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AN IMPROVEMENT OF THE MIXING RATES IN A COUNTER-EXAMPLE TO THE WEAK INVARIANCE PRINCIPLE

DAVIDE GIRAUDDO

ABSTRACT. In [1], the authors gave an example of absolutely regular strictly stationary process which satisfies the central limit theorem but not the weak invariance principle. For each $q < 1/2$, the process can be constructed with mixing rates of order N^{-q} . The goal of this note is to show that actually the same construction can give mixing rates of order N^{-q} for a given $q < 1$.

RÉSUMÉ. Dans [1], les auteurs ont fourni un exemple de processus strictement stationnaire β -mélangeant vérifiant le théorème limite central mais pas le principe d'invariance faible. Pour tout $q < 1/2$, le processus peut être construit avec des taux de mélange de l'ordre de N^{-q} . L'objectif de cette note est de montrer que la même construction peut fournir des taux de mélange de l'ordre de N^{-q} pour un $q < 1$ donné.

1. NOTATIONS AND MAIN RESULT

We recall some notations in order to make this note more self-contained. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. If $T: \Omega \rightarrow \Omega$ is one-to-one, bi-measurable and measure preserving (in sense that $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{F}$), then the sequence $(f \circ T^k)_{k \in \mathbb{Z}}$ is strictly stationary for any measurable $f: \Omega \rightarrow \mathbb{R}$. Conversely, each strictly stationary sequence can be represented in this way.

For a zero mean square integrable $f: \Omega \rightarrow \mathbb{R}$, we define $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j$, $\sigma_n^2(f) := \mathbb{E}(S_n(f)^2)$ and $S_n^*(f, t) := S_{[nt]}(f) + (nt - [nt])f \circ T^{[nt]}$, where $[x]$ is the greatest integer which is less than or equal to x .

Define the β -mixing coefficients by

$$(1) \quad \beta(\mathcal{A}, \mathcal{B}) := \frac{1}{2} \sup \sum_{i=1}^I \sum_{j=1}^J |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|,$$

where the supremum is taken over the finite partitions $\{A_i, 1 \leq i \leq I\}$ and $\{B_j, 1 \leq j \leq J\}$ of Ω of elements of \mathcal{A} (respectively of \mathcal{B}). They were introduced by Volkonskii and Rozanov [4].

For a strictly stationary sequence $(X_k)_{k \in \mathbb{Z}}$ and $n \geq 0$ we define $\beta_X(n) = \beta(n) = \beta(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty)$ where \mathcal{F}_u^v is the σ -algebra generated by X_k with $u \leq k \leq v$ (if $u = -\infty$ or $v = \infty$, the corresponding inequality is strict).

Theorem. *Let $\delta > 0$. There exists a strictly stationary real valued process $Y = (Y_k)_{k \geq 0} = (f \circ T^k)_{k \geq 0}$ satisfying the following conditions:*

- a) *the central limit theorem with normalization \sqrt{n} takes place;*
- b) *the weak invariance principle with normalization \sqrt{n} does not hold;*
- c) *$\sigma_N(f)^2 \asymp N$;*
- d) *for some positive C and each integer N , $\beta_Y(N) \leq C \cdot N^{-1+\delta}$;*
- e) *$Y_0 \in \mathbb{L}^p$ for any $p > 0$.*

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We refer the reader to Remark 2 of [1] for a comparison with existing results about the weak invariance principle for strictly stationary mixing sequences.

2. PROOF

We recall the construction given in [1]. Let us consider an increasing sequence of positive integers $(n_k)_{k \geq 1}$ such that

$$(2) \quad n_1 \geq 2 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{n_k} < \infty,$$

and for each integer $k \geq 1$, let A_k^-, A_k^+ be disjoint measurable sets such that $\mu(A_k^-) = 1/(2n_k^2) = \mu(A_k^+)$.

Let the random variables e_k be defined by

$$(3) \quad e_k(\omega) := \begin{cases} 1 & \text{if } \omega \in A_k^+, \\ -1 & \text{if } \omega \in A_k^-, \\ 0 & \text{otherwise.} \end{cases}$$

We can choose the dynamical system $(\Omega, \mathcal{F}, \mu, T)$ and the sets A_k^+, A_k^- in such a way that the family $(e_k \circ T^i)_{k \geq 1, i \in \mathbb{Z}}$ is independent. We define $A_k := A_k^+ \cup A_k^-$ and

$$(4) \quad h_k := \sum_{i=0}^{n_k-1} U^{-i} e_k - U^{-n_k} \sum_{i=0}^{n_k-1} U^{-i} e_k, \quad h := \sum_{k=1}^{+\infty} h_k.$$

Let $i(N)$ denote the unique integer such that $n_{i(N)} \leq N < n_{i(N)+1}$. We shall show the following intermediate result.

