \\ y(x, 0) = y_0(x), \quad z(x, 0) = z_0(x) & \text{in } (0, 2\pi) \end{cases} \end{aligned} \quad (5.37)$$

is  $\Pi_1$ -null controllable, more particularly for the initial conditions  $y(0) = \bar{p}_0$  and  $z(0) = \bar{q}_0$  in  $(0, 2\pi)$ , there exists a control  $u$  in  $L^2((0, 2\pi) \times (0, T))$  such that the solution  $(y, z)$  to System (5.37) satisfies  $y(T) \equiv 0$  in  $(0, 2\pi)$ . We remark now that  $(p, q) := (y|_{(0, \pi)}, z|_{(0, \pi)})$  is a solution of (5.36) with  $(p(0), q(0)) = (p_0, q_0)$  in  $(0, \pi)$ ,  $v(t) = y(\pi, t)$  in  $(0, T)$  and satisfying  $p(T) \equiv 0$  in  $(0, \pi)$ . This contradicts that for any control  $v \in L^2(0, T)$  the solution  $(p, q)$  to System (5.36) can not be identically equal to zero at time T.  $\square$

### 5.3 Numerical illustration

In this section, we illustrate numerically the results obtained previously in Sections 5.1 and 5.2. We adapt the HUM method to our control problem. For all penalty parameter  $\varepsilon > 0$  we compute the control that minimizes the penalized HUM functional  $F_\varepsilon$  given by

$$F_\varepsilon(u) := \frac{1}{2} \|u\|_{L^2(\omega \times (0, T))}^2 + \frac{1}{2\varepsilon} \|y(T; y_0, u)\|_{L^2(\Omega)}^2,$$

where  $y$  is the solution to (5.1). We can find in [7] the argument relating the null/approximate controllability and this kind of functional. Using the Fenchel-Rockafellar theory (see [10] p.59) we know that the minimum of  $F_\varepsilon$  is equal to the opposite of the minimum of  $J_\varepsilon$ , the so-called dual functional, defined for all  $\varphi_0 \in L^2(\Omega)$  by

$$J_\varepsilon(\varphi_0) := \frac{1}{2}\|\varphi\|_{L^2(Q_T)}^2 + \frac{\varepsilon}{2}\|\varphi_0\|_{L^2(Q_T)}^2 + \langle y(T; y_0, 0), \varphi_0 \rangle_{L^2(\Omega)},$$

where  $\varphi$  is the solution to the backward System (5.38). Moreover the minimizers  $u_\varepsilon$  and  $\varphi_{\varepsilon,0}$  of the functionals  $F_\varepsilon$  and  $J_\varepsilon$  respectively, are related through the equality  $u_\varepsilon = \mathbf{1}_\omega \varphi_\varepsilon$ , where  $\varphi_\varepsilon$  is the solution to the backward System (5.38) with the initial data  $\varphi(T) = \varphi_{\varepsilon,0}$ . A simple computation leads to

$$\nabla J_\varepsilon(\varphi_0) = \Lambda \varphi_0 + \varepsilon \varphi_0 + y(T; y_0, 0),$$

with the Gramian operator  $\Lambda$  defined as follows

$$\begin{aligned} \Lambda : L^2(\Omega) &\mapsto L^2(\Omega), \\ \varphi_0 &\rightarrow w(T), \end{aligned}$$

where  $w$  is the solution to the following backward and forward systems

$$\begin{cases} -\partial_t \varphi = \Delta \varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(T) = \varphi_0 & \text{in } \Omega \end{cases} \quad (5.38)$$

and

$$\begin{cases} \partial_t w = \Delta w + \mathbf{1}_\omega \varphi & \text{in } Q_T, \\ w = 0, & \text{on } \Sigma_T, \\ w(0) = 0 & \text{in } \Omega. \end{cases} \quad (5.39)$$

