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# AN IMPROVEMENT OF THE MIXING RATES IN A COUNTER-EXAMPLE TO THE WEAK INVARIANCE PRINCIPLE

DAVIDE GIRAUDDO

ABSTRACT. In [1], the authors gave an example of absolutely regular strictly stationary process which satisfies the central limit theorem but not the weak invariance principle. For each  $q < 1/2$ , the process can be constructed with mixing rates of order  $N^{-q}$ . The goal of this note is to show that actually the same construction can give mixing rates of order  $N^{-q}$  for a given  $q < 1$ .

RÉSUMÉ. Dans [1], les auteurs ont fourni un exemple de processus strictement stationnaire  $\beta$ -mélangeant vérifiant le théorème limite central mais pas le principe d'invariance faible. Pour tout  $q < 1/2$ , le processus peut être construit avec des taux de mélange de l'ordre de  $N^{-q}$ . L'objectif de cette note est de montrer que la même construction peut fournir des taux de mélange de l'ordre de  $N^{-q}$  pour un  $q < 1$  donné.

## 1. NOTATIONS AND MAIN RESULT

We recall some notations in order to make this note more self-contained. Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space. If  $T: \Omega \rightarrow \Omega$  is one-to-one, bi-measurable and measure preserving (in sense that  $\mu(T^{-1}(A)) = \mu(A)$  for all  $A \in \mathcal{F}$ ), then the sequence  $(f \circ T^k)_{k \in \mathbb{Z}}$  is strictly stationary for any measurable  $f: \Omega \rightarrow \mathbb{R}$ . Conversely, each strictly stationary sequence can be represented in this way.

For a zero mean square integrable  $f: \Omega \rightarrow \mathbb{R}$ , we define  $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j$ ,  $\sigma_n^2(f) := \mathbb{E}(S_n(f)^2)$  and  $S_n^*(f, t) := S_{[nt]}(f) + (nt - [nt])f \circ T^{[nt]}$ , where  $[x]$  is the greatest integer which is less than or equal to  $x$ .

Define the  $\beta$ -mixing coefficients by

$$(1) \quad \beta(\mathcal{A}, \mathcal{B}) := \frac{1}{2} \sup \sum_{i=1}^I \sum_{j=1}^J |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|,$$

where the supremum is taken over the finite partitions  $\{A_i, 1 \leq i \leq I\}$  and  $\{B_j, 1 \leq j \leq J\}$  of  $\Omega$  of elements of  $\mathcal{A}$  (respectively of  $\mathcal{B}$ ). They were introduced by Volkonskii and Rozanov [4].

For a strictly stationary sequence  $(X_k)_{k \in \mathbb{Z}}$  and  $n \geq 0$  we define  $\beta_X(n) = \beta(n) = \beta(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty)$  where  $\mathcal{F}_u^v$  is the  $\sigma$ -algebra generated by  $X_k$  with  $u \leq k \leq v$  (if  $u = -\infty$  or  $v = \infty$ , the corresponding inequality is strict).

**Theorem.** *Let  $\delta > 0$ . There exists a strictly stationary real valued process  $Y = (Y_k)_{k \geq 0} = (f \circ T^k)_{k \geq 0}$  satisfying the following conditions:*

- a) *the central limit theorem with normalization  $\sqrt{n}$  takes place;*
- b) *the weak invariance principle with normalization  $\sqrt{n}$  does not hold;*
- c)  *$\sigma_N(f)^2 \asymp N$ ;*
- d) *for some positive  $C$  and each integer  $N$ ,  $\beta_Y(N) \leq C \cdot N^{-1+\delta}$ ;*
- e)  *$Y_0 \in \mathbb{L}^p$  for any  $p > 0$ .*

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We refer the reader to Remark 2 of [1] for a comparison with existing results about the weak invariance principle for strictly stationary mixing sequences.

## 2. PROOF

We recall the construction given in [1]. Let us consider an increasing sequence of positive integers  $(n_k)_{k \geq 1}$  such that

$$(2) \quad n_1 \geq 2 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{n_k} < \infty,$$

and for each integer  $k \geq 1$ , let  $A_k^-, A_k^+$  be disjoint measurable sets such that  $\mu(A_k^-) = 1/(2n_k^2) = \mu(A_k^+)$ .

