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Stability of the electron cyclotron resonance

Joachim Asch^{*†} Olivier Bourget[‡] Cédric Meresse[§]

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Abstract

We consider the magnetic AC Stark effect for the quantum dynamics of a single particle in the plane under the influence of an oscillating homogeneous electric and a constant perpendicular magnetic field. We prove that the electron cyclotron resonance is insensitive to impurity potentials.

1 Introduction

For $T > 0$ let $E \in C^0(\mathbb{R}, \mathbb{R}^2)$ be a T periodic function and V be a T periodic multiplication operator by a real valued bounded function in $L^2(\mathbb{R}^2, dq)$. The time dependent family of Hamiltonians under investigation is

$$H(t) = H_0(t) + V(t) \quad (t \in \mathbb{R})$$

with

$$H_0(t) = H_{La} - \langle E(t), q \rangle \text{ and } H_{La} = \frac{1}{2} \left(-i\nabla - \frac{q^\perp}{2} \right)^2,$$

where a^\perp denotes the direct perpendicular of a vector a . $H_{La}, H_0(t), H(t)$ are essentially selfadjoint on the Schwartz space $\mathcal{S}(\mathbb{R}^2)$.

$H(\cdot)$ describes the dynamics of a quantum particle of mass m and charge e for a magnetic field of strength B with $eB > 0$, in units of magnetic length $\sqrt{\frac{\hbar}{eB}}$, gyration time $\frac{m}{eB}$, and energy $\frac{\hbar eB}{m}$. E defines the strength of the electric field and V the impurity potential.

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Electron cyclotron resonance means the growth of the system's kinetic energy if the frequency of the electric field is in resonance with the cyclotron frequency defined by the magnetic field and the particle: the spectrum of H_{La} is pure point and equals $\mathbb{N} + \frac{1}{2}$; thus $e^{-i2\pi H_{La}} = e^{-i\pi \mathbb{I}}$, all orbits generated by H_{La} are periodic with cyclotron frequency equal to 1; in case of resonance, i.e. if T is an integer multiple of 2π , the Floquet operator of H_0 turns out to be a product of phase space translation operators, see (6), which may accelerate the system.

We shall prove that this acceleration and also the asymptotic velocity is insensitive to the impurity V in a large class of potentials provided that the force field created by V vanishes at spatial infinity.

We state this more precisely. Denote the Fréchet space of smooth bounded functions whose derivatives are bounded

$$S(1) := \left\{ f \in C^\infty(\mathbb{R}^2; \mathbb{R}); \sup_{q \in \mathbb{R}^2} |\partial^\alpha f(q)| < \infty, \quad \alpha \in \mathbb{N}_0^2 \right\}.$$

We always make the assumption

(A): Let $T > 0$. Suppose that $E \in C^0(\mathbb{R}; \mathbb{R}^2)$ and $V \in C^0(\mathbb{R}; S(1))$ are T periodic functions.

The family $H(\cdot)$ then defines a unitary propagator U , see Theorem 2.1, and is T periodic so one knows from Floquet's Theorem that $U(t+T, t_0+T) = U(t, t_0)$ for all t, t_0 . Denote $U(T) := U(T, 0)$ its Floquet operator,

$$R(t) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

the rotation matrix of angle t , q the (vector) multiplication operator by the coordinate, and the operator of asymptotic velocity

$$v_{asy} := s - \lim_{n \rightarrow \infty} \frac{1}{nT} (U^*(nT)qU(nT) - q)$$

on its maximal domain.

We summarize our main results in two theorems. If the force ∇V vanishes at spatial infinity we have

Theorem 1.1 Assume **(A)** and suppose $|\nabla V(t, q)| \rightarrow_{|q| \rightarrow \infty} 0$, uniformly in t .

If $T \in 2\pi\mathbb{N}$ and $\left| \int_0^T E(t) dt \right| + \left| \int_0^T R(t)E(t) dt \right| > 0$ then

1. the spectrum of $U(T)$ is purely absolutely continuous except possibly for a finite number of eigenvalues;

2. v_{asy} has discrete spectrum independent of V and :

$$v_{asy} = \frac{1}{T} \int_0^T \left(R(t)E^\perp(t) - E^\perp(t)dt \right) P_{ac}(U(T))$$

P_{ac} denoting the projector on the absolutely continuous subspace;

3. for ψ in the domain of H_{La} it holds

$$\lim_{n \rightarrow \infty} \frac{1}{(nT)^2} \langle U(nT)\psi, H_{La}U(nT)\psi \rangle = \rho \|P_{ac}(U(T))\psi\|^2$$

with the V -independent rate

$$\rho = \frac{1}{2} \left| \frac{1}{T} \int_0^T R(t)E(t) dt \right|^2.$$

If $T \notin 2\pi\mathbb{N}$ it holds

1. if $\int_0^T E(t) dt \neq 0$

(a) the spectrum of $U(T)$ is purely absolutely continuous except possibly for a finite number of eigenvalues;

(b) v_{asy} has discrete spectrum independent of V :

$$v_{asy} = \left(-\frac{1}{T} \int_0^T E^\perp(t)dt \right) P_{ac}(U(T));$$

2. if $\int_0^T E(t) dt = 0$ then $v_{asy} = 0$.

Remark that we adopt the convention to count eigenvalues with their multiplicities; the absolutely continuous subspace in Theorem 1.1 is of finite codimension.

If ∇V is small with respect to the external field E or if V is local in space we have

Theorem 1.2 Assume **(A)**.

1. Suppose $\left| \int_0^T E(t) dt \right| > 0$ and $\sup_{t,q} |\nabla V(t,q)| < \left| \frac{1}{T} \int_0^T E \right|$ or $\left| \int_0^T R(t)E(t) dt \right| > 0$ and $\sup_{t,q} |\nabla V(t,q)| < \left| \frac{1}{T} \int_0^T RE \right|$ then the spectrum of $U(T)$ is purely absolutely continuous.

2. Suppose $V(t,q) \rightarrow_{q \rightarrow \infty} 0$ uniformly in t . If $T \in 2\pi\mathbb{Q}$ and $\int_0^T E = 0 = \int_0^T R(s)E(s) ds$ then the spectrum of $U(T)$ is pure point.

Several remarks are in order.