Then the minimizer  $u_\varepsilon$  of  $F_\varepsilon$  will be computed with the help of the minimizer  $\varphi_{0,\varepsilon}$  of  $J_\varepsilon$  which is the solution to the linear problem

$$(\Lambda + \varepsilon)\varphi_{0,\varepsilon} = -y(T; y_0, 0).$$

*Remark 1.* The proof of Theorem 1.7 in [7] can be adapted to prove that

- (i) System (5.1) is  $\Pi_1$ -null controllable if and only if  $\sup_{\varepsilon > 0} \left( \inf_{L^2(\omega \times (0,T))} F_\varepsilon \right) < \infty$ ,
- (ii) System (5.1) is  $\Pi_1$ -approximately controllable if and only if  $y_\varepsilon(T) \xrightarrow{\varepsilon \rightarrow 0} 0$ ,

where  $y_\varepsilon$  is the solution to System (5.1) for the control  $u_\varepsilon$ .

System (5.1) with  $T = 0.005$ ,  $\Omega := (0, 2\pi)$ ,  $\omega := (0, \pi)$  and  $y_0 := 100 \sin(x)$  has been considered. We take the two expressions below for the coupling coefficient  $\alpha$  that correspond respectively to Cases (1)-(2) and (3) in Theorem 1.3:

- (a)  $\alpha(x) = 1$ ,
- (b)  $\alpha(x) = \sum_{p \geq 0} \frac{1}{p^2} \cos(15px)$ .

Systems (5.1) and (5.38)-(5.39) are discretized with backward Euler time-marching scheme (time step  $\delta t = 1/400$ ) and piecewise linear Lagrange finite elements on a uniform mesh of size  $h$  successively equal to  $2\pi/50$ ,  $2\pi/100$ ,  $2\pi/200$  and  $2\pi/300$ . We follow the methodology of F. Boyer (see [7]) that introduces a penalty parameter  $\varepsilon = \phi(h) := h^4$ . We denote by  $E_h$ ,  $U_h$  and  $L^2_{\delta t}(0, T; U_h)$  the fully-discretized spaces associated to  $L^2(\Omega)$ ,  $L^2(\omega)$  and  $L^2(Q_T)$ .  $F_\varepsilon^{h,\delta t}$  is the discretization of  $F_\varepsilon$  and  $(y_\varepsilon^{h,\delta t}, z_\varepsilon^{h,\delta t}, u_\varepsilon^{h,\delta t})$  the solution to the corresponding fully-discrete problem of minimisation. For more details on the fully-discretization of System (5.1) and Gramian  $\Lambda$  (used to the minimisation of  $F_\varepsilon$ ), we refer to Section 3 in [7] and in [15, p. 37] respectively. The results are depicted Figure 1 and 2.

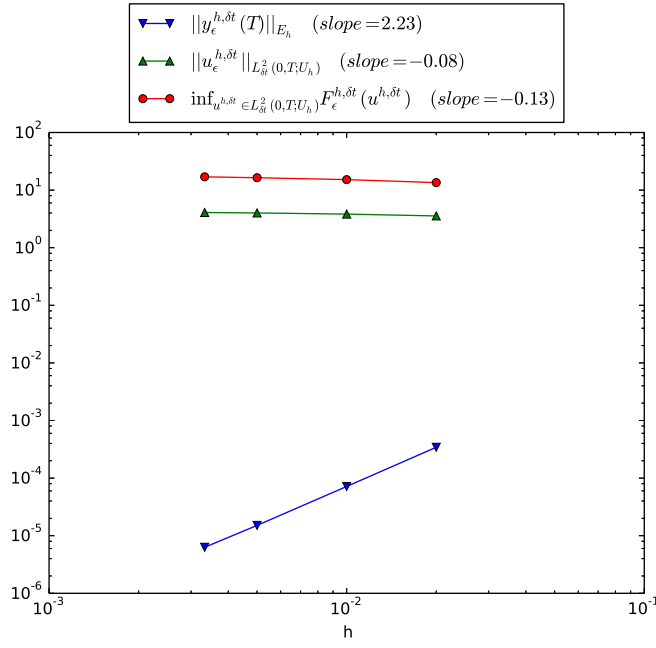


Figure 1: Distance to the target  $\|y_\epsilon^{h,\delta t}(T)\|_{E_h}$ , norm of the control  $\|u_\epsilon^{h,\delta t}\|_{L^2_{\delta t}(0,T;U_h)}$  and minimal value of the functional  $\inf_{u^{h,\delta t} \in L^2_{\delta t}(0,T;U_h)} F_\epsilon^{h,\delta t}(u^{h,\delta t})$  in Case (a).

