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ABOUT THE CONVOLUTION OF DISTRIBUTIONS ON GROUPOIDS

JEAN-MARIE LESCURE , DOMINIQUE MANCHON, STÉPHANE VASSOUT

ABSTRACT. We review the properties of transversality of distributions with respect to submersions. This allows us to construct a convolution product for a large class of distributions on Lie groupoids. We get a unital algebra $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ enlarging the convolution algebra $C_c^\infty(G, \Omega^{1/2})$ associated with any Lie groupoid G . We prove that G -operators are convolution operators by transversal distributions. We also investigate the microlocal aspects of the convolution product. We give conditions on wave front sets sufficient to compute the convolution product and we show that the wave front set of the convolution product of two distributions is essentially the product of their wave front sets in the cotangent groupoid T^*G of Costes-Dazord-Weinstein. This also leads to a subalgebra $\mathcal{E}'_a(G, \Omega^{1/2})$ of $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ which contains for instance the algebra of pseudodifferential operators on G .

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1. INTRODUCTION

The motivation of this paper is twofold. Firstly, we wish to clarify the notions of distributions on a Lie groupoid and their convolution product, and to understand the relationship between these notions and the so-called G -operators. Secondly, we would like to set up a neat framework in order to investigate in a future work the notions of Fourier integrals and Fourier integral operators on a groupoid.

The notion of C^∞ longitudinal family of distributions in the framework of groupoids appears in [12, 13, 16] in order to define right invariant pseudodifferential operators. Also, in the works of B. Monthubert [11], these families are considered from the point of view of distributions on the whole groupoid, so that the action of the corresponding pseudodifferential operators on C^∞ functions is given by a convolution product. Here, we carry on this idea by exploring the correspondance between C^∞ longitudinal families of distributions and single distributions on the whole underlying manifold of the groupoid and by studying the convolution product of distributions on groupoids. This is achieved at two levels.

The first level is based on the notion of transversality of distributions with respect to a submersion $\pi : M \rightarrow B$ [1]. It appears that the space $\mathcal{D}'_{\pi}(M)$ of such distributions is actually isomorphic to the space of C^{∞} family of distributions in the fibers of π and to the space of continuous $C^{\infty}(B)$ -linear maps from $C_c^{\infty}(M)$ to $C^{\infty}(B)$ as well. Furthermore, operations such as push-forwards and fibered-products of distributions behave well on transversal distributions and these operations allow to define the convolution product of distributions on groupoids, as soon as these distributions satisfy some transversality assumptions with respect to source or target maps. Distributions on a groupoid which are transversal both to the source and target maps are called bi-transversal and they give rise to an involutive unital algebra $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ for the convolution product. Then, one has the necessary tools to prove that G -operators on a groupoid are in 1 to 1 correspondance with transversal distributions acting by convolution and that bi-transversal distributions are in 1 to 1 correspondance with adjointable G -operators.

The second level is a microlocal refinement of the first one and consists in using the wave front set of distributions. A basic observation, due to Weinstein [17], is that the convolution of functions on a groupoid G is itself given by a distribution \mathcal{C} on G^3 , living on the graph G_m of the multiplication map of G . Hence, the convolution product, viewed as a distribution, has a wave front set which is exactly the conormal space N^*G_m . Moreover, it appears that the conormal space N^*G_m is itself the graph of the multiplication map of another groupoid, the so-called cotangent groupoid $T^*G \rightrightarrows A^*G$ of Costes-Dazord-Weinstein [3]. This non trivial structure of groupoid on T^*G encodes fine properties of the differential of the multiplication map of G .

These observations combined with the usual conditions of Hormander to define the product of distributions bring in natural conditions on wave front sets of distributions on a groupoid allowing to define their convolution product. The class of distributions permitted in this second approach is slightly different from the one arising from transversality. However, there is a large natural class of transversal distributions for which the second approach applies, consisting of distributions whose wave front sets are transversal with respect to the (differential of) source or target maps. Actually, the transversality of the wave front set is a strictly stronger assumption than the transversality condition of the distribution itself and thus, distributions with bi-transversal wave front sets provide a sub-space $\mathcal{E}'_a(G, \Omega^{1/2})$ of the algebra $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$.

Using the Costes-Dazord-Weinstein groupoid structure of T^*G , we show that $\mathcal{E}'_a(G, \Omega^{1/2})$ is a unital involutive sub-algebra of $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ and we compute the wave front set of a convolution product in term of the wave front sets of the factors, whenever this convolution product is defined by the second approach. For instance, when $u, v \in \mathcal{E}'_a(G, \Omega^{1/2})$, we obtain

$$(1) \quad \text{WF}(u * v) \subset \text{WF}(u) \circ \text{WF}(v),$$

where \circ is the composition law in the Costes-Dazord-Weinstein groupoid T^*G . We would like to add that the corresponding formula of Hormander for the wave front set of composition of kernels [6, 7] makes the above formula quite predictable. Indeed, given a manifold X , the composition of kernels corresponds to convolution in the pair groupoid $X \times X$ and the composition law that Hormander defines on $T^*(X \times X)$ to compute wave front sets of composition of kernels is precisely the multiplication map of the Costes-Dazord-Weinstein cotangent groupoid $T^*(X \times X)$.

The paper is organized as follows. In section 2, we review the notion of distributions transversal with respect to a submersion and study natural operations available on them. In section 3, we apply the results of section 2 to the case of groupoids. We then define the convolution product of (transversal) distributions and obtain the unital algebra $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ of bi-transversal distributions.

In section 4, we link the notion of G -operators with the one of transversal distributions and we obtain a 1 to 1 correspondance between the space of adjointable compactly supported G -operators and $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$. In section 5 we show that natural transversality conditions on the wave front set of a distribution imply transversality with respect to a submersion, but that these notions do not coincide in general. This leads naturally to the question of the convolution of distributions with transversal wave front sets on a groupoid. After reviewing the groupoid structure of Costes-Dazord-Weinstein [3] in section 6, convolution is thus revisited in section 7 in terms of wave front sets. The cotangent groupoid structure is used to establish a formula for the wave front set of the convolution product of distributions. In section 8 we end up by studying the meaning of bi-transversality of the wave front set in the framework of G -operators.

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2. DISTRIBUTIONS, SUBMERSIONS, TRANSVERSALITY

Before dealing with distributions on groupoids, it is useful to go back to the behavior of distributions on total spaces of submersions. The notion of transversality we shall recall in this part is borrowed from [1] and it extends the condition of semi-regularity given by Trèves [15, p.532].

In this paper, we will handle spaces of distributions and spaces of continuous linear maps between locally convex vector spaces. Unless otherwise stated, distribution spaces will be provided with the strong topology and spaces of continuous linear maps with the topology of bounded convergence.

Let $X \subset \mathbb{R}^{n_x}$ and $Y \subset \mathbb{R}^{n_y}$ denote open subsets and for any $u \in \mathcal{D}'(X \times Y)$, we denote by $P_u : C_c^\infty(Y) \rightarrow \mathcal{D}'(X)$ the continuous linear operator associated with u through the Schwartz kernel theorem. The projections on the first and second factors of $X \times Y$ are noted p_1, p_2 .

Definition 1. *We say that $u \in \mathcal{D}'(X \times Y)$ is transversal to p_1 (or semi-regular with respect to x) if*

$$P_u(C_c^\infty(Y)) \subset C^\infty(X)$$

We note $\mathcal{D}'_{p_1}(X \times Y)$ the vector space of p_1 -transversal distributions.