- Remarks 1.3**
1. *Our results apply in particular to the case of constant E which describes the static Quantum Hall Effect. We refer to [G] for a review of known mathematical results. These concern mainly the small E limit of vanishing exterior electric field and potentials V which are random or periodic.*
 2. *In Theorem 1.1 no decay of V is assumed, on the other hand, due to the decay assumption on ∇V we do not say anything for random or periodic V .*
 3. *In Theorem 1.2.1 smallness of ∇V with respect to E implies that the spectrum of $U(T)$ is purely absolutely continuous. For fixed V the condition is not fulfilled when E vanishes. We have no explicit information on asymptotic acceleration or velocity in this case.*
 4. *v_{asy} may be considered as proportional to the time averaged Hall-current. If E is constant we have in particular $v_{asy} = -E^\perp P_{ac}(U(T))$. Remark that the time dependency of the electric field can suppress v_{asy} in the resonant case; as an example : if $E(t) := E_0 + R(-t)E_0$, for an $E_0 \in \mathbb{R}^2$, then $v_{asy} = 0$.*
 5. *It suffices to assume $\int_0^T E \neq 0$ and $\left\langle \int_0^T E^\perp, \nabla V(t, q) \right\rangle \rightarrow_{|q| \rightarrow \infty} 0$ uniformly in t to conclude that the absolutely continuous subspace is of finite codimension. This is a corollary of the proof of Theorem 1.1. The condition is, however, not sufficient so deduce information on v_{asy} in the direction of $\int_0^T E$.*
 6. *In Theorem 1.2.2 we prove that the spectrum of $U(T)$ is pure point in the resonant case. So local potentials are not likely to accelerate the system. We guess that this remains true in the non resonant case; however, no method to prove this seems to be available. The problem of stability under non-resonant time periodic local perturbations was raised in [EV]. Progress on this difficult problem has recently been achieved for perturbations of the harmonic oscillator in one dimension [W, GT].*

In [GY] the electric AC Stark effect was discussed for perturbations of the isotropic harmonic oscillator. They proved for a sinusoidal electric field that the spectrum of the Floquet hamiltonian is purely absolutely continuous if ∇V is small.

[BF] observed 1.2.1 for the case $|\nabla V| < \left| \frac{1}{T} \int_0^T E \right|$.

The eigenvalues of the asymptotic velocities were examined in [AK] for $-\frac{\hbar^2}{2}\Delta + V$ and space-periodic V in the Floquet-Bloch decomposition.

We shall discuss general properties of the propagator in section 2. Then in section 3 we establish Theorem 3.1, a dynamical compactness result which is our main technical contribution. We prove a Mourre estimate in Theorem 3.4; this allows us to apply a unitary Mourre Theorem which implies the spectral results. We finish the paper with the proof of Theorems 1.1, 1.2.

2 The propagator

$H_0(\cdot)$ is quadratic in the canonical operators; as in the case of the Stark effect [AH, KY] its propagator can be determined explicitly. We present it here in terms of center and velocity operators. Although this is folklore we provide an explicit proof. See [Fo] for the metaplectic representation in general.

In the following, if not otherwise specified, operators and identities are to be understood as acting on Schwartz-space $\mathcal{S}(\mathbb{R}^2)$. Denote $D := -i\nabla$. The (vector-) operators of center c and velocity v are

$$v = (v_1, v_2) := D - \frac{1}{2}q^\perp; \quad c = (c_1, c_2) := \frac{1}{2}q - D^\perp.$$

Relevant commutation relations are

$$\begin{aligned} [v_1, v_2] &= i, & [c_2, c_1] &= i, & [c_j, v_k] &= 0 \quad \forall j, k \\ [c, H_{La}] &= 0, & [v^\perp, H_{La}] &= iv. \end{aligned} \tag{1}$$

To relate to the usual position and momentum operators note that

$$q = c + v^\perp, \quad D = \frac{1}{2}(c^\perp + v).$$

While the operator of kinetic energy is $H_{La} = \frac{1}{2}v^2$ the name center operator is motivated by the relation

$$e^{itH_{La}} q e^{-itH_{La}} = c + R(-t)v^\perp \tag{2}$$

which follows from (1).

Denote for $a \in \mathbb{R}^2$ and a vector operator w :

$$\langle a, w \rangle := a_1 w_1 + a_2 w_2.$$

This an abuse of notation as we also use $\langle \cdot, \cdot \rangle$ for the scalar product in $L^2(\mathbb{R}^2)$ in the statement of Theorem 1.1.

Note that the domain of $H(t)$ is time dependent in general so some care is needed to assure the existence of its propagator. We have

Theorem 2.1 *Assume (A). For $t \in \mathbb{R}$ consider the operators*

$$H_{La} := \frac{1}{2} \left(D - \frac{1}{2}q^\perp \right)^2, \quad H_0(t) := H_{La} - \langle E(t), q \rangle, \quad H(t) = H_0(t) + V(t)$$

which are all essentially selfadjoint on $\mathcal{S}(\mathbb{R}^2)$. We then have:

1. the family $H(\cdot)$ generates a unitary propagator U which leaves $\mathcal{S}(\mathbb{R}^2)$ invariant. Furthermore for $\psi \in \mathcal{S}(\mathbb{R}^2)$, $t \mapsto U(t, t_0)\psi$ is the solution of

$$i\partial_t \psi(t) = H(t)\psi(t), \quad \psi(t_0) = \psi;$$

2. S defined by

$$S(t, t_0) := e^{i\langle \int_{t_0}^t E, c \rangle} e^{i\langle \int_{t_0}^t RE, v^\perp \rangle} e^{-i\varphi(t, t_0)}$$

with

$$\varphi(t, t_0) := \int_{t_0}^t \left(\frac{1}{2} \left\langle E(s), \int_0^s E^\perp \right\rangle - \frac{1}{2} \left\langle R(s)E(s), \int_0^s RE^\perp \right\rangle \right) ds,$$

is a unitary propagator;

3. for the propagator U_0 generated by $H_0(\cdot)$ it holds:

$$U_0(t, t_0) = e^{-itH_{L^a}} S(t, t_0) e^{it_0H_{L^a}}; \quad (3)$$

4. U is given by

$$U(t, t_0) = U_0(t, 0)\Omega(t, t_0)U_0(0, t_0) \quad (4)$$

where Ω is the propagator generated by the family of bounded operators

$$V(t, x(t, q, D))$$

which is the Weyl quantization of the symbol $V(t, x(t, q, p))$ with

$$x(t, q, p) := \frac{1}{2}q - p^\perp - \int_0^t E^\perp + R(-t) \left(\frac{1}{2}q + p^\perp + \int_0^t RE^\perp \right) \quad (5)$$

for $q, p \in \mathbb{R}^2$. In particular it holds for $T = n2\pi$, $n \in \mathbb{N}$

$$\begin{aligned} U_0(T) &:= U_0(T, 0) = e^{i\alpha_n} e^{i\langle \int_0^T E, c \rangle} e^{i\langle \int_0^T RE, v^\perp \rangle} \\ &= e^{i\beta_n} e^{\frac{i}{2}\langle \int_0^T (\mathbb{I}+R)E, q \rangle} e^{i\langle \int_0^T (\mathbb{I}-R)E^\perp, D \rangle} \end{aligned} \quad (6)$$

with $i\alpha_n := -i\pi n - i\varphi(T)$,

$$i\beta_n := -i\pi n - i\varphi(T) + \frac{i}{4} \left\langle \int_0^T (\mathbb{I} - R)E^\perp, \int_0^T (\mathbb{I} + R)E \right\rangle,$$

and \mathbb{I} the 2×2 identity matrix.