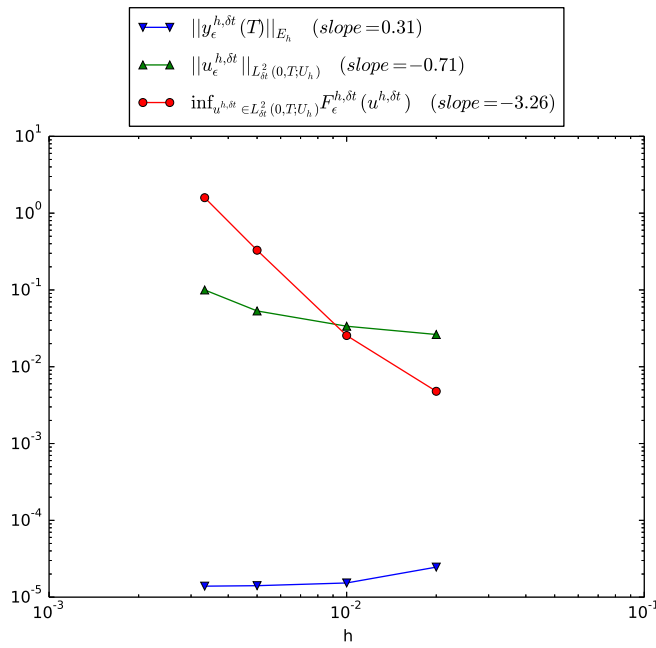


Figure 2: Distance to the target  $\|y_\epsilon^{h,\delta t}(T)\|_{E_h}$ , norm of the control  $\|u_\epsilon^{h,\delta t}\|_{L^2_{\delta t}(0,T;U_h)}$  and minimal value of the functional  $\inf_{u^{h,\delta t} \in L^2_{\delta t}(0,T;U_h)} F_\epsilon^{h,\delta t}(u^{h,\delta t})$  in Case (b).

As mentioned in the introduction of the present article (see Theorem 1.3), in both situations (a) and (b), System (5.1) is  $\Pi_1$ -approximately controllable and we observe indeed in Figure 1 and 2 that the norm of the numerical solution to System (5.1) at time  $T$  ( $-\blacktriangledown-$ ) is decreasing when reducing the penalty parameter  $\varepsilon = h^4$ .

In Figure 1, the minimal value of the functional  $F_\varepsilon^{h,\delta t}$  ( $-\bullet-$ ) as well as the  $L^2$ -norm of the control  $u_\varepsilon^{h,\delta t}$  ( $-\blacktriangle-$ ) remain roughly constant whatever is the value of  $h$  (and  $\varepsilon = h^4$ ). This appears in agreement with the results (1)-(2) of Theorem 1.3, that state the  $\Pi_1$ -null controllability of System (5.1) in Case (a) of a constant coupling coefficient  $\alpha$  (see Remark 1 (i)). Furthermore the convergence to the null target is approximately of order 2 (slope of 2.23). This is in agreement with the convergence rate established in [7, Proposition 2.2], which should be  $h^2$  for  $\varepsilon = h^4$  (this result should be in fact slightly adapted to consider  $\Pi_1$ -null controllability).

At the opposite, in Figure 2, the minimal value of the functional  $F_\varepsilon^{h,\delta t}$  as well as the  $L^2$ -norm of the control  $u_\varepsilon^{h,\delta t}$  are strongly increasing whenever  $h$  (and  $\varepsilon$ ) become smaller. This coincides with point (3) of Theorem 1.3: for the chosen value of the coupling coefficient  $\alpha$  in Case (b), no  $\Pi_1$ -null controllability of System (5.1) is expected. Moreover, convergence to the null target is quite slow, with a slope of approximately 0.31.

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