Let the random variables  $e_k$  be defined by

$$(3) \quad e_k(\omega) := \begin{cases} 1 & \text{if } \omega \in A_k^+, \\ -1 & \text{if } \omega \in A_k^-, \\ 0 & \text{otherwise.} \end{cases}$$

We can choose the dynamical system  $(\Omega, \mathcal{F}, \mu, T)$  and the sets  $A_k^+, A_k^-$  in such a way that the family  $(e_k \circ T^i)_{k \geq 1, i \in \mathbb{Z}}$  is independent. We define  $A_k := A_k^+ \cup A_k^-$  and

$$(4) \quad h_k := \sum_{i=0}^{n_k-1} U^{-i} e_k - U^{-n_k} \sum_{i=0}^{n_k-1} U^{-i} e_k, \quad h := \sum_{k=1}^{+\infty} h_k.$$

Let  $i(N)$  denote the unique integer such that  $n_{i(N)} \leq N < n_{i(N)+1}$ .

We shall show the following intermediate result.

**Proposition 1.** *Assume the sequence  $(n_k)_{k \geq 1}$  satisfies (2) and the following condition:*

$$(5) \quad \text{there exists } \eta > 0 \text{ such that for each } k, \quad n_{k+1} \geq n_k^{1+\eta}.$$

Then:

- a')  $n^{-1/2} S_n(h) \rightarrow 0$  in probability;
- b') the process  $(N^{-1/2} S_N^*(h, \cdot))_{N \geq 1}$  is not tight in  $C[0, 1]$ ;
- c')  $\sigma_N(h)^2 \lesssim N$ ;
- d') for some positive  $C$ ,  $N \cdot \beta_Y(N) \leq C n_{i(N)+1} / n_{i(N)}$ ;
- e')  $h \in \mathbb{L}^p$  for any  $p > 0$ .

Adding a mean-zero nondegenerate independent sequence  $(m \circ T^i)_{i \in \mathbb{Z}}$  with moments of any order greater than 2, the variance of the  $N$ th partial sum of  $((m+h) \circ T^i)_{i \geq 1}$  is bounded above and below by a quantity proportional to  $N$ . Defining  $f := m+h$ , we have the following corollary:

**Corollary 2.** *Assume the sequence  $(n_k)_{k \geq 1}$  satisfies (5). Then  $(f \circ T^i)_{i \geq 0}$  satisfies a), b), c), d') and e).*

For  $k \geq 1$  and  $N \geq n_k$ , the  $N$  partial sum of  $h_k$  admits the expression

$$(6) \quad S_N(h_k) = \sum_{j=1}^{n_k} j U^{j+N-2n_k} e_k + \sum_{j=1}^{n_k-1} (n_k - j) U^{j+N-n_k} e_k \\ - \sum_{j=1}^{n_k} j U^{j-2n_k} e_k - \sum_{j=1}^{n_k-1} (n_k - j) U^{j-n_k} e_k.$$

Let us prove Proposition 1. Item a') follows from the fact that  $h$  is a coboundary (see the explanation before Section 2.2 of [1]).

For b'), we recall the following lemma (Lemma 10, [1]).

**Lemma 3.** *There exists  $N_0$  such that*

$$(7) \quad \mu \left\{ \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq n_k \right\} > 1/4$$

whenever  $n_k \geq N_0$ .

The following proposition improves Lemma 11 of [1] since the condition on the sequence  $(n_k)_{k \geq 1}$  (namely, (5)) is weaker than both conditions (11) and (12) of [1].