Proposition 1. *If $P : \mathcal{D}'(X \times Y) \rightarrow \mathcal{L}(C_c^\infty(Y), \mathcal{D}'(X))$ denotes the Schwartz kernel theorem isomorphism, then*

$$P(\mathcal{D}'_{p_1}(X \times Y)) = \mathcal{L}(C_c^\infty(Y), C^\infty(X)).$$

We will provide $\mathcal{D}'_{p_1}(X \times Y)$ with the topology turning

$$(2) \quad P : \mathcal{D}'_{p_1}(X \times Y) \xrightarrow{\simeq} \mathcal{L}(C_c^\infty(Y), C^\infty(X)).$$

into a topological isomorphism.

Proof. It is a consequence of the closed graph theorem on LF spaces, also called Gotthendieck-Köthe theorem [8, Cor 1.2.20, p. 22] that for any $u \in \mathcal{D}'_{p_1}(X \times Y)$, the linear map

$$P_u : C_c^\infty(Y) \rightarrow C^\infty(X)$$

is continuous for the natural topologies of $C_c^\infty(Y)$ and $C^\infty(X)$. Indeed, P_u has a closed graph in $C_c^\infty(Y) \times \mathcal{D}'(X)$, and its graph is, by hypothesis, included into $C_c^\infty(Y) \times C^\infty(X)$. If a net (f_α, g_α) in the graph of P_u converges to (f, g) in $C_c^\infty(Y) \times C^\infty(X)$, then it converges to (f, g) in $C_c^\infty(Y) \times \mathcal{D}'(X)$. It follows that (f, g) is in the graph of P_u , so that the graph of P_u is closed in $C_c^\infty(Y) \times C^\infty(X)$. This proves that $P(\mathcal{D}'_{p_1}(X \times Y)) \subset \mathcal{L}(C_c^\infty(Y), C^\infty(X))$.

The converse inclusion is obvious. \square

Since we also have canonical (topological) isomorphisms [15, prop. 50.5, thm 44.1, see also p. 532]

$$\mathcal{L}(C_c^\infty(Y), C^\infty(X)) \simeq C^\infty(X) \widehat{\otimes} \mathcal{D}'(Y) \simeq C^\infty(X, \mathcal{D}'(Y))$$

we get that the space of p_1 -transversal distributions on $X \times Y$ is canonically isomorphic to the space of C^∞ functions on X valued in $\mathcal{D}'(Y)$. To be concrete, the isomorphism

$$I : C^\infty(X, \mathcal{D}'(Y)) \longrightarrow \mathcal{D}'_{p_1}(X \times Y)$$

is given by

$$(3) \quad \langle I(u), f \rangle = \int_X \langle u(x), f(x, \cdot) \rangle_Y dx, \quad f \in C_c^\infty(X \times Y)$$

and its inverse is given by

$$\langle I^{-1}(v)(x), h \rangle_Y = P_v(h)(x), \quad h \in C_c^\infty(Y).$$

If $u \in \mathcal{D}'(X \times Y)$ and $f \in C^\infty(X, C_c^\infty(Y))$, the push-forward distribution $u_f = p_{1*}(u \cdot f) \in \mathcal{D}'(X)$ is given by

$$\langle u_f, g \rangle_X = \langle u, f \cdot g \circ p_1 \rangle_{X \times Y}, \quad g \in C_c^\infty(X)$$

and the map

$$p_{1*}(u \cdot) : C^\infty(X, C_c^\infty(Y)) \rightarrow \mathcal{D}'(X), \quad f \mapsto u_f$$

is continuous. Again by the closed graph theorem between LF-spaces, if u is such that u_f is C^∞ for any f then

$$\begin{array}{ccc} C^\infty(X, C_c^\infty(Y)) & \longrightarrow & C^\infty(X) \\ f & \longmapsto & u_f \end{array}$$

is continuous for the natural topologies.

Proposition 2. *Let $u \in \mathcal{D}'(X \times Y)$. Then $u \in \mathcal{D}'_{p_1}(X \times Y)$ if and only if $u_f \in C^\infty(X)$ for any $f \in C^\infty(X, C_c^\infty(Y))$. Moreover the map*

$$\begin{array}{ccc} \tilde{P} : \mathcal{D}_{p_1}(X \times Y) & \longrightarrow & \mathcal{L}_{C^\infty(X)}(C^\infty(X, C_c^\infty(Y)), C^\infty(X)) \\ u & \longmapsto & (f \mapsto u_f) \end{array}$$

is a topological isomorphism. The range space is the topological subspace of $\mathcal{L}(C^\infty(X, C_c^\infty(Y)), C^\infty(X))$ consisting of $C^\infty(X)$ linear maps.

Proof. We first recall that there exists a canonical topological isomorphism

$$(4) \quad E : \mathcal{L}(C_c^\infty(Y), C^\infty(X)) \xrightarrow{\simeq} \mathcal{L}_{C^\infty(X)}(C^\infty(X, C_c^\infty(Y)), C^\infty(X))$$

which is given as follows. The natural extension of $u \in \mathcal{L}(C_c^\infty(Y), C^\infty(X))$ into

$$\tilde{u} : C^\infty(X) \otimes C_c^\infty(Y) \longrightarrow C^\infty(X); \quad g \otimes h \mapsto u(h)g$$

is continuous and $C^\infty(X)$ linear. By nuclearity for instance of $C^\infty(X)$, theorem 44.1 of [15] gives the isomorphism

$$C_c^\infty(Y) \widehat{\otimes} C^\infty(X) \simeq C^\infty(X, C_c^\infty(Y)).$$

Thus \tilde{u} has a unique continuous extension into a linear map $C^\infty(X, C_c^\infty(Y)) \rightarrow C^\infty(X)$, which is automatically $C^\infty(X)$ linear, and this defines $E(u)$. The map E is clearly injective and the formulas

$$E(u)(f)(x) = u(f(x, \cdot))(x), \quad \forall x \in X \text{ and } f \in C^\infty(X, C_c^\infty(Y)),$$

$$\partial_{x_j} E(u)(f)(x) = u(\partial_{x_j} f(x, \cdot))(x) + \partial_{x_j} u(f(x', \cdot))(x), \quad x' = x, \quad j = 1, \dots, n_X,$$

hold.

Conversely, any $U \in \mathcal{L}_{C^\infty(X)}(C^\infty(X, C_c^\infty(Y)), C^\infty(X))$, being characterized by its restriction to $C_c^\infty(Y) \otimes 1$, defines a unique $u \in \mathcal{L}(C_c^\infty(Y), C^\infty(X))$ such that $u(h) = U(h \otimes 1)$ for any $h \in C_c^\infty(Y)$, which yields easily that $E(u) = U$. Hence E is bijective.

