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ABOUT THE CONVOLUTION OF DISTRIBUTIONS ON GROUPOIDS

JEAN-MARIE LESCURE , DOMINIQUE MANCHON, STÉPHANE VASSOUT

ABSTRACT. We review the properties of transversality of distributions with respect to submersions. This allows us to construct a convolution product for a large class of distributions on Lie groupoids. We get a unital involutive algebra $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ enlarging the convolution algebra $C_c^\infty(G, \Omega^{1/2})$ associated with any Lie groupoid G . We prove that G -operators are convolution operators by transversal distributions. We also investigate the microlocal aspects of the convolution product. We give conditions on wave front sets sufficient to compute the convolution product and we show that the wave front set of the convolution product of two distributions is essentially the product of their wave front sets in the symplectic groupoid T^*G of Costes-Dazord-Weinstein. This also leads to a subalgebra $\mathcal{E}'_a(G, \Omega^{1/2})$ of $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ which contains for instance the algebra of pseudodifferential operators on G .

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1. INTRODUCTION

The motivation of this paper is twofold. Firstly, we wish to clarify the notions of distributions on a Lie groupoid and their convolution product, and to understand the relationship between these notions and the so-called G -operators. Secondly, we would like to set up a neat framework in order to investigate in a future work the notions of Fourier integrals and Fourier integral operators on a groupoid.

The notion of C^∞ longitudinal family of distributions in the framework of groupoids appears in [12, 13, 16] in order to define right invariant pseudodifferential operators. Also, in the works of B. Monthubert [11], these families are considered from the point of view of distributions on the whole groupoid, so that the action of the corresponding pseudodifferential operators on C^∞ functions is given by a convolution product. Here, we carry on this idea by exploring the correspondance between C^∞ longitudinal families of distributions and single distributions on the whole underlying manifold of the groupoid and by studying the convolution product of distributions on groupoids. This is achieved at two levels.

The first level is based on the notion of transversality of distributions with respect to a submersion $\pi : M \rightarrow B$ [1]. It appears that the space $\mathcal{D}'_{\pi}(M)$ of such distributions is actually isomorphic to the space of C^{∞} family of distributions in the fibers of π and to the space of continuous $C^{\infty}(B)$ -linear maps from the space $C^{\infty}_{c-\pi}(M)$ of C^{∞} functions compactly supported in the fibers to $C^{\infty}(B)$. Furthermore, operations such as push-forwards and fibered-products of distributions behave well on transversal distributions and these operations allow to define the convolution product of distributions on groupoids, as soon as these distributions satisfy some transversality assumptions with respect to source or target maps. Distributions on a groupoid which are transversal both to the source and target maps are called bi-transversal and they give rise to an involutive unital algebra $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ for the convolution product. Then, one has the necessary tools to prove that G -operators on a groupoid are in 1 to 1 correspondance with transversal distributions acting by convolution and that bi-transversal distributions are in 1 to 1 correspondance with adjointable G -operators.

The second level is a microlocal refinement of the first one and consists in using the wave front set of distributions. A basic observation, due to Costes, Dazord and Weinstein [4], is that the cotangent manifold T^*G of any Lie groupoid G carries a non trivial structure of symplectic groupoid which is intimately related to the multiplication of G and then to the convolution on $C^{\infty}_c(G)$. This groupoid combined with the classical calculus of wave front sets developped by Hörmander bring in natural conditions on wave front sets of distributions on a groupoid allowing to define their convolution product and to compute the corresponding wave front set using the law of T^*G .

The class of distributions permitted in this second approach is slightly different from the one arising from transversality. However, there is a large natural class of transversal distributions for which the second approach applies, consisting of distributions whose wave front sets are transversal with respect to the (differential of) source or target maps. Actually, the transversality of the wave front set is a strictly stronger assumption than the transversality condition of the distribution itself and thus, distributions with bi-transversal wave front sets provide a sub-space $\mathcal{E}'_a(G, \Omega^{1/2})$ of the algebra $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$. Still using the Costes-Dazord-Weinstein groupoid structure of T^*G , we show that $\mathcal{E}'_a(G, \Omega^{1/2})$ is a unital involutive sub-algebra of $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ and for any $u, v \in \mathcal{E}'_a(G, \Omega^{1/2})$, we obtain

$$(1) \quad \text{WF}(u * v) \subset \text{WF}(u) * \text{WF}(v),$$

where $*$ is the composition law in the Costes-Dazord-Weinstein groupoid T^*G . We would like to add that the corresponding formula of Hormander for the wave front set of composition of kernels [7, 8] makes the above formula quite predictable. Indeed, given a manifold X , the composition of kernels corresponds to convolution in the pair groupoid $X \times X$ and the composition law that Hörmander defines on $T^*(X \times X)$ to compute wave front sets of composition of kernels is precisely the multiplication map of the Costes-Dazord-Weinstein symplectic groupoid $T^*(X \times X)$.

The paper is organized as follows. In section 2, we review the notion of distributions transversal with respect to a submersion and study natural operations available on them. In section 3, we apply the results of section 2 to the case of groupoids. We then define the convolution product of (transversal) distributions and obtain the unital algebra $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ of bi-transversal distributions. In section 4, we link the notion of G -operators with the one of transversal distributions and we obtain a 1 to 1 correspondance between the space of adjointable compactly supported G -operators and $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$. In section 5 we show that natural transversality conditions on the wave front set of a distribution imply transversality with respect to a submersion, but that these notions do not

coincide in general. This leads naturally to the question of the convolution of distributions with transversal wave front sets on a groupoid. After reviewing the groupoid structure of Costes-Dazord-Weinstein [4] in section 6, convolution is thus revisited in section 7 in terms of wave front sets. The symplectic groupoid structure is used to establish a formula for the wave front set of the convolution product of distributions. In section 8 we end up by studying the meaning of bi-transversality of the wave front set in the framework of G -operators.

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2. DISTRIBUTIONS, SUBMERSIONS, TRANSVERSALITY

Before dealing with distributions on groupoids, it is useful to go back to the behavior of distributions on total spaces of submersions. The notion of transversality we shall recall in this part is borrowed from [1] and it extends the condition of semi-regularity given by Trèves [15, p.532].

Unless otherwise stated, distribution spaces will be provided with the strong topology and spaces of continuous linear maps between locally convex vector spaces with the topology of bounded convergence.

Let $X \subset \mathbb{R}^{n_X}$ and $Y \subset \mathbb{R}^{n_Y}$ denote open subsets and pr_1, pr_2 the projections onto the first and second factors of $X \times Y$. When f is a function on $X \times Y$, partial maps $y \mapsto f(x, y)$ and $x \mapsto f(x, y)$ are usually noted f_x and f_y respectively.

The Schwartz kernel theorem asserts that the map $P : \mathcal{D}'(X \times Y) \longrightarrow \mathcal{L}(C_c^\infty(Y), \mathcal{D}'(X))$ given by

$$(2) \quad \mathcal{D}'(X \times Y) \ni u \longmapsto P(u) : f \longmapsto \int_Y u(x, y) f(y) dy \in \mathcal{L}(C_c^\infty(Y), \mathcal{D}'(X))$$

is a topological isomorphism. Note that $P(u)(f) = (\text{pr}_1)_*(u.f) \in \mathcal{D}'(X)$ is the push-forward of the distribution $u.f$, that is, the distribution on X obtained by "integrating in the fibers of pr_1 " the distribution $u.f$. This operation still makes sense if we now take f in the LF space

$$(3) \quad C_{c-\text{pr}_1}^\infty(X \times Y) = \{f \in C^\infty(X \times Y) ; \forall g \in C_c^\infty(X), f.g \circ \text{pr}_1 \in C_c^\infty(X \times Y)\}$$

and we get a linear map

$$(4) \quad \begin{aligned} \tilde{P} : \mathcal{D}'(X \times Y) &\longrightarrow \mathcal{L}_{C^\infty(X)}(C_{c-\text{pr}_1}^\infty(X \times Y), \mathcal{D}'(X)) \\ u &\longmapsto (\tilde{P}(u) : f \longmapsto (\text{pr}_1)_*(u.f)). \end{aligned}$$

Here $\mathcal{L}_{C^\infty(X)}(C_{c-\text{pr}_1}^\infty(X \times Y), \mathcal{D}'(X))$ denotes the topological subspace of $\mathcal{L}(C_{c-\text{pr}_1}^\infty(X \times Y), \mathcal{D}'(X))$ consisting of $C^\infty(X)$ -linear maps. It is then easy to check that the map \tilde{P} is also a topological isomorphism with inverse given by

$$\mathcal{L}_{C^\infty(X)}(C_{c-\text{pr}_1}^\infty(X \times Y), \mathcal{D}'(X)) \ni U \longmapsto P^{-1}(U|_{1 \otimes C_c^\infty(Y)}) \in \mathcal{D}'(X \times Y).$$

Equivalently, we have a topological isomorphism $\mathcal{L}_{C^\infty(X)}(C_{c-\text{pr}_1}^\infty(X \times Y), \mathcal{D}'(X)) \simeq \mathcal{L}(C_c^\infty(Y), \mathcal{D}'(X))$. That (4) is an isomorphism is another formulation of the Schwartz kernel theorem.

We are interested in the subspace of $\mathcal{D}'(X \times Y)$ consisting of distributions for which the "integration along the fibers" (4) produces C^∞ functions.

Definition 1. *We say that $u \in \mathcal{D}'(X \times Y)$ is transversal to pr_1 (or semi-regular with respect to x) if*

$$P(u)(C_c^\infty(Y)) \subset C^\infty(X)$$

We note $\mathcal{D}'_{\text{pr}_1}(X \times Y)$ (resp. $\mathcal{E}'_{\text{pr}_1}(X \times Y)$) the linear space of pr_1 -transversal (resp. compactly supported and pr_1 -transversal) distributions.

This is of course equivalent to the requirement that $\tilde{P}(u)$ maps $C_{c-\text{pr}_1}^\infty(X \times Y)$ into $C^\infty(X)$.

Proposition 1. *The following equalities hold*

$$P(\mathcal{D}'_{\text{pr}_1}(X \times Y)) = \mathcal{L}(C_c^\infty(Y), C^\infty(X)), \quad \tilde{P}(\mathcal{D}'_{\text{pr}_1}(X \times Y)) = \mathcal{L}_{C^\infty(X)}(C_{c-\text{pr}_1}^\infty(X \times Y), C^\infty(X)).$$

Proof. We only check the first equality, the second one is then analogous. If $u \in \mathcal{D}'_{\text{pr}_1}(X \times Y)$, then $P(u) : C_c^\infty(Y) \rightarrow C^\infty(X)$ is a linear map which is a priori continuous when $C^\infty(X)$ has the topology of $\mathcal{D}'(X)$. It is then a consequence of the closed graph theorem on LF spaces, also called Grothendieck-Köthe theorem [9, Cor 1.2.20, p. 22] that it is also continuous for the natural topology of $C^\infty(X)$.

Indeed, $P(u)$ has a closed graph in $C_c^\infty(Y) \times \mathcal{D}'(X)$, and its graph is, by hypothesis, included into $C_c^\infty(Y) \times C^\infty(X)$. If a net (f_α, g_α) in the graph of $P(u)$ converges to (f, g) in $C_c^\infty(Y) \times C^\infty(X)$, then it converges to (f, g) in $C_c^\infty(Y) \times \mathcal{D}'(X)$. It follows that (f, g) is in the graph of $P(u)$, so that the graph of $P(u)$ is closed in $C_c^\infty(Y) \times C^\infty(X)$. This proves that $P(\mathcal{D}'_{\text{pr}_1}(X \times Y)) \subset \mathcal{L}(C_c^\infty(Y), C^\infty(X))$. The other inclusion is obvious. \square

Using the bijection P , we give to $\mathcal{D}'_{\text{pr}_1}(X \times Y)$ the topology of $\mathcal{L}(C_c^\infty(Y), C^\infty(X))$. This is the same topology as the one obtained from the bijection \tilde{P} since we have a topological isomorphism

$$(5) \quad \mathcal{L}_{C^\infty(X)}(C_{c-\text{pr}_1}^\infty(X \times Y), C^\infty(X)) \simeq \mathcal{L}(C_c^\infty(Y), C^\infty(X))$$

given by $U \mapsto U|_{1 \otimes C_c^\infty(Y)}$.

There is a last interpretation of transversal distributions, using the topological isomorphisms [15, prop. 50.5, thm 44.1, see also p. 532]

$$\mathcal{L}(C_c^\infty(Y), C^\infty(X)) \simeq C^\infty(X) \hat{\otimes} \mathcal{D}'(Y) \simeq C^\infty(X, \mathcal{D}'(Y)).$$

This means that pr_1 -transversal distributions on $X \times Y$ are, in a canonical way, identified with C^∞ functions on X with values in distributions on Y . Concretely, this identification

$$I : C^\infty(X, \mathcal{D}'(Y)) \xrightarrow{\simeq} \mathcal{D}'_{\text{pr}_1}(X \times Y)$$

is given by

$$(6) \quad \langle I(u), f \rangle = \int_X \langle u(x), f(x, \cdot) \rangle_Y dx, \quad f \in C_{c-\text{pr}_1}^\infty(X \times Y),$$

with inverse given by

$$\langle I^{-1}(v)(x), h \rangle_Y = P_v(h)(x), \quad h \in C_c^\infty(Y).$$

In summary, we have exhibited several canonical isomorphisms of topological vector spaces

$$(7) \quad \mathcal{D}'_{\text{pr}_1}(X \times Y) \simeq C^\infty(X, \mathcal{D}'(Y)) \simeq \mathcal{L}(C_c^\infty(Y), C^\infty(X)) \simeq \mathcal{L}_{C^\infty(X)}(C_{c-\text{pr}_1}^\infty(X \times Y), C^\infty(X)).$$

The discussion can be carried out along the same lines when conditions on the support of transversal distributions are added. For instance:

$$(8) \quad \mathcal{E}'_{\text{pr}_1}(X \times Y) \simeq C_c^\infty(X, \mathcal{E}'(Y)) \simeq \mathcal{L}(C^\infty(Y), C_c^\infty(X)) \simeq \mathcal{L}_{C^\infty(X)}(C^\infty(X \times Y), C_c^\infty(X)).$$

More importantly, we can consider the space of distributions on $X \times Y$ with compact supports in the fibers of pr_1

$$(9) \quad \mathcal{D}'_{c-\text{pr}_1}(X \times Y) = \{u \in \mathcal{D}'(X \times Y); \forall g \in C_c^\infty(X), g.u \in \mathcal{E}'\} = (C_{c-\text{pr}_2}^\infty(X \times Y))'.$$

Then, setting

$$(10) \quad \mathcal{P}'_{\text{pr}_1}(X \times Y) = \mathcal{D}'_{c-\text{pr}_1}(X \times Y) \cap \mathcal{D}'_{\text{pr}_1}(X \times Y)$$

we get

$$(11) \quad \mathcal{P}'_{\text{pr}_1}(X \times Y) \simeq C^\infty(X, \mathcal{E}'(Y)) \simeq \mathcal{L}(C^\infty(Y), C^\infty(X)) \simeq \mathcal{L}_{C^\infty(X)}(C^\infty(X \times Y), C^\infty(X)).$$

We now replace $\text{pr}_1 : X \times Y \rightarrow X$ by a surjective submersion $\pi : M \rightarrow B$. We will use bundles of densities in order to obtain canonical statements. If M is a manifold, we note Ω_M^α or $\Omega^\alpha(M)$ the bundle of α -densities over M . Since the base manifold B is not necessarily compact, the natural analog of $C_{c-\text{pr}_1}^\infty(X \times Y)$ is the LF space

$$(12) \quad C_{c-\pi}^\infty(M) = \{f \in C^\infty(M) ; \forall g \in C_c^\infty(B), f.g \circ \pi \in C_c^\infty(M)\}$$

or its variant for C^∞ sections of vector bundles as well. We now introduce the spaces generalizing the ones appearing in (7),(8), (9) and (10).