Proof. We use the commutation relations (1). Note that for observables w_ℓ which are linear polynomials in D_j, q_k it holds

$$e^{-i(w_1+w_2)} = e^{-iw_1} e^{-iw_2} e^{\frac{1}{2}[w_1, w_2]} \quad (7)$$

which for the case $[w_1, w_2] = i$ and $a \in C^1(\mathbb{R}; \mathbb{R}^2)$ implies for $\psi \in \mathcal{S}(\mathbb{R}^2)$

$$i\partial_t e^{-i\langle a, w \rangle} \psi = \left(\langle \dot{a}, w \rangle + \frac{1}{2} \langle \dot{a}, a^\perp \rangle \right) e^{-i\langle a, w \rangle} \psi. \quad (8)$$

Furthermore $e^{itH_{La}}v^\perp e^{-itH_{La}} = R(-t)v^\perp$ because the derivatives of both functions coincide as well as their values at $t = 0$. This also proves (2).

To apply formula (8) we write

$$S = e^{-i\langle a, c \rangle} e^{-i\langle b, v^\perp \rangle} e^{-i\varphi}$$

with $a(t) := -\int_{t_0}^t E$, $b(t) := -\int_{t_0}^t RE$ and conclude for $\psi \in \mathcal{S}(\mathbb{R}^2)$:

$$\begin{aligned} i(\partial_t S)S^{-1}\psi &= \left(\langle \dot{a}, c \rangle + \langle \dot{b}, v^\perp \rangle - \frac{1}{2} \langle \dot{a}, a^\perp \rangle + \frac{1}{2} \langle \dot{b}, b^\perp \rangle + \dot{\varphi} \right) \psi \\ &= \left(-\langle E, c \rangle - \langle RE, v^\perp \rangle \right) \psi \end{aligned}$$

by definition of φ .

Also:

$$\begin{aligned} i\partial_t e^{-itH_{La}} S(t, 0)\psi &= (H_{La} + e^{-itH_{La}} i\partial_t S S^{-1}(t) e^{itH_{La}}) e^{-itH_{La}} S(t, 0)\psi \\ &= \left(H_{La} - \langle E(t), c \rangle - \langle R(t)E(t), R(t)v^\perp \rangle \right) e^{-itH_{La}} S(t, 0)\psi \\ &= H_0(t) e^{-itH_{La}} S(t, 0)\psi. \end{aligned}$$

It follows that $U_0(t, 0) = e^{-itH_{La}} S(t, 0)$; (4) results from $U_0(t, t_0) = U_0(t, 0)(U_0(t_0, 0))^{-1}$.

The formula for $U_0(T)$ in terms of q, D follows from the identity

$$\langle a, c \rangle + \langle b, v^\perp \rangle = \left\langle \frac{a+b}{2}, q \right\rangle + \langle (a-b)^\perp, D \rangle$$

and (7).

Now define $\Omega(t, 0) := U_0^*(t, 0)U(t, 0)$ then

$$i\partial_t \Omega(t, 0)\psi = U_0^*(t, 0)V(t)U_0(t, 0)\Omega(t, 0)\psi;$$

to prove the equality $U_0^*VU_0(t) = V(t, x(t, q, D))$ note that for $a \in \mathbb{R}^2$ it follows from (1):

$$e^{i\langle a, v^\perp \rangle} v^\perp e^{-i\langle a, v^\perp \rangle} = v^\perp - a^\perp, \quad e^{i\langle a, c \rangle} c e^{-i\langle a, c \rangle} = c + a^\perp.$$

Together with (2) this yields

$$\begin{aligned} U_0^*(t, 0)qU_0(t, 0) &= c - \int_0^t E^\perp + R(-t) \left(v^\perp + \int_0^t RE^\perp \right) \\ &= x(t, q, D) = \frac{1}{2}q - D^\perp - \int_0^t E^\perp + R(-t) \left(\frac{1}{2}q + D^\perp + \int_0^t RE^\perp \right). \end{aligned} \quad (9)$$

The equality $U_0^*VU_0(t) = V(t, x(t, q, D))$ now follows from Egorov's Theorem because H_0 is a quadratic polynomial in q and D , cf [Fo]. By the Calderon Vaillancourt Theorem the operator $V(t, x(t, q, D))$ is bounded thus the propagator Ω is well defined. The invariance of Schwartz space now follows from $i\partial_t \psi(t) = V(t, x(t, q, D))\psi(t)$ and the fact that the pseudodifferential operator $V(t, x(t, q, D))$ leaves $\mathcal{S}(\mathbb{R}^2)$ invariant. ■

Note the free evolution of the observables c, v^\perp, H_{La} :

Corollary 2.2 For $\psi \in \mathcal{S}(\mathbb{R}^2)$ and $t \in \mathbb{R}$ we have

$$U_0^*(t, 0)cU_0(t, 0)\psi = c\psi - \int_0^t E^\perp \psi \quad (10)$$

$$U_0^*(t, 0)v^\perp U_0(t, 0)\psi = R(-t) \left(v^\perp \psi + \int_0^t RE^\perp \psi \right) \quad (11)$$

$$U_0^*(t, 0)H_{La}U_0(t, 0)\psi = \frac{1}{2} \left(v^\perp + \int_0^t RE^\perp \right)^2 \psi.$$

By Floquet's Theorem it holds

$$U(t) = M(t) (U(T))^{\frac{t}{T}}$$

for a unitary T -periodic family M . Eigenvectors of $U(T)$ give rise to periodic orbits whereas for ψ in its absolutely continuous subspace the function $n \mapsto \langle \psi, U(nT)\psi \rangle$ decays. Define

$$a_c(t) := - \int_0^t E^\perp, \quad a_v(t) := \int_0^t RE^\perp \quad (12)$$

By (6), for $T \in 2\pi\mathbb{N}$, the operator $U_0(T) = e^{i\alpha_n} e^{i\langle a_c^\perp, c \rangle} e^{-i\langle a_v^\perp, v^\perp \rangle}$ is, except for a phase, the product of the two commuting phase space translation operators; its spectrum is absolutely continuous if $a_c(T)$ or $a_v(T)$ is non zero.

