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# THE CATEGORY OF COCOMMUTATIVE HOPF ALGEBRAS IS SEMI-ABELIAN

CHRISTINE VESPA AND MARC WAMBST

ABSTRACT. We prove that the category of cocommutative Hopf algebras over a field is semi-abelian and show that its abelian core is the category of commutative and cocommutative Hopf algebras.

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*Keywords:* Hopf algebra, abelian category, semi-abelian category

## INTRODUCTION

It is a classical result that the category of commutative and cocommutative Hopf algebras is an abelian category (see for example [Tak72, Corollary 4.16] or [New75, Theorem 4.3]). It is also known that this is no more the case for the category of cocommutative (resp. commutative) Hopf algebras since the coproduct and the product are not equivalent in each of these categories.

In 2002, the more general notion of semi-abelian category emerges in category theory [JMT02]. In a semi-abelian category, classical diagram lemmas (five lemma, snake lemma) are valid. Among the examples of semi-abelian category we have the categories of groups, ring without unit, Lie algebras (and more generally algebras over a reduced linear operad) and sheaves or presheaves of these. Abelian categories are also examples of semi-abelian categories. In fact, a category  $\mathcal{C}$  is abelian precisely when both  $\mathcal{C}$  and  $\mathcal{C}^{op}$  are semi-abelian. Since then, semi-abelian categories become widely-known as the good generalization of the category of groups just as abelian categories is the good generalization of the category of abelian groups.

The principal result of this paper is the following theorem.

**Theorem 0.1.** *The category of cocommutative Hopf algebras over a field is semi-abelian.*

A category is semi-abelian if it has a zero object and finite products and is Barr-exact [Bar71] and protomodular in the sense of Bourn [Bou91]. For more details on exact, protomodular and semi-abelian categories, we refer the reader to the excellent book by Borceux and Bourn [BB04].

In this paper, we use the following characterization of semi-abelian categories given by Hartl and Loiseau in [HL11]. Namely, a category  $\mathcal{C}$  is semi-abelian if and only if the following four axioms are satisfied.

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- A1. The category  $\mathcal{C}$  is pointed, finitely complete and finitely cocomplete.
- A2. For any split epimorphism  $p : X \rightarrow Y$  with section  $s : Y \rightarrow X$  and with kernel  $\kappa : K \hookrightarrow X$ , the arrow  $\langle \kappa, s \rangle : K \amalg Y \rightarrow X$  is a cokernel.
- A3. The pullback of a cokernel is a cokernel.
- A4. The image of a kernel by a cokernel is a kernel.

The proof of Theorem 0.1 is heavily based on a result of Newman [New75] (see also [Mas91]) giving a natural correspondence between left ideal which are also two-sided coideals of a cocommutative Hopf algebra and its sub-Hopf algebras. We use this result in order to prove axioms A2. A3. and A4.

In the last section we compute the abelian core of the category of cocommutative Hopf algebras (*i.e.* the subcategory of abelian objects) to obtain the following result:

**Theorem 0.2.** *The abelian core of the category of cocommutative Hopf algebras over a field is the category of commutative and cocommutative Hopf algebras.*

Roughly speaking the abelian core of a semi-abelian category  $\mathcal{C}$  is the biggest abelian subcategory of  $\mathcal{C}$ . In particular, we recover as a corollary of this theorem the fact that the category of commutative and cocommutative Hopf algebras is abelian. It also proves that the category of cocommutative Hopf algebras is not abelian.

The category of commutative Hopf algebras is not semi-abelian. However, we will prove in [BVW] that the opposite of this category is semi-abelian.

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## 1. CONVENTIONS AND PREREQUISITES.

In the whole article,  $\mathbb{k}$  is a commutative field. By *module* we will understand module over  $\mathbb{k}$ . The unadorned symbol  $\otimes$  between two  $\mathbb{k}$ -modules will stand for  $\otimes_{\mathbb{k}}$ . We denote by  $\mathcal{H}^{\text{coco}}$  the category of cocommutative Hopf algebras over  $\mathbb{k}$  and by  $\mathcal{H}^{\text{co-coco}}$  the category of commutative and cocommutative Hopf algebras over  $\mathbb{k}$ .

Let  $H$  be a Hopf algebra. Its structure maps will be denoted as follows: multiplication  $\mu_H : H \otimes H \rightarrow H$ , comultiplication  $\Delta_H : H \rightarrow H \otimes H$ , unit  $\eta_H : \mathbb{k} \rightarrow H$ , counit  $\varepsilon_H : H \rightarrow \mathbb{k}$  and antipode  $S_H : H \rightarrow H$ . Moreover, for any  $a, b \in H$ , we will denote  $\mu_H(a \otimes b)$  by  $ab$ . The unity  $\eta_H(1)$  will be denoted by  $1_A$  or simply  $1$ . We also adopt the Sweedler-Heyneman notation  $\Delta_H(a) = a_1 \otimes a_2$ . More generally, a generic element in a tensor product of  $\mathbb{k}$ -modules  $A \otimes B$ , will be denoted by  $a \otimes b$ , the summation sign being omitted.