Recall that on a manifold M , the space $\mathcal{D}'(M, \Omega_M^\alpha)$ (resp. $\mathcal{E}'(M, \Omega_M^\alpha)$) is the topological dual of the space $C_c^\infty(M, \Omega_M^{1-\alpha})$ (resp. $C^\infty(M, \Omega_M^{1-\alpha})$). We also set

$$(13) \quad \mathcal{D}'_{c-\pi}(M, \Omega_M^\alpha) = \{u \in \mathcal{D}'(M, \Omega_M^\alpha), \forall g \in C_c^\infty(B), g \circ \pi.u \in \mathcal{E}'\}.$$

The latter can be viewed as the topological dual of the LF-space of C^∞ densities u of degree $1 - \alpha$ on M such that $\pi(\text{supp}(u))$ is compact. The submersion π also gives a $C^\infty(B)$ -module structure to the spaces of C^∞ functions on M , hence the spaces of $C^\infty(B)$ -linear continuous maps :

$$\mathcal{L}_{C^\infty(B)}(C_{c-\pi}^\infty(M), \mathcal{D}'(B)), \quad \mathcal{L}_{C^\infty(B)}(C^\infty(M), C^\infty(B)), \dots$$

make sense and are considered as topological subspaces of the corresponding spaces of \mathbb{C} -linear continuous maps.

The push-forward of distributions by π makes sense :

$$\langle \pi_*(u.f), g \rangle_B = \langle u, f.g \circ \pi \rangle_M,$$

and the previous discussion about the Schwartz kernel theorem can be rephrased in the submersion case. (We set $\alpha = 0$ for simplicity).

Theorem 2 (Schwartz kernel theorem, submersion case.). *The maps*

$$\begin{aligned} \mathcal{D}'(M) &\longrightarrow \mathcal{L}_{C^\infty(B)}(C_{c-\pi}^\infty(M, \Omega_M), \mathcal{D}'(B, \Omega_B)), \\ \mathcal{E}'(M) &\longrightarrow \mathcal{L}_{C^\infty(B)}(C^\infty(M, \Omega_M), \mathcal{E}'(B, \Omega_B)), \\ \mathcal{D}'_{c-\pi}(M) &\longrightarrow \mathcal{L}_{C^\infty(B)}(C^\infty(M, \Omega_M), \mathcal{D}'(B, \Omega_B)), \end{aligned}$$

given by

$$u \longrightarrow (\tilde{P}(u) : f \longmapsto \pi_*(u.f))$$

are topological isomorphisms.

Bundles can be added in the following statements but we omit them for the sake of simplicity.

Definition 2. ([1] *Androulidakis-Skandalis*). *Let $\pi : M \rightarrow B$ be a C^∞ submersion onto B . A distribution $u \in \mathcal{D}'(M)$ is said transversal to π if for any $f \in C_{c-\pi}^\infty(M, \Omega_M)$, the distribution $\pi_*(u.f) \in \mathcal{D}'(B, \Omega_B)$ is C^∞ . We denote by $\mathcal{D}'_\pi(M)$ the space of π -transversal distributions. We also set*

$$\mathcal{E}'_\pi(M) = \mathcal{D}'_\pi(M) \cap \mathcal{E}'(M) \text{ and } \mathcal{P}'_\pi(M) = \mathcal{D}'_\pi(M) \cap \mathcal{D}'_{c-\pi}(M)$$

Definition 3. Let $\pi : M \longrightarrow B$ be a C^∞ submersion onto B . A family $u = (u_x)_{x \in B}$ of distributions $u_x \in \mathcal{D}'(\pi^{-1}(x))$ is C^∞ if for any local trivialization of π

$$U \subset M, X \subset B, \kappa : U \xrightarrow{\cong} X \times Y, \pi|_U = \pi_X \circ \kappa,$$

we have

$$\kappa_*(u|_U) \in C^\infty(X, \mathcal{D}'(Y)).$$

The space of C^∞ families is noted $C_\pi^\infty(B, \mathcal{D}'(M))$. The spaces $C_{\pi, \text{cpct}}^\infty(B, \mathcal{E}'(M))$ and $C_\pi^\infty(B, \mathcal{D}'_{c-\pi}(M))$ are defined accordingly.

Using a covering of M by local trivializations and a partition of unity argument, we can use the seminorms of the spaces $C^\infty(X, \mathcal{D}'(Y))$ to build a complete Hausdorff locally convex vector space structure on $C_\pi^\infty(B, \mathcal{D}'(M))$.

The discussion about transversal distributions in the product case can be easily extended to general submersions.

Proposition 3. The maps

$$(14) \quad \begin{array}{ccc} C_\pi^\infty(B, \mathcal{D}'(M)) & \xrightarrow{I} & \mathcal{D}'_\pi(M) \\ C_{\pi, \text{cpct}}^\infty(B, \mathcal{E}'(M)) & \longrightarrow & \mathcal{E}'_\pi(M) \\ C_\pi^\infty(B, \mathcal{D}'_{c-\pi}(M)) & \longrightarrow & \mathcal{P}'_\pi(M) \end{array}$$

given by

$$u \longmapsto (f \mapsto \int_B \langle u_x, f_x \rangle)$$

and the maps

$$(15) \quad \begin{array}{ccc} \mathcal{D}'_\pi(M) & \xrightarrow{\tilde{P}} & \mathcal{L}_{C^\infty(B)}(C_{c-\pi}^\infty(M, \Omega_M), C^\infty(B, \Omega_B)) \\ \mathcal{E}'_\pi(M) & \longrightarrow & \mathcal{L}_{C^\infty(B)}(C^\infty(M, \Omega_M), C_c^\infty(B, \Omega_B)) \\ \mathcal{P}'_\pi(M) & \longrightarrow & \mathcal{L}_{C^\infty(B)}(C^\infty(M, \Omega_M), C^\infty(B, \Omega_B)) \end{array}$$

given by

$$u \longmapsto (f \mapsto \pi_*(u.f))$$

are all bijective. The topologies on each space $\mathcal{D}'_\pi, \mathcal{E}'_\pi, \mathcal{P}'_\pi$, deduced from the corresponding bijections coincide.

To be concrete, the inverse of the first map in (14) is the map sending $u \in \mathcal{D}'_\pi(M)$ to the C^∞ family $u_x, x \in B$ given by

$$(16) \quad \langle u_x, f \rangle = (\pi_*(u.\tilde{f}))(x) = \int_{m \in \pi^{-1}(x)} u(m)f(m);$$

where $f \in C_c^\infty(\pi^{-1}(x), \Omega_M|_{\pi^{-1}(x)})$ and $\tilde{f} \in C_c^\infty(M, \Omega_M)$ is any C^∞ extension of f . Also, the inverse of the first map in (15) is given by

$$(17) \quad \langle \tilde{P}^{-1}(U), f \rangle = \int_B U(f), \quad f \in C_{c-\pi}^\infty(M, \Omega_M).$$

The other cases are given by the same formula, only the spaces of functions involved has to be changed.

Remark 4. Sometimes, we need to handle vector bundles E over M and F over B . All the previous isomorphisms adapt to this case. For instance, we embed $\mathcal{D}'_\pi(M, E)$ into $\mathcal{D}'_\pi(M, E \otimes \text{End}(\pi^*F))$ for which we have canonical isomorphisms

$$(18) \quad \widetilde{P}_F : \mathcal{D}'_\pi(M, E \otimes \text{End}(\pi^*F)) \longrightarrow \mathcal{L}_{C^\infty(B)}(C_{c-\pi}^\infty(M, \Omega_M \otimes E^* \otimes \pi^*F), C^\infty(B, \Omega_B \otimes F))$$

and

$$(19) \quad I_F : C^\infty(B, \mathcal{D}'(M, E \otimes \text{End}(\pi^*F))) \longrightarrow \mathcal{D}'_\pi(M, E \otimes \text{End}(\pi^*F)).$$

We now list several operations available on transversal distributions.

Proposition 5. Let $\pi : M \rightarrow B$ a surjective submersion and $C \subset B$ a submanifold. Let E be a C^∞ vector bundle over M and E_C its restriction onto C . We denote $\pi_C : M_C = \pi^{-1}(C) \rightarrow C$ the submersion obtained by restriction. The restriction of distributions

$$\text{Rest}_C : \mathcal{D}'_\pi(M, E) \longrightarrow \mathcal{D}'_{\pi_C}(M_C, E_C)$$

is well defined and continuous.

This is obvious.

Proposition 6. Let $\rho : Z \rightarrow M$ and $\pi : M \rightarrow B$ be surjective submersions. Let E be a C^∞ vector bundle over M . The pull back of distributions restricts to a continuous map

$$\rho^* : \mathcal{D}'_\pi(M, E) \longrightarrow \mathcal{D}'_{\pi \circ \rho}(Z, \rho^*E).$$

Proof. Since ρ is a submersion, the map

$$\begin{aligned} \rho_* : C_{c-\pi \circ \rho}^\infty(Z, \Omega_Z \otimes \rho^*E^*) &\longrightarrow C_{c-\pi}^\infty(M, \Omega_M \otimes E^*) \\ f &\longmapsto (m \mapsto \int_{\rho^{-1}(m)} f) \end{aligned}$$

is well defined and general theorems of basic calculus show that it is continuous. Thus ρ^* identifies with the map

$$\rho^* : \mathcal{L}_{C^\infty(B)}(C_{c-\pi}^\infty(M, \Omega_M \otimes E^*), C^\infty(B, \Omega_B)) \longrightarrow \mathcal{L}_{C^\infty(B)}(C_{c-\pi \circ \rho}^\infty(Z, \Omega_Z \otimes \rho^*E^*), C^\infty(B, \Omega_B))$$

$$U \longmapsto \rho^*(U) : f \mapsto U(\rho_*(f))$$

which is clearly continuous. \square

When two submersions $\pi_i : M_i \rightarrow B$, $i = 1, 2$ are given over the same base, we note

$$M = M_1 \times_{\pi} M_2 = \{(m_1, m_2) \in M_1 \times M_2 ; \pi_1(m_1) = \pi_2(m_2)\}$$

and

$$\pi : M \longrightarrow B, \quad (m_1, m_2) \longmapsto \pi_1(m_1)$$

Also, the natural projections $M \rightarrow M_i$, $i = 1, 2$ are denoted by pr_i . We thus have a commutative square of submersions

$$(20) \quad \begin{array}{ccc} M & \xrightarrow{\text{pr}_2} & M_2 \\ \downarrow \text{pr}_1 & & \downarrow \pi_2 \\ M_1 & \xrightarrow{\pi_1} & B \end{array}$$

Proposition 7. With the notations above, the pull-back of distributions $\text{pr}_1^* : \mathcal{D}'(M_1) \rightarrow \mathcal{D}'(M)$ restricts to a continuous map

$$(21) \quad \text{pr}_1^* : \mathcal{D}'_{\pi_1}(M_1) \longrightarrow \mathcal{D}'_{\text{pr}_2}(M)$$

Proof. We use a partition of unity argument and local trivializations of the submersions π_1, π_2 to replace M_i by a product $U_i = X \times Y_i$ of open subsets in \mathbb{R}^{n_i} , so that π_i is the first projection and pr_i is the obvious projection map of $X \times Y_1 \times Y_2$ onto $X \times Y_i$, for $i = 1, 2$. Then

$$(22) \quad \text{pr}_1^* : \mathcal{D}'(X \times Y_1) \longrightarrow \mathcal{D}'(X \times Y_1 \times Y_2)$$

$$(23) \quad u \longmapsto u \otimes 1.$$

The assertion then follows easily when one considers transversal distributions as C^∞ families. Actually, one can also invoke the Schwartz kernel theorem to get

$$(24) \quad \begin{array}{ccc} \mathcal{D}'(X \times Y_1) & \xrightarrow{\text{pr}_1^*} & \mathcal{D}'(X \times Y_1 \times Y_2) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathcal{L}(C_c^\infty(Y_1), \mathcal{D}'(X)) & \xrightarrow{\pi_2^* \circ \cdot} & \mathcal{L}(C_c^\infty(Y_1), \mathcal{D}'(X \times Y_2)), \end{array}$$

Since $\pi_2^* \circ \cdot$ maps continuously $\mathcal{L}(C_c^\infty(Y_1), C^\infty(X))$ to $\mathcal{L}(C_c^\infty(Y_1), C^\infty(X \times Y_2))$, we get that pr_1^* maps continuously $\mathcal{D}'_{\pi_1}(X \times Y_1)$ to $\mathcal{D}'_{\text{pr}_2}(X \times Y_1 \times Y_2)$. \square

Remark 8. (1) *The assertion of the previous proposition holds for commutative square of surjective submersions*

$$(25) \quad \begin{array}{ccc} M & \xrightarrow{\text{pr}_2} & M_2 \\ \downarrow \text{pr}_1 & & \downarrow \pi_2 \\ M_1 & \xrightarrow{\pi_1} & B \end{array}$$

such that any point of M , $\ker d\pi_1 \circ \text{pr}_1 = \ker d\text{pr}_1 + \ker d\text{pr}_2$ or, equivalently, such that $\text{pr}_1 : \text{pr}_2^{-1}(m_2) \rightarrow \pi_1^{-1}(b)$, $b = \pi_2(m_2)$, is a submersion for any $m_2 \in M_2$.

(2) *One can also prove in the same way that the restriction $\text{pr}_1^* : \mathcal{P}'_{\pi_1}(M_1) \rightarrow \mathcal{P}'_{\text{pr}_2}(M)$ is continuous, observing that pr_1^* restricts to $\mathcal{D}'_{c-\pi_1}(X \times Y_1) \rightarrow \mathcal{D}'_{c-\text{pr}_2}(X \times Y_1 \times Y_2)$.*

Then one can, as above, extend this result to more general commutative squares, provided the additional condition that the map $M \rightarrow M_1 \times_{\pi} M_2, m \mapsto (\text{pr}_1(m), \text{pr}_2(m))$ has compact fibers.

When a finite set \mathcal{I} of submersions is given on M , we introduce

$$(26) \quad \mathcal{D}'_{\mathcal{I}}(M) = \bigcap_{\rho \in \mathcal{I}} \mathcal{D}'_{\rho}(M) \subset \mathcal{D}'(M).$$

The space $\mathcal{D}'_{\mathcal{I}}(M, E)$ is given the topology generated by the union of the induced topologies by each $\mathcal{D}'_{\rho}(M)$, $\rho \in \mathcal{I}$. We adapt similar convention for the spaces $\mathcal{E}'_{\mathcal{I}}(M)$ and $\mathcal{D}'_{c-\mathcal{I}}(M)$. The previous proposition is now used to define fibered product of distributions.