In order to prove spectral properties for $U(T)$ we shall use the following result of unitary Mourre theory which was proven in [ABCF], Theorem 3.3. Consult also [ABC], Theorem 2.3, for an optimal version.

Theorem 2.3 Let U be unitary and A selfadjoint in a separable Hilbert space such that

$$(U^*AU - A) \quad \text{and} \quad [(U^*AU - A), A]$$

are densely defined on an U -invariant core of A and can be extended as bounded operators. Suppose that

$$(U^*AU - A) \geq c\mathbb{1} + K$$

for a $c > 0$ and K a compact operator. Then the spectrum of U is absolutely continuous except possibly for a finite number of eigenvalues of finite multiplicity. If $K = 0$ the spectrum is purely absolutely continuous.

Define the Mourre operators

$$A_c := \langle a_c(T), c \rangle, \quad A_v := \langle a_v(T), v^\perp \rangle. \quad (13)$$

By the explicit evolution of observables in the free case we have

Proposition 2.4 Let $T > 0$, suppose $E \in C^0(\mathbb{R}, \mathbb{R}^2)$ and T periodic. Denote σ the spectrum of $U_0(T)$. It holds:

$$(U_0^*(T)A_cU_0(T) - A_c) = |a_c(T)|^2. \quad (14)$$

1. If $a_c(T) \neq 0$ then σ is purely absolutely continuous.
2. If $a_c(T) = 0$ then
 - (a) if $T \notin 2\pi\mathbb{N}$ then σ is pure point
 - (b) if $T \in 2\pi\mathbb{N}$ then

$$(U_0^*(T)A_vU_0(T) - A_v) = |a_v(T)|^2, \quad (15)$$

if $a_v(T) \neq 0$ then σ is purely absolutely continuous, if $a_v(T) = 0$ then σ is pure point.

Proof. The identities (14), (15) follow from Corollary 2.2. The second order commutators, which are densely defined on $\mathcal{S}(\mathbb{R}^2)$, vanish:

$$[(U_0^*(T)A_cU_0(T) - A_c), A_c] = [(U_0^*(T)A_vU_0(T) - A_v), A_v] = 0.$$

It follows from Theorem 2.3 that the spectrum of $U_0(T)$ is purely absolutely continuous if $a_c(T) \neq 0$ for any T or $a_v(T) \neq 0$ for $T \in 2\pi\mathbb{N}$.

Concerning the remaining cases, remark that for $\psi \in \mathcal{S}(\mathbb{R}^2)$ and $t \in \mathbb{R}$ it holds

$$\sup_{t \in \mathbb{R}} \| (\mathbb{I} + c^2 + v^2) U_0(t)\psi \| = \sup_{t \in \mathbb{R}} \| \left(\mathbb{I} + (c + a_c(t))^2 + (v^\perp + a_v(t))^2 \right) \psi \|.$$

This supremum is finite iff a_c and a_v are bounded functions of time. On the other hand $(1 + c^2 + v^2)^{-1}$ is a compact operator and thus the trajectory $\{U_0(t)\psi, t \in \mathbb{R}\}$ is relatively compact iff a_c and a_v are bounded. If all trajectories are relatively compact one knows, see [EV], that σ is pure point.

E being T periodic $a_c(T) = 0$ implies that $a_c(\cdot)$ is a bounded function. If $T \in 2\pi\mathbb{N}$ then $R(\cdot)E^\perp(\cdot)$ is T periodic and so $a_v(\cdot)$ is bounded if $a_v(T) = 0$. Remark that if both vanish then $U_0(T)$ is just multiplication by a phase factor.

If $T \notin 2\pi\mathbb{N}$ then the product $R(\cdot)E^\perp(\cdot)$ is an almost periodic function whose associated Fourier exponents are of the form $\lambda_{mn} = (n + m\frac{2\pi}{T})$ with $(m, n) \in \mathbb{Z} \times \{-1, 1\}$. Now $\inf_{\mathbb{Z} \times \{-1, 1\}} |\lambda_{mn}| > 0$ which implies boundedness of $a_v(\cdot)$, see [F], Theorem 4.12. ■

3 Compactness and the proof of Theorems 1.1 and 1.2

In order to go from the unperturbed to the full propagator, we shall prove the following dynamical compactness result which is our main technical contribution.

Theorem 3.1 Let $T_1 < T_2$ be real, let $E \in C^0(\mathbb{R}; \mathbb{R}^2)$ and $V \in C^0(\mathbb{R}; S(1))$. Denote U the propagator defined by $H(\cdot)$ in Theorem 2.1.

1. For $f \in C^0([T_1, T_2] \times \mathbb{R}^2, \mathbb{R})$ and $f(t, q) \rightarrow_{|q| \rightarrow \infty} 0$ uniformly in $t \in [T_1, T_2]$, it holds for the multiplication operator by f

$$\int_{T_1}^{T_2} U^*(t) f(t) U(t) dt \text{ is compact.}$$

2. For $f \in C^\infty([T_1, T_2] \times \mathbb{R}^2, \mathbb{R})$ such that for an $\varepsilon > 0$ and all $\alpha \in \mathbb{N}_0^2$, $\sup_t \sup_{q \in \mathbb{R}^2} |\langle q \rangle^{(1+\varepsilon)} \partial_q^\alpha f(t, q)| < \infty$ we have

$$\int_{T_1}^{T_2} U_0^*(t) f(t) U_0(t) dt \text{ belongs to the } \alpha\text{-th Schatten class for } \alpha > 4.$$

The proof is built on the following lemma

Lemma 3.2 $T_1 < T_2$ be real, let $E \in C^0(\mathbb{R}; \mathbb{R}^2)$ and $x(t, q, p)$ defined in equation (5).