We will call *Hopf ideal* of a Hopf algebra  $H$  any two-sided ideal  $I$  of the algebra  $H$  which is also a two-sided coideal of the coalgebra  $H$  (*i.e.*  $\Delta_H(I) \subset I \otimes H + H \otimes I$  and  $\varepsilon_H(I) = 0$ ) such, moreover, one has  $S_H(I) \subset I$ . In particular, the structure on  $H$  induces a Hopf algebra structure on the quotient  $H/I$ .

A sub-Hopf algebra  $A$  of a Hopf algebra  $H$  will be called *normal* if, for any  $a \in A$  and  $y \in H$ , one has  $y_1 a S(y_2) \in A$ .

We will denote by  $\text{im}(\varphi)$  the linear image and  $\text{ker}(\varphi)$  the linear kernel of any morphism of Hopf algebras  $\varphi$ . A morphism of Hopf algebras  $\varphi$  is injective if  $\text{ker}(\varphi) = 0$ . The kernel  $\text{ker}(\varepsilon_H)$  of the counit of a Hopf algebra  $H$  will more specifically be denoted by  $H^+$ .

## 2. COMPLETENESS AND COCOMPLETENESS

In this section we prove Axiom A1.

**Theorem 2.1.** *The category  $\mathcal{H}^{\text{coco}}$  is pointed, finitely complete and finitely cocomplete.*

*Proof.* First, we remark that the category  $\mathcal{H}^{\text{coco}}$  is pointed. Indeed, its zero object is the ground field  $\mathbb{k}$  with initial and terminal morphisms given by the unit  $\eta_H : H \rightarrow \mathbb{k}$  and the counit  $\varepsilon_H : \mathbb{k} \rightarrow H$ .

Finite (co)completeness is the existence of finite (co)limits which is equivalent to the existence of finite (co)products and (co)equalizer. For the finite (co)products, as the category is pointed, it is in fact sufficient to prove the existence of binary (co)products. Details can be found in [Mac98, V.2].

The explicit descriptions of the binary (co)products and (co)equalizer in  $\mathcal{H}^{\text{coco}}$  are given below. The reader may check, by straightforward computations, that the given constructions fulfill the definitions.  $\square$

The definition of equalizers for morphisms of general Hopf algebras was first given by Andruskiewitsch and Devoto in [AD96] generalizing the notions of kernel sooner given in [Swe69] or [BCM86]. The same authors give explicit description of coequalizers and cokernels. For finite coproducts of Hopf algebras we refer to [Par02, §2] and for products to [Ago11b], [BW03], [Ago11a]. We simply follow the cited authors. It happens that their constructions for Hopf algebras restricts to  $\mathcal{H}^{\text{coco}}$ .

### 2.1. Equalizers, kernels and products.

First, we give the constructions of equalizers and kernels in  $\mathcal{H}^{\text{coco}}$ . By [AD96, Lemma 1.1.3],

for any two morphisms  $A \begin{array}{c} \xrightarrow{f} \\ \rightrightarrows \\ \xrightarrow{g} \end{array} B$  of  $\mathcal{H}^{\text{coco}}$  the set

$$\text{Heq}(f, g) = \{x \in A \mid f(x_1) \otimes x_2 = g(x_1) \otimes x_2\} = \{x \in A \mid x_1 \otimes f(x_2) = x_1 \otimes g(x_2)\}$$

is a sub-Hopf algebra of  $A$ . The equalizer of  $A \begin{array}{c} \xrightarrow{f} \\ \rightrightarrows \\ \xrightarrow{g} \end{array} B$  in  $\mathcal{H}^{\text{coco}}$  is the inclusion morphism  $\text{heq}(f, g) : \text{Heq}(f, g) \rightarrow A$ .

We denote by  $\text{hker}(f)$  the kernel of a morphism  $A \xrightarrow{f} B$  in  $\mathcal{H}^{\text{coco}}$  which is, by definition, the equalizer  $\text{heq}(f, \eta_B \circ \varepsilon_A)$ . Explicitly,  $\text{hker}(f)$  is the inclusion  $\text{Hker}(f) \rightarrow A$  with

$$\text{Hker}(f) = \{x \in A \mid x_1 \otimes f(x_2) = x \otimes 1\} = \{x \in A \mid f(x_1) \otimes x_2 = 1 \otimes x\}.$$

It can be easily check by straightforward computation that the kernel  $\text{Hker}(f)$  of a morphism  $A \xrightarrow{f} B$  is a normal sub-Hopf algebra of  $A$ .

The direct product of two objects  $A$  and  $B$  in  $\mathcal{H}^{\text{coco}}$  is given by the tensor product over  $\mathbb{k}$ . Indeed the product is  $\langle \pi_A, \pi_B \rangle$  where the projections

$$A \xleftarrow{\pi_A} A \otimes B \xrightarrow{\pi_B} B$$

are given by  $\pi_A(a \otimes b) = \varepsilon_B(b)a$  and  $\pi_B(a \otimes b) = \varepsilon_A(a)b$  for  $a \otimes b \in A \otimes B$ . For any two morphisms  $f : H \rightarrow A$  and  $g : H \rightarrow B$  of  $\mathcal{H}^{\text{coco}}$ , the morphism  $\varphi : H \rightarrow A \otimes B$  fulfilling the universal property of the product is defined by  $\varphi(x) = f(x_1) \otimes g(x_2)$ .