Let us prove the continuity of E . Let K be a compact subset of X and k be an integer. Let us note p the semi-norm on $C^\infty(X)$ defined by $p(g) = \sum_{|\alpha| \leq k} \sup_{x \in K} |\partial^\alpha f(x)|$. Let \mathcal{B} be a bounded subset of $C^\infty(X, C_c^\infty(Y))$ and $p_{\mathcal{B}}$ the semi-norm on $\mathcal{L}(C^\infty(X, C_c^\infty(Y)), C^\infty(X))$ defined by

$$p_{\mathcal{B}}(U) = \sup_{f \in \mathcal{B}} p(U(f)).$$

We recall that the topology of bounded convergence is precisely the one given by this family of semi-norms. Observe also that the set

$$\mathcal{B}_{k,K} = \{\partial_x^\alpha f(x, \cdot) \in C_c^\infty(Y) ; x \in K, f \in \mathcal{B}, |\alpha| \leq k\}$$

is a bounded subset of $C_c^\infty(Y)$. We get

$$\begin{aligned} p_{\mathcal{B}}(E(u)) &= \sup_{f \in \mathcal{B}} p(E(u)(f)) \\ &= \sup_{f \in \mathcal{B}} \sum_{|\alpha| \leq k} \sup_{x \in K} |\partial_x^\alpha (u(f(x, \cdot)))(x)| \\ &\leq C \sup_{f \in \mathcal{B}} \sum_{|\alpha| \leq k} \sum_{\beta + \gamma = \alpha} \sup_{x \in K} |\partial_x^\beta (u(\partial_x^\gamma f(x, \cdot)))(x)| \\ &\leq C \sum_{|\alpha| \leq k} \sup_{h \in \mathcal{B}_{k,K}} |\partial_x^\alpha u(h)(x)| = C \cdot q(u) \end{aligned}$$

and q so defined is a continuous semi-norm of $\mathcal{L}(C_c^\infty(Y), C^\infty(X))$. This proves that E is continuous and the continuity of the inverse map is easy since $E^{-1}(U) = U|_{1 \otimes C_c^\infty(Y)}$.

Now, Let $u \in \mathcal{D}'(X \times Y)$ and assume that $u_f \in C^\infty(X)$ for any $f \in C^\infty(X, C_c^\infty(Y))$. Since $P_u(h) = u_{h \otimes 1} \in C^\infty(X)$ for any $h \in C_c^\infty(Y)$, we get $u \in \mathcal{D}'_{p_1}(X \times Y)$.

Conversely, assume that $u \in \mathcal{D}'_{p_1}(X \times Y)$. The equality of distributions

$$E(P_u)(f) = u_f$$

holds for any $f \in C^\infty(X, C_c^\infty(Y))$, as these are continuous and coincide on the dense subset $C_c^\infty(Y) \otimes C^\infty(X)$. This proves both that u_f is C^∞ and that $\tilde{P} = E \circ P$. \square

In summary, we have exhibited several canonical isomorphisms of topological vector spaces

$$(5) \quad \mathcal{D}'_{p_1}(X \times Y) \simeq C^\infty(X, \mathcal{D}'(Y)) \simeq \mathcal{L}(C_c^\infty(Y), C^\infty(X)) \simeq \mathcal{L}_{C^\infty(X)}(C^\infty(X, C_c^\infty(Y)), C^\infty(X)).$$

We now replace $p_1 : X \times Y \rightarrow X$ by a surjective submersion $\pi : M \rightarrow B$. We will use bundles of densities in order to obtain canonical statements. If M is a manifold, we note Ω_M^α or $\Omega^\alpha(M)$ the bundle of α -densities over M and if $\pi : M \rightarrow B$ is a submersion, we note $\Omega_\pi^\alpha = \Omega^\alpha(\ker d\pi)$ the vector bundle over M of α densities along the fibers of π . In the following statements, the base manifold B is assumed to be compact for simplicity, but all of them extend easily to the non compact case when we replace $C_c^\infty(M)$ by

$$(6) \quad C_{c\pi}^\infty(M) = \{f \in C^\infty(M) ; \forall g \in C_c^\infty(B), f \cdot g \circ \pi \in C_c^\infty(M)\}$$

or by its variant for C^∞ sections of vector bundles as well. We now introduce the spaces generalizing the ones appearing in (5) and show how to relate them. Recall that if E is a fibre bundle on a manifold M , the space $\mathcal{D}'(M, E)$ is the topological dual of the space $C_c^\infty(M, \Omega_M \otimes E^*)$.

Definition 2. ([1] *Androulidakis-Skandalis*). Let $\pi : M \longrightarrow B$ be a C^∞ submersion onto B , E any C^∞ vector bundle over M and $u \in \mathcal{D}'(M, E)$. For any $f \in C_c^\infty(M, \Omega_M \otimes E^*)$, we denote by u_f the distribution in $\mathcal{D}'(B, \Omega_B)$ defined by

$$\langle u_f, g \rangle_B = \langle u, f \cdot g \circ \pi \rangle_M, \quad g \in C^\infty(B).$$

We say that u is transversal to π if u_f is C^∞ for any f . We denote by $\mathcal{D}'_\pi(M, E)$ the space of π -transversal distributions.

Typical choices for E will be Ω_M^α or $\Omega_\pi^\alpha \otimes \pi^* \Omega_B^\beta$.

Definition 3. Let $\pi : M \longrightarrow B$ be a C^∞ submersion onto B , let E be a C^∞ vector bundle over M and note E_x the restriction of E onto the fiber $\pi^{-1}(x)$. A family $u = (u_x)_{x \in B}$ of distributions $u_x \in \mathcal{D}'(\pi^{-1}(x), E_x)$ is said C^∞ if for any local trivialization of π

$$U \subset M, \quad X \subset B, \quad \kappa : U \xrightarrow{\simeq} X \times Y, \quad \pi|_U = \pi_X \circ \kappa$$

and any local trivialization of E , we have

$$\kappa_*(u|_U) \in C^\infty(X, \mathcal{D}'(Y, \mathbb{R}^{\text{rk}(E)})).$$

The spaces of such C^∞ families are noted $C_\pi^\infty(B, \mathcal{D}'(M, E))$.

Using a covering of M by local trivializations and a partition of unity argument, we can use the seminorms of the spaces $C^\infty(X, \mathcal{D}'(Y))$ to build a complete Hausdorff locally convex vector space structure on $C_\pi^\infty(B, \mathcal{D}'(M, E))$.

The third type of spaces of interest are the spaces of continuous $C^\infty(B)$ -linear maps from $C_c^\infty(M, \Omega_M \otimes E^*)$ to $C^\infty(B, \Omega_B)$

$$(7) \quad \mathcal{L}_{C^\infty(B)}(C_c^\infty(M, \Omega_M \otimes E^*), C^\infty(B, \Omega_B)),$$

which is considered as a topological subspace of $\mathcal{L}(C_c^\infty(M, \Omega_M \otimes E^*), C^\infty(B, \Omega_B))$.

Propositions 1 and 2 have an immediate generalization to the submersion case.

Proposition 3. *The maps*

$$(8) \quad \begin{array}{ccc} I : C_\pi^\infty(B, \mathcal{D}'(M, E)) & \longrightarrow & \mathcal{D}'_\pi(M, E) \\ u & \longmapsto & (f \mapsto \langle I(u), f \rangle = \int_B \langle u(x), f_x \rangle) \end{array}$$

and

$$(9) \quad \begin{array}{ccc} P : \mathcal{D}'_\pi(M, E) & \longrightarrow & \mathcal{L}_{C^\infty(B)}(C_c^\infty(M, \Omega_M \otimes E^*), C^\infty(B, \Omega_B)) \\ u & \longmapsto & (f \longmapsto \pi_*(u \cdot f)) \end{array}$$

are isomorphisms. The topologies coincide in the sense that if $\mathcal{D}'_\pi(M, E)$ is provided with the topology turning I into a topological isomorphism, then P is a topological isomorphism.