If $f \in C^\infty([T_1, T_2] \times \mathbb{R}^2, \mathbb{R})$ such that for an $\varepsilon > 0$ and all $\alpha \in \mathbb{N}_0^2$, $\sup_t \sup_{q \in \mathbb{R}^2} |\langle q \rangle^{(1+\varepsilon)} \partial_q^\alpha f(t, q)| < \infty$, then it holds for

$$f_{av}(q, p) := \int_{T_1}^{T_2} f(t, x(t, q, p)) dt :$$

$$\sup_{(q, p) \in \mathbb{R}^4} |(1 + q^2 + p^2)^{1/2} \partial_q^\alpha \partial_p^\beta f_{av}(q, p)| < \infty \quad \forall \alpha, \beta \in \mathbb{N}_0^2;$$

Proof of Lemma 3.2.

Let h_{La}, h_0 be the hamiltonian symbols associated to the partial differential operators $H_{La}, H_0(\cdot)$

$$h_{La}(q, p) := \frac{1}{2} \left(p - \frac{q^\perp}{2} \right)^2, \quad h_0(t, q, p) := h_{La}(q, p) - \langle E(t), q \rangle.$$

Denote Φ_{h_0} the hamiltonian flow generated by h_0 . $x(t, q, p)$ is the projection of $\Phi_{h_0}(t, q, p)$ to the q -component, i.e. the configuration space part of the trajectory, originating in q, p .

$$x(t, q, p) = \chi_{La}(t, q, p) + \mathbf{b}(t)$$

for a function $\mathbf{b} \in C^0(\mathbb{R}, \mathbb{R}^2)$ and $\chi_{La}(t, q, p)$ the projection to the q -component of the trajectory generated by h_{La} . $\chi_{La}(t, q, p)$ is linear in q, p and, with the symbols of the center and velocity operators

$$v(q, p) := p - \frac{q^\perp}{2}, \quad c(q, p) := q - v^\perp,$$

one has

$$\chi_{La}(t, q, p) = c(q, p) + R(-t)v^\perp(q, p)$$

describing the cyclotron orbit with center c and radius $|v|$.

Denote $z = (q, p) \in \mathbb{R}^2 \times \mathbb{R}^2$, $\langle z \rangle := (1 + q^2 + p^2)^{1/2}$.

b being bounded on $[T_1, T_2]$ the boundedness of $\langle q \rangle^{1+\varepsilon} f(t, q)$ implies

$$|f(t, x(t, z))| \leq \frac{cte}{(1 + (\chi_{La}(t, z))^2)^{\frac{1+\varepsilon}{2}}} \quad (16)$$

for a $cte > 0$, $t \in [T_1, T_2]$ and all z .

Denote $\widehat{z}(q, p) := (c(q, p), v^\perp(q, p)) \in \mathbb{R}^2 \times \mathbb{R}^2$ and remark that

$$\frac{1}{\sqrt{2}}|z| \leq |\widehat{z}| \leq \sqrt{2}|z|.$$

Because of the time periodicity of χ_{La} we may assume $[T_1, T_2] \subset [0, 2\pi]$. In order to estimate the decay of $\int_{T_1}^{T_2} |f(\chi(t, z))| dt$ we separate the time events "orbit is close to the origin" and "orbit is far from the origin" as follows:

For a fixed number $d \in (0, \frac{1}{\sqrt{2}})$ define for $z \in \mathbb{R}^4$

$$I_>(z) := \{t \in [0, 2\pi], |\chi_{La}(t, z)| \geq d|z|\}, \quad I_<(z) := [0, 2\pi] \setminus I_>$$

and estimate the contributions to $f_{av}(z)$ of these sets. By (16):

$$\sup_z \langle z \rangle \int_{[T_1, T_2] \cap I_>(z)} |f(t, x(t, z))| dt \leq \sup_z \frac{\langle z \rangle cte |T_2 - T_1|}{(1 + (d|z|)^2)^{\frac{1+\varepsilon}{2}}} < \infty.$$

Now for z fixed let $t_0 \in [0, 2\pi]$ be the point for which the distance to the origin $t \mapsto |\chi_{La}(t, z)|$ is minimal.

If $I_<(z) \neq \emptyset$ then $t_0 \in I_<(z)$. It holds

$$\partial_t^2 |\chi_{La}(t, z)|^2 = -2\langle c, R(-t)v^\perp \rangle$$

and thus for $t \in I_<(z)$:

$$|\chi_{La}(t, z)|^2 = c^2 + v^2 + 2\langle c, R(-t)v^\perp \rangle \leq 2d^2|\widehat{z}|^2 =: \widehat{d}^2|\widehat{z}|^2, \text{ which implies}$$

$$\partial_t^2 |\chi_{La}(t, z)|^2 \geq (1 - \widehat{d}^2)|\widehat{z}|^2. \text{ It follows that for a } \tau_t \in [t_0, t]$$

$$|\chi_{La}(t, z)|^2 = |\chi_{La}(t_0, z)|^2 + \partial_t^2 |\chi_{La}(\tau_t)|^2 \frac{(t - t_0)^2}{2} \geq \frac{(1 - \widehat{d}^2)}{2} |\widehat{z}|^2 (t - t_0)^2 \text{ and}$$

$$\sup_z \langle z \rangle \int_{[T_1, T_2] \cap I_<(z)} |f(t, x(t, z))| dt \leq \sup_z \int_{T_1}^{T_2} \frac{cte \langle z \rangle}{\left(1 + \left(\frac{1 - \widehat{d}^2}{2}\right) |\widehat{z}|^2 (t - t_0)^2\right)^{\frac{1+\varepsilon}{2}}} dt < \infty$$

and the claim is proven for f . Analogously the assertion that the derivatives decay follows from the assumptions on the derivatives of f and the affine character of $z \mapsto x(t, z)$. \blacksquare

Remark 3.3 *Intuitively Lemma 3.2 means that the sojourn time of a classical particle in the "support" of f decays as $1/(1+q^2+p^2)^{1/2}$ in the initial data because the center and the radius of the cyclotron orbit grow linearly in these data.*

Proof of Theorem 3.1.

We use pseudodifferential Weyl Calculus, see [Fo, Zw] and references therein. We first prove the second assertion.

By Egorov's Theorem $U_0^*(t)f(t)U_0(t)$ (respectively its integral) is the Weyl quantization of the symbol $z \mapsto f(x(t, z))$ (respectively $\int_{T_1}^{T_2} f(x(t, z)) dt$). Lemma 3.2 means that

$$\int_{T_1}^{T_2} f(t, x(t, \cdot)) dt \in S(m, g),$$

the Hörmander class with respect to the euclidean metric

$$g = \sum_i dq_i^2 + dp_i^2$$

and weight function

$$m(z) = \langle z \rangle = (1 + q^2 + p^2)^{\frac{1}{2}}.$$

m and $\int_{T_1}^{T_2} f(t, x(t, \cdot)) dt$ are both in $L^\alpha(\mathbb{R}^4)$ for $\alpha > 4$. Thus the second claim follows from Theorem 2.1 of [BT], see also their Proposition 4.2.