This form of the product is very specific of the cocommutative case. It is a consequence of the fact that the comultiplication  $\Delta_H$  of a Hopf algebra  $H$  is a morphism of coalgebras if and only if  $H$  is cocommutative.

**2.2. Coequalizers, cokernels and coproducts.** We now give explicit description of coequalizers and cokernels in  $\mathcal{H}^{\text{coco}}$ . Let  $A$  and  $B$  be two cocommutative Hopf algebras. For

any two morphisms  $A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$ , set

$$J = \{f(x) - g(x) \mid x \in A\} \quad \text{and} \quad \text{Hcoeq}(f, g) = B / BJB$$

The projection  $\text{hcoeq}(f, g) : B \rightarrow \text{Hcoeq}(f, g)$  is the coequalizer of  $A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$ , set

$$J = \{f(x) - g(x) \mid x \in A\} \quad \text{and} \quad \text{Hcoeq}(f, g) = B / BJB$$

in  $\mathcal{H}^{\text{coco}}$ . In the sequel, for simplicity of notations, we sometimes denote the Hopf ideal  $BJB$  by  $\langle J \rangle$ .

The cokernel  $\text{hcoker}(f)$  of a morphism  $A \xrightarrow{f} B$  in  $\mathcal{H}^{\text{coco}}$  is, by definition, the coequalizer  $\text{hcoeq}(f, \eta_B \circ \varepsilon_A)$ . Notice that, when  $g = \eta_B \circ \varepsilon_A$ , the set  $J$  reduces to  $J = f(A^+)$  where  $A^+ = \ker(\varepsilon_A)$  is the linear kernel of the counit of  $A$ . So the cokernel of  $f$  is the projection map

$$\text{hcoker}(f) : B \rightarrow B / \langle f(A^+) \rangle.$$

We set  $\text{Hcoker}(f) = B / \langle f(A^+) \rangle$ .

Let  $A$  and  $B$  be two objects in  $\mathcal{H}^{\text{coco}}$ . The coproduct object  $A \amalg B$  of  $A$  and  $B$  in  $\mathcal{H}^{\text{coco}}$  has the following explicit description ([Par02] or [Ago11a]). The coproduct  $A \amalg B$  is the module spanned over  $\mathbb{k}$  as an algebra by the elements  $1, t_a$  and  $t_b$  with  $a \in A$  and  $b \in B$  submitted to the relations

$$t_\lambda = \lambda, \quad t_{\lambda a + b} = \lambda t_a + t_b, \quad t_{aa'} = t_a t_{a'}, \quad t_{bb'} = t_b t_{b'}$$

with  $\lambda \in \mathbb{k}$ , with  $a, a' \in A$ , and  $b, b' \in B$ .

The coproduct  $A \amalg B$  is endowed with a Hopf algebra structure given by:

$$\Delta(t_a) = \sum t_{a_1} \otimes t_{a_2}, \quad \varepsilon(t_a) = \varepsilon(a), \quad S(t_a) = t_{S(a)}.$$

with  $a \in A$  or  $a \in B$ . The coproduct diagram is given by

$$A \xrightarrow{\iota_A} A \amalg B \xleftarrow{\iota_B} B$$

with  $\iota_A(a) = t_a$  and  $\iota_B(b) = t_b$  for  $a \in A$ , and  $b \in B$ . This construction satisfies the universal property of coproduct. Indeed, for any two morphisms  $f : A \rightarrow H$  and  $g : B \rightarrow H$  of  $\mathcal{H}^{\text{coco}}$ ,

the unique morphism  $h : A \amalg B \rightarrow H$  such as one has  $h \circ \iota_A = f$  and  $h \circ \iota_B = g$  is defined by  $h(t_a) = f(a)$  and  $h(t_b) = g(b)$  on the generators of  $A \amalg B$  with  $a \in A$  and  $b \in B$ .

### 3. THE NEWMAN CORRESPONDENCE, THE SEMI-DIRECT PRODUCT

In this section, we recall some constructions and results involving kernel and cokernels which we will use in the sequel.

**3.1. The Newman correspondence.** The following result is crucial in the sequel of the paper

**Theorem 3.1.** [New75] *For any cocommutative Hopf algebra over a field, there is a one-to-one correspondence between its sub-Hopf algebras and its left ideals which are also two-sided coideals.*

If  $G$  is a sub-Hopf algebra of a Hopf algebra  $H$ , Newman associates the ideal  $\tau(G) = HG^+$ . He proves that  $\tau$  is bijective with reciprocal  $\sigma(I) = \text{Hker}(H \rightarrow H/I)$ .

We state three lemmas directly induced by this result.

**Lemma 3.2.** *Let  $H$  be a cocommutative Hopf algebra over a field. For any Hopf ideal  $I$  of  $H$ , it exists a sub-coalgebra  $G$  of  $H$  such that one has the isomorphism of cocommutative Hopf algebras*

$$H/I \cong H/HG^+H.$$

In other words the projection map  $H \rightarrow H/I$  is a cokernel in the category  $\mathcal{H}^{\text{coco}}$ .