Proposition 9. *Let $\pi_i : M_i \rightarrow B$, $i = 1, 2$, $\rho : M_1 \rightarrow A$, $\sigma : M_2 \rightarrow C$ be submersions and set $M = M_1 \times_{\pi} M_2$, $\text{pr}_i : M \rightarrow M_i$, $i = 1, 2$, $\pi : M_1 \times M_2 \rightarrow B$, $(m_1, m_2) \mapsto \pi_1(m_1)$.*

$$\begin{array}{ccccc} M & \xrightarrow{\text{pr}_2} & M_2 & \xrightarrow{\sigma} & C \\ \downarrow \text{pr}_1 & \searrow \pi & \downarrow \pi_2 & & \\ M_1 & \xrightarrow{\pi_1} & B & & \\ \downarrow \rho & & & & \\ A & & & & \end{array}$$

The fibered product of C^∞ functions $(f_1, f_2) \mapsto f_1 \otimes f_2|_M$ extends uniquely to separately continuous bilinear maps

$$(27) \quad \begin{array}{ccc} \mathcal{D}'_{\pi_1}(M_1) \times \mathcal{D}'_{\sigma}(M_2) & \longrightarrow & \mathcal{D}'_{\sigma \circ \text{pr}_2}(M) \\ (u_1, u_2) & \longmapsto & u_1 \times_{\pi_1} u_2 \end{array}$$

If $u_1 \in \mathcal{D}'_{\pi_1}(M_1)$ and $u_2 \in \mathcal{D}'_{\pi_2}(M_2)$, then the equality

$$(28) \quad u_1 \times_{\pi_1} u_2 = u_1 \times_{\pi_2} u_2$$

holds and the previous maps restrict to a separately continuous bilinear map

$$(29) \quad \begin{array}{ccc} \mathcal{D}'_{\rho, \pi_1}(M_1) \times \mathcal{D}'_{\pi_2, \sigma}(M_2) & \longrightarrow & \mathcal{D}'_{\rho \circ \text{pr}_1, \pi, \sigma \circ \text{pr}_2}(M) \\ (u_1, u_2) & \longmapsto & u_1 \times_{\pi_1} u_2. \end{array}$$

Remark 10. The above map in (27) restricts to a separately continuous map

$$(30) \quad \mathcal{P}'_{\pi_1}(M_1) \times \mathcal{D}'_{\sigma}(M_2) \longrightarrow \mathcal{D}'_{\sigma \circ \text{pr}_2}(M) \cap \mathcal{D}'_{c-\text{pr}_2}(M).$$

There are analogous statements for the fibered product over π_2 and the conditions on supports can be interchanged.

Proof. Let $u_1 \in \mathcal{D}'_{\pi_1}(M_1)$ and $u_2 \in \mathcal{D}'(M_2)$. Then we define the distribution $u_1 \times_{\pi_1} u_2$ on M by

$$(31) \quad \langle u_1 \times_{\pi_1} u_2, f \rangle = \int_{M_2} \left(\int_{M_1 \pi_2(m_2)} u_1 \pi_2(m_2)(m_1) f(m_1, m_2) \right) u_2(m_2)$$

where the integrals are taken in the distributional sense. This clearly coincides with the fibered product of functions when u_1, u_2 are C^∞ . To justify that $u_1 \times_{\pi_1} u_2$ is $\sigma \circ \text{pr}_2$ -transversal and the continuity with respect to u_1 and u_2 , we apply Proposition 7 to u_1 . This gives $\text{pr}_1^*(u_1) \in \mathcal{L}_{C^\infty(M_2)}(C_{c-\text{pr}_2}^\infty(M, \Omega_M), C^\infty(M_2, \Omega_{M_2}))$ and it is clear from the definition that $\text{pr}_1^*(u_1)$ restricts to give an element of $\mathcal{L}_{C^\infty(M_2)}(C_{c-\sigma \circ \text{pr}_2}^\infty(M, \Omega_M), C_{c-\sigma}^\infty(M_2, \Omega_{M_2}))$ which depends continuously on u_1 . Then

$$(32) \quad u_1 \times_{\pi_1} u_2 = u_2 \circ \text{pr}_1^*(u_1) \in \mathcal{L}_{C^\infty(C)}(C_{c-\sigma \circ \text{pr}_2}^\infty(M, \Omega_M), C^\infty(C, \Omega_C))$$

is continuous in u_1 and u_2 since the composition of continuous linear maps is separately continuous.

When both fibered products are available, we have

$$(33) \quad \begin{aligned} \langle u_1 \times_{\pi_1} u_2, f \rangle &= \int_{M_2} \int_{M_1 \pi_2(m_2)} u_1 \pi_2(m_2)(m_1) f(m_1, m_2) u_2(m_2) \\ &= \int_{y \in B} \int_{M_{2y}} \int_{M_{1y}} u_{1y}(m_1) f(m_1, m_2) u_{2y}(m_2) \\ &= \int_{y \in B} \int_{M_{1y}} \int_{M_{2y}} u_{1y}(m_1) f(m_1, m_2) u_{2y}(m_2) \\ &= \int_{M_1} u_1(m_1) \int_{M_2 \pi_1(m_1)} f(m_1, m_2) u_2 \pi_1(m_1)(m_2) \\ &= \langle u_1 \times_{\pi_2} u_2, f \rangle. \end{aligned}$$

We have noted for convenience $M_{iy} = \pi_i^{-1}(y)$, $u_{iy} = I^{-1}(u_i)(y) \in \mathcal{D}'(M_{iy})$, $i = 1, 2$.

This equality allows us to take into account the π_2, σ -transversality of u_2 and the ρ -transversality of u_1 to conclude by the previous argument that $u_1 \times_{\pi_1} u_2$ is transversal with respect to $\rho \circ \text{pr}_1, \pi$ and $\sigma \circ \text{pr}_2$ and depends continuously in u_1 and u_2 .

□

Now let $f : M \rightarrow N$ be a C^∞ map between smooth manifolds. Then the push-forward of distributions

$$(34) \quad \langle f_* u, g \rangle = \langle u, g \circ f \rangle$$

is well defined as a continuous map $\mathcal{E}'(M, \Omega_M) \rightarrow \mathcal{E}'(N, \Omega_N)$. It follows easily from the definition that if $\rho : M \rightarrow B$, $\rho' : N \rightarrow B$ are two submersions onto B such that $\rho' \circ f = \rho$, then this maps restricts naturally to a continuous map

$$(35) \quad f_* : \mathcal{E}'_\rho(M, \Omega_M) \rightarrow \mathcal{E}'_{\rho'}(N, \Omega_N).$$

Note that if $f : M \rightarrow N$ is proper then the push-forward of distributions naturally extends to a continuous map $f_* : \mathcal{D}'(M, \Omega_M) \rightarrow \mathcal{D}'(N, \Omega_N)$ which naturally restricts to $f_* : \mathcal{D}'_\pi(M, \Omega_M) \rightarrow \mathcal{D}'_\rho(N, \Omega_N)$. However, this assumption of properness is too strong in general and we will need some refined statement.

Proposition 11. *Assume M is the fibered product of M_1 and M_2 as in (20) and that the smooth map $f : M \rightarrow N$ satisfies that for any compact $K \subset M_1$ the restriction of the map f to $(K \times M_2) \cap M$ is proper (that is, for any compact $L \subset N$, the set $f^{-1}(L) \cap K \times M_2$ is compact in M). Then for any compactly supported map $\varphi \in C_c^\infty(M_1)$ and any distribution $u \in \mathcal{D}'(M, \Omega_M)$, the push forward of $u \cdot (\varphi \circ \text{pr}_1)$ by f is a well-defined distribution on N .*

Moreover the map $u \mapsto f_(u \cdot (\varphi \circ \text{pr}_1))$ from $\mathcal{D}'(M, \Omega_M)$ to $\mathcal{D}'(N, \Omega_N)$ is continuous, and, if $\rho : M \rightarrow B$, $\rho' : N \rightarrow B$ are two submersions onto B such that $\rho' \circ f = \rho$, it restricts to a continuous map*

$$\mathcal{D}'_\pi(M, \Omega_M) \rightarrow \mathcal{D}'_\rho(N, \Omega_N).$$

Proof. The proof is straightforward from the hypotheses : observe that for any $g \in C_c^\infty(N)$, the smooth map $m \mapsto (g \circ f)(m)(\varphi \circ \text{pr}_1)(m)$ is compactly supported as it vanishes outside $f^{-1}(L) \cap (K \times M_2)$ which is compact by assumption if L is the support of g and K the support of φ . Hence $f_*(u \cdot (\varphi \circ \text{pr}_1))$ is well defined by

$$g \mapsto \langle u, (g \circ f) \cdot (\varphi \circ \text{pr}_1) \rangle$$

and this formula depends continuously on u for fixed f, φ . For the restriction map, let us denote by u_φ the distribution $u \cdot (\varphi \circ \text{pr}_1)$. Then observe that for any $u \in \mathcal{D}'_\rho(M, \Omega_M)$ and $g \in C^\infty(N)$, we have $f_*(u_\varphi) \cdot g = f_*(u_\varphi \cdot f^*g)$, hence

$$(36) \quad \rho'_*(f_*(u_\varphi) \cdot g) = \rho'_* \circ f_*(u_\varphi \cdot f^*g) = \rho_*(u_\varphi \cdot f^*g).$$

This proves that $f_*(u_\varphi) \in \mathcal{D}'_\rho(N, \Omega_N)$ and the continuity of the map $u \mapsto f_*(u_\varphi)$. □

3. CONVOLUTION OF TRANSVERSAL DISTRIBUTIONS ON GROUPOIDS

We apply these observations in the context of groupoids. We provide some of the details of the construction of the canonical convolution algebra $C_c^\infty(G, \Omega^{1/2})$ [3, 5] associated with any Lie groupoid G . The product of convolution

$$(37) \quad C_c^\infty(G, \Omega^{1/2}) \times C_c^\infty(G, \Omega^{1/2}) \xrightarrow{*} C_c^\infty(G, \Omega^{1/2})$$

is given by the integral

$$(38) \quad f * g(\gamma) = \int_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1)g(\gamma_2), \quad \gamma \in G$$

which is well defined and gives an internal operation as soon as we take

$$(39) \quad \Omega^{1/2} = \Omega^{1/2}(\ker dr) \otimes \Omega^{1/2}(\ker ds) = \Omega^{1/2}(\ker dr \oplus \ker ds).$$

To understand this point, we recall

Lemma 12. [3, 5]. *Denoting by m the multiplication map of G and by $\text{pr}_1, \text{pr}_2 : G \times G \rightarrow G$ the natural projection maps, we have a canonical isomorphism*

$$(40) \quad \text{pr}_1^*(\Omega^{1/2}) \otimes \text{pr}_2^*(\Omega^{1/2})|_{G^{(2)}} \simeq \Omega_m \otimes m^*(\Omega^{1/2}).$$

Proof. We note $G_\gamma^{(2)}$ the fiber of m at γ , that is $G_\gamma^{(2)} = \{(\gamma_1, \gamma_2) \in G^{(2)} ; \gamma_1\gamma_2 = \gamma\}$.

Now, the restricted map $\text{pr}_1 : G_\gamma^{(2)} \rightarrow G^{r(\gamma)}$ being a diffeomorphism, we have a canonical isomorphism of vector bundles

$$TG_\gamma^{(2)} \xrightarrow{\simeq} \text{pr}_1^*(TG^{r(\gamma)}) = \text{pr}_1^*(\ker dr)|_{G_\gamma^{(2)}}, \quad (\gamma_1, \gamma_2, X_1, X_2) \longmapsto (\gamma_1, \gamma_2, X_1).$$

Similarly, $TG_\gamma^{(2)} \simeq \text{pr}_2^*(TG_{s(\gamma)})$. Moreover the map

$$\text{pr}_1^*(\ker ds)|_{G_\gamma^{(2)}} \xrightarrow{\simeq} G_\gamma^{(2)} \times T_\gamma G_{s(\gamma)}, \quad (\gamma_1, \gamma_2, X_1) \longmapsto (\gamma_1, \gamma_2, (dR_{\gamma_2})_{\gamma_1}(X_1))$$

provides a canonical trivialisation of the vector bundle $\text{pr}_1^*(\ker ds)|_{G_\gamma^{(2)}}$. The same holds for

$$\text{pr}_2^*(\ker dr)|_{G_\gamma^{(2)}} \simeq G_\gamma^{(2)} \times T_\gamma G^{r(\gamma)}.$$

With these isomorphisms in hand, we get

$$\begin{aligned} \text{pr}_1^*(\Omega^{1/2})|_{G_\gamma^{(2)}} \otimes \text{pr}_2^*(\Omega^{1/2})|_{G_\gamma^{(2)}} &\simeq \Omega^{1/2}(\text{pr}_1^*(\ker dr \oplus \ker ds)|_{G_\gamma^{(2)}} \oplus \text{pr}_2^*(\ker dr \oplus \ker ds)|_{G_\gamma^{(2)}}) \\ &\simeq \Omega^{1/2}(TG_\gamma^{(2)} \oplus T_\gamma G_{s(\gamma)} \oplus T_\gamma G^{r(\gamma)} \oplus TG_\gamma^{(2)}) \\ &\simeq \Omega(TG_\gamma^{(2)}) \otimes \Omega^{1/2}(T_\gamma G_{s(\gamma)} \oplus T_\gamma G^{r(\gamma)}). \end{aligned}$$

This gives the canonical isomorphism (40). □

Since in the basic formula (38) the function under sign of integration

$$G_\gamma^{(2)} \ni (\gamma_1, \gamma_2) \mapsto f(\gamma_1)g(\gamma_2) \in \left(\text{pr}_1^*(\Omega^{1/2}) \otimes \text{pr}_2^*(\Omega^{1/2}) \right)_{(\gamma_1, \gamma_2)}$$

is thus a C^∞ section of the bundle $(\text{pr}_1^*(\Omega^{1/2}) \otimes \text{pr}_2^*(\Omega^{1/2}))|_{G_\gamma^{(2)}}$, the previous lemma shows that (38) is the integral of a one density, canonically associated with f, g , over the manifold $G_\gamma^{(2)} = \{(\gamma_1, \gamma_2), \gamma_1\gamma_2 = \gamma\}$ and that the result is a C^∞ section of $\Omega^{1/2}$. Further computations on densities show that the statement

$$(41) \quad f * g(\gamma) = \int_{G^{r(\gamma)}} f(\gamma_1)g(\gamma_1^{-1}\gamma) = \int_{G_{s(\gamma)}} f(\gamma\gamma_2^{-1})g(\gamma_2)$$

makes sense and is true. The involution on $C_c^\infty(G, \Omega^{1/2})$ is also natural in terms of densities

$$f^* = \overline{i^*(f)}, \quad f \in C_c^\infty(G, \Omega^{1/2})$$

where i is the induced vector bundle isomorphism over the inversion map of G

$$\ker dr \oplus \ker ds \longrightarrow \ker dr \oplus \ker ds, \quad (\gamma, X_1, X_2) \longmapsto (\gamma^{-1}, di(X_2), di(X_1)).$$

The spaces $C^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$ and $C_c^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$ are endowed with their usual Fréchet and \mathcal{LF} topological vector space structures and we denote by $\mathcal{E}'(G, \Omega^{1/2})$ and $\mathcal{D}'(G, \Omega^{1/2})$ their topological duals. The choice of densities is made so that we have canonical embeddings

$$C^\infty(G, \Omega^{1/2}) \hookrightarrow \mathcal{D}'(G, \Omega^{1/2}) \text{ and } C_c^\infty(G, \Omega^{1/2}) \hookrightarrow \mathcal{E}'(G, \Omega^{1/2}).$$

For simplicity, we assume in the sequel that $G^{(0)}$ is compact, thus $\mathcal{E}'(G, \Omega^{1/2}) = \mathcal{D}'_{c-\pi}(G, \Omega^{1/2})$ and $\mathcal{E}'_{\pi}(G, \Omega^{1/2}) = \mathcal{P}'_{\pi}(G, \Omega^{1/2})$ if $\pi \in \{r, s\}$.

Theorem 13. *The convolution product on $C_c^{\infty}(G, \Omega^{1/2})$ extends to a separately continuous bilinear map*

$$(42) \quad \mathcal{E}'_s(G, \Omega^{1/2}) \times \mathcal{D}'(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{D}'(G, \Omega^{1/2})$$

$$(u, v) \longmapsto u * v = m_* \left(u \times_s v \right).$$

The above map can also be defined on

$$(43) \quad \mathcal{D}'_s \times \mathcal{E}', \quad \mathcal{E}' \times \mathcal{D}'_r, \quad \mathcal{D}' \times \mathcal{E}'_r$$

using the fibered product over r . The convolution product gives by restriction a separately continuous bilinear map

$$(44) \quad \mathcal{E}'_{\pi}(G, \Omega^{1/2}) \times \mathcal{D}'_{\pi}(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{D}'_{\pi}(G, \Omega^{1/2})$$

with $\pi = r$ or $\pi = s$ and $(\mathcal{E}'_{\pi}(G, \Omega^{1/2}), *)$ is an associative algebra with unit given by

$$(45) \quad \langle \delta, f \rangle = \int_{G^{(0)}} f, \quad f \in C^{\infty}(G, \Omega^{-1/2} \otimes \Omega_G).$$

In particular $(\mathcal{E}'_{r,s}(G, \Omega^{1/2}), *)$ is an associative unital algebra with involution given by

$$(46) \quad u^* = \overline{i^*(u)}.$$

Proof. Applying Proposition 9 to the case $M_1 = M_2 = G$, $B = G^{(0)}$, $\pi_1 = s$, $\pi_2 = r$ and $\sigma = Id$, one gets a distribution $u \times_s v \in \mathcal{D}'_{\text{pr}_2}(G, \Omega^{1/2})$.