Now $C_0^\infty(\mathbb{R})$ is uniformly dense in the continuous functions vanishing at infinity and the space of compact operators in the bounded operators is norm closed. Thus we conclude that

$$\int_{T_1}^{T_2} U_0^*(t)f(t)U_0(t)dt$$

is compact for $f \in C^0([T_1, T_2] \times \mathbb{R}^2, \mathbb{R})$ and $f(t, q) \rightarrow_{|q| \rightarrow \infty} 0$ uniformly in t . Now suppose that $B(t)$ ($t \in [0, T]$) is a norm continuous family of operators. We prove:

$$\int_s^t U_0^*(\tau)B(\tau)U_0(\tau) d\tau \text{ compact } (\forall s, t \in [0, T]) \implies \int_0^T U^*(\tau)B(\tau)U(\tau) d\tau \text{ is compact.}$$

We follow an argument of [EV] : denote $\Omega(t) := U_0^*(t)U(t)$; $[0, T]$ is the disjoint union of $I_j = \frac{T}{N}[j, j+1)$, $\{j \in 0, \dots, N-1\}$. Note that

$$\|\Omega(t) - \Omega(s)\| \leq |t - s| \|V\|.$$

Now use that for $t, s \in I_j$, $\tau \in (t, s)$ it holds $\int_s^t \|B\| = \mathcal{O}\left(\frac{1}{N}\right)$ and $\|\Omega(\tau) - \Omega(s)\| = \mathcal{O}\left(\frac{1}{N}\right)$ thus

$$\int_s^t U^* B U(\tau) d\tau = \Omega^*(s) \left(\int_s^t U_0^* B U_0(\tau) d\tau \right) \Omega(s) + \mathcal{O}\left(\frac{1}{N^2}\right).$$

It follows

$$\left\| \int_0^T U^* B U - \sum_{j=0}^{N-1} \Omega^*(jT/N) \left(\int_{I_j} U_0^* B U_0 \right) \Omega(jT/N) \right\| = \mathcal{O}\left(\frac{1}{N}\right).$$

Since $\int_{I_j} U_0^* B U_0$ is compact $\forall j$, we conclude that $\int_0^T U^* B U$ is compact as the space of compact operators is norm closed.

In particular $\int_0^T U^*(t) K(t) U(t) dt$ is compact so the theorem is proven. \blacksquare

We now prove the Mourre estimate needed to apply Theorem 2.3.

Theorem 3.4 *Assume (A). Denote U the propagator defined by $H(\cdot)$ in Theorem 2.1 and the quantities a_c, A_c, a_v, A_v defined in (12,13).*

1. Suppose $\int_0^T E \neq 0$ and $\left\langle \int_0^T E^\perp, \nabla V(t, q) \right\rangle \rightarrow_{|q| \rightarrow \infty} 0$ uniformly in t .

Then there exists a compact operator \mathcal{C} on $L^2(\mathbb{R}^2)$ such that

$$\begin{aligned} (U^*(T) A_c U^*(T) - A_c) &= |a_c(T)|^2 \mathbb{I} + \mathcal{C}, \\ [(U^*(T) A_c U(T) - A_c), A_c] &\text{ is bounded.} \end{aligned}$$

2. Suppose $T \in 2\pi\mathbb{N}$ and $|\nabla V(t, q)| \rightarrow_{|q| \rightarrow \infty} 0$ uniformly in t . If $\int_0^T R(s) E(s) ds \neq 0$ then there exists a compact operator \mathcal{C} on $L^2(\mathbb{R}^2)$ such that

$$\begin{aligned} (U^*(T) A_v U^*(T) - A_v) &= |a_v(T)|^2 \mathbb{I} + \mathcal{C}, \\ [(U^*(T) A_v U(T) - A_v), A_v] &\text{ is bounded.} \end{aligned}$$

Proof. Throughout this proof the symbols a, A without subscript denote one of the a_c, A_c or a_v, A_v defined in (12,13). To prove the results concerning the first commutator we first determine a continuous family of bounded operators K such that

$$(U^*(T) A U(T) - A) - |a|^2 = -i \int_0^T U^*(t) K(t) U(t) dt,$$

and such that $\int_0^T U^*(t) K(t) U(t) dt$ is compact, then we shall conclude by applying Theorem 2.3.

Each of the following computations is to be understood first on $\mathcal{S}(\mathbb{R}^2)$ then by extension to $L^2(\mathbb{R}^2)$; recall Theorem 2.1 and in particular that all the involved operators leave $\mathcal{S}(\mathbb{R}^2)$ invariant.

Suppose $T \in 2\pi\mathbb{N}$. By (14, 15) : $(U_0^*(T)AU_0(T) - A) = |a|^2$. It follows

$$\begin{aligned} U^*(T)AU(T) - |a|^2 - A &= U^*U_0(T)AU_0^*U(T) - A \\ &= -i \int_0^T U^*(t) \underbrace{[U_0(t)AU_0^*(t), V]}_{=:K(t)} U(t) dt. \end{aligned} \quad (17)$$

Remark that for the case A_c, a_c (17) holds for any T .

From the explicit free evolution of observables, Corollary 2.2, one gets

$$K_c(t) = [U_0(t)A_cU_0^*(t), V(t)] = \left[\left\langle a_c(T), -D^\perp \right\rangle, V(t) \right] = i \left\langle a_c(T), \nabla V^\perp(t) \right\rangle$$

and

$$K_v(t) = \left[\left\langle a_v(T), R(-t)D^\perp \right\rangle, V(t) \right] = -i \left\langle R(t)a_v(T), \nabla V^\perp(t) \right\rangle.$$

By the decay assumption on ∇V Theorem 3.1 is applicable which in both cases implies the compactness of $\int_0^T U^*(t)K(t)U(t)dt$. Note that (17) implies in particular that $(\Omega^*(T)A\Omega(T) - A)$, and hence $[A, \Omega(T)]$, is bounded.