*Proof.* The two-sided ideal  $I$  is in particular a left ideal. So after [New75, Corollary 3.4], one has  $I \cong HG^+$  for  $G = \text{Hker}(H \rightarrow H/I)$ . As  $I$  is also a right ideal, we deduce  $I = IH \cong HG^+H$ .  $\square$

As a consequence, we have:

**Lemma 3.3.** *Let  $f : H \rightarrow H'$  be a surjective map of cocommutative Hopf algebras over a field. The map  $f$  is a cokernel in the category  $\mathcal{H}^{\text{coco}}$ .*

*Proof.* Consider the linear ideal  $I = \ker(f)$  of  $f : H \rightarrow H'$ . It is well know that  $I$  is a Hopf ideal. Moreover, one has  $H/I \cong H'$ . After Lemma 3.2, the map is a cokernel.  $\square$

**Lemma 3.4.** *Let  $H$  be a cocommutative Hopf algebra over a field and  $G$  one of its normal sub-Hopf algebras. Then we have  $G \cong \text{Hker}(H \rightarrow H/HG^+H)$ .*

In other words any inclusion of a normal sub-Hopf algebra into a Hopf algebra is a kernel.

*Proof.* As  $G$  is a normal sub-Hopf algebra, the usual trick  $gh = h_1(S(h_2)gh_3)$  for  $g \in G$  and  $h \in H$  shows that  $HG^+H = HG^+$ . Moreover one has  $HG^+ = H(\text{Hker}(H \rightarrow H/HG^+H))^+$  by [New75, Corollary 3.4] and, as  $\tau$  is injective by [New75, Corollary 2.5], on has  $G \cong \text{Hker}(H \rightarrow H/HG^+H)$ .  $\square$

We also point out an important lemma which can be found as a part of the proof of [New75, Theorem 4.4]

**Lemma 3.5.** *A monomorphism in  $\mathcal{H}^{\text{coco}}$  is injective.*

*Proof.* Let  $m : X \rightarrow Y$  be a monomorphism in  $\mathcal{H}^{\text{coco}}$ . The linear kernel  $\ker(m)$  of  $m$  is a Hopf ideal of  $X$  by [Swe69, Theorem 4.17]. By Lemma 3.2, it exists a Hopf algebra inclusion  $G \xrightarrow{\iota} X$  such as  $X\iota(G^+)X = \ker(m)$ . This implies  $(m \circ \iota)(g) = \varepsilon_X(g)1_Y = (m \circ \eta_Y \circ \varepsilon_X \circ \iota)(g)$  for  $g \in G$ . So we get  $\iota = \eta_Y \circ \varepsilon_X \circ \iota$  and thus we have  $G = \mathbb{k}$ . Finally, one has  $\ker(m) = X\mathbb{k}^+X = \{0\}$ .  $\square$

**3.2. The semi-direct product of Hopf algebras.** We will need a notion of semi-direct product of Hopf algebras. The construction we will recall for our purpose is very special case of the now classical semi-direct product introduced in the article [BCM86] to which we refer for proofs. Anyway properties of the product we state here may also be checked through direct calculation.

Let  $Y$  and  $K$  be two Hopf algebras, we will say that  $K$  is a  $Y$ -Hopf algebra if  $Y$  acts on  $K$ . In other words if it exists an action map  $\rightarrow - : Y \otimes K \rightarrow K$  such as the following axioms are satisfied:

$$\begin{aligned} y \rightarrow (ab) &= (y_1 \rightarrow a)(y_2 \rightarrow b) & 1_Y \rightarrow a &= a \\ (yy') \rightarrow a &= y \rightarrow (y' \rightarrow a) & y \rightarrow 1_K &= \varepsilon_Y(y)1_K \end{aligned}$$

with  $y, y' \in Y$  and  $a, b \in K$ . We denote by  $Y\text{-}\mathcal{H}^{\text{coco}}$  the subcategory of  $\mathcal{H}^{\text{coco}}$  having as objects the  $Y$ -Hopf algebras and as morphisms those of  $\mathcal{H}^{\text{coco}}$  compatible with the action.

Given an object  $K$  in  $Y\text{-}\mathcal{H}^{\text{coco}}$ , one may define a semi-direct product  $K\#Y$  of  $K$  and  $Y$ . As a module it is  $K \otimes Y$  on which a structure of Hopf algebras is endowed by

$$\begin{aligned} (a \otimes y)(b \otimes y') &= a(y_1 \rightarrow b) \otimes y_2 y' \\ \eta(1) &= 1 \otimes 1 \\ \Delta(a \otimes y) &= (a_1 \otimes y_1) \otimes (a_2 \otimes y_2) \\ \varepsilon(a \otimes y) &= \varepsilon(a)\varepsilon(y) \\ S(x \otimes y) &= (S(y_1) \rightarrow S(a)) \otimes S(y_2) \end{aligned}$$

given for  $a, b \in K$  and  $y, y' \in Y$ . This product is nothing else than the product  $K\#_{\sigma}Y$  of [BCM86] with cocycle  $\sigma = \eta_Y \circ (\varepsilon_K \otimes \varepsilon_K) : K \otimes K \rightarrow Y$ . In the sequel, an element  $a \otimes y \in K\#Y$  will be denoted by  $a\#y$ .

Consider the category  $\mathcal{P}t_Y$  of pointed objects over an object  $Y$  of  $\mathcal{H}^{\text{coco}}$ . Its objects are the couples of maps  $(p, s) : X \xrightleftharpoons[p]{s} Y$  of  $\mathcal{H}^{\text{coco}}$  such that  $s$  is a section of  $p$  (i.e.  $p \circ s = \text{id}_Y$ ).