Then assuming, that u is compactly supported on G is the same as saying that $u = u \cdot \varphi$ where φ is compactly supported in G . Then one has that

$$(u \cdot \varphi) \times_s v = u \times_s v \cdot (\varphi \circ \text{pr}_1).$$

We can then apply Proposition 11 to m , noting that the multiplication map $m : G^{(2)} \rightarrow G$ fulfills the hypotheses of the proposition (which comes easily by using the inverse map). Then $u * v$ is well-defined by

$$u * v = m_* \left(u \times_s v \cdot (\varphi \circ \text{pr}_1) \right) = m_* \left((u \cdot \varphi) \times_s v \right) = m_* \left(u \times_s v \right).$$

The other compositions in (43) are obtained in the same way (moving around the compactly supported map φ).

For the proof of (44) for $\pi = s$ we apply again Proposition 9 to the case $M_1 = M_2 = G$, $B = G^{(0)}$, $\pi_1 = s$, $\pi_2 = r$ and $\sigma = s$ and the second part of Proposition 11 to $\rho' = s$ and $\rho = s \circ \text{pr}_2$ to get the result.

For associativity, first note that for $u, v \in C_c^{\infty}$, the distribution $u * v$ is given by

$$\langle u * v, f \rangle = \int_{G^{(2)}} u(\gamma_1) v(\gamma_2) f(\gamma_1 \gamma_2)$$

and the associativity of $*$ for $C_c^{\infty}(G, \Omega^{1/2})$ boils down to the application of Fubini's theorem in the integral

$$\langle u * v * w, f \rangle = \int_{G^{(3)}} u(\gamma_1) v(\gamma_2) w(\gamma_3) f(\gamma_1 \gamma_2 \gamma_3),$$

where $G^{(3)} = \{(\gamma_1, \gamma_2, \gamma_3) \in G^3 ; s(\gamma_1) = r(\gamma_2), s(\gamma_2) = r(\gamma_3)\}$. Using the density of $C_c^{\infty}(G, \Omega^{1/2})$ in $\mathcal{E}'_s(G, \Omega^{1/2})$, $\mathcal{E}'_r(G, \Omega^{1/2})$ and $\mathcal{E}'(G, \Omega^{1/2})$, and the continuity of the convolution products, one obtains the associativity of $*$ on $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$.

We check that the integral defining δ has an intrinsic meaning and gives a unit in $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$. Since $TG|_{G^{(0)}} = TG^{(0)} \oplus \ker ds$, we have $\Omega(G)|_{G^{(0)}} = \Omega(G^{(0)}) \otimes \Omega(\ker ds)$. On the other hand the inversion gives a canonical isomorphism between the bundles $\ker ds|_{G^{(0)}}$ and $\ker dr|_{G^{(0)}}$, thus

$$\Omega(\ker ds) = \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker ds) \simeq \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dr) = \Omega^{1/2}$$

Through these canonical identifications, any $f \in C^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$ gives by restriction at $G^{(0)}$ a one density on $G^{(0)}$, which gives a well defined meaning to $\delta(f)$. Now a direct computation shows that

$$r_*(\delta) = s_*(\delta) = 1 \in C^\infty(G^{(0)}) \subset \mathcal{D}'(G^{(0)}),$$

thus $\delta \in \mathcal{E}'_{r,s}(G, \Omega^{1/2})$.

Let us consider the C^∞ -family δ^x , $x \in G^{(0)}$ coming from the r -transversality of δ . For any test function $\phi \in C_c^\infty(G^x)$ and C^∞ function $\tilde{\phi} \in C_c^\infty(G)$ such that $\tilde{\phi}|_{G^x} = \phi$ we have

$$\langle \delta^x, \phi \rangle = r_*(\delta\tilde{\phi})(x)$$

and obviously $r_*(\delta\tilde{\phi})(x) = \tilde{\phi}(x) = \phi(x)$. It follows that for any $f \in C^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$,

$$\begin{aligned} \langle u * \delta, f \rangle &= \int_{x \in G^{(0)}} \langle u_x \otimes \delta^x, (f \circ m)|_{G_x \times G^x} \rangle \\ &= \int_{x \in G^{(0)}} \langle u_x, f|_{G_x} \rangle \\ &= \langle u, f \rangle \end{aligned}$$

Thus $u * \delta = u$. The proof of the equality $\delta * u = u$ is similar. The assertion about the involution is obvious. \square

In particular, when one of the two factors is in C_c^∞ , the convolution product is defined without any restriction on the other factor. We give a sufficient condition for the result to be C^∞ .

Proposition 14. *The convolution product gives by restriction a bilinear separately continuous map*

$$\mathcal{D}'_r(G, \Omega^{1/2}) \times C_c^\infty(G, \Omega^{1/2}) \xrightarrow{*} C^\infty(G, \Omega^{1/2}).$$

*The analogous statement with C^∞ functions on the left and s -transversal distributions on the right also holds. The map $u \mapsto u * \cdot$ mapping $\mathcal{D}'_r(G, \Omega^{1/2})$ to $\mathcal{L}(C_c^\infty(G, \Omega^{1/2}), C^\infty(G, \Omega^{1/2}))$ is injective.*

Proof. If $u = (u^y)_y \in \mathcal{D}'_r$, the map

$$(47) \quad \gamma_2 \mapsto \langle u^{r(\gamma_2)}(\cdot), f((\cdot)^{-1}\gamma_2) \rangle$$

is C^∞ and we check from the definition of the convolution product that

$$\langle u * f, \phi \rangle = \int_{\gamma_2 \in G} \langle u^{r(\gamma_2)}(\cdot), f((\cdot)^{-1}\gamma_2) \rangle \phi(\gamma_2).$$

Thus $u * f$ coincides with the C^∞ function

$$(48) \quad u * f(\gamma) = \langle u^{r(\gamma)}(\cdot), f((\cdot)^{-1}\gamma) \rangle.$$

Now, the vanishing of $u * f$ for any f and the previous expression for $u * f$ shows that $u^x = 0$, for any x , and thus $u = 0$. \square

Remark 15. *Note that if in the previous proposition we suppose that u has compact support $K \subset G$, then $u * f$ can be defined for any map $f \in C^\infty(G, \Omega^{1/2})$. Moreover for any $f \in C_c^\infty(G, \Omega^{1/2})$, then $u * f$ is also compactly supported and $\text{supp}(u * f) \subset K.\text{supp}(f)$.*

4. G-OPERATORS

We recall the definition of G -operators given in [12].

Definition 4. A (left) G -operator is a continuous linear map $P : C_c^\infty(G, \Omega^{1/2}) \rightarrow C^\infty(G, \Omega^{1/2})$ such that there exists a family $P_x : C_c^\infty(G_x, \Omega_{G_x}^{1/2}) \rightarrow C^\infty(G_x, \Omega_{G_x}^{1/2})$, $x \in G^{(0)}$ of operators such that

$$(49) \quad P(f)|_{G_x} = P_x(f|_{G_x})$$

and

$$(50) \quad P_{r(\gamma)} \circ R_\gamma = R_\gamma \circ P_{s(\gamma)}$$

for any $x \in G^{(0)}$ and $\gamma \in G$. A G -operator is supported in K when for all $f \in C_c^\infty(G, \Omega^{1/2})$, $\text{supp}(P(f)) \subset K.\text{supp}(f)$. We denote by Op_G the linear space of G -operators and by $\text{Op}_{G,c}$ the subspace of compactly supported G -operators.

We now give a bijection between Op_G and $\mathcal{D}'_s(G, \Omega^{1/2})$. We know that i^* is an isomorphism between $\mathcal{D}'_s(G, \Omega^{1/2})$ and $\mathcal{D}'_r(G, \Omega^{1/2})$. If $v \in \mathcal{D}'_s(G, \Omega^{1/2})$, we are going to show that the convolution on the left by $u = i^*(v) \in \mathcal{D}'_r(G, \Omega^{1/2})$ defines a G -operator. First, observe that Proposition 14 shows that $P(u) : f \mapsto u * f$ is a linear continuous map from $C_c^\infty(G, \Omega^{1/2})$ to $C^\infty(G, \Omega^{1/2})$. Moreover, the formula (48) shows that the restriction of $u * f$ to G_x depends only on G_x , hence, for any x we get an operator $P(u)_x : C_c^\infty(G_x) \rightarrow C^\infty(G_x)$. For the equivariance, let $\gamma \in G_x^y$, then

$$[P(u)_y(f(\cdot\gamma))](\gamma') = u * (R_\gamma f)(\gamma') = [R_\gamma(u * f)](\gamma') = [P(u)_x f](\gamma'\gamma).$$

This shows that the left convolution by u defines a G -operator $P(u)$. Moreover if u is compactly supported, from remark 15, we know that $P(u)$ is compactly supported.

Let us note that the Schwartz kernel $k_{u,x}$ of $P(u)_x$ on $G_x \times G_x$ can be recovered as follows. For any $x \in G^{(0)}$, Let us denote by $O(x)$ the orbit of x in $G^{(0)}$, that is, the submanifold of $G^{(0)}$ defined by $O(x) = r(s^{-1}(x))$. We note $G(x) = r^{-1}(O(x))$ and j_x the map

$$j_x : G_x \times G_x \rightarrow G(x), \quad (\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2^{-1}.$$

This map is a surjective submersion, hence Proposition 5 gives a distribution $\tilde{u} = \text{Rest}_{O(x)}(u)$ in $\mathcal{D}'_r(G(x), \Omega_{G(x)}^{1/2})$. Next, use Proposition 6 to pull-back the distribution \tilde{u} to $j_x^*(\tilde{u})$ as a distribution on $G_x \times G_x$. This gives

$$(51) \quad j_x^*(\tilde{u}) = k_{u,x} \in \mathcal{D}'(G_x \times G_x).$$

Note that the pull-back of the distribution u can be done globally using the submersion

$$j : G \times_s G \rightarrow G, \quad (\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2^{-1},$$

where we have denoted $G \times_s G = \{(\gamma_1, \gamma_2) \in G^2 ; s(\gamma_1) = s(\gamma_2)\}$. On the other hand, if $K_u \in \mathcal{D}'(G \times G)$ denotes the Schwartz kernel of $P(u)$, we obtain using (49) a well defined distribution $k_u \in \mathcal{D}'(G \times G)$ by the formula

$$(52) \quad \langle k_u, f \rangle = \langle K_u, \tilde{f} \rangle, \quad \forall f \in C_c^\infty(G \times_s G), \tilde{f} \in C_c^\infty(G \times G), \tilde{f}|_{G \times_s G} = f,$$

and it follows that

$$(53) \quad j^*(u) = k_u \in \mathcal{D}'(G \times_s G).$$

Conversely, let P be a left G -operator. Its Schwartz kernel K belongs to $\mathcal{D}'_{\text{pr}_1}(G^2)$ and as above, there exists $k \in \mathcal{D}'_{s\text{opr}_1}(G \times_s G)$ such that

$$\langle K, f \rangle = \langle k, f|_{G \times_s G} \rangle, \quad f \in C_c^\infty(G^2).$$

The distribution k is given by the C^∞ family $x \mapsto k_x \in \mathcal{D}'(G_x \times G_x)$ where k_x is the Schwartz kernel of P_x . Still by assumption, each k_x belongs to $\mathcal{D}'_{\text{pr}_1}(G_x \times G_x)$ and we have, for any $f \in C_c^\infty(G, \Omega^{1/2})$, that the map

$$\gamma_1 \longmapsto \langle K(\gamma_1, \cdot), f(\cdot) \rangle = \langle k_{s(\gamma_1)}(\gamma_1, \cdot), f|_{G_{s(\gamma_1)}}(\cdot) \rangle$$

is C^∞ . This proves that the map

$$G^{(0)} \ni x \mapsto k_P(x)(\cdot) = k_x(x, \cdot) \in \mathcal{D}'(G_x)$$

is C^∞ and we obtain a s -transversal distribution $k_P \in \mathcal{D}'_s(G)$.

Moreover, for any $f \in C_c^\infty(G, \Omega^{1/2})$, $x, y \in G^{(0)}$ and $\gamma \in G_x^y$, we have

$$\begin{aligned} (54) \quad P(f)(\gamma) &= P_x(f)(\gamma) = P_x(f)(y \cdot \gamma) \\ &= P_y(R_\gamma f)(y) = \langle k_P(y), f(\cdot \gamma) \rangle_{G_y} \\ &= \langle i^* k_P(y), f((\cdot)^{-1} \gamma) \rangle_{G^y} \\ (55) \quad &= (i^* k_P) * f(\gamma). \end{aligned}$$

This proves that the action of P on $C_c^\infty(G, \Omega^{1/2})$ is given by convolution on the left by the r -transversal distribution $i^* k_P$. The distribution k_P is called *the convolution distributional kernel* of P . Note that if P has compact support K , then the distribution k_P has also support in K . We have proved

Theorem 16. *The map $P \mapsto k_P$ gives a bijection between the space Op_G and $\mathcal{D}'_s(G, \Omega^{1/2})$, and a bijection between $\text{Op}_{G,c}$ and $\mathcal{E}'_s(G, \Omega^{1/2})$.*

We also consider the notion of adjointable left G -operators. For that purpose, consider the $C_c^\infty(G, \Omega^{1/2})$ -valued pre-hilbertian product defined by

$$(56) \quad (f|g) = f^* * g; \quad f, g \in C_c^\infty(G, \Omega^{1/2}),$$

where f^* is defined by $f^*(\gamma) = \overline{f(\gamma^{-1})}$, that is $f^* = \overline{i^* f}$. Then a left G -operator P is said *adjointable* if there exists a left G -operator Q such that

$$(57) \quad (P(f)|g) = (f|Q(g)); \quad f, g \in C_c^\infty(G, \Omega^{1/2}).$$

If an adjoint of P exists, then it is unique (since $(u * f)^* * g = f^* * u^* * g = 0$ for any f, g implies $u = 0$) and it is then denoted by P^* .

The space of adjointable left G -operators will be noted Op_G^* .

If $k_P \in \mathcal{D}'_{r,s}(G, \Omega^{1/2})$ then P is obviously adjointable and $k_{P^*} = (k_P)^*$. Conversely, if P as an adjoint Q then

$$(58) \quad (i^* k_P * f)^* * g = (f^* * \overline{k_P}) * g = f^* * (i^* k_Q * g); \quad f, g \in C_c^\infty(G, \Omega^{1/2}),$$

hence $\overline{k_P} = i^* k_Q \in \mathcal{D}'_s(G, \Omega^{1/2}) \cap \mathcal{D}'_r(G, \Omega^{1/2})$. This proves that adjointable G -operators are those with convolution distributional kernel belonging to $\mathcal{D}'_{r,s}(G, \Omega^{1/2})$.

There is another way to characterize adjointable G -operators. Note that for G -operators with $k_P \in \mathcal{D}'_{r,s}(G, \Omega^{1/2})$, it follows from Theorem 13 that P can be extended into a continuous linear operator $\mathcal{E}'(G, \Omega^{1/2}) \longrightarrow \mathcal{D}'(G, \Omega^{1/2})$ which maps the subspace $\mathcal{E}'_{r,s}$ to $\mathcal{D}'_{r,s}$, by using the left convolution with $i^* k_P$. We are going to check that the converse is true and we conclude

Proposition 17. *A left G -operator P is adjointable if and only if it has a continuous extension to a linear map*

$$\tilde{P} : \mathcal{E}'(G, \Omega^{1/2}) \longrightarrow \mathcal{D}'(G, \Omega^{1/2})$$

such that $\tilde{P}(\mathcal{E}'_{r,s}(G, \Omega^{1/2})) \subset \mathcal{D}'_{r,s}(G, \Omega^{1/2})$.