Concerning the double commutator observe :

$$\begin{aligned} [(U^*(T)AU(T) - A), A] &=_{(17)} [(\Omega^*(T)A\Omega(T) - A), A] = \\ &[\Omega^*(T)[A, \Omega(T)], A] = \Omega^*(T)[[A, \Omega(T)], A] + \Omega^*(T)[A, \Omega(T)]\Omega^*(T)[A, \Omega(T)]. \end{aligned}$$

Thus it is sufficient to prove that $[A, \Omega(t)]$ and $[A, [A, \Omega(t)]]$ admit bounded extensions to infer that $[(U^*(T)AU(T) - A), A]$ can be extended to a bounded operator. By Theorem 2.1.4 Ω is the solution of

$$i\partial_t\Omega(t) = V(t, x(t, x, D)\Omega(t), \quad \Omega(0) = \mathbb{I},$$

thus denoting $L(t) := -V(t, x(t, x, D)$ the operator triple

$$OT(t) := (\Omega(t), [A, \Omega(t)], [A, [A, \Omega(t)]])$$

is a fixed point of the system

$$\begin{aligned} \Omega(t) &= \mathbb{I} + i \int_0^t L\Omega \\ [A, \Omega(t)] &= i \int_0^t ([A, L]\Omega + L[A, \Omega]) \\ [A, [A, \Omega(t)]] &= i \int_0^t ([A, [A, L]]\Omega + 2[A, L][A, \Omega] + L[A, [A, \Omega]]). \end{aligned} \quad (18)$$

The right hand side as applied to OT is a Volterra operator in the Banach space \mathbb{K}^3 where $\mathbb{K} := C^0([0, T], \mathbb{B}(L^2(\mathbb{R}^2)))$ with the norm $\sup_t \|\cdot\|$ provided that $L, [A, L]$ and $[A, [A, L]]$ belong to \mathbb{K} which we will show now.

V is bounded thus $L \in \mathbb{K}$. $[A, L] = -U_0^* [U_0 A U_0^*, V] U_0 \stackrel{(17)}{=} -U_0^* K(t) U_0$ and it follows that $[A, L] \in \mathbb{K}$. Finally

$$[A, [A, L]] = -U_0^* [U_0 A U_0^*, K] U_0.$$

Using the explicit expressions for K we get

$$[U_0 A_c U_0^*, K_c] = \left[\left\langle a_c(T), D^\perp \right\rangle, \left[\left\langle a_c(T), D^\perp \right\rangle, V \right] \right] = \left\langle a_c(T)^\perp, Hess(V) a_c(T)^\perp \right\rangle$$

$$\begin{aligned} [U_0 A_v U_0^*, K_v](t) &= \left[\left\langle R(t) a_v(T), D^\perp \right\rangle, \left[\left\langle R(t) a_v(T), D^\perp \right\rangle, V(t) \right] \right] \\ &= \left\langle R(t) a_v(T)^\perp, Hess(V(t)) R(t) a_v(T)^\perp \right\rangle. \end{aligned}$$

Thus from our assumptions of boundedness of the second derivatives of V it follows that $[A, [A, L]] \in \mathbb{K}$. We conclude that the fixed point $OT \in \mathbb{K}^3$ so the double commutators admit bounded extensions. Theorem 2.3 is applicable and the proof of Theorem 3.4 is completed. ■

We now have all elements to provide the proofs of Theorems 1.1 and 1.2

Proof. of Theorem 1.1

The spectral results, i.e. point 1. for the case $T \in 2\pi\mathbb{N}$ and point 1.(a) for $T \notin 2\pi\mathbb{N}$, follow from Theorem 2.3 and Theorem 3.4 because $|a_c(T)| + |a_v(T)| \neq 0$.

To prove the results on v_{asy} , recall equations (9,10,11) for the free evolution of observables.

Let $\psi \in \mathcal{S}(\mathbb{R}^2)$. As in equation (17) one calculates:

$$\begin{aligned} U^*(t) v^\perp U(t) \psi &= U^* U_0(t) \left(R(-t) v^\perp + \int_0^t R(s-t) E^\perp(s) ds \right) U_0^* U(t) \psi \\ &= \int_0^t R(s-t) E^\perp(s) ds \psi + R(-t) \left(v^\perp \psi + i \int_0^t U^*(s) [R(s) D^\perp, V(s)] U(s) \psi ds \right) \\ &= \int_0^t R(s-t) E^\perp(s) ds \psi + R(-t) v^\perp \psi + \int_0^t U^*(s) R(s) \nabla V^\perp(s) U(s) \psi ds \end{aligned}$$

and

$$U^*(t) c U(t) \psi = c \psi - \int_0^t E^\perp(s) \psi + \int_0^t U^*(s) \left(-\nabla V^\perp(s) \right) U(s) \psi ds,$$

$$U^*(t) q U(t) \psi = U_0^*(t) q U_0(t) \psi + \underbrace{\int_0^t U^*(s) (R(s-t) - \mathbb{I}) \nabla V^\perp(s) U(s) \psi ds}_{=: \mathcal{C}_q(t) \psi} \tag{19}$$

It follows from $U(t + T, t_0 + T) = U(t, t_0)$ that

$$\mathcal{C}_q(nT) = \sum_{j=0}^{n-1} U^*(jT) \mathcal{C}_q(T) U(jT) \quad (n \in \mathbb{N}).$$

By the decay hypothesis on ∇V and Theorem 3.1 we have that $\mathcal{C}_q(T)$ is compact. Now

$$\begin{aligned} \frac{1}{nT} (U^*(nT) q U(nT) - q) \psi &= \frac{1}{nT} (R(-nT) - \mathbb{I}) v^\perp \psi \\ &+ \frac{1}{nT} \int_0^{nT} (R(s - nT) - \mathbb{I}) E^\perp(s) ds \psi + \frac{1}{nT} \sum_{j=0}^{n-1} U^*(jT) \mathcal{C}_q(T) U(jT) \psi. \end{aligned} \quad (20)$$

The first term on the right hand side of (20) converges to zero when n goes to infinity. For the third term it holds by Wiener's theorem, see Chapter 5.4 in [CFKS]:

$$s - \lim_{n \rightarrow \infty} \frac{1}{nT} \sum_{j=0}^{n-1} U^*(jT) \mathcal{C}_q(T) U(jT) = \sum_{\lambda \in \sigma_d} P_\lambda \mathcal{C}_q(T) P_\lambda,$$

where σ_d denotes the (finite) set of eigenvalues of $U(T)$ and P_λ the eigenprojection associated to $\lambda \in \sigma_d$.