The morphisms of  $\mathcal{P}t_Y$  between two objects  $X \xrightleftharpoons[p]{s} Y$  and  $X' \xrightleftharpoons[p']{s'} Y$  are the maps  $f : X \rightarrow X'$  such as one has  $p' \circ f = p$  and  $f \circ s = s'$ .

**Lemma 3.6.** *Let  $Y$  be an object of  $\mathcal{H}^{\text{coco}}$ . The categories  $\mathcal{P}t_Y$  and  $Y\text{-}\mathcal{H}^{\text{coco}}$  are equivalent.*

*Proof.* Details may be found in [BCM86], we will only describe the correspondence between objects. To an action  $- \rightharpoonup - : Y \otimes K \rightarrow K$  one associates the maps

$$p = \varepsilon_K \otimes \text{id}_Y : K \# Y \rightarrow Y \quad \text{and} \quad s = \eta_K \otimes \text{id}_Y : Y \rightarrow K \# Y.$$

On the other hand, given the data  $X \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{p} \end{array} Y$ , one sets  $K = \text{Hker}(p)$ . It is easy to check that  $y \rightharpoonup k = s(y_1)ks(S_Y(y_2))$  defines an action of  $Y$  on  $K$ .

The equivalence is based on the isomorphism between  $K \# Y$  and  $X$  given by

$$\begin{aligned} \mathcal{F} : X &\rightarrow K \# Y & \mathcal{G} : Y \# K &\rightarrow X \\ x &\mapsto x_1 s(S p(x_2)) \# p(x_3) & k \# y &\mapsto ky \end{aligned}$$

□

#### 4. PULLBACKS OF COKERNELS, REGULARITY

This section is devoted to the proof of Axiom A3. As a consequence, we prove that  $\mathcal{H}^{\text{coco}}$  is a regular category and a homological category.

From the constructions of products and equalizers one easily derives the one for pullbacks in  $\mathcal{H}^{\text{coco}}$ . Let  $A, B, C$  be cocommutative Hopf algebras and  $f : A \rightarrow C$  and  $g : B \rightarrow C$  be morphisms of Hopf algebras. The pullback object of  $A$  and  $B$  over  $C$  is given by

$$A \amalg_C B = \{a \otimes b \in A \otimes B \mid a_1 \otimes f(a_2) \otimes b = a \otimes g(b_1) \otimes b_2\}.$$

It is a sub-Hopf algebra of the product Hopf algebra  $A \otimes B$ . One has the commutative diagram

$$\begin{array}{ccc} A \amalg_C B & \xrightarrow{\pi_B} & B \\ \pi_A \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

with  $\pi_B(a \otimes b) = \varepsilon(a)b$  and  $\pi_A(a \otimes b) = \varepsilon(b)a$ .

The universal property of pullbacks is given in the following way. For  $H$  a cocommutative Hopf algebra and  $\gamma : H \rightarrow B$  and  $\varphi : H \rightarrow A$  two morphisms, there exists a unique morphism  $F : H \rightarrow A \amalg_C B$  making commutative the diagram

$$\begin{array}{ccccc} H & & & & \\ & \searrow \gamma & & & \\ & & A \amalg_C B & \xrightarrow{\pi_B} & B \\ & \searrow F & \downarrow \pi_A & & \downarrow g \\ & & A & \xrightarrow{f} & C \\ & \searrow \varphi & & & \end{array}$$

The morphism  $F$  is defined by  $F(d) = \varphi(d_1) \otimes \gamma(d_2)$  for any  $d \in H$ .

**Proposition 4.1.** *In the category  $\mathcal{H}^{\text{coco}}$ , the pullback of a cokernel is a cokernel.*



*Proof.*— Consider the map  $A \xrightarrow{f} B$  in  $\mathcal{H}^{\text{coco}}$ . For notation convenience, we set  $\langle f(A^+) \rangle = Bf(A^+)B$  and  $B/\langle f(A^+) \rangle = B'$ . One has  $\text{Hcoker}(f) = B'$ . The cokernel of  $f$  is the projection map  $\text{hcoker}(f) = \pi : B \rightarrow B'$ .

We wish to show that the pullback  $B \amalg_{B'} D \xrightarrow{\pi_D} D$  is still a cokernel. First, we remark that, in the pullback diagram,

$$\begin{array}{ccc} B' \amalg_{B'} D & \xrightarrow{\pi'_D} & D \\ \pi_{B'} \downarrow & & \downarrow \gamma \\ B' & \xrightarrow{\text{id}_{B'}} & B' \end{array}$$

the map  $\pi'_D = \varepsilon_{B'} \otimes \text{id}_D$  is an isomorphism. It is a consequence of the universal property of pullbacks. In this particular case, one can also prove it directly by checking that its inverse is the map  $D \rightarrow B' \amalg_{B'} D$  defined by  $d \mapsto 1 \otimes d$ .