In that case, \tilde{P} is given by left convolution with $i^*k_P = \tilde{P}(\delta) \in \mathcal{D}'_{r,s}(G, \Omega^{1/2})$.

Proof. For any $u \in \mathcal{E}'(G, \Omega^{1/2})$ and $f \in C_c^\infty(G, \Omega^{1/2})$ we have, using a sequence $(u_n) \subset C_c^\infty(G, \Omega^{1/2})$ converging to u in \mathcal{E}'

$$\tilde{P}(u * f) = \lim P(u_n * f) = \lim P(u_n) * f = \tilde{P}(u) * f.$$

Thus \tilde{P} is automatically $C_c^\infty(G, \Omega^{1/2})$ -right linear. For any $u \in \mathcal{E}'_{r,s}(G, \Omega^{1/2})$, $v \in \mathcal{E}'(G, \Omega^{1/2})$ and using a sequence $(v_n) \subset C_c^\infty(G, \Omega^{1/2})$ converging to v in \mathcal{E}' , we then have

$$\tilde{P}(u * v) = \lim \tilde{P}(u * v_n) = \lim \tilde{P}(u) * v_n = \tilde{P}(u) * v.$$

Now we can write the equalities

$$i^*k_P * f = P(f) = P(\delta * f) = \tilde{P}(\delta) * f$$

for any $f \in C_c^\infty(G, \Omega^{1/2})$, which proves that $i^*k_P = \tilde{P}(\delta) \in \mathcal{D}'_{r,s}(G, \Omega^{1/2})$ and the remaining assertion is obvious. \square

Remark 18. *Rephrasing the previous results, we have, for instance*

$$\text{Op}_G \simeq \mathcal{L}_s(C_c^\infty(G, \Omega^{1/2}), C^\infty(G^{(0)})).$$

where we have replaced $\mathcal{L}_{C^\infty(G^{(0)})}$ by \mathcal{L}_s to emphasize that the $C^\infty(G^{(0)})$ -module structure on $C_c^\infty(G, \Omega^{1/2})$ is given by s . Also

$$\text{Op}_G^* \simeq \mathcal{L}_{r,s}(C_c^\infty(G, \Omega^{1/2}), C^\infty(G^{(0)})).$$

where $\mathcal{L}_{r,s} = \mathcal{L}_s \cap \mathcal{L}_r$. Thinking again in terms of Schwartz kernel theorems for submersions, we see that G -operators thus appear as “semi-regular” distributions in the terminology of Treves [15, p.532] since, for $\pi = s$ or $\pi = r$

$$\mathcal{D}'(G, \Omega^{1/2}) \simeq \mathcal{L}_\pi(C_c^\infty(G, \Omega^{1/2}), \mathcal{D}'(G^{(0)})).$$

5. WAVE FRONT SETS AND TRANSVERSALITY

We now turn to some microlocal aspects of distributions on groupoids. We are going to compare transversality conditions for distributions with transversality conditions for their wave front sets. We begin with the case of submersions.

Given a submersion $\pi : M \longrightarrow B$, a sufficient condition of π -transversality is provided by the behavior of the wave front set. We denote

$$(59) \quad \mathcal{M}(\pi) = \ker d\pi^\perp = \{(x, \xi) \in T^*M, ; \langle \xi, t \rangle = 0, \forall t \in \ker d\pi_x\}.$$

Observe that $\mathcal{M}(\pi)$ is the union of the conormal bundles of the fibers of π . If $F \xrightarrow{\iota} M$ is an immersion, we denote by $N^*(\iota)$ or $N^*(F)$ the conormal bundle of F in M , which is defined by $N^*(F) = \{(m, \xi) \in T^*M \mid m \in F; \langle \xi, t \rangle = 0, \forall t \in T_m F\}$. Then

$$(60) \quad \mathcal{M}(\pi) = \bigcup_{b \in B} N^*(\pi^{-1}(b)).$$

We recall that, for any closed cone $W \subset T^*M \setminus 0$, $\mathcal{D}'_W(M)$ is the space of distributions u on M such that $\text{WF}(u) \subset W$.

Proposition 19. *Let $\pi : M \rightarrow B$ be a C^∞ submersion onto B and $W \subset T^*M \setminus 0$ a closed cone such that $W \cap \mathcal{M}(\pi) = \emptyset$. Then*

$$\mathcal{D}'_W(M) \subset \mathcal{D}'_\pi(M).$$

Proof. We apply the formula (3.6) p. 328 of [6] to $u_f = \pi_*(u.f)$:

$$\text{WF}(\pi_*(u.f)) \subset (d\pi)_*(\text{WF}(u.f)) \subset (d\pi)_*(\text{WF}(u)) = \{(x, \xi) ; x = \pi(m), (m, {}^t d\pi_m(\xi)) \in \text{WF}(u)\}$$

Since $\mathcal{M}(\pi) = \{(m, \zeta) ; \zeta \in \text{Im}({}^t d\pi_m)\}$, we obtain $\text{WF}(\pi_*(u.f)) = \emptyset$, that is, u_f is C^∞ . □

Remark 20. The reverse inclusion does not hold. It is possible to find a distribution in \mathcal{D}'_π the wave front set of which does not avoid $\mathcal{M}(\pi)$. To see this, we suppose $M = \mathbb{R}^p \times \mathbb{R}^q$ and $B = \mathbb{R}^p$, with π equal to the first projection. Let us consider the distribution u on M , with Fourier transform equal to:

$$\widehat{u}(\xi, \eta) = v(\xi, \eta) = \chi(\eta)a(\xi, \eta).$$

Here a is positively homogeneous of degree zero and smooth away from the origin, and χ is a smooth function on \mathbb{R}^q such that $\chi(\eta) = 0$ for $\|\eta\| \leq 1$ and $\chi(\eta) = 1$ for $\|\eta\| \geq 2$. Hence v is well-defined as a smooth function on M . The function v defines a tempered distribution, hence $u = \mathcal{F}^{-1}(v)$ is also tempered. We see that the wave front set $WF(u)$ contains the nonempty subset $\{(0, 0; \xi, 0), \xi \in \mathbb{R}^p\}$ of $\mathcal{M}(\pi) = \{(x, y; \xi, \eta) \in T^*M, \eta = 0\}$. For example, one can choose $a(\xi, \eta) = \exp(-\frac{\|\xi\|^2}{\|\eta\|^2})$, which yields the estimates:

$$(61) \quad |\widehat{u}(\xi, \eta)| \leq C_N \sqrt{1 + \|\xi\|^2 / \|\eta\|^2}^{-N} = C_N \|\eta\|^N \|\xi, \eta\|^{-N}, \quad N = 1, 2, 3, \dots$$

We shall suppose from now on that the estimates (61) are verified, and use the shorthand notation $\Lambda(-) := \sqrt{1 + \|\cdot\|^2}$. To prove $u \in \mathcal{D}'_\pi$ we have to show, in view of Definition 2, that the push-forward $\pi_*(fu)$ is a C^∞ function on B for any $f \in C_c^\infty(M)$. This will result from the following computation with a test function $\varphi \in C_c^\infty(B)$:

$$\begin{aligned} \langle \pi_*(fu), \varphi \rangle &= \langle fu, \varphi \circ \pi \rangle = \langle \widehat{fu}, \widehat{\varphi \circ \pi} \rangle \\ &= \iint (\widehat{f} * \widehat{u})(\xi, \eta) \widehat{\varphi \circ \pi}(\xi, \eta) d\xi d\eta \\ &= \iiint \widehat{u}(\xi', \eta') \widehat{f}(\xi - \xi', \eta - \eta') \widehat{\varphi}(\xi) \delta(\eta) d\xi d\xi' d\eta d\eta' \\ &= \iiint \widehat{u}(\xi', \eta') \widehat{f}(\xi - \xi', -\eta') \widehat{\varphi}(\xi) d\xi d\xi' d\eta' \\ &= \int \left(\int \left(\iint \widehat{u}(\xi', \eta') \widehat{f}(\xi - \xi', -\eta') d\eta' d\xi' \right) e^{-i\langle \xi, x \rangle} d\xi \right) \varphi(x) dx. \end{aligned}$$

We split the innermost double integral $I(\xi)$ into two terms, the first one on the domain A defined by $\Lambda(\xi - \xi') \leq \frac{1}{4}\Lambda(\xi)$, the second one on the complementary A' . An elementary straightforward computation shows the following double inequality for any $\xi' \in A$:

$$(62) \quad \frac{\sqrt{2}}{2}\Lambda(\xi) \leq \Lambda(\xi') \leq \frac{5}{4}\Lambda(\xi).$$

The Fourier transform \widehat{f} is rapidly decreasing, which yields the estimates:

$$(63) \quad \|\widehat{f}(\xi - \xi', -\eta')\| \leq D_N \Lambda(\xi' - \xi, \eta')^{-N}, \quad N = 1, 2, 3, \dots$$

We have then, using (61), (62), (63) and the fact that $\|\eta'\| \geq 1$ for any (ξ', η') in the support of \widehat{u} :

$$\begin{aligned} |I(\xi)| &\leq \iint_A C_{2N} D_M \|\eta'\|^{2N} \|(\xi', \eta')\|^{-2N} \Lambda(\xi' - \xi, \eta')^{-M} d\xi' d\eta' + \iint_{A'} C_N D_{2M} \|\eta'\|^N \|(\xi', \eta')\|^{-N} \Lambda(\xi' - \xi, \eta')^{-2M} d\xi' d\eta' \\ &\leq 2^{\frac{N}{2}} C_{2N} D_M \iint \Lambda(\eta')^{2N-M} \Lambda(\xi')^{-N} \Lambda(\xi)^{-N} d\xi' d\eta' + 5^M C_N D_{2M} \iint \Lambda(\eta')^{N-M} \Lambda(\xi')^{-N} \Lambda(\xi)^{-M} d\xi' d\eta'. \end{aligned}$$

As this holds for arbitrary M and N , the innermost double integral $I(\xi)$ converges and defines a rapidly decreasing function of ξ . Hence the inner triple integral defines a smooth function of x , which proves the claim.

It is not obvious to us how to characterize transversal distributions whose wave front set avoids $\mathcal{M}(\pi)$. We give in the following lemma a sufficient condition.

Lemma 21. *Let $v \in \mathcal{D}'_{\pi_Y}(X \times Y)$ and assume that there exists constants $d \in \mathbb{N}$ and $\delta \in [0, 1)$ such that for any compact subset K of X and multi-index $\beta \in \mathbb{N}^{n_Y}$, one can find a constant $C_{K\beta}$ such that*

$$(64) \quad |\langle \partial^\beta v_y, f \rangle| \leq C_{K\beta} \|f\|_{K, d+\delta|\beta|}, \quad \forall f \in C_c^\infty(X), \quad y \in Y.$$

Here, we have set $\|f\|_{K, d+\delta|\beta|} = \sum_{|\alpha| \leq d+\delta|\beta|} \sup_K |\partial^\alpha f|$. Then

$$(65) \quad \text{WF}(v) \subset \mathcal{M}(\pi_X).$$

In particular, $\text{WF}(v) \cap \mathcal{M}(\pi_Y) = \emptyset$.

Proof. Let us fix $(x_0, y_0, \xi_0, \eta_0) \notin \mathcal{M}(\pi_X)$, that is, $\eta_0 \neq 0$ and assume that $|(\xi_0, \eta_0)| = 1$. We work below in a conic neighborhood Γ of $(x_0, y_0, \xi_0, \eta_0)$ such that for all $(x, y, \xi, \eta) \in \Gamma$ with $|(\xi, \eta)| = 1$, we have $|\eta_j| \geq |\eta_{0j}|/2$ for some fixed j such that $\eta_{0j} \neq 0$.

Let $(x, y, \xi, \eta) \in \Gamma$ be such that $|(\xi, \eta)| = 1$ and $\varphi(x, y)$ be supported in a compact neighborhood $K \times L$ of (x_0, y_0) in $X \times Y$. Denoting $\varphi_y = \varphi(\cdot, y)$, we have for any $N > 0$

$$\begin{aligned} |\langle v, \varphi e^{-it\langle(\cdot, \cdot), (\xi, \eta)\rangle} \rangle| &= \left| \int \langle v_y, \varphi_y e^{-it\langle(\cdot, y), (\xi, \eta)\rangle} \rangle dy \right| \\ &= \left| \int \widehat{\varphi_y v_y}(t\xi) e^{-it\langle y, \eta \rangle} dy \right| \\ (66) \quad &\leq C \cdot \left(\sum_{|\alpha| \leq N} \sup_{y \in L} |\partial_y^\alpha \widehat{\varphi_y v_y}(t\xi)| |\eta|^{|\alpha| - 2N} \right) t^{-N} \text{ by [8, Theorem 7.7.1].} \end{aligned}$$

Moreover, since $v : y \mapsto v_y$ is C^∞ , we have

$$\partial_{y_j}^N \widehat{\varphi_y v_y}(t\xi) = \partial_{y_j}^N \langle \varphi_y v_y, e^{-it\langle \cdot, \xi \rangle} \rangle = \langle \partial_{y_j}^N \varphi_y v_y, e^{-it\langle \cdot, \xi \rangle} \rangle = \partial_{y_j}^N \widehat{\varphi_y v_y}(t\xi).$$

We note $K_\epsilon = \{x + y; x \in K, |y| < \epsilon\}$ for any $\epsilon > 0$ and let $\chi_\epsilon \in C_c^\infty(K_\epsilon)$ be such that $\chi_\epsilon = 1$ on $K_{\epsilon/2}$. If $H(\xi)$ denotes the supporting function of K [8, 4.3.1], we get using the assumption (64) and the proof of the Paley-Wiener-Schwartz Theorem in [8, 7.3.1]

$$\begin{aligned} |\partial_{y_j}^N \widehat{\varphi_y v_y}(\xi)| &= |\partial_{y_j}^N \varphi_y v_y(\chi_\epsilon e^{-i\langle \cdot, \xi \rangle})| \\ &\leq C_{K_\epsilon N} \sum_{|\beta| \leq d+\delta N} \sup |\partial^\beta (\chi_\epsilon e^{-i\langle \cdot, \xi \rangle})| \\ &\leq C_{K_\epsilon N} \cdot C \cdot e^{H(0)} \cdot \sum_{|\beta| \leq d+\delta N} \epsilon^{-\beta} (1 + |\xi|)^{d+\delta N - |\beta|}. \end{aligned}$$

With $\epsilon = 1/(1 + |\xi|)$ and using the inequalities $C_{K_\epsilon N} \leq C_{K_{\epsilon'} N}$ if $\epsilon < \epsilon'$, we obtain

$$|\widehat{\partial_{y_j}^N \varphi_y v_y}(\xi)| \leq C_{K_1 N} \cdot C \cdot (1 + |\xi|)^{d+\delta N} \leq C'_{K_1 N} (1 + |\xi|)^{d+\delta N}.$$

Using uniform estimates $|\eta| \geq c_1 > 0$ and $(1 + |t\xi|) \leq c_2 t$ for $(\xi, \eta) \in \Gamma$, $|(\xi, \eta)| = 1$ and the estimate (67) applied to (66), we get

$$|\langle v, \varphi e^{-it\langle(\cdot, \cdot), (\xi, \eta)\rangle} \rangle| \leq C \cdot t^{d+(\delta-1)N}.$$

since $\delta - 1 < 0$, we conclude that $(x_0, y_0, \xi_0, \eta_0) \notin \text{WF}(v)$. \square

In view of Proposition 19, it is natural to set

Definition 5. Let G be a C^∞ groupoid. A subset $W \subset T^*G \setminus 0$ is r -transversal if

$$(67) \quad W \cap N^*(G^x) = \emptyset \text{ for all } x \in G^{(0)},$$

A subset $W \subset T^*G \setminus 0$ is s -transversal if

$$(68) \quad W \cap N^*(G_x) = \emptyset \text{ for all } x \in G^{(0)}.$$

We call bi-transversal a set which is r -transversal and s -transversal. We denote by $\mathcal{D}'_a(G, \Omega^{1/2})$ the subset of $\mathcal{D}'(G, \Omega^{1/2})$ of distributions whose wavefront set $\text{WF}(u)$ is bi-transversal, and by $\mathcal{E}'_a(G, \Omega^{1/2})$ the subset of compactly supported distributions in $\mathcal{D}'_a(G, \Omega^{1/2})$

Example 1. We obviously get that $A^*G \setminus 0$ is a bi-transversal subset of $T^*G \setminus 0$. Remember that the space $\Psi(G)$ of G -pseudodifferential operators can be defined as the space of distributions on G which are conormal to $G^{(0)}$ (see [11]). Since such distributions have wave front sets included in $A^*G \setminus 0 = N^*(G^{(0)}) \setminus 0$, we get

$$(69) \quad \Psi(G) \subset \mathcal{D}'_a(G, \Omega^{1/2}) \text{ and } \Psi_c(G) \subset \mathcal{E}'_a(G, \Omega^{1/2}).$$

In our previous notation (59), condition (67) reads $W \cap \mathcal{M}(r) = \emptyset$ and condition (68) reads $W \cap \mathcal{M}(s) = \emptyset$. Observe that Proposition 19 gives :

- If W is r -transversal, $\mathcal{D}'_W(G, \Omega^{1/2}) \subset \mathcal{D}'_r(G, \Omega^{1/2})$.
- If W is s -transversal, $\mathcal{D}'_W(G, \Omega^{1/2}) \subset \mathcal{D}'_s(G, \Omega^{1/2})$.
- If W is bi-transversal, $\mathcal{D}'_W(G, \Omega^{1/2}) \subset \mathcal{D}'_{r,s}(G, \Omega^{1/2})$.