To be concrete, let us give the inverse of I and P .

$$(10) \quad \langle I^{-1}(v)(x), f \rangle = P(v)(\tilde{f})(x);$$

where $v \in \mathcal{D}'_\pi(M, E)$, $x \in B$, $f \in C_c^\infty(\pi^{-1}(x), (\Omega_M \otimes E^*)_x)$ and $\tilde{f} \in C_c^\infty(M, \Omega_M \otimes E^*)$ is any C^∞ extension of f .

$$(11) \quad \langle P^{-1}(u), f \rangle = \int_B u(f), \quad f \in C_c^\infty(M, \Omega_M \otimes E^*).$$

Remark 4. Sometimes, we will need to also consider a vector bundle F over B . In that cases it is natural to consider $\mathcal{D}'_\pi(M, E)$ as a subspace of $\mathcal{D}'_\pi(M, E \otimes \text{End}(\pi^*F))$ for which we have canonical isomorphisms

$$(12) \quad P_F : \mathcal{D}'_\pi(M, E \otimes \text{End}(\pi^*F)) \longrightarrow \mathcal{L}_{C^\infty(B)}(C_c^\infty(M, \Omega_M \otimes E^* \otimes \pi^*F), C^\infty(B, \Omega_B \otimes F))$$

and

$$(13) \quad I_F : C^\infty(B, \mathcal{D}'_\pi(M, E \otimes \text{End}(\pi^*F))) \longrightarrow \mathcal{D}'_\pi(M, E \otimes \text{End}(\pi^*F)).$$

We now list several operations available on transversal distributions.

Proposition 5. Let $\pi : M \rightarrow B$ a surjective submersion and $C \subset B$ a submanifold. Let E be a C^∞ vector bundle over M and E_C its restriction onto C . We denote $\pi_C : M_C = \pi^{-1}(C) \rightarrow C$ the submersion obtained by restriction. The restriction of distributions

$$\text{Res}_C : \mathcal{D}'_\pi(M, E) \longrightarrow \mathcal{D}'_{\pi_C}(M_C, E_C)$$

is well defined and continuous.

This is obvious.

Proposition 6. Let $\rho : Z \rightarrow M$ and $\pi : M \rightarrow B$ be surjective submersions. Let E be a C^∞ vector bundle over M . The pull back of distributions restricts to a continuous map

$$\rho^* : \mathcal{D}'_\pi(M, E) \longrightarrow \mathcal{D}'_{\pi \circ \rho}(Z, \rho^*E).$$

Proof. Since ρ is a submersion, the map

$$\begin{aligned} \rho_* : C_c^\infty(Z, \Omega_Z \otimes \rho^*E^*) &\longrightarrow C_c^\infty(M, \Omega_M \otimes E^*) \\ f &\longmapsto (m \mapsto \int_{\rho^{-1}(m)} f) \end{aligned}$$

is well defined and general theorems of basic calculus show that it is continuous. Thus ρ^* identifies with the map

$$\begin{aligned} \rho^* : \mathcal{L}_{C^\infty(B)}(C^\infty(M, \Omega_M \otimes E^*), C^\infty(B, \Omega_B)) &\longrightarrow \mathcal{L}_{C^\infty(B)}(C^\infty(Z, \Omega_Z \otimes \rho^*E^*), C^\infty(B, \Omega_B)) \\ U &\longmapsto \rho^*(U) : f \mapsto U(\rho_*(f)) \end{aligned}$$

which is clearly continuous. □

When two submersions $\pi_i : M_i \rightarrow B$, $i = 1, 2$ are given over the same base, we note

$$M_1 \times_{\pi} M_2 = \{(m_1, m_2) \in M_1 \times M_2 ; \pi_1(m_1) = \pi_2(m_2)\}$$

and

$$\pi : M_1 \times_{\pi} M_2 \longrightarrow B, \quad (m_1, m_2) \longmapsto \pi_1(m_1)$$

Also, the natural projections $M_1 \times_{\pi} M_2 \rightarrow M_i$, $i = 1, 2$ are denoted by pr_i .

Proposition 7. Let

$$(14) \quad \begin{array}{ccc} M & \xrightarrow{p_1} & M_1 \\ \downarrow p_2 & & \downarrow \pi_1 \\ M_2 & \xrightarrow{\pi_2} & B \end{array}$$

be a commutative square of surjective submersions satisfying the condition, at any point of M

$$\ker d\phi = \ker dp_1 + \ker dp_2$$

where we have denoted $\phi = \pi_2 \circ p_2 = \pi_1 \circ p_1$. Let E_1 be a C^∞ vector bundle over M_1 . Then the pull-back of distributions $p_1^* : \mathcal{D}'(M_1, E_1) \longrightarrow \mathcal{D}'(M, p_1^*E_1)$ restricts to a continuous map

$$(15) \quad p_1^* : \mathcal{D}'_{\pi_1}(M_1, E_1) \longrightarrow \mathcal{D}'_{p_2}(M, p_1^*E_1).$$

Proof. It is no restriction to omit the bundle E_1 in the proof. Consider first the particular case where

$$M = M_1 \times_{\pi} M_2 = \{(m_1, m_2) \in M_1 \times M_2 ; \pi_1(m_1) = \pi_2(m_2)\}.$$

that is, we consider the diagram

$$(16) \quad \begin{array}{ccc} M_1 \times M_2 & \xrightarrow{\text{pr}_1} & M_1 \\ \pi \downarrow & & \downarrow \pi_1 \\ M_2 & \xrightarrow{\pi_2} & B \end{array}$$

Let $u \in \mathcal{D}'_{\pi_1}(M_1)$ and set, for any $f \in C_c^\infty(M, \Omega_M)$ a simple computation shows the equality of distributions

$$\text{pr}_{2*}(\text{pr}_1^*(u)f)(m_2) = \int_{M_1 \pi_2(m_2)} u(\pi_2(m_2))(m_1) f(m_1, m_2) \in \mathcal{D}'(M_2, \Omega_{M_2})$$

and that the right hand side is C^∞ in $m_2 \in M_2$. Thus $\text{pr}_1^*(u) \in \mathcal{D}'_{\text{pr}_2}(M)$. Using local trivializations and reusing the trick employed for the continuity of E in the proof of Proposition 2, we get that for any bounded subset \mathcal{B} of $C_c^\infty(M)$ and any semi-norm p of $C^\infty(M_2)$, there exists a bounded subset \mathcal{C} of $C^\infty(M_1)$ and a semi-norm q of $C^\infty(B)$ such that, with similar notations, $p_{\mathcal{B}}(\text{pr}_1^*u) \leq q_{\mathcal{C}}(u)$, which proves the desired continuity, via the identifications between distributions and continuous linear maps

$$\mathcal{D}'_{\pi_1}(M_1) \simeq \mathcal{L}_{C^\infty(B)}(C^\infty(M_1, \Omega_{M_1}), C^\infty(B, \Omega_B)) \xrightarrow{\text{pr}_1^*u} \mathcal{L}_{C^\infty(M_2)}(C^\infty(M, \Omega_M), C^\infty(M_2, \Omega_{M_2})) \simeq \mathcal{D}'_{\text{pr}_2}(M).$$

For the general case, observe first that $\ker d\phi = \ker dp_1 + \ker dp_2$ gives

$$\ker d\pi_1 = dp_1(\ker d\phi) = dp_1(\ker dp_1 + \ker dp_2) = dp_1(\ker dp_2).$$

In other words, $p_1 : p_2^{-1}(m_2) \rightarrow \pi_1^{-1}(b)$, $b = \pi_2(m_2)$ is a submersion. It follows that we can trivialize locally the commutative square into the diagram

$$(17) \quad \begin{array}{ccc} Z \times X_1 \times X_2 \times Y & \longrightarrow & X_1 \times Y \\ \downarrow & & \downarrow \\ X_2 \times Y & \longrightarrow & Y \end{array}$$

and it proves that the map

$$p : M \longrightarrow M_1 \times_{\pi} M_2, \quad m \longmapsto (p_1(m), p_2(m))$$

is again a submersion. Then, using the particular case and Proposition 6 we get the continuity of the map

$$p_1^* = p^* \circ \text{pr}_1^* : \mathcal{D}'_{\pi_1}(M_1) \longrightarrow \mathcal{D}'_{\text{pr}_2}(M_1 \times_{\pi} M_2) \longrightarrow \mathcal{D}'_{p_2}(M).$$