The map  $B \amalg_{B'} D \xrightarrow{\pi_D} D$  factors by  $B \amalg_{B'} D \xrightarrow{\pi \otimes \text{id}_D} B' \amalg_{B'} D \xrightarrow{\pi'_D} D$ . As  $\pi : B \rightarrow B'$  is surjective, so is  $\pi \otimes \text{id}_D$ . After Lemma 3.3, the latter map is a cokernel.  $\square$

## 5. COPRODUCTS AND SPLIT EPIMORPHISMS

The following proposition proves the Axiom A.2 for the category  $\mathcal{H}^{\text{coco}}$ .

**Proposition 5.1.** *Let  $p$  be a morphism in  $\mathcal{H}^{\text{coco}}$ ,  $s$  one of its sections (i.e.  $pos = \text{id}_Y$ ) in  $\mathcal{H}^{\text{coco}}$  and  $\kappa : K \rightarrow X$  be the kernel of  $p$  in  $\mathcal{H}^{\text{coco}}$  :*

$$K \xrightarrow{\kappa} X \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{p} \end{array} Y$$

*The arrow  $\langle \kappa, s \rangle : K \amalg Y \rightarrow X$  is a cokernel in  $\mathcal{H}^{\text{coco}}$ .*

*Proof.* First remark that if an element  $x \in X$ , is also an element of  $K = \text{Hker}(p)$ , then, for any  $y \in Y$ , on also has  $s(y_1)xs(S_Y(y_2)) \in K$ . It can be easily check by straightforward computation that the formula  $y \mapsto x = s(y_1)xs(S_Y(y_2))$  defines an action of  $Y$  over  $K$ .

Now consider the two linear maps  $f, g : K \otimes Y \rightarrow K \amalg Y$  defined by  $f(x \otimes y) = t_{y_1 \rightarrow x} t_{y_2}$  and  $g(x \otimes y) = t_y t_x$  with  $y \in Y$  and  $x \in K$ . We denote by  $L$  the linear image  $\text{im}(f - g)$ .

Note that both maps preserve the coalgebra structure so we have

$$\Delta \circ (f - g) = ((f - g) \otimes f + g \otimes (f - g)) \circ \Delta.$$

This later relation yields that  $L$  is a two-sided coideal of  $K \amalg Y$ . We set  $U = (K \amalg Y)L(K \amalg Y)$  which is both a two-sided ideal a two-sided coideal.

Moreover, for any  $x \in K$  and  $y \in Y$ , on computes

$$\begin{aligned} S(t_{y_1 \rightarrow x} t_{y_2} - t_y t_x) &= t_{S(y_1)} t_{y_2 \rightarrow S(x)} - t_{S(x)} t_{S(y)} \\ &= t_{S(y_1)} t_{y_2 \rightarrow S(x)} t_{y_3} t_{S(y_3)} - t_{S(y_1)} t_{y_2} t_{S(x)} t_{S(y_3)} \\ &= t_{S(y_1)} ((f - g)(S(x) \otimes y_2)) t_{S(y_3)} \end{aligned}$$

Notice that for the first equality, we used cocommutativity and the relation  $S(y \dashv x) = y \dashv S(x)$  which is a consequence of  $S^2 = \text{id}$ . (Remember that the antipode of a cocommutative Hopf algebra is involutive (cf. [Swe69, Proposition 4.0.1])). Our computation proves  $S(U) \subset U$  and consequently that  $U$  is a Hopf ideal.

One clearly has  $(K \amalg Y)/U \simeq K \# Y$  which is isomorphic to  $X$  after the proof of Lemma 3.6. Moreover, after Lemma 3.2,  $(K \amalg Y) \rightarrow (K \amalg Y)/U$  is a cokernel.  $\square$

**Corollary 5.2.** *The category  $\mathcal{H}^{\text{coco}}$  is finitely cocomplete homological.*

*Proof.* The category satisfies to Axioms A1, A2 and A3 (cf [HL11]).  $\square$

**Corollary 5.3.** *The category  $\mathcal{H}^{\text{coco}}$  is regular.*

*Proof.* The result of Proposition 5.1 combined with the existence of finite limits and coequalizers fulfills the definitions axioms of regular categories.  $\square$

## 6. IMAGES OF KERNELS

It remains to check Axiom A.4. As the category  $\mathcal{H}^{\text{coco}}$  is regular, the image of a morphism is canonically defined as the coequalizer of its kernel pair (see [Bar71]).

Let  $f : X \rightarrow Y$  be a morphism in  $\mathcal{H}^{\text{coco}}$ . The coequaliser object of the kernel pair of  $f$  is  $\text{Hcoeq}(\pi_1, \pi_2)$  where  $\pi_1$  and  $\pi_2$  are the canonical maps of the pullback diagram

$$\begin{array}{ccc} X \amalg_Y X & \xrightarrow{\pi_1} & X \\ \pi_2 \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

We have  $X \amalg_Y X = \{x \otimes x' \in X \otimes X \mid x_1 \otimes f(x_2) \otimes x' = x \otimes f(x'_1) \otimes x'_2\}$  and  $\pi_1(x \otimes x') = \varepsilon(x')x$  and  $\pi_2(x \otimes x') = \varepsilon(x)x'$ .

One has  $\text{Hcoeq}(\pi_1, \pi_2) = X/XJX$  where  $J = \{\varepsilon(x')x - \varepsilon(x)x' \mid x \otimes x' \in X \otimes_Y X\}$ . It is easy to see that  $J$  is in fact a Hopf ideal of  $X$  so  $XJX = J$ . The image object  $X/J$  of  $f$  will be denoted by  $\text{HIm}(f)$ .