In particular, one has $\mathcal{E}'_a(G, \Omega^{1/2}) \subset \mathcal{E}'_{r,s}(G, \Omega^{1/2})$. Using Theorem 13 and Proposition 14, we obtain

Corollary 22. Let W_1 and W_2 be closed conic subsets $T^*G \setminus 0$ which are respectively s -transversal and r -transversal. The convolution product restricts to bilinear maps

$$\begin{aligned} \mathcal{E}'_{W_1}(G, \Omega^{1/2}) \times \mathcal{D}'(G, \Omega^{1/2}) &\xrightarrow{*} \mathcal{D}'(G, \Omega^{1/2}), \\ \mathcal{D}'_{W_2}(G, \Omega^{1/2}) \times C_c^\infty(G, \Omega^{1/2}) &\xrightarrow{*} C^\infty(G, \Omega^{1/2}), \\ \mathcal{D}'(G, \Omega^{1/2}) \times \mathcal{E}'_{W_2}(G, \Omega^{1/2}) &\xrightarrow{*} \mathcal{D}'(G, \Omega^{1/2}), \\ C_c^\infty(G, \Omega^{1/2}) \times \mathcal{D}'_{W_1}(G, \Omega^{1/2}) &\xrightarrow{*} C^\infty(G, \Omega^{1/2}). \\ \mathcal{E}'_a(G, \Omega^{1/2}) \times \mathcal{E}'_a(G, \Omega^{1/2}) &\xrightarrow{*} \mathcal{E}'_{r,s}(G, \Omega^{1/2}). \end{aligned}$$

The next natural question is to understand the wave front set of the convolution product of two distributions with bi-transversal wave front sets. To do so, we will need a better understanding of the structure of T^*G .

6. THE COTANGENT GROUPOID OF COSTES-DAZORD-WEINSTEIN

We recall the definition of the cotangent groupoid of Costes-Dazord-Weinstein. We have tried to make more concise the construction of the source and target map given in [4] and to enlight the role played by the transpose of the differential of the multiplication map of G . More general and conceptual approaches can also be found in [14, 10].

Let G be a Lie groupoid whose multiplication is denoted by m , source and target by s, r and inversion by i . It is straightforward that, differentiating all the structure maps of G , we get that $TG \rightrightarrows TG^{(0)}$ is a Lie groupoid whose multiplication is given by dm , source and target by ds, dr and inversion by di . Hence, it is natural to try to transpose everything to get a groupoid structure on T^*G . Following this idea, it is natural to define $(\gamma, \xi) = (\gamma_1, \xi_1) \cdot (\gamma_2, \xi_2) \in T^*G$ by $\gamma = \gamma_1 \gamma_2$ and ξ by the equation

$$(70) \quad {}^t dm_{(\gamma_1, \gamma_2)}(\xi) = (\xi_1, \xi_2).$$

The point is that $m : G^{(2)} \rightarrow G$ is a submersion, so that, at any point $(\gamma_1, \gamma_2) \in G^{(2)}$, the linear map ${}^t dm_{(\gamma_1, \gamma_2)}$ is injective. Thus, the solution ξ of (70) exists, and then is unique, if and only if

$$(71) \quad (\xi_1, \xi_2) \in \text{Im } {}^t dm_{(\gamma_1, \gamma_2)}.$$

In that case, we note

$$(72) \quad \xi = \xi_1 \oplus \xi_2 = \mu(\xi_1, \xi_2).$$

Since $\text{Im } {}^t dm_{(\gamma_1, \gamma_2)} = (\ker dm_{(\gamma_1, \gamma_2)})^\perp$, the condition (71) is equivalent to

$$(73) \quad \forall (X_1, X_2) \in T_{(\gamma_1, \gamma_2)} G^{(2)}, \quad dm_{(\gamma_1, \gamma_2)}(X_1, X_2) = 0 \Rightarrow \xi_1(X_1) + \xi_2(X_2) = 0.$$

We are going to reformulate condition (73) in term of composability of (ξ_1, ξ_2) in the expected groupoid structure on T^*G and thus we will clarify what are the source map, target map and unit space of T^*G . For that purpose, we collect useful information on the kernel of dm . We will use the following classical notations for left and right multiplication : for any $\gamma \in G$, we denote :

$$L_\gamma : G^{s(\gamma)} \rightarrow G^{r(\gamma)}, \gamma' \mapsto \gamma \gamma' \text{ and } R_\gamma : G_{r(\gamma)} \rightarrow G_{s(\gamma)}, \gamma' \mapsto \gamma' \gamma$$

Lemma 23. *Let $(X_1, X_2) \in T_{(\gamma_1, \gamma_2)} G^{(2)}$ and $y = s(\gamma_1) = r(\gamma_2)$, $Y = ds(X_1) = dr(X_2)$. Then the equivalences below hold.*

$$\begin{aligned} & (X_1, X_2) \in \ker dm_{(\gamma_1, \gamma_2)} \\ \Leftrightarrow & X_1 \in \ker dr, X_2 \in \ker ds \text{ and } X_1 = (dL_{\gamma_1})(Y_1), X_2 = (dR_{\gamma_2})(Y_2) \text{ with } (Y_1, Y_2) \in \ker dm_{(y, y)} \\ \Leftrightarrow & X_1 \in \ker dr, X_2 \in \ker ds \text{ and } X_1 = (dL_{\gamma_1})(Y_1), X_2 = (dR_{\gamma_2})(Y_2) \text{ with } Y_1 + Y_2 = Y \\ \Leftrightarrow & X_1 \in \ker dr, X_2 \in \ker ds \text{ and } X_1 = (dL_{\gamma_1})(Y_1), X_2 = (dR_{\gamma_2})(Y_2) \text{ with } di(Y_1) = Y_2, \\ \Leftrightarrow & X_1 \in \ker dr \text{ and } X_2 = dR_{\gamma_2} \circ di \circ dL_{\gamma_1}^{-1}(X_1), \\ \Leftrightarrow & X_2 \in \ker ds \text{ and } X_1 = dL_{\gamma_1} \circ di \circ dR_{\gamma_2}^{-1}(X_2). \end{aligned}$$

are true.

Proof. Let $(X_1, X_2) \in T_{(\gamma_1, \gamma_2)} G^{(2)}$ such that $dm_{(\gamma_1, \gamma_2)}(X_1, X_2) = 0$. We write $y = s(\gamma_1) = r(\gamma_2)$ and $Y = ds_{\gamma_1}(X_1) = dr_{\gamma_2}(X_2) \in T_y G^{(0)} \subset T_y G$.

Since $r \circ m = r \circ \text{pr}_1$ and $s \circ m = s \circ \text{pr}_2$ if $p_i : G^2 \rightarrow G$ denote the canonical projections, we have

$$dr_{\gamma_1}(X_1) = ds_{\gamma_2}(X_2) = 0 \text{ that is } X_1 \in T_{\gamma_1} G^{r(\gamma_1)} \text{ and } X_2 \in T_{\gamma_2} G_{s(\gamma_2)}.$$

We introduce the vectors $Y_1 \in T_y G^y$ and $Y_2 \in T_y G_y$ such that

$$X_1 = d(L_{\gamma_1})_y(Y_1) \text{ and } X_2 = d(R_{\gamma_2})_y(Y_2).$$

The equality $m \circ (L_{\gamma_1}, R_{\gamma_2}) = L_{\gamma_1} R_{\gamma_2} \circ m$ then gives the equivalence

$$dm_{(\gamma_1, \gamma_2)}(X_1, X_2) = 0 \iff dm_{(y, y)}(Y_1, Y_2) = 0.$$

The relations $s \circ L_\gamma = s$ and $r \circ R_\gamma = r$ give

$$ds_y(Y_1) = dr_y(Y_2) = Y.$$

A simple computation shows that $dm_{(y, y)}(Y, Y) = Y$ and since $Y_1 - Y \in \ker ds$, $Y_2 - Y \in \ker dr$, we also have

$$dm_{(y, y)}(Y_1 - Y, Y_2 - Y) = d(R_y)_y(Y_1 - Y) + d(L_y)_y(Y_2 - Y) = Y_1 - Y + Y_2 - Y = Y_1 + Y_2 - 2Y.$$

This implies the equality

$$(74) \quad dm_{(y, y)}(Y_1, Y_2) = dm_{(y, y)}(Y, Y) + dm_{(y, y)}(Y_1 - Y, Y_2 - Y) = Y_1 + Y_2 - Y$$

which proves the second equivalence of the lemma.

Differentiating the relation $\gamma^{-1}\gamma = s(\gamma)$ at $\gamma = y$, we get $di_y + \text{id} = ds_y + dr_y$, hence

$$di_y(Y_1) - Y_2 = ds_y(Y_1) + dr_y(Y_1) - Y_1 - Y_2 = Y - Y_1 - Y_2,$$

which proves the third equivalence. The last ones are obvious. \square

Going back to (73), we have under the assumption $dm(X_1, X_2) = 0$ and using the notation of the lemma

$$\xi_1(X_1) + \xi_2(X_2) = {}^t di_y {}^t d(L_{\gamma_1})_y(\xi_1)(Y_2) + {}^t d(R_{\gamma_2})_y(\xi_2)(Y_2)$$

Thus (73) holds if and only if

$$(75) \quad {}^t di_y {}^t d(L_{\gamma_1})_y(\xi_1) + {}^t d(R_{\gamma_2})_y(\xi_2) = 0.$$

This equality in $A_y^* G$ (viewed as the space of linear forms on $T_y G_y$ that is, $A^* G = (AG)^*$ with $AG = \ker ds|_{G(0)}$) leads to the definition of source and target maps for $\Gamma = T^* G$:

Definition 6. [4]. *Let G be a C^∞ groupoid. We denote by $\Gamma = T^* G$ the cotangent space of G endowed with the C^∞ groupoid structure given by*

- unit space $\Gamma^{(0)} = A^* G$;
- source map

$$(76) \quad s_\Gamma(\gamma, \xi) = (s(\gamma), \tilde{s}(\xi)) = (s(\gamma), -{}^t di_{s(\gamma)} {}^t d(L_\gamma)_{s(\gamma)}(\xi)) \in A^* G;$$

- target map

$$(77) \quad r_\Gamma(\gamma, \xi) = (r(\gamma), \tilde{r}(\xi)) = (r(\gamma), {}^t d(R_\gamma)_{r(\gamma)}(\xi)) \in A^* G;$$

- product map

$$(78) \quad m_\Gamma(\gamma_1, \xi_1, \gamma_2, \xi_2) = (\gamma_1 \gamma_2, \mu(\xi_1, \xi_2)), \quad \mu(\xi_1, \xi_2) = ({}^t dm_{(\gamma_1, \gamma_2)})^{-1}(\xi_1, \xi_2);$$

- inversion map

$$(79) \quad i_\Gamma(\gamma, \xi) = (\gamma^{-1}, -({}^t di_\gamma)^{-1}(\xi)).$$

Remark 24. The Lie algebroid of G is defined above by $AG = \ker ds|_{G^{(0)}}$. It is also often defined by $AG = (TG|_{G^{(0)}})/TG^{(0)}$. Both bundles are canonically isomorphic since $\ker ds|_{G^{(0)}} \oplus TG^{(0)} = TG|_{G^{(0)}}$. Looking at AG as the normal bundle of $G^{(0)}$ into G gives an alternative and useful definition of source and target maps of T^*G .

Let $\xi \in T_\gamma^*G$ and $x = s(\gamma)$. The extension of the linear form

$$\xi \circ d(L_\gamma)_x : T_x G^x \longrightarrow \mathbb{R},$$

by 0 on the subspace $T_x G^{(0)}$ of $T_x G$ gives an element of $A_x^*G = (T_x G/T_x G^{(0)})^*$ denoted by $\bar{s}(\xi)$. Similarly, with $y = r(\gamma)$, we denote $\bar{r}(\xi)$ the extension by 0 on the subspace $T_y G^{(0)}$ of $T_y G$ of the linear form

$$\xi \circ d(R_\gamma)_y : T_y G_y \longrightarrow \mathbb{R},$$

If we use again the relation $di_y + \text{id} = ds_y + dr_y$, we obtain that (75) is equivalent to

$$(80) \quad \bar{s}(\xi_1) = \bar{r}(\xi_2),$$

hence we can also set

$$(81) \quad s_\Gamma(\gamma, \xi) = (s(\gamma), \bar{s}(\xi)) ; r_\Gamma(\gamma, \xi) = (r(\gamma), \bar{r}(\xi)).$$

Remark 25. An obvious computation shows that the graph Γ_μ of the map $(\gamma_1, \xi_1, \gamma_2, \xi_2) \mapsto (\gamma_1 \gamma_2, \xi_1 \oplus \xi_2)$ is canonically isomorphic to the conormal bundle N^*G_m of the graph of the multiplication of G

$$(\gamma, \xi, \gamma_1, \xi_1, \gamma_2, \xi_2) \in \Gamma_\mu \Leftrightarrow (\gamma, -\xi, \gamma_1, \xi_1, \gamma_2, \xi_2) \in N^*G_m = (TG_m)^\perp.$$

Since N^*G_m is lagrangian in $(T^*G)^3$, this remark proves that the groupoid structure of $\Gamma = T^*G$ which follows from these considerations is automatically symplectic.

Remark 26. Observe that $\Gamma = T^*G \xrightarrow{\pi} G$ is a vector bundle and that μ is a vector bundle homomorphism over m . Actually, the diagram

$$(82) \quad \begin{array}{ccc} \Gamma^{(2)} & \xrightarrow{\mu} & \Gamma \\ \downarrow \pi \times \pi & & \downarrow \pi \\ G^{(2)} & \xrightarrow{m} & G \end{array}$$

commutes and μ is linear in the fibers. We observe that

$$(83) \quad N^*G^{(2)} = \ker \mu \subset \Gamma^{(2)}.$$

We finish this review with two basic examples, the second one being certainly the initial motivation of [4].