□

Consider again the commutative square (16), let E_1, E_2 be C^∞ vector bundles over M_1, M_2 respectively and denote

$$M = M_1 \times_{\pi} M_2 \quad ; \quad E = \text{pr}_1^*E_1 \otimes \text{pr}_2^*E_2|_M.$$

Given $u_1 \in \mathcal{D}'_{\pi_1}(M_1, E_1)$ and $u_2 \in \mathcal{D}'(M_2, E_2)$, we get a distribution $u_1 \times_{\pi_1} u_2 \in \mathcal{D}'(M, E)$ by the formula

$$(18) \quad u_1 \times_{\pi_1} u_2 = u_2 \circ \text{pr}_1^*(u_1) \in \mathcal{L}(C_c^\infty(M, \Omega_M \otimes E^*), \mathbb{C}) = \mathcal{D}'(M, E).$$

The composition is licit, for we have applied remark 4 to u_1 with $F = E_2^*$. To be concrete, this is the distribution given by

$$(19) \quad \langle u_1 \times_{\pi_1} u_2, f \rangle = \int_{M_2} \left(\int_{M_1 \pi_2(m_2)} u_1 \pi_2(m_2)(m_1) f(m_1, m_2) \right) u_2(m_2)$$

where the integrals are taken in the distributional sense. Since $u_1 \rightarrow \text{pr}_1^*(u_1)$ is continuous and since the composition of continuous linear maps is separately continuous, we have proved

Corollary 8. *With the notations above, the map*

$$(20) \quad \begin{array}{ccc} \mathcal{D}'_{\pi_1}(M_1, E_1) \times \mathcal{D}'(M_2, E_2) & \longrightarrow & \mathcal{D}'(M, E) \\ (u_1, u_2) & \longmapsto & u_1 \times_{\pi_1} u_2 \end{array}$$

is well defined, separately continuous and extends the natural map

$$C^\infty(M_1, E_1) \times C^\infty(M_2, E_2) \rightarrow C^\infty(M, E).$$

There is an analogous statement using π_2 -transversality on u_2 . When a finite set \mathcal{I} of submersions is given on M , we write

$$(21) \quad \mathcal{D}'_{\mathcal{I}}(M, E) = \bigcap_{\rho \in \mathcal{I}} \mathcal{D}'_{\rho}(M, E) \subset \mathcal{D}'(M, E).$$

The space $\mathcal{D}'_{\mathcal{I}}(M, E)$ is given the topology generated by the union of the induced topologies by each $\mathcal{D}'_{\rho}(M, E)$, $\rho \in \mathcal{I}$.

Proposition 9. *We keep the notation of corollary 8. For any $u_1 \in \mathcal{D}'_{\pi_1}(M_1, E_1)$ and $u_2 \in \mathcal{D}'_{\pi_2}(M_2, E_2)$, the equality*

$$(22) \quad u_1 \times_{\pi_1} u_2 = u_1 \times_{\pi_2} u_2$$

holds and the distribution is transversal to $\pi : M = M_1 \times_{\pi} M_2 \rightarrow B, (m_1, m_2) \mapsto \pi_1(m_1)$. The map

$$(23) \quad \begin{array}{ccc} \mathcal{D}'_{\pi_1}(M_1, E_1) \times \mathcal{D}'_{\pi_2}(M_2, E_2) & \longrightarrow & \mathcal{D}'_{\pi}(M, E) \\ (u_1, u_2) & \longmapsto & u_1 \times_{\pi_1} u_2 \end{array}$$

is separately continuous. Moreover, if supplementary submersions $\rho : M_1 \rightarrow A$ and $\sigma : M_2 \rightarrow C$ are given, then the previous map restricts to a separately continuous map

$$(24) \quad \begin{array}{ccc} \mathcal{D}'_{\rho, \pi_1}(M_1, E_1) \times \mathcal{D}'_{\pi_2, \sigma}(M_2, E_2) & \longrightarrow & \mathcal{D}'_{\tilde{\rho}, \pi, \tilde{\sigma}}(M, E) \\ (u_1, u_2) & \longmapsto & u_1 \times_{\pi_1} u_2. \end{array}$$

Here we have denoted $\tilde{\rho} = \rho \circ \text{pr}_1$ and $\tilde{\sigma} = \sigma \circ \text{pr}_2$.

Proof. One can check the equality of linear maps $u_2 \circ \text{pr}_1^*(u_1) = u_1 \circ \text{pr}_2^*(u_2)$ and then (18) gives (22). Actually, one can also write this in terms of distributions, that is

$$\begin{aligned}
(25) \quad \langle u_1 \times_{\pi_1} u_2, f \rangle &= \int_{M_2} \int_{M_{1\pi_2(m_2)}} u_{1\pi_2(m_2)}(m_1) f(m_1, m_2) u_2(m_2) \\
&= \int_{y \in B} \int_{M_{2y}} \int_{M_{1y}} u_{1y}(m_1) f(m_1, m_2) u_{2y}(m_2) \\
&= \int_{y \in B} \int_{M_{1y}} \int_{M_{2y}} u_{1y}(m_1) f(m_1, m_2) u_{2y}(m_2) \\
&= \int_{M_1} u_1(m_1) \int_{M_{2\pi_1(m_1)}} f(m_1, m_2) u_{2\pi_1(m_1)}(m_2) \\
&= \langle u_1 \times_{\pi_2} u_2, f \rangle.
\end{aligned}$$

We have noted $M_{iy} = \pi_i^{-1}(y)$, $u_{iy} = I^{-1}(u_i)(y) \in \mathcal{D}'(M_{iy})$, $i = 1, 2$. Taking into account both transversalities of u_1 and u_2 we get instead of (18)

$$u_1 \times_{\pi_1} u_2 = u_2 \circ \text{pr}_1^*(u_1) \in \mathcal{L}_{C^\infty(B)}(C_c^\infty(M, \Omega_M \otimes E^*), C^\infty(B, \Omega_B)).$$

This proves that $u_1 \times_{\pi_1} u_2 \in \mathcal{D}'_\pi(M, E)$ and this justifies the separate continuity in u_1 and u_2 of the map (23). Note that as a π -transversal distribution, $u_1 \times_{\pi_1} u_2$ is simply given by the family $(u_{1y} \otimes u_{2y})_{y \in B}$, as it clear from (25).