Now

$$\frac{1}{nT} \int_0^{nT} (R(s - nT) - \mathbb{I}) E^\perp(s) ds \rightarrow_{n \rightarrow \infty} \begin{cases} \frac{1}{T} \int_0^T (R(s) - \mathbb{I}) E^\perp(s) ds & T \in 2\pi\mathbb{N} \\ -\frac{1}{T} \int_0^T E^\perp(s) ds & T \notin 2\pi\mathbb{N} \end{cases}$$

because $\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau R E^\perp = 0$ if $T \notin 2\pi\mathbb{N}$ as remarked in the proof of Proposition 2.4.

Using the symbol $q(nT) := U^*(nT) q U(nT)$ we have shown that for $\psi \in \mathcal{S}(\mathbb{R}^2)$:

$$\lim_{n \rightarrow \infty} \frac{1}{nT} (q(nT) - q) \psi = \lim_{n \rightarrow \infty} \frac{1}{nT} q(nT) \psi = v_{asy} \psi$$

with the bounded operator

$$v_{asy} := \begin{cases} \frac{1}{T} \int_0^T (R(s) - \mathbb{I}) E^\perp(s) ds + \sum_{\lambda \in \sigma_d} P_\lambda \mathcal{C}_q(T) P_\lambda & T \in 2\pi\mathbb{N} \\ -\frac{1}{T} \int_0^T E^\perp(s) ds + \sum_{\lambda \in \sigma_d} P_\lambda \mathcal{C}_q(T) P_\lambda & T \notin 2\pi\mathbb{N}. \end{cases}$$

In order to show the claimed result on v_{asy} we shall now prove that a virial theorem holds, meaning here: $P_\lambda v_{asy} P_\lambda = 0 \quad \forall \lambda \in \sigma_d$. This is not trivial because $\frac{1}{nT} (q(nT) - q)$ is unbounded in general.

Let $0 \neq x \in \mathbb{R}^2$. Then for $\psi \in \mathcal{S}(\mathbb{R}^2)$, $\varphi = (i + \langle x, v_{asy} \rangle) \psi$

$$\begin{aligned} & \left((i + \langle x, q(nT)/nT \rangle)^{-1} - (i + \langle x, v_{asy} \rangle)^{-1} \right) \varphi = \\ & (i + \langle x, q(nT)/nT \rangle)^{-1} \langle x, (v_{asy} - q(nT)/nT) \rangle \psi \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Now v_{asy} is bounded implying that $(i + \langle x, v_{asy} \rangle) \mathcal{S}(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2, dq)$, $(i + \langle x, v_{asy} \rangle)^{-1}$ and $(i + \langle x, q(nT)/nT \rangle)^{-1}$ are bounded by 1 so it follows that

$$s - \lim_{n \rightarrow \infty} (i + \langle x, q(nT)/nT \rangle)^{-1} = (i + \langle x, v_{asy} \rangle)^{-1}.$$

On the other hand

$$\begin{aligned} P_\lambda (i + \langle x, q(nT)/nT \rangle)^{-1} P_\lambda &= P_\lambda (i + \langle x, q/nT \rangle)^{-1} P_\lambda \xrightarrow{n \rightarrow \infty} \\ &= 0 = P_\lambda (i + \langle x, v_{asy} \rangle)^{-1} P_\lambda. \end{aligned}$$

This is true for all x so it follows that $P_\lambda v_{asy} P_\lambda = 0$ and we have finished the proof that

$$v_{asy} = \begin{cases} \frac{1}{T} \int_0^T (R(s) - \mathbb{I}) E^\perp(s) ds P_{ac}(U(T)) & T \in 2\pi\mathbb{N} \\ -\frac{1}{T} \int_0^T E^\perp(s) ds P_{ac}(U(T)) & T \notin 2\pi\mathbb{N}. \end{cases}$$

It remains to prove the energy growth in the case $T \in 2\pi\mathbb{N}$. By the same argument as before we have for the asymptotic acceleration:

$$s - \lim_{n \rightarrow \infty} \frac{1}{nT} \left(\underbrace{U^*(nT)v^\perp U(nT)}_{=: v^\perp(nT)} - v^\perp \right) = s - \lim_{n \rightarrow \infty} \frac{1}{nT} v^\perp(nT) = \frac{1}{T} \int_0^T RE^\perp P_{ac}(U(T)).$$

Now for $\psi \in \mathcal{S}(\mathbb{R}^2)$:

$$\begin{aligned} & \frac{1}{(nT)^2} \langle U^*(nT)\psi, H_{La}U(nT)\psi \rangle = \frac{1}{2} \sum_{j=1}^2 \left\| \frac{1}{nT} v_j^\perp(nT)\psi \right\|^2 \\ & \xrightarrow{n \rightarrow \infty} \frac{1}{2} \left| \frac{1}{T} \int_0^T RE^\perp \right|^2 \|P_{ac}(U(T))\psi\|^2 \end{aligned}$$

■

Proof. of Theorem 1.2

1) From the proof of Theorem 3.4 we have

$$(U^*(T)A_c U^*(T) - A_c) = |a_c(T)|^2 \mathbb{I} + \int_0^T U^*(t) \langle a_c(T), \nabla V^\perp(t) \rangle U(t) dt.$$

$$(U^*(T)A_vU^*(T)-A_v) = |a_v(T)|^2\mathbb{I} - \int_0^T U^*(t) \left\langle R(t)a_v(T), \nabla V^\perp(t) \right\rangle U(t) dt.$$

The assumption of smallness of ∇V implies that one of right hand sides is a positive operator and the result follows from Theorem 2.3

2) Theorem 3.1 and the assumed decay of V imply compactness of

$$\int_0^T U_0^*(t)V(t)U_0(t) dt$$

which as in the proof of Theorem 3.1 implies compactness of

$$\int_0^T U_0^*(t)V(t)U(t) dt = U_0^*(T)U(T) - \mathbb{I}.$$

Thus $U(T) - U_0(T)$ is compact and the essential spectra of $U(T)$ and $U_0(T)$ coincide. By formula (4) $U_0(T) = e^{-i\varphi(T)}e^{-iT H_{L^a}}$ so its spectrum is the set of points $\left(e^{-i\varphi(T)}e^{-i(n+\frac{1}{2})T} \right)_{n \in \mathbb{N}_0}$ which is discrete as $T \in 2\pi\mathbb{Q}$ which implies that the spectrum of $U(T)$ is pure point and Theorem 1.2 is proven. ■

Remark that this argument on stability of the pure point spectrum for resonant perturbations was applied in [ADE] to the electric AC Stark effect.

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