Example 2. We take $G = X \times X \times Z \rightrightarrows X \times Z$ (cartesian product of the pair groupoid $X \times X$ with the space Z). Here we have

$$\Gamma^{(0)} = A^*G = \{(x, \xi, x, -\xi, 0) ; (x, \xi) \in T^*X\}.$$

Let $\gamma = (x, y, z)$ and $\xi = (\zeta, \eta, \sigma) \in T_\gamma^*G$. Then $\bar{s}(\xi)$ is the element in $T_{(y,y,z)}^*X \times X \times Z$ obtained from $\eta \in T_y^*X \simeq 0 \times T_y^*X \times 0$ after extension by 0 into the subspace of vectors of the form (u, u, w) . In the same way, $\bar{r}(\xi) \in T_{(x,x,z)}^*X \times X \times Z$ is obtained from $\zeta \in T_x^*X \simeq T_x^*X \times 0 \times 0$ after extension by 0 into the subspace of vectors of the form (u, u, w) . Writing

$$(u, v, w) = (u - v, 0, 0) + (v, v, w) = (0, v - u, 0) + (u, u, w)$$

we see that $(x_1, y_1, z_1; \zeta_1, \eta_1, \sigma_1)$ and $(x_2, y_2, z_2; \zeta_2, \eta_2, \sigma_2)$ are composable in Γ if and only if

$$y_1 = x_2 ; z_1 = z_2 ; \eta_1 = -\zeta_2 ;$$

and the product is then given by (γ, ξ) with $\gamma = \gamma_1\gamma_2$ and

$$\xi = (\zeta_1, \eta_2, \sigma_1 + \sigma_2).$$

When Z is a point, the groupoid $\Gamma = T^*(X \times X)$ is isomorphic to the pair groupoid of T^*X , with isomorphism given by

$$T^*X \times T^*X \longrightarrow T^*X \times T^*X ; (x, \zeta, y, \eta) \mapsto (x, \zeta, y, -\eta).$$

Example 3. We suppose that the Lie groupoid G is a Lie group with Lie algebra \mathfrak{g} . Two elements (g, ξ) and (h, η) in T^*G are composable if and only if :

- g and h are composable (this condition is void in a Lie group),
- $L_g^*\xi = R_h^*\eta$ (equality in \mathfrak{g}^*).

In this case, the composition is defined by:

$$(g, \xi)(h, \eta) = (gh, \xi \oplus \eta)$$

with:

$$\langle {}^t dm_{(g,h)} \xi \oplus \eta, (X, Y) \rangle = \langle \xi, X \rangle + \langle \eta, Y \rangle$$

for any $X \in T_g G$ and $Y \in T_h G$. This can also be written as:

$$(84) \quad \langle \xi \oplus \eta, (L_g)_* Y + (R_h)_* X \rangle = \langle \xi, X \rangle + \langle \eta, Y \rangle.$$

Let us remark at this stage that we obtain:

$$(85) \quad \xi \oplus \eta = R_{h^{-1}}^* \xi = L_{g^{-1}}^* \eta,$$

as we can see by setting respectively $Y = 0$ and $X = 0$ in (84). Using the compossability condition and the fact that left and right translations commute, we can directly check:

$$\begin{aligned} R_{h^{-1}}^* \xi &= R_{h^{-1}}^* L_{g^{-1}}^* L_g^* \xi \\ &= R_{h^{-1}}^* L_{g^{-1}}^* R_h^* \eta \\ &= L_{g^{-1}}^* \eta. \end{aligned}$$

Now, as the cotangent bundle of a Lie group is trivial, we can identify T^*G with the product $G \times \mathfrak{g}^*$ via the map:

$$\begin{aligned} \Phi : T^*G &\longrightarrow G \times \mathfrak{g}^* \\ (g, \xi) &\longmapsto (g, R_g^* \xi). \end{aligned}$$

We will denote $R_g^* \xi$ by $\bar{\xi}$. The compossability condition can the be rewritten as:

$$L_g^* R_{g^{-1}}^* \bar{\xi} = R_h^* R_{h^{-1}}^* \bar{\eta}$$

hence:

$$\text{Ad}^* g. \bar{\xi} = \bar{\eta}.$$

The composition law is then rewritten as:

$$\begin{aligned}
(g, \bar{\xi})(h, \bar{\eta}) &= \Phi((g, R_{g^{-1}}^* \bar{\xi})(h, R_{h^{-1}}^* \bar{\eta})) \\
&= \phi(gh, R_{h^{-1}}^* R_{g^{-1}}^* \bar{\xi}) \\
&= (gh, R_{gh}^* R_{h^{-1}}^* R_{g^{-1}}^* \bar{\xi}) \\
&= (gh, \bar{\xi}).
\end{aligned}$$

The cotangent groupoid T^*G is thus identified via Φ with the transformation groupoid $G \times \mathfrak{g}^*$, with source, target and composition defined by:

$$\begin{aligned}
s(g, \bar{\xi}) &= \text{Ad}^* g \cdot \bar{\xi}, \\
r(g, \bar{\xi}) &= \bar{\xi}, \\
(g, \bar{\xi})(h, \text{Ad}^* g \cdot \bar{\xi}) &= (gh, \bar{\xi}).
\end{aligned}$$

7. CONVOLUTION AND WAVE FRONT SETS ON GROUPOIDS

We revisit the convolution of distributions on a groupoid in terms of their wave front sets. The operations on distributions and their wave front sets used below can be found in [8, 6], see also [2] for a detailed discussion about continuity, and [17] which has inspired the results of the present section.

Let $u_1, u_2 \in \mathcal{D}'(G, \Omega^{1/2})$ and assume that at least one is compactly supported, say u_1 . The restriction of $u_1 \otimes u_2 \in \mathcal{D}'(G^2, \Omega^{1/2} \boxtimes \Omega^{1/2})$ to $G^{(2)}$, that is, the pull back of $u_1 \otimes u_2$ by $\rho : G^{(2)} \hookrightarrow G^2$ is allowed if

$$(86) \quad \text{WF}(u_1 \otimes u_2) \cap N^*G^{(2)} = \emptyset.$$

In that case, Theorem 8.2.4 of [8] ensures that the wave front set of $u_1 \otimes u_2|_{G^{(2)}} = \rho^*(u_1 \otimes u_2)$ satisfies

$$(87) \quad \text{WF}(u_1 \otimes u_2|_{G^{(2)}}) \subset \rho^*(\text{WF}(u_1 \otimes u_2))$$

where $\rho^* : T^*G^2 \rightarrow T^*G^{(2)}$ is the canonical projection. Since $\text{supp}(u_1)$ is compact, m is proper on $\text{supp}(u_1) \times G \cap G^{(2)}$ so that

$$(88) \quad u_1 * u_2 = m_*(u_1 \otimes u_2|_{G^{(2)}}) \in \mathcal{D}'(G, \Omega^{1/2})$$

is well defined and, using formula (3.6) p. 328 of [6], the wave front set of $u_1 * u_2$ satisfies, writing $\widetilde{W} = \rho^*(\text{WF}(u_1 \otimes u_2))$,

$$(89) \quad \text{WF}(u_1 * u_2) \subset m_*(\widetilde{W}) = \{(\gamma, \xi) \in T^*G \setminus 0, \exists(\gamma_1, \gamma_2) \in G_\gamma^{(2)}, (\gamma_1, \gamma_2, {}^t dm_{\gamma_1, \gamma_2}(\xi)) \in \widetilde{W} \cup G^{(2)} \times 0\}.$$

Let us clarify what is the set $m_*(\widetilde{W})$. Firstly, since m is a submersion, ${}^t dm_{\gamma_1, \gamma_2}$ is injective and the term $G^{(2)} \times 0$ can be removed. Secondly, it follows from the definition of the symplectic groupoid $\Gamma = T^*G$ that

$$(91) \quad (\gamma_1, \gamma_2, {}^t dm_{\gamma_1, \gamma_2}(\xi)) \in \rho^*(\text{WF}(u_1 \otimes u_2))$$

is equivalent to

$$(92) \quad \exists(\xi_1, \xi_2) \in T_{\gamma_1, \gamma_2}^* G^2, (\gamma_1, \xi_1, \gamma_2, \xi_2) \in \Gamma^{(2)} \cap \text{WF}(u_1 \otimes u_2) \text{ and } \xi = \xi_1 \oplus \xi_2$$

by definition of the composition in the cotangent groupoid (see (71), (72)). In other words,

$$(93) \quad m_*(\rho^*(\text{WF}(u_1 \otimes u_2))) = m_\Gamma(\text{WF}(u_1 \otimes u_2) \cap \Gamma^{(2)}).$$

Let us also analyse the condition (86). We know (Theorem 8.2.9 of [8]) that

$$(94) \quad \text{WF}(u_1 \otimes u_2) \subset \text{WF}(u_1) \times \text{WF}(u_2) \cup \text{WF}(u_1) \times (G \times \{0\}) \cup (G \times \{0\}) \times \text{WF}(u_2).$$

Let $(\gamma_1, \xi_1, \gamma_2, 0) \in \text{WF}(u_1) \times (G \times \{0\}) \cap \Gamma^{(2)}$. Since $\text{WF}(u_1) \subset T^*G \setminus 0$, we can choose $X_1 \in T_{\gamma_1}G$ such that $\xi_1(X_1) \neq 0$. Using a local section β of r such that $\beta(s(\gamma_1)) = \gamma_2$, we set $X_2 = d\beta ds(X_1) \in T_{\gamma_2}G$. Then $(X_1, X_2) \in T_{(\gamma_1, \gamma_2)}G^{(2)}$ and $\xi_1(X_1) + 0(X_2) \neq 0$, which proves that $(\gamma_1, \xi_1, \gamma_2, 0) \notin \ker \mu = N^*G^{(2)}$. Since $N^*G^{(2)} \subset \Gamma^{(2)}$, this proves that $\text{WF}(u_1) \times (G \times \{0\}) \cap N^*G^{(2)} = \emptyset$, and similarly $(G \times \{0\}) \times \text{WF}(u_2) \cap N^*G^{(2)} = \emptyset$. Thus condition (86) is equivalent to

$$(95) \quad \text{WF}(u_1) \times \text{WF}(u_2) \cap N^*G^{(2)} = \emptyset.$$

Remark 27. *The last computation can be rephrased in a more conceptual way. The map*

$$\phi : \gamma \longmapsto \gamma \cdot \beta(s(\gamma))$$

is defined in a neighborhood of γ_1 , maps γ_1 to $\gamma_1\gamma_2$ and is a local diffeomorphism at γ_1 . We then check easily the equality

$$\xi_1 \oplus 0 = {}^t(d\phi_{\gamma_1})(\xi_1).$$

From this, we recover that $\xi_1 \neq 0$ implies $(\xi_1, 0) \notin \ker \mu$.

For any subsets $W_1, W_2 \in T^*G \setminus 0$, we write $W_1 \bar{\times} W_2 = W_1 \times W_2 \cup W_1 \times 0 \cup 0 \times W_2$, $W_1 * W_2 = m_\Gamma(W_1 \times W_2 \cap \Gamma^{(2)})$ and $W_1 \bar{*} W_2 = m_\Gamma(W_1 \bar{\times} W_2 \cap \Gamma^{(2)})$.

Let W_1, W_2 be two closed cones in $T^*G \setminus 0$ such that $W_1 \times W_2 \cap N^*G^{(2)} = \emptyset$. Then the bilinear map

$$(96) \quad \begin{aligned} \mathcal{D}'_{W_1}(G, \Omega^{1/2}) \times \mathcal{D}'_{W_2}(G, \Omega^{1/2}) &\longrightarrow \mathcal{D}'_{\rho^*(W_1 \bar{\times} W_2)}(G^{(2)}, \Omega^{1/2}) \\ (u_1, u_2) &\longmapsto u_1 \otimes u_2|_{G^{(2)}} \end{aligned}$$

is sequentially separately continuous for the natural notion of convergence of sequences in the spaces \mathcal{D}'_W [8, 6], and separately continuous for the normal topology of these spaces [2]. To apply m_* continuously, we need to impose conditions on supports since m is not proper in general. Since we are not interested by optimal results, we impose compact supports on both factors and we get continuity in the above senses for the composition

$$(97) \quad \mathcal{E}'_{W_1}(G, \Omega^{1/2}) \times \mathcal{E}'_{W_2}(G, \Omega^{1/2}) \xrightarrow{\otimes} \mathcal{E}'_{W_1 \bar{\times} W_2}(G^2, \Omega^{1/2}) \xrightarrow{|\cdot|_{G^{(2)}}} \mathcal{E}'_{\rho^*(W_1 \bar{\times} W_2)}(G^{(2)}, \Omega^{1/2}) \xrightarrow{m_*} \mathcal{E}'_{W_1 \bar{*} W_2}(G, \Omega^{1/2}).$$

In conclusion, we have proved

Theorem 28. *Let $u_1, u_2 \in \mathcal{D}'(G, \Omega^{1/2})$ be two distributions such that*

$$(98) \quad \text{WF}(u_1) \times \text{WF}(u_2) \cap \ker \mu = \emptyset$$

*and such that at least one is compactly supported. Then the convolution product $u_1 * u_2 \in \mathcal{D}'(G, \Omega^{1/2})$ is well defined by (88). If W_1, W_2 are closed cones in $T^*G \setminus 0$ such that $W_1 \times W_2 \cap \ker \mu = \emptyset$, then the map*

$$(99) \quad \mathcal{E}'_{W_1}(G, \Omega^{1/2}) \times \mathcal{E}'_{W_2}(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{E}'_{W_1 \bar{*} W_2}(G, \Omega^{1/2})$$

extends continuously the convolution product on $C_c^\infty(G, \Omega^{1/2})$.

If u_1 or u_2 is smooth then $\text{WF}(u_1) \times \text{WF}(u_2)$ is empty and (98) is trivially satisfied, thus

Corollary 29. *The convolution product of Theorem 28 gives by restriction the maps*

$$(100) \quad \mathcal{E}'(G, \Omega^{1/2}) \times C_c^\infty(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{E}'_{s_\Gamma^{-1}(0)}(G, \Omega^{1/2})$$

and

$$(101) \quad C_c^\infty(G, \Omega^{1/2}) \times \mathcal{E}'(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{E}'_{r_\Gamma^{-1}(0)}(G, \Omega^{1/2})$$

where $s_\Gamma^{-1}(0)$ and $r_\Gamma^{-1}(0)$ denote the preimages of the zero section in A^*G under the source and target maps of $\Gamma = T^*G$.

Now we take into account assumptions of transversality on the wave front sets of distributions. Firstly, we can use the groupoid structure of T^*G to rephrase r and s transversality of subsets of T^*G .

Proposition 30. *Let W be a subset of $T^*G \setminus 0$. Then W is r -transversal if and only if*

$$s_\Gamma(W) \subset A^*G \setminus 0.$$

Similarly, W is s -transversal if and only if

$$r_\Gamma(W) \subset A^*G \setminus 0.$$

Proof. Let $W \subset T^*G \setminus 0$ be a set. For any $(\gamma, \xi) \in W$, the non vanishing of

$$\xi \circ d(L_\gamma \circ i)_{s(\gamma)} : T_{s(\gamma)}G_{s(\gamma)} \rightarrow \mathbb{R}$$

is equivalent to the non vanishing of $\xi \circ d(L_\gamma)_{s(\gamma)} : T_{s(\gamma)}G^{s(\gamma)} \rightarrow \mathbb{R}$ which is equivalent to

$$\xi|_{T_\gamma G^{r(\gamma)}} \neq 0.$$

Thus r -transversality is equivalent to (67). The other case is similar. \square

Using the equalities $s_\Gamma \circ m_\Gamma = s_\Gamma \circ \text{pr}_2$ and $r_\Gamma \circ m_\Gamma = r_\Gamma \circ \text{pr}_1$, we get a class of examples for which condition (98) is automatically satisfied.