Next, if u_1 is also ρ -transversal and u_2 is σ -transversal, then we obtain

$$(26) \quad u_1 \times_{\pi_2} u_2 = u_1 \circ \text{pr}_2^*(u_2) \in \mathcal{L}_{C^\infty(A)}(C_c^\infty(M, \Omega_M \otimes E^*), C^\infty(A, \Omega_A)),$$

and

$$(27) \quad u_1 \times_{\pi_1} u_2 = u_2 \circ \text{pr}_1^*(u_1) \in \mathcal{L}_{C^\infty(C)}(C_c^\infty(M, \Omega_M \otimes E^*), C^\infty(C, \Omega_C)),$$

which proves both that $u_1 \times_{\pi_2} u_2$ belongs to $\mathcal{D}'_{\tilde{\rho}, \pi, \tilde{\sigma}}(M, E)$ and that the corresponding map is separately continuous. □

Lemma 10. *Let $f : M \rightarrow N$ be a C^∞ map and $\pi : M \rightarrow B$, $\rho : N \rightarrow B$ be two submersions onto B such that $\rho \circ f = \pi$. Let F be a C^∞ vector bundle over N . Then the push-forward of (compactly supported) distributions :*

$$(28) \quad \langle f_* u, g \rangle = \langle u, g \circ f \rangle, \quad u \in \mathcal{E}'(M, f^*(F) \otimes \Omega_M), \quad g \in C^\infty(N, F^*)$$

restricts to a continuous map

$$(29) \quad f_* : \mathcal{E}'_\pi(M, \Omega_M \otimes f^*(F)) \longrightarrow \mathcal{D}'_\rho(N, \Omega_N \otimes F).$$

Proof. For any $u \in \mathcal{E}'_\pi(M, f^*(F) \otimes \Omega_M)$ and $g \in C_c^\infty(N, F^*)$, we have $f_*(u).g = f_*(u.f^*g)$, hence

$$(30) \quad (f_* u)_g = (\rho)_*(f_* u.g) = (\rho)_* \circ f_*(u.f^*g) = (\pi)_*(u.f^*g) = u_{f^*g}.$$

This proves that $f_* u \in \mathcal{D}'_\rho(N, F)$ and the continuity of the map $u \mapsto f_* u$ as well. □

Remark 11. *If f is a submersion (the case of interest in the next section), then the push-forward map f_* can be reformulated in this way*

$$(31) \quad f_* : \mathcal{E}'_\pi(M, \Omega_f \otimes f^*(F)) \longrightarrow \mathcal{D}'_\rho(N, F).$$

This is essentially the same map, as we can see using the canonical isomorphism $\Omega_M \simeq \Omega_f \otimes f^(\Omega_N)$.*

3. CONVOLUTION OF TRANSVERSAL DISTRIBUTIONS ON GROUPOIDS

We apply these observations in the context of groupoids. We provide some of the details of the construction of the canonical convolution algebra $C_c^\infty(G, \Omega^{1/2})$ [2, 4] associated with any Lie groupoid G . The product of convolution

$$(32) \quad C_c^\infty(G, \Omega^{1/2}) \times C_c^\infty(G, \Omega^{1/2}) \xrightarrow{*} C_c^\infty(G, \Omega^{1/2})$$

is given by the integral

$$(33) \quad f * g(\gamma) = \int_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2), \quad \gamma \in G$$

which is well defined and gives an internal operation as soon as we take

$$(34) \quad \Omega^{1/2} = \Omega^{1/2}(\ker dr) \otimes \Omega^{1/2}(\ker ds) = \Omega^{1/2}(\ker dr \oplus \ker ds).$$

To understand this point, we recall

Lemma 12. [2, 4]. *Denoting by m the multiplication map of G and by $\text{pr}_1, \text{pr}_2 : G \times G \rightarrow G$ the natural projection maps, we have a canonical isomorphism*

$$(35) \quad \text{pr}_1^*(\Omega^{1/2}) \otimes \text{pr}_2^*(\Omega^{1/2})|_{G^{(2)}} \simeq \Omega_m \otimes m^*(\Omega^{1/2}).$$

Proof. We note $G_\gamma^{(2)}$ the fiber of m at γ , that is $G_\gamma^{(2)} = \{(\gamma_1, \gamma_2) \in G^{(2)} ; \gamma_1 \gamma_2 = \gamma\}$.

Now, the restricted map $\text{pr}_1 : G_\gamma^{(2)} \rightarrow G^{r(\gamma)}$ being a diffeomorphism, we have a canonical isomorphism of vector bundles

$$TG_\gamma^{(2)} \xrightarrow{\simeq} \text{pr}_1^*(TG^{r(\gamma)}) = \text{pr}_1^*(\ker dr)|_{G_\gamma^{(2)}}, \quad (\gamma_1, \gamma_2, X_1, X_2) \mapsto (\gamma_1, \gamma_2, X_1).$$

Similarly, $TG_\gamma^{(2)} \simeq \text{pr}_2^*(TG_{s(\gamma)})$. Moreover the map

$$\text{pr}_1^*(\ker ds)|_{G_\gamma^{(2)}} \xrightarrow{\simeq} G_\gamma^{(2)} \times T_\gamma G_{s(\gamma)}, \quad (\gamma_1, \gamma_2, X_1) \mapsto (\gamma_1, \gamma_2, (dR_{\gamma_2})_{\gamma_1}(X_1))$$

provides a canonical trivialisation of the vector bundle $\text{pr}_1^*(\ker ds)|_{G_\gamma^{(2)}}$. The same holds for

$$\text{pr}_2^*(\ker dr)|_{G_\gamma^{(2)}} \simeq G_\gamma^{(2)} \times T_\gamma G^{r(\gamma)}.$$

With these isomorphisms in hand, we get

$$\begin{aligned} \text{pr}_1^*(\Omega^{1/2})|_{G_\gamma^{(2)}} \otimes \text{pr}_2^*(\Omega^{1/2})|_{G_\gamma^{(2)}} &\simeq \Omega^{1/2}(\text{pr}_1^*(\ker dr \oplus \ker ds)|_{G_\gamma^{(2)}} \oplus \text{pr}_2^*(\ker dr \oplus \ker ds)|_{G_\gamma^{(2)}}) \\ &\simeq \Omega^{1/2}(TG_\gamma^{(2)} \oplus T_\gamma G_{s(\gamma)} \oplus T_\gamma G^{r(\gamma)} \oplus TG_\gamma^{(2)}) \\ &\simeq \Omega(TG_\gamma^{(2)}) \otimes \Omega^{1/2}(T_\gamma G_{s(\gamma)} \oplus T_\gamma G^{r(\gamma)}). \end{aligned}$$

This gives the canonical isomorphism (35). □

Since in the basic formula (33) the function under sign of integration

$$G_\gamma^{(2)} \ni (\gamma_1, \gamma_2) \mapsto f(\gamma_1) g(\gamma_2) \in \left(\text{pr}_1^*(\Omega^{1/2}) \otimes \text{pr}_2^*(\Omega^{1/2}) \right)_{(\gamma_1, \gamma_2)}$$

is thus a C^∞ section of the bundle $(\text{pr}_1^*(\Omega^{1/2}) \otimes \text{pr}_2^*(\Omega^{1/2}))|_{G_\gamma^{(2)}}$, the previous lemma shows that (33) is the integral of a one density, canonically associated with f, g , over the manifold $G_\gamma^{(2)} = \{(\gamma_1, \gamma_2), \gamma_1 \gamma_2 = \gamma\}$ and that the result is a C^∞ section of $\Omega^{1/2}$. Further computations on densities show that the statement

$$(36) \quad f * g(\gamma) = \int_{G^{r(\gamma)}} f(\gamma_1) g(\gamma_1^{-1} \gamma) = \int_{G_{s(\gamma)}} f(\gamma \gamma_2^{-1}) g(\gamma_2)$$

makes sense and is true. The involution on $C_c^\infty(G, \Omega^{1/2})$ is also natural in terms of densities

$$f^* = \overline{i^*(f)}, \quad f \in C_c^\infty(G, \Omega^{1/2})$$

where i is the induced vector bundle isomorphism over the inversion map of G

$$\ker dr \oplus \ker ds \longrightarrow \ker dr \oplus \ker ds, \quad (\gamma, X_1, X_2) \longmapsto (\gamma^{-1}, di(X_2), di(X_1)).$$

The spaces $C^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$ and $C_c^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$ are endowed with their usual Fréchet and \mathcal{LF} topological vector space structures and we denote by $\mathcal{E}'(G, \Omega^{1/2})$ and $\mathcal{D}'(G, \Omega^{1/2})$ their topological duals. The choice of densities is made so that we have canonical embeddings

$$C^\infty(G, \Omega^{1/2}) \hookrightarrow \mathcal{D}'(G, \Omega^{1/2}) \text{ and } C_c^\infty(G, \Omega^{1/2}) \hookrightarrow \mathcal{E}'(G, \Omega^{1/2}).$$

We denote by $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ and $\mathcal{D}'_{r,s}(G, \Omega^{1/2})$ the subspaces of distributions which are transversal to both $r : G \rightarrow G^{(0)}$ and $s : G \rightarrow G^{(0)}$. For simplicity, we assume that $G^{(0)}$ is compact.