**Proposition 4.** *Assume that the sequence  $(n_k)_{k \geq 1}$  satisfies (5). Then we have for  $k$  large enough*

$$(8) \quad \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq 1/2 \right\} \geq 1/8.$$

*Proof.* Let us fix an integer  $k$ . Let us define the events

$$(9) \quad A := \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{2} \right\},$$

$$(10) \quad B := \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j \geq k} h_j \right) \right| \geq 1 \right\} \text{ and}$$

$$(11) \quad C := \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j \leq k-1} h_j \right) \right| \leq \frac{1}{2} \right\}.$$

Since the family  $\{e_k \circ T^i, k \geq 1, i \in \mathbb{Z}\}$  is independent, the events  $B$  and  $C$  are independent. Notice that  $B \cap C \subset A$  hence

$$\mu(A) = \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{2} \right\} \geq \mu(B)\mu(C).$$

In order to give a lower bound for  $\mu(B)$ , we define  $E_k := \bigcup_{N=2n_k}^{n_k^2} \bigcup_{j \geq k+1} \{S_N(h_j) \neq 0\}$ ; then

$$(12) \quad \mu(B) \geq \mu(B \cap E_k^c)$$

$$(13) \quad = \mu \left( \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq 1 \right\} \cap E_k^c \right)$$

$$(14) \quad \geq \mu \left( \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq 1 \right\} \right) - \mu(E_k).$$

Let us give an estimate of the probability of  $E_k$ . As noted in [1] (proof of Lemma 11 therein), the inclusion

$$(15) \quad \bigcup_{N=2n_k}^{n_k^2} \{S_N(h_j) \neq 0\} \subset \bigcup_{i=-2n_j+1}^{n_k^2} T^{-i} A_j$$

takes place for  $j > k$ , hence

$$(16) \quad \mu \left( \bigcup_{N=2n_k}^{n_k^2} \{S_N(h_j) \neq 0\} \right) \leq \frac{n_k^2 + 2n_j}{n_j^2},$$

and it follows that

$$(17) \quad \mu(E_k) \leq \sum_{j=k+1}^{+\infty} \frac{2n_k}{n_j}.$$

By (5), we have  $n_k \leq n_j^{1/(1+\eta)}$  for  $j > k$ , hence by (17),

$$(18) \quad \mu(E_k) \leq 2 \sum_{j=k+1}^{+\infty} n_j^{-\frac{\eta}{1+\eta}}.$$

As condition (5) implies that  $n_k \geq 2^k$  for  $k$  large enough, we conclude that the following inequality holds for  $k$  large enough:

$$(19) \quad \mu(E_k) \leq 2 \sum_{j=k+1}^{+\infty} 2^{-j\frac{\eta}{1+\eta}}$$

Thus, by Lemma 3 and (19), we have for  $k$  large enough

$$(20) \quad \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{2} \right\} \\ \geq \left( \frac{1}{4} - 2 \sum_{j=k+1}^{+\infty} 2^{-j\frac{\eta}{1+\eta}} \right) \left( 1 - \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j \leq k-1} h_j \right) \right| > \frac{1}{2} \right\} \right).$$

Defining  $c_k := \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j \leq k-1} h_j \right) \right| > \frac{1}{2} \right\}$ , it is enough to prove that

$$(21) \quad \lim_{k \rightarrow \infty} c_k = 0.$$

Using (6) (accounting  $N \geq 2n_k \geq n_j$  for  $j < k$ ), we get the inequalities

$$(22) \quad c_k \leq \sum_{j=1}^{k-1} \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_j)| > \frac{1}{2(k-1)} \right\} \\ (23) \quad \leq \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{8k} \right\} + \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\} \\ + \sum_{j=1}^{k-1} \mu \left\{ \max_{2n_k \leq N \leq n_k^2} U^N \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{8k} \right\} + \\ + \sum_{j=1}^{k-1} \mu \left\{ \max_{2n_k \leq N \leq n_k^2} U^N \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\} \\ (24) \quad \leq n_k^2 \left( \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{8k} \right\} + \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\} \right).$$

Notice that for each  $j \leq k-1$ ,

$$(25) \quad \mu \left\{ \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\} \leq \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{16k} \right\} + \mu \left\{ |n_j U^{n_j} e_j| > \frac{n_k}{16k} \right\}.$$