Corollary 31. *Let W_1, W_2 be any subsets of T^*G .*

*If W_1 is s -transversal (resp. W_2 is r -transversal) then so is $W_1 * W_2$.*

In both cases, we have in particular

$$W_1 \times W_2 \cap \ker \mu = \emptyset.$$

Proof. Suppose W_1 is s -transversal (the other case follows identically). One has $r_\Gamma \circ m_\Gamma = r_\Gamma \circ \text{pr}_1$, hence

$$r_\Gamma(W_1 * W_2) = r_\Gamma \circ m_\Gamma(W_1 \times W_2 \cap \Gamma^{(2)}) = r_\Gamma \circ \text{pr}_1(W_1 \times W_2 \cap \Gamma^{(2)}) \subset r_\Gamma(W_1) \subset A^*G \setminus 0.$$

which proves at once that $W_1 * W_2$ is s -transversal and that $(W_1 \times W_2) \cap \ker \mu = \emptyset$. Indeed if there exists $(\gamma_1, \xi_1, \gamma_2, \xi_2) \in (W_1 \times W_2) \cap \ker \mu$, then $m_\Gamma((\gamma_1, \xi_1), (\gamma_2, \xi_2)) = (\gamma_1 \gamma_2, 0)$ hence $0 \in r_\Gamma(W_1 * W_2) \subset r_\Gamma(W_1)$, thus W_1 would not be s -transversal. \square

We conclude this section by proving, as announced, that $\mathcal{E}'_a(G, \Omega^{1/2})$ is a subalgebra of $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$.

Let $u_1, u_2 \in \mathcal{E}'_a(G, \Omega^{1/2})$. We have by Corollary 31 that $\text{WF}(u_1) \times \text{WF}(u_2) \cap \ker \mu = \emptyset$ so the convolution product of Theorem 28 can be computed for u_1 and u_2 . By r -transversality of $\text{WF}(u_1)$, we have $s_\Gamma(\text{WF}(u_1)) \subset A^*G \setminus 0$, so

$$\text{WF}(u_1) \times G \cap \Gamma^2 = \emptyset.$$

Similarly, s -transversality of $\text{WF}(u_2)$ gives $G \times \text{WF}(u_2) \cap \Gamma^2 = \emptyset$. We obtain

$$\text{WF}(u_1) \bar{*} \text{WF}(u_2) = [\text{WF}(u_1) * \text{WF}(u_2)] \cup [\text{WF}(u_1) * G \times \{0\}] \cup [G \times \{0\} * \text{WF}(u_2)] = \text{WF}(u_1) * \text{WF}(u_2)$$

Moreover, Corollary 31 also gives the bi-transversality of $\text{WF}(u_1) * \text{WF}(u_2)$.

Thus, the convolution product provided by Theorem 28 is well defined on $\mathcal{E}'_a(G, \Omega^{1/2})$ and gives a map

$$\mathcal{E}'_a(G, \Omega^{1/2}) \times \mathcal{E}'_a(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{E}'_a(G, \Omega^{1/2}).$$

Since $\text{WF}(\delta) = A^*G \setminus 0$ is bi-transversal, we get $\delta \in \mathcal{E}'_a(G, \Omega^{1/2})$. Also, since $\text{WF}(i^*u) = {}^t(di)(\text{WF}(u))$ and ${}^t(di)$ is the inversion in the groupoid T^*G , we get that the involution of $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ preserves the bi-transversality of the wave front sets. In conclusion,

Theorem 32. *The space $\mathcal{E}'_a(G, \Omega^{1/2})$ is a unital involutive subalgebra of $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$.*

Remark 33. The convolution products defined by Theorems 13 and 28 coincide when both make sense. Indeed, remember that $\ker \mu = N^*G^{(2)}$ and that

$$\text{WF}(u_1 \otimes u_2) \cap \ker \mu = (\text{WF}(u_1) \times \text{WF}(u_2)) \cap \ker \mu = \emptyset.$$

This proves that the restriction $u_1 \otimes u_2|_{G^{(2)}} \in \mathcal{D}'(G^{(2)})$, that is, the pull-back of $u_1 \otimes u_2$ by the map $G^{(2)} \hookrightarrow G^2$ exists, ([8, Theorem 8.2.4]) and it follows that the product given by Theorem 28 is given by

$$(102) \quad \langle u_1 * u_2, \phi \rangle = \langle u_1 \otimes u_2|_{G^{(2)}}, \phi \circ m \rangle.$$

Since u_1 is s -transversal and u_2 is r -transversal, the distribution $u_1 \otimes u_2|_{G^{(2)}}$ is also given by the family

$$u_{1x} \otimes u_2^x \quad x \in G^{(0)}.$$

and we then see that the pairing (102) coincides with the one provided by Theorem 13.

8. G -OPERATORS AND BI-TRANSVERSALITY

In view of the classical result on the composition of operators [8, Theorem 8.2.13] on a manifold M , one may say that the wave front set of the Schwarz kernel K_P of an operator P acting on $C_c^\infty(M)$ is *bi-transversal* whenever one has $\text{WF}(K_P) \in (T^*M \setminus 0) \times (T^*M \setminus 0)$. Indeed, we know that the composition of two such operators is again an operator with $\text{WF}(K_{P \circ Q}) \in (T^*M \setminus 0) \times (T^*M \setminus 0)$. We have seen in section 4 that adjointable compactly supported G -operators are in 1 to 1 correspondence with kernels in $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ and hence the composition of two such operators, P and Q is still an adjointable compactly supported G -operator $P \circ Q$. Using the fact that for any $x \in G^{(0)}$ one has $(P \circ Q)_x = P_x \circ Q_x$, we can thus compose operators acting on $C_c^\infty(G_x)$. However, the conditions of bi-transversality we assume do not imply that the induced operators on the fibers G_x have bi-transversal wave front sets, as seen in remark 20.

In fact, we will see right now that the bi-transversality of the wave front set of the distributional kernel k_P of an adjointable compactly supported G -operator is equivalent to the bi-transversality of the wave front sets of all operators G_x .

First we translate the conditions of r -transversality or s -transversality of a set in $T^*G \setminus 0$ with corresponding transversality properties in $T^*G_x \times T^*G_x$

Proposition 34. *For any $x \in G^{(0)}$, we note $j_x : G_x \times G_x \longrightarrow G$ the map $(\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2^{-1}$. Let $W \subset T^*G \setminus 0$ be any set.*

- (1) W is r -transversal if and only if $j_x^*(W)$ is r -transversal as a subset of $T^*G_x \times T^*G_x$, that is $j_x^*(W) \subset T^*G_x \times (T^*G_x \setminus 0)$, for all $x \in G^{(0)}$.
- (2) W is s -transversal if and only if $j_x^*(W)$ is s -transversal, that is $j_x^*(W) \subset (T^*G_x \setminus 0) \times T^*G_x$, for all $x \in G^{(0)}$.

Proof. Recall that if $A \subset T^*G$ then

$$j_x^*(A) = \{(\gamma_1, \gamma_2, {}^t d(j_x)_{(\gamma_1, \gamma_2)}(\eta)) ; (\gamma_1, \gamma_2) \in G_x \times G_x (\gamma_1 \gamma_2^{-1}, \eta) \in A\}$$

We decompose ${}^t d(j_x)_{(\gamma_1, \gamma_2)}(\eta) = (\xi_1, \xi_2)$ in the product space $T_{\gamma_1}^*G_x \times T_{\gamma_2}^*G_x$. Observe that

$$T_{\gamma_1}^*G_x \times T_{\gamma_2}^*G_x \longrightarrow T_{\gamma_1}^*G_x, (\xi_1, \xi_2) \longmapsto \xi_1$$

is the transpose of the differential of the map

$$\ell_{\gamma_2} : G_x \longrightarrow G_x \times G_x, \gamma \longmapsto (\gamma, \gamma_2)$$

Since $j_x \circ \ell_{\gamma_2}(\gamma) = R_{\gamma_2^{-1}}(\gamma)$ we get

$$\xi_1 = {}^t d(\ell_{\gamma_2})_{\gamma_1} \circ {}^t d(j_x)_{(\gamma_1, \gamma_2)}(\eta) = {}^t d(R_{\gamma_2^{-1}})_{\gamma_1}(\eta).$$

Similarly we get

$$\xi_2 = {}^t d(L_{\gamma_1} \circ i)_{\gamma_2}(\eta).$$

Thus we get

$$j_x^*(A) = \{(\gamma_1, {}^t d(R_{\gamma_2^{-1}})_{\gamma_1}(\eta), \gamma_2, {}^t d(L_{\gamma_1} \circ i)_{\gamma_2}(\eta)) ; (\gamma_1, \gamma_2) \in G_x \times G_x (\gamma_1 \gamma_2^{-1}, \eta) \in A\}$$

Since $\ker {}^t d(R_{\gamma_2^{-1}})_{\gamma_1} = (\text{Im } d(R_{\gamma_2^{-1}})_{\gamma_1})^\perp = (T_{\gamma_1 \gamma_2^{-1}}G_{r(\gamma_2)})^\perp$, we have for instance the equivalences

$$\begin{aligned} & A \text{ is } s\text{-transversal} \\ \Leftrightarrow & \forall (\gamma, \eta) \in A, \eta \notin (T_\gamma G_{s(\gamma)})^\perp \\ \Leftrightarrow & \forall x \in G^{(0)}, \forall (\gamma_1, \gamma_2) \in G_x \times G_x \text{ such that } (\gamma_1 \gamma_2^{-1}, \eta) \in A, \eta \notin (T_{\gamma_1 \gamma_2^{-1}} G_{r(\gamma_2)})^\perp \\ \Leftrightarrow & \forall x \in G^{(0)}, \forall (\gamma_1, \gamma_2) \in G_x \times G_x \text{ such that } (\gamma_1 \gamma_2^{-1}, \eta) \in A, {}^t d(R_{\gamma_2^{-1}})_{\gamma_1}(\eta) \neq 0 \\ \Leftrightarrow & \forall x \in G^{(0)}, j_x^*(A) \subset (T^*G_x \setminus 0) \times T^*G_x. \end{aligned}$$

The other case is identical. □

Using this, we have the following

Corollary 35. *Let P be a compactly supported G -operator. Then the wave front set of the distributional kernel k_P of P is bi-transversal in T^*G if and only if for any $x \in G^{(0)}$, the wave front set of the Schwartz kernel of P_x is bi-transversal in $T^*(G_x \times G_x)$.*

Remark 36. Using this approach, one could give another proof for Theorem 32, via families of operators by means of the following lemma.

Lemma 37. *Let A_1, A_2 be two subsets of T^*G . Then for any $x \in G^{(0)}$ we have*

$$(103) \quad j_x^*(A_1 * A_2) = j_x^*(A_1) * j_x^*(A_2)$$

where the composition $*$ in the right hand side is the one of $T^*(G_x \times G_x)$.

Suppose now we are given two compactly supported G -operators P and Q such that for any $x \in G^{(0)}$, the wave front sets of the Schwartz kernels of P_x and Q_x are both bi-transversal in $T^*(G_x \times G_x)$. From the classical result [8, Theorem 8.2.14], we then know that the wave front set of the Schwartz kernel of the composition $P_x \circ Q_x$ is again bi-transversal in $T^*(G_x \times G_x)$. Thus, using the lemma $j_x^*(W_P) * j_x^*(W_Q) = j_x^*(W_P * W_Q)$ and the corollary 35, we recover the result of Theorem 32.

Proof of the lemma. Let A be any subset of T^*G . For any $x \in G^{(0)}$ we have seen in the proof of the Proposition 34 that

$$(104) \quad j_x^*(A) = \{(\gamma_1, {}^t d(R_{\gamma_2^{-1}})_{\gamma_1}(\eta), \gamma_2, {}^t d(L_{\gamma_1} \circ i)_{\gamma_2}(\eta)) ; (\gamma_1, \gamma_2) \in G_x \times G_x (\gamma_1 \gamma_2^{-1}, \eta) \in A\}$$

Identifying A^*G with $\ker ds|_{G^{(0)}}$, we have for any $\eta \in T_{\gamma_1 \gamma_2^{-1}}^* G$

$$\begin{aligned} \tilde{s}(\eta) &= {}^t d(L_{\gamma_1 \gamma_2^{-1}} \circ i)_{r(\gamma_2)}(\eta) = {}^t d(L_{\gamma_1} \circ L_{\gamma_2^{-1}} \circ i)_{r(\gamma_2)}(\eta) \\ &= {}^t d(L_{\gamma_1} \circ i \circ R_{\gamma_2})_{r(\gamma_2)}(\eta) = {}^t d(R_{\gamma_2})_{r(\gamma_2)} \circ {}^t d(L_{\gamma_1} \circ i)_{\gamma_2}(\eta), \end{aligned}$$

which gives $\xi_2 = ({}^t d(R_{\gamma_2})_{r(\gamma_2)})^{-1}(\tilde{s}(\eta)) = {}^t d(R_{\gamma_2^{-1}})_{\gamma_2}(\tilde{s}(\eta))$. We also have

$$\xi_1 = ({}^t d(R_{\gamma_1})_{r(\gamma_1)})^{-1}(\tilde{r}(\eta)) = {}^t d(R_{\gamma_1^{-1}})_{\gamma_1}(\tilde{r}(\eta)).$$

We thus have

$$(105) \quad j_x^*(A) = \{(\gamma_1, {}^t d(R_{\gamma_1^{-1}})_{\gamma_1}(\tilde{r}(\eta)), \gamma_2, {}^t d(R_{\gamma_2^{-1}})_{\gamma_2}(\tilde{s}(\eta))) ; \gamma_i \in G_x, (\gamma_1 \gamma_2^{-1}, \eta) \in A\}.$$

Next, we apply this formula to $A_1 * A_2$:

$$(106) \quad j_x^*(A_1 * A_2) = \{(\gamma_1, {}^t d(R_{\gamma_1^{-1}})_{\gamma_1}(\tilde{r}(\eta_1)), \gamma_2, {}^t d(R_{\gamma_2^{-1}})_{\gamma_2}(\tilde{s}(\eta_2))) ;$$

$$(107) \quad \forall \gamma_i \in G_x, (\delta_1, \eta_1, \delta_2, \eta_2) \in (A_1 \times A_2)^{(2)}, \text{ s.t. } \gamma_1 \gamma_2^{-1} = \delta_1 \delta_2\}.$$

On the other hand, the composition of $j_x^*(A_1)$ and $j_x^*(A_2)$ gives

$$(108) \quad j_x^*(A_1) * j_x^*(A_2) = \{(\gamma_1, {}^t d(R_{\gamma_1^{-1}})_{\gamma_1}(\tilde{r}(\eta_1)), \gamma_2, {}^t d(R_{\gamma_2^{-1}})_{\gamma_2}(\tilde{s}(\eta_2))) ;$$

$$(109) \quad \forall \gamma_i \in G_x, \gamma_3 \in G_x, \text{ s.t. } (\gamma_1 \gamma_3^{-1}, \eta_1, \gamma_3 \gamma_2^{-1}, \eta_2) \in (A_1 \times A_2)^{(2)}\}.$$

The inclusion $j_x^*(A_1) * j_x^*(A_2) \subset j_x^*(A_1 * A_2)$ follows from $\gamma_1 \gamma_3^{-1} \gamma_3 \gamma_2^{-1} = \gamma_1 \gamma_2^{-1}$ and conversely observe that if $(\delta_1, \eta_1, \delta_2, \eta_2) \in (A_1 \times A_2)^{(2)}$ satisfies $\gamma_1 \gamma_2^{-1} = \delta_1 \delta_2$, we get using $\gamma_3 = \delta_2 \gamma_2 \in G_x$:

$$\delta_1 = \gamma_1 \gamma_3^{-1}, \quad \delta_2 = \gamma_3 \gamma_2^{-1}.$$

□

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