Theorem 13. *The convolution product on $C_c^\infty(G, \Omega^{1/2})$ extends to separately continuous bilinear maps*

$$(37) \quad \mathcal{E}'_s(G, \Omega^{1/2}) \times \mathcal{E}'(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{E}'(G, \Omega^{1/2})$$

and

$$(38) \quad \mathcal{E}'(G, \Omega^{1/2}) \times \mathcal{E}'_r(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{E}'(G, \Omega^{1/2}),$$

still called convolution product, by the formulas $u * v = m_*(u \times v)$ in the first case and $u * v = m_* \begin{smallmatrix} u \times v \\ r \end{smallmatrix}$ in the second one. Both maps give by restriction a well defined separately continuous bilinear map

$$\mathcal{E}'_{r,s}(G, \Omega^{1/2}) \times \mathcal{E}'_{r,s}(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{E}'_{r,s}(G, \Omega^{1/2})$$

and $(\mathcal{E}'_{r,s}(G, \Omega^{1/2}), *)$ is an associative involutive unital algebra. The unit δ is given by

$$(39) \quad \langle \delta, f \rangle = \int_{G^{(0)}} f, \quad f \in C^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$$

and the involution is given by

$$(40) \quad u^* = \overline{i^*(u)}.$$

Proof. For the first two maps, we apply Corollary 8 to the case $M_1 = M_2 = G$, $B = G^{(0)}$, $\pi_1 = s$ and $\pi_2 = r$ and then Lemma 10 with $M_1 = G^{(2)}$, $M_2 = G$ and $f = m$ the multiplication map of G . The third map is obtained by applying Proposition 9 to the case $M_1 = M_2 = G$, $B = G^{(0)}$, $\pi_1 = s$ and $\pi_2 = r$ and Lemma 10 again. For associativity, first note that for $u, v \in C_c^\infty$, the distribution $u * v$ is given by

$$\langle u * v, f \rangle = \int_{G^{(2)}} u(\gamma_1)v(\gamma_2)f(\gamma_1\gamma_2)$$

and the associativity of $*$ for $C_c^\infty(G, \Omega^{1/2})$ boils down to the application of Fubini's theorem in the integral

$$\langle u * v * w, f \rangle = \int_{G^{(3)}} u(\gamma_1)v(\gamma_2)w(\gamma_3)f(\gamma_1\gamma_2\gamma_3),$$

where $G^{(3)} = \{(\gamma_1, \gamma_2, \gamma_3) \in G^3 ; s(\gamma_1) = r(\gamma_2), s(\gamma_2) = r(\gamma_3)\}$. Using the density of $C_c^\infty(G, \Omega^{1/2})$ in $\mathcal{E}'_s(G, \Omega^{1/2})$, $\mathcal{E}'_r(G, \Omega^{1/2})$ and $\mathcal{E}'(G, \Omega^{1/2})$, and the continuity of the convolution products (37), (38), one obtains the associativity of $*$ on $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$.

We check that the integral defining δ has an intrinsic meaning and gives a unit in $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$. Since $TG|_{G^{(0)}} = TG^{(0)} \oplus \ker ds$, we have $\Omega(G)|_{G^{(0)}} = \Omega(G^{(0)}) \otimes \Omega(\ker ds)$. On the other hand the inversion gives a canonical isomorphism between the bundles $\ker ds|_{G^{(0)}}$ and $\ker dr|_{G^{(0)}}$, thus

$$\Omega(\ker ds) = \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker ds) \simeq \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dr) = \Omega^{1/2}$$

Through these canonical identifications, any $f \in C^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$ gives by restriction at $G^{(0)}$ a one density on $G^{(0)}$, which gives a well defined meaning to $\delta(f)$. Now a direct computation shows that

$$r_*(\delta) = s_*(\delta) = 1 \in C^\infty(G^{(0)}) \subset \mathcal{D}'(G^{(0)}),$$

thus $\delta \in \mathcal{E}'_{r,s}(G, \Omega^{1/2})$.

Let us consider the C^∞ -family δ^x , $x \in G^{(0)}$ coming from the r -transversality of δ . For any test function $\phi \in C_c^\infty(G^x)$ and C^∞ function $\tilde{\phi} \in C_c^\infty(G)$ such that $\tilde{\phi}|_{G^x} = \phi$ we have

$$\langle \delta^x, \phi \rangle = r_*(\delta\tilde{\phi})(x)$$

and obviously $r_*(\delta\tilde{\phi})(x) = \tilde{\phi}(x) = \phi(x)$. It follows that for any $f \in C^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$,

$$\begin{aligned} \langle u * \delta, f \rangle &= \int_{x \in G^{(0)}} \langle u_x \otimes \delta^x, (f \circ m)|_{G_x \times G^x} \rangle \\ &= \int_{x \in G^{(0)}} \langle u_x, f|_{G_x} \rangle \\ &= \langle u, f \rangle \end{aligned}$$

Thus $u * \delta = u$. The proof of the equality $\delta * u = u$ is similar. The assertion about the involution is obvious. \square

In particular, when one of the two factors is in C_c^∞ , the convolution product is defined without any restriction on the other factor. We give a sufficient condition for the result to be C^∞ .

Proposition 14. *The convolution product gives by restriction a map*

$$\mathcal{E}'_r(G, \Omega^{1/2}) \times C_c^\infty(G, \Omega^{1/2}) \xrightarrow{*} C_c^\infty(G, \Omega^{1/2}).$$

*The analogous statement with C^∞ functions on the left and s -transversal distributions on the right also holds. The map $u \mapsto u * \cdot$ sending $\mathcal{E}'_r(G, \Omega^{1/2})$ into the left multipliers of $C_c^\infty(G, \Omega^{1/2})$ is injective.*

Proof. Since $(u^y)_y$ is a C^∞ family, the map

$$(41) \quad \gamma_2 \mapsto \langle u^{r(\gamma_2)}(\cdot), f((\cdot)^{-1}\gamma_2) \rangle$$

is C^∞ and we check from the definition of the convolution product that

$$\langle u * f, \phi \rangle = \int_{\gamma_2 \in G} \langle u^{r(\gamma_2)}(\cdot), f((\cdot)^{-1}\gamma_2) \rangle \phi(\gamma_2).$$

Thus $u * f$ coincides with the C^∞ function

$$(42) \quad u * f(\gamma) = \langle u^{r(\gamma)}(\cdot), f((\cdot)^{-1}\gamma) \rangle.$$

Now, the vanishing of $u * f$ for any f and the previous expression for $u * f$ shows that $u^x = 0$, for any x , and thus $u = 0$. Starting with $u \in \mathcal{E}'_s(G, \Omega^{1/2})$, we get similar conclusions for $\cdot * u$. \square

4. G-OPERATORS

We recall the definition of G -operators given in [12].