Proposition 1. *Assume the sequence $(n_k)_{k \geq 1}$ satisfies (2) and the following condition:*

$$(5) \quad \text{there exists } \eta > 0 \text{ such that for each } k, \quad n_{k+1} \geq n_k^{1+\eta}.$$

Then:

- a') $n^{-1/2} S_n(h) \rightarrow 0$ in probability;
- b') the process $(N^{-1/2} S_N^*(h, \cdot))_{N \geq 1}$ is not tight in $C[0, 1]$;
- c') $\sigma_N(h)^2 \lesssim N$;
- d') for some positive C , $N \cdot \beta_Y(N) \leq C n_{i(N)+1} / n_{i(N)}$;
- e') $h \in \mathbb{L}^p$ for any $p > 0$.

Adding a mean-zero nondegenerate independent sequence $(m \circ T^i)_{i \in \mathbb{Z}}$ with moments of any order greater than 2, the variance of the N th partial sum of $((m+h) \circ T^i)_{i \geq 1}$ is bounded above and below by a quantity proportional to N . Defining $f := m+h$, we have the following corollary:

Corollary 2. *Assume the sequence $(n_k)_{k \geq 1}$ satisfies (5). Then $(f \circ T^i)_{i \geq 0}$ satisfies a), b), c), d') and e).*

For $k \geq 1$ and $N \geq n_k$, the N partial sum of h_k admits the expression

$$(6) \quad S_N(h_k) = \sum_{j=1}^{n_k} j U^{j+N-2n_k} e_k + \sum_{j=1}^{n_k-1} (n_k - j) U^{j+N-n_k} e_k - \sum_{j=1}^{n_k} j U^{j-2n_k} e_k - \sum_{j=1}^{n_k-1} (n_k - j) U^{j-n_k} e_k.$$

Let us prove Proposition 1. Item a') follows from the fact that h is a coboundary (see the explanation before Section 2.2 of [1]).

For b'), we recall the following lemma (Lemma 10, [1]).

Lemma 3. *There exists N_0 such that*

$$(7) \quad \mu \left\{ \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq n_k \right\} > 1/4$$

whenever $n_k \geq N_0$.

The following proposition improves Lemma 11 of [1] since the condition on the sequence $(n_k)_{k \geq 1}$ (namely, (5)) is weaker than both conditions (11) and (12) of [1].

Proposition 4. *Assume that the sequence $(n_k)_{k \geq 1}$ satisfies (5). Then we have for k large enough*

$$(8) \quad \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq 1/2 \right\} \geq 1/8.$$

Proof. Let us fix an integer k . Let us define the events

$$(9) \quad A := \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{2} \right\},$$

$$(10) \quad B := \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left(\sum_{j \geq k} h_j \right) \right| \geq 1 \right\} \text{ and}$$

$$(11) \quad C := \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left(\sum_{j \leq k-1} h_j \right) \right| \leq \frac{1}{2} \right\}.$$

Since the family $\{e_k \circ T^i, k \geq 1, i \in \mathbb{Z}\}$ is independent, the events B and C are independent. Notice that $B \cap C \subset A$ hence

$$\mu(A) = \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{2} \right\} \geq \mu(B)\mu(C).$$

In order to give a lower bound for $\mu(B)$, we define $E_k := \bigcup_{N=2n_k}^{n_k^2} \bigcup_{j \geq k+1} \{S_N(h_j) \neq 0\}$; then

$$(12) \quad \mu(B) \geq \mu(B \cap E_k^c)$$

$$(13) \quad = \mu \left(\left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq 1 \right\} \cap E_k^c \right)$$

$$(14) \quad \geq \mu \left(\left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq 1 \right\} \right) - \mu(E_k).$$

Let us give an estimate of the probability of E_k . As noted in [1] (proof of Lemma 11 therein), the inclusion

$$(15) \quad \bigcup_{N=2n_k}^{n_k^2} \{S_N(h_j) \neq 0\} \subset \bigcup_{i=-2n_j+1}^{n_k^2} T^{-i} A_j$$

takes place for $j > k$, hence

$$(16) \quad \mu \left(\bigcup_{N=2n_k}^{n_k^2} \{S_N(h_j) \neq 0\} \right) \leq \frac{n_k^2 + 2n_j}{n_j^2},$$

and it follows that

$$(17) \quad \mu(E_k) \leq \sum_{j=k+1}^{+\infty} \frac{2n_k}{n_j}.$$

By (5), we have $n_k \leq n_j^{1/(1+\eta)}$ for $j > k$, hence by (17),

$$(18) \quad \mu(E_k) \leq 2 \sum_{j=k+1}^{+\infty} n_j^{-\frac{\eta}{1+\eta}}.$$

As condition (5) implies that $n_k \geq 2^k$ for k large enough, we conclude that the following inequality holds for k large enough:

$$(19) \quad \mu(E_k) \leq 2 \sum_{j=k+1}^{+\infty} 2^{-j\frac{\eta}{1+\eta}}$$

Thus, by Lemma 3 and (19), we have for k large enough

$$(20) \quad \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{2} \right\} \\ \geq \left(\frac{1}{4} - 2 \sum_{j=k+1}^{+\infty} 2^{-j\frac{\eta}{1+\eta}} \right) \left(1 - \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left(\sum_{j \leq k-1} h_j \right) \right| > \frac{1}{2} \right\} \right).$$