After [BB04] the morphism  $f$  factorizes as a product of the regular epimorphism  $\pi = \text{hcoeq}(\pi_1, \pi_2)$  and a monomorphism  $\iota$ . One has the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi \downarrow & \nearrow \iota & \\ \text{HIm}(f) & & \end{array}$$

the morphism  $\iota$  being induced by  $f : X \rightarrow Y$ . In our case, in fact, the notion of image and linear image coincide.

**Lemma 6.1.** *In the category  $\mathcal{H}^{\text{coco}}$ , for any morphism  $f$ , one has  $\text{HIm}(f) \cong \text{im}(f)$ .*

*Proof.* The following factorization diagram derives from the construction of  $\iota$ :

$$\begin{array}{ccc} X & \xrightarrow{f} & \text{im}(f) \xrightarrow{\subset} Y \\ \pi \downarrow & \nearrow \iota' & \nearrow \iota \\ \text{HIm}(f) & & \end{array}$$

where  $\iota'$  is still a monomorphism but is also surjective. After Lemma 3.5 a monomorphism is injective.  $\square$

We can now prove that Axiom A.4 is fulfilled.

**Proposition 6.2.** *In  $\mathcal{H}^{\text{coco}}$  the image of a kernel is a kernel.*

*Proof.* Consider the following commutative diagram in  $\mathcal{H}^{\text{coco}}$ :

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow f & & \\ \text{Hker}(g) & \xrightarrow{\text{hker}(g)} & X & \xrightarrow{g} & Z \\ \downarrow \pi & \searrow \pi' & \downarrow \text{hcoker}(f) & & \\ \text{HIm}(\pi') & \xrightarrow{\iota} & \text{Hcoker}(f) & & \end{array}$$

As  $\text{Hker}(g)$  is a normal sub-Hopf algebra of  $X$ , its linear image, through the projection  $X \rightarrow \text{Hcoker}(f)$  is a normal sub-Hopf algebra of  $\text{Hcoker}(f)$ . The later linear image is nothing else than  $\text{HIm}(\pi')$ . After Lemma 3.4, it is a kernel object under our assumptions.  $\square$

At this point, we proved that all axioms A1, A2, A3 and A4 are fulfilled for  $\mathcal{H}^{\text{coco}}$  and so proved Theorem 0.1.

## 7. THE ABELIAN CORE, THE CATEGORICAL SEMI-ABELIAN PRODUCT

This section is widely inspired by [Bor04]. In a first time, we determine the abelian core of  $\mathcal{H}^{\text{coco}}$ . In a second time, we prove that the categorical semi-direct product in  $\mathcal{H}^{\text{coco}}$  is nothing else than the semi-direct product defined in section 3.

**Lemma 7.1.** *Let  $A$  be a sub-algebra of a cocommutative Hopf algebra  $H$ . The sub-Hopf algebra is normal if and only if the inclusion  $A \rightarrow H$  is normal in  $\mathcal{H}^{\text{coco}}$ .*

*Proof.* Consider a normal map  $A \rightarrow H$  in  $\mathcal{H}^{\text{coco}}$ . The sub-object  $A$  is a sub-Hopf algebra of  $H$  such as it exists a morphism  $\varphi : H \rightarrow H'$  and  $A = \text{Hker}(\varphi)$ . We already noticed that kernel objects are normal sub-Hopf algebras. The converse assertion is Lemma 3.4.  $\square$

The following proposition is Theorem 0.2.

**Proposition 7.2.** *The full sub-category of abelian objects of  $\mathcal{H}^{\text{coco}}$  is  $\mathcal{H}^{\text{co-coco}}$ .*

*Proof.* We use the characterization of [Bor04, Theorem 6.9] which states that an object  $C$  in a semi-abelian category is abelian if and only if its diagonal  $C \rightarrow C \otimes C$  is normal. In our case, if  $C$  is an object of  $\mathcal{H}^{\text{coco}}$ , the diagonal map is nothing else than the comultiplication  $\Delta_C$ . So after Lemma 7.1, it suffices to prove that  $C$  is abelian if and only if  $\text{im}(\Delta_C)$  is a normal sub-Hopf algebra of  $C \otimes C$ .

If  $C$  is commutative, so is  $C \otimes C$  and as sub-Hopf algebras of a commutative algebra are normal, it follows that  $\text{im}(\Delta_C)$  is.

On the other hand suppose that  $\text{im}(\Delta_C)$  is a normal sub-Hopf algebra of  $C \otimes C$ . For any two elements  $a, c \in C$  it exists  $d \in C$  such as we have

$$\Delta(d) = (c_1 \otimes 1)(a_1 \otimes a_2)S(c_2 \otimes 1) = c_1 a_1 S(c_2) \otimes a_2.$$

Successively applying  $\varepsilon_C$  to each tensor factor of the previous equality, we get

$$d = \varepsilon_C(c)a = c_1 a S(c_2).$$

As the identity is true for any  $a \otimes c \in C$ , we may apply it to the first tensor factor of  $a \otimes c_1 \otimes c_2$  and get

$$\varepsilon_C(c_1)a \otimes c_2 = c_1 a S(c_2) \otimes c_3 \implies \varepsilon_C(c_1)ac_2 = c_1 a S(c_2)c_3 \implies ac = ca.$$

Thus,  $C$  is commutative. □

We retrieve known results: the category  $\mathcal{H}^{\text{co-coco}}$  is abelian and the category  $\mathcal{H}^{\text{coco}}$  is not abelian.

To end the article, we prove that the semi-direct product defined in Section 3 is the semi-direct product in  $\mathcal{H}^{\text{coco}}$  in the categorical sense defined in [Bor04]. We will follow the latter reference.

For any object  $Y$  in  $\mathcal{H}^{\text{coco}}$  we have a pair of adjoint functors:

$$\begin{array}{ccc} \text{Hker} : & \mathcal{P}t_Y \longrightarrow \mathcal{H}^{\text{coco}} & \mathcal{H}^{\text{coco}} \longrightarrow \mathcal{P}t_Y \\ & X \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{p} \end{array} Y \mapsto \text{Hker}(p) & \text{and} \quad K \mapsto K \amalg Y \begin{array}{c} \xleftarrow{\iota_Y} \\ \xrightarrow{(\varepsilon_K, \text{id}_Y)} \end{array} Y \end{array}$$

where the functor  $\text{Hker}$  is monadic. Then one can consider the monad  $\mathbb{T}_Y$  associated to  $\text{Hker}$ . By definition, the semi-direct product of an algebra  $(K, \xi)$  for the monad  $\mathbb{T}_Y$  and the object  $Y$  is the domain of the pointed object  $(p, s) : X \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{p} \end{array} Y$  corresponding to  $(K, \xi)$  via the equivalence  $\mathcal{P}t_Y \cong (\mathcal{H}^{\text{coco}})^{\mathbb{T}_Y}$ .

**Theorem 7.3.** *Let  $Y$  be an object in  $\mathcal{H}^{\text{coco}}$ . Let  $(K, \xi)$  be an algebra for the monad  $\mathbb{T}_Y$ . The semi-direct product of an algebra  $(K, \xi)$  and  $Y$  is  $K \# Y$ .*

*Proof.* The proof given for the category of groups in [Bor04, Section 5] is still valid in our case if one replaces [Bor04, Proposition 5.7] by Lemma 3.6. □

## REFERENCES

- [AD96] N. Andruskiewitsch and J. Devoto. Extensions of hopf algebras. *St. Petersburg Mat. J.*, 7(1):17–52, 1996.
- [Ago11a] A. L. Agore. Categorical constructions for Hopf algebras. *Comm. Algebra*, 39:1476–1481, 2011.
- [Ago11b] A. L. Agore. Limits of coalgebras, bialgebras and Hopf algebras. *Proc. Amer. Math. Soc.*, 139(3):855–863, 2011.
- [Bar71] Michael Barr. Exact categories. In *in.*, volume 236 of *Lecture Notes in Mathematics*, pages 1–120. Springer, Berlin, 1971.
- [BB04] F. Borceux and D. Bourn. *Mal'cev, protomodular, homological and semi-abelian categories*, volume 566 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2004.
- [BCM86] R. J. Blattner, M. Cohen, and S. Montgomery. Crossed products and inner actions of Hopf algebras. *Trans. Amer. Math. Soc.*, 298(2):671–711, 1986.
- [Bor04] F. Borceux. A survey of semi-abelian categories. In *Galois theory, Hopf algebras, and semiabelian categories*, volume 43 of *Fields Inst. Commun.*, pages 27–60. Amer. Math. Soc., Providence, RI, 2004.
- [Bou91] Dominique Bourn. Normalization equivalence, kernel equivalence and affine categories. In *Category theory (Como, 1990)*, volume 1488 of *Lecture Notes in Math.*, pages 43–62. Springer, Berlin, 1991.
- [BVW] Dominique Bourn, Christine Vespa, and Marc Wambst. The category of cocommutative Hopf algebras is a semi-abelian category of internal groups. (In preparation).
- [BW03] T. Brzezinski and R. Wisbauer. *Corings and comodules*, volume 309 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.
- [HL11] M. Hartl and B. Loiseau. A characterization of finite cocomplete homological and of semi-abelian categories. *Cah. Topol. Géom. Différ. Catég.*, 52(1):77–80, 2011.
- [JMT02] George Janelidze, László Márki, and Walter Tholen. Semi-abelian categories. *J. Pure Appl. Algebra*, 168(2-3):367–386, 2002. Category theory 1999 (Coimbra).
- [Mac98] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [Mas91] Akira Masuoka. On Hopf algebras with cocommutative coradicals. *J. Algebra*, 144(2):451–466, 1991.
- [New75] K. Newman. A correspondence between bi-ideals and sub-hopf algebras in cocommutative hopf algebras. *Journal of Algebra*, 36(1):1 – 15, 1975.
- [Par02] B. Pareigis. Lectures on quantum groups and noncommutative geometry. 2002.
- [Swe69] M. E. Sweedler. *Hopf algebras*. Princeton University Press, 1969.
- [Tak72] Mitsuhiro Takeuchi. A correspondence between Hopf ideals and sub-Hopf algebras. *Manuscripta Math.*, 7:251–270, 1972.

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