Definition 4. A (left) G -operator is a continuous linear map $P : C_c^\infty(G, \Omega^{1/2}) \rightarrow C^\infty(G, \Omega^{1/2})$ such that there exists a family $P_x : C_c^\infty(G_x) \rightarrow C^\infty(G_x)$, $x \in G^{(0)}$ of operators such that

$$(43) \quad P(f)|_{G_x} = P_x(f|_{G_x})$$

and

$$(44) \quad P_{r(\gamma)} \circ R_\gamma = R_\gamma \circ P_{s(\gamma)}$$

for any $x \in G^{(0)}$ and $\gamma \in G$ (right G -operators are defined accordingly). We say that a G -operator is compactly supported when it maps $C_c^\infty(G, \Omega^{1/2})$ to itself. We denote by Op_G the linear space of left G -operators and by $\text{Op}_{G,c}$ the subspace of compactly supported G -operators.

We now give a bijection between $\text{Op}_{G,c}$ and $\mathcal{E}'_s(G, \Omega^{1/2})$. We know that i^* is an isomorphism between $\mathcal{E}'_s(G, \Omega^{1/2})$ and $\mathcal{E}'_r(G, \Omega^{1/2})$. If $v \in \mathcal{E}'_s(G, \Omega^{1/2})$, we are going to show that the convolution on the left by $u = i^*(v) \in \mathcal{E}'_r(G, \Omega^{1/2})$ defines a compactly supported G -operator. First, observe that Proposition 14 shows that $P(u) : f \mapsto u * f$ is a linear continuous map from $C_c^\infty(G, \Omega^{1/2})$ to itself. Moreover, the formula (42) shows that the restriction of $u * f$ to G_x depends only on G_x , hence, for any x we get an operator $P(u)_x$ mapping $C_c^\infty(G_x)$ to itself. For the equivariance, let $\gamma \in G_x^y$, then

$$[P(u)_y(f(\cdot\gamma))](\gamma') = u * (R_\gamma f)(\gamma') = [R_\gamma(u * f)](\gamma') = [P(u)_x f](\gamma'\gamma).$$

This shows that the left convolution by u defines a left compactly supported G -operator $P(u)$. Let us note that the Schwartz kernel $k_{u,x}$ of $P(u)_x$ on $G_x \times G_x$ can be recovered as follows. For any $x \in G^{(0)}$, let us denote by $O(x)$ the orbit of x in $G^{(0)}$, that is, the submanifold of $G^{(0)}$ defined by $O(x) = r(s^{-1}(x))$. We note $G(x) = r^{-1}(O(x))$ and j_x the map

$$j_x : G_x \times G_x \rightarrow G(x), \quad (\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2^{-1}.$$

This map is a surjective submersion, hence Proposition 5 gives a distribution $\tilde{u} = \text{Res}_{O(x)}(u)$ in $\mathcal{E}'_r(G(x), \Omega_{G(x)}^{1/2})$. Next, use Proposition 6 to pull-back the distribution \tilde{u} to $j_x^*(\tilde{u})$ as a distribution on $G_x \times G_x$. This gives

$$(45) \quad j_x^*(\tilde{u}) = k_{u,x} \in \mathcal{D}'(G_x \times G_x).$$

Note that the pull-back of the distribution u can be done globally using the submersion

$$j : G \times_s G \rightarrow G, \quad (\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2^{-1},$$

where we have denoted $G \times_s G = \{(\gamma_1, \gamma_2) \in G^2 ; s(\gamma_1) = s(\gamma_2)\}$. On the other hand, if $K_u \in \mathcal{D}'(G \times G)$ denotes the Schwartz kernel of P_u , we obtain using (43) a well defined distribution $k_u \in \mathcal{D}'(G \times_s G)$ by the formula

$$(46) \quad \langle k_u, f \rangle = \langle K_u, \tilde{f} \rangle, \quad \forall f \in C_c^\infty(G \times_s G), \tilde{f} \in C_c^\infty(G \times G), \tilde{f}|_{G \times_s G} = f,$$

and it follows that

$$(47) \quad j^*(u) = k_u \in \mathcal{D}'(G \times_s G).$$

Conversely, let P be a left G -operator with compact support. Its Schwartz kernel K belongs to $\mathcal{E}'_{\text{pr}_1}(G^2)$ and as above, there exists $k \in \mathcal{E}'_{s\text{opr}_1}(G \times_s G)$ such that

$$\langle K, f \rangle = \langle k, f|_{G \times_s G} \rangle, \quad f \in C_c^\infty(G^2).$$

The distribution k is given by the C^∞ family $x \mapsto k_x \in \mathcal{E}'(G_x \times G_x)$ where k_x is the Schwartz kernel of P_x . Still by assumption, each k_x belongs to $\mathcal{E}'_{\text{pr}_1}(G_x \times G_x)$ and we have, for any $f \in C_c^\infty(G, \Omega^{1/2})$, that the map

$$\gamma_1 \longmapsto \langle K(\gamma_1, \cdot), f(\cdot) \rangle = \langle k_{s(\gamma_1)}(\gamma_1, \cdot), f|_{G_{s(\gamma_1)}}(\cdot) \rangle$$

is C^∞ . This proves that the map

$$G^{(0)} \ni x \mapsto k_P(x)(\cdot) = k_x(x, \cdot) \in \mathcal{E}'(G_x)$$

is C^∞ and we obtain a s -transversal distribution $k_P \in \mathcal{E}'_s(G)$.

Moreover, for any $f \in C_c^\infty(G, \Omega^{1/2})$, $x, y \in G^{(0)}$ and $\gamma \in G_x^y$, we have

$$\begin{aligned} (48) \quad P(f)(\gamma) &= P_x(f)(\gamma) = P_x(f)(y \cdot \gamma) \\ &= P_y(R_\gamma f)(y) = \langle k_P(y), f(\cdot \gamma) \rangle_{G_y} \\ &= \langle i^* k_P(y), f((\cdot)^{-1} \gamma) \rangle_{G^y} \\ (49) \quad &= (i^* k_P) * f(\gamma). \end{aligned}$$

This proves that the action of P on $C_c^\infty(G, \Omega^{1/2})$ is given by convolution on the left by the r -transversal distribution $i^* k_P$. The distribution k_P is called *the convolution distributional kernel* of P . We have proved

Theorem 15. *The map $P \mapsto k_P$ gives a bijection between the space $\text{Op}_{G,c}$ of compactly supported left G -operators and $\mathcal{E}'_s(G, \Omega^{1/2})$.*

We also consider the notion of compactly supported adjointable left G -operators. For that purpose, consider the $C_c^\infty(G, \Omega^{1/2})$ -valued pre-hilbertian product defined by

$$(50) \quad (f|g) = f^* * g; \quad f, g \in C_c^\infty(G, \Omega^{1/2}).$$

Then a left G -operator P is said *adjointable* if there exists a left G -operator Q such that

$$(51) \quad (P(f)|g) = (f|Q(g)); \quad f, g \in C_c^\infty(G, \Omega^{1/2}).$$

If an adjoint of P exists, then it is unique (since $(u * f)^* * g = f^* * u^* * g = 0$ for any f, g implies $u = 0$) and it is then denoted by P^* . If $k_P \in \mathcal{E}'_{r,s}(G, \Omega^{1/2})$ then P is obviously adjointable and $k_{P^*} = (k_P)^*$. Conversely, if P as an adjoint Q then

$$(52) \quad (i^* k_P * f)^* * g = (f^* * \overline{k_P}) * g = f^* * (i^* k_Q * g); \quad f, g \in