Condition (5) implies the inequality  $16k \cdot n_{k-1} < n_k$  for  $k$  large enough, hence keeping in mind that  $U^{n_j} e_j$  is bounded by 1, inequality (25) becomes for such  $k$ 's,

$$(26) \quad \mu \left\{ \left| \sum_{i=1}^{n_j-1} iU^i e_j \right| > \frac{n_k}{8k} \right\} \leq \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{16k} \right\}.$$

Combining (24) with (26), we obtain

$$(27) \quad c_k \leq 2n_k^2 \sum_{j=1}^{k-1} \mu \left\{ \left| \sum_{i=1}^{n_j} iU^i e_j \right| > \frac{n_k}{16k} \right\}$$

$$(28) \quad \leq 2n_k^2 \frac{(16k)^p}{n_k^p} \sum_{j=1}^{k-1} \mathbb{E} \left| \sum_{i=1}^{n_j} iU^i e_j \right|^p,$$

where  $p > 2 + 1/\eta$ . By Rosenthal's inequality (see [3], Theorem 1), we have

$$(29) \quad \mathbb{E} \left| \sum_{i=1}^{n_j} iU^i e_j \right|^p \leq C_p \left( \sum_{i=1}^{n_j} i^p \mathbb{E} |e_j| + \left( \sum_{i=1}^{n_j} \mathbb{E} [i^2 e_j^2] \right)^{p/2} \right)$$

$$(30) \quad \leq C_p (n_j^{p+1-2} + n_j^{3p/2}/n_j^p)$$

$$(31) \quad \leq 2C_p n_j^{p-1}$$

as  $p > 2$ . Therefore, for some constant  $K$  depending only on  $p$ ,

$$(32) \quad c_k \leq K \cdot n_k^{2-p} k^p \sum_{j=1}^{k-1} n_j^{p-1} \leq K \cdot k^{p+1} \frac{n_{k-1}^{p-1}}{n_k^{p-2}},$$

and by (5),

$$(33) \quad c_k \leq K \cdot k^{p+1} n_{k-1}^{p-1-(p-2)(1+\eta)}.$$

Since  $p-1-(p-2)(1+\eta) = 1-(p-2)\eta < 0$  and  $n_{k-1} \geq 2^{k-1}$  for each  $k \geq 2$ , we get

$$(34) \quad c_k \leq K \cdot k^{p+1} 2^{(1-(p-2)\eta)(k-1)}.$$

This concludes the proof of Proposition 4 hence that of b').  $\square$

For c'), we follow the computation in the proof of Proposition 13 of [1], using the fact that  $\sup_k \sum_{j=1}^{k-1} n_j/n_k$  is finite.

We now provide a bound for the mixing rates. Corollary 6 of [1] states the following.

**Proposition 5.** *For each integer  $k$ , we have*

$$(35) \quad \beta(N) \leq \sum_{j: 2n_j \geq N} \frac{4}{n_j}.$$

Then d') follows from the bounds

$$(36) \quad \beta(2N) \leq \frac{4}{n_{i(N)}} + \sum_{k \geq i(N)} \frac{4}{n_{k+1}} = \frac{4}{n_{i(N)}} \left( 1 + \sum_{j \geq 1} \frac{n_j}{n_{j+1}} \right).$$

In Proposition 14 of [1], it was proved that for each  $q \geq 2$ , there exists a constant  $C_q$  such that for each  $k \geq 1$ ,  $\|h_k\|_q \leq C_q n_k^{-1/q}$ . Condition (5) implies that  $n_k \geq 2^k$  for  $k$  large enough, hence e') is satisfied.

This concludes the proof of Proposition 1 and that of Corollary 2.

In order to prove the main result, we shall make some particular choices of sequence  $(n_k)_{k \geq 1}$  which satisfy conditions (2) and (5).

We define for a positive  $\delta$

$$(37) \quad n_k := \lfloor 2^{(1+\eta)k+1} \rfloor.$$

The sequence  $(n_k)_{k \geq 1}$  satisfies (5) and by Proposition 11 of [2], we have  $\beta(N) \leq CN^{-1/(1+\eta)}$  for some universal constant  $C$ .

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