Defining $c_k := \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left(\sum_{j \leq k-1} h_j \right) \right| > \frac{1}{2} \right\}$, it is enough to prove that

$$(21) \quad \lim_{k \rightarrow \infty} c_k = 0.$$

Using (6) (accounting $N \geq 2n_k \geq n_j$ for $j < k$), we get the inequalities

$$(22) \quad c_k \leq \sum_{j=1}^{k-1} \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_j)| > \frac{1}{2(k-1)} \right\} \\ (23) \quad \leq \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{8k} \right\} + \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\} \\ + \sum_{j=1}^{k-1} \mu \left\{ \max_{2n_k \leq N \leq n_k^2} U^N \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{8k} \right\} + \\ + \sum_{j=1}^{k-1} \mu \left\{ \max_{2n_k \leq N \leq n_k^2} U^N \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\} \\ (24) \quad \leq n_k^2 \left(\sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{8k} \right\} + \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\} \right).$$

Notice that for each $j \leq k-1$,

$$(25) \quad \mu \left\{ \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\} \leq \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{16k} \right\} + \mu \left\{ |n_j U^{n_j} e_j| > \frac{n_k}{16k} \right\}.$$

Condition (5) implies the inequality $16k \cdot n_{k-1} < n_k$ for k large enough, hence keeping in mind that $U^{n_j} e_j$ is bounded by 1, inequality (25) becomes for such k 's,

$$(26) \quad \mu \left\{ \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\} \leq \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{16k} \right\}.$$

Combining (24) with (26), we obtain

$$(27) \quad c_k \leq 2n_k^2 \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{16k} \right\}$$

$$(28) \quad \leq 2n_k^2 \frac{(16k)^p}{n_k^p} \sum_{j=1}^{k-1} \mathbb{E} \left| \sum_{i=1}^{n_j} iU^i e_j \right|^p,$$

where $p > 2 + 1/\eta$. By Rosenthal's inequality (see [3], Theorem 1), we have

$$(29) \quad \mathbb{E} \left| \sum_{i=1}^{n_j} iU^i e_j \right|^p \leq C_p \left(\sum_{i=1}^{n_j} i^p \mathbb{E} |e_j| + \left(\sum_{i=1}^{n_j} \mathbb{E} [i^2 e_j^2] \right)^{p/2} \right)$$

$$(30) \quad \leq C_p (n_j^{p+1-2} + n_j^{3p/2}/n_j^p)$$

$$(31) \quad \leq 2C_p n_j^{p-1}$$

as $p > 2$. Therefore, for some constant K depending only on p ,

$$(32) \quad c_k \leq K \cdot n_k^{2-p} k^p \sum_{j=1}^{k-1} n_j^{p-1} \leq K \cdot k^{p+1} \frac{n_{k-1}^{p-1}}{n_k^{p-2}},$$

and by (5),

$$(33) \quad c_k \leq K \cdot k^{p+1} n_{k-1}^{p-1-(p-2)(1+\eta)}.$$

Since $p-1-(p-2)(1+\eta) = 1-(p-2)\eta < 0$ and $n_{k-1} \geq 2^{k-1}$ for each $k \geq 2$, we get

$$(34) \quad c_k \leq K \cdot k^{p+1} 2^{(1-(p-2)\eta)(k-1)}.$$

This concludes the proof of Proposition 4 hence that of b'). \square

For c'), we follow the computation in the proof of Proposition 13 of [1], using the fact that $\sup_k \sum_{j=1}^{k-1} n_j/n_k$ is finite.

We now provide a bound for the mixing rates. Corollary 6 of [1] states the following.

Proposition 5. *For each integer k , we have*

$$(35) \quad \beta(N) \leq \sum_{j: 2n_j \geq N} \frac{4}{n_j}.$$

Then d') follows from the bounds

$$(36) \quad \beta(2N) \leq \frac{4}{n_{i(N)}} + \sum_{k \geq i(N)} \frac{4}{n_{k+1}} = \frac{4}{n_{i(N)}} \left(1 + \sum_{j \geq 1} \frac{n_j}{n_{j+1}} \right).$$

In Proposition 14 of [1], it was proved that for each $q \geq 2$, there exists a constant C_q such that for each $k \geq 1$, $\|h_k\|_q \leq C_q n_k^{-1/q}$. Condition (5) implies that $n_k \geq 2^k$ for k large enough, hence e') is satisfied.

This concludes the proof of Proposition 1 and that of Corollary 2.

In order to prove the main result, we shall make some particular choices of sequence $(n_k)_{k \geq 1}$ which satisfy conditions (2) and (5).

We define for a positive δ

$$(37) \quad n_k := \lfloor 2^{(1+\eta)k+1} \rfloor.$$

The sequence $(n_k)_{k \geq 1}$ satisfies (5) and by Proposition 11 of [2], we have $\beta(N) \leq CN^{-1/(1+\eta)}$ for some universal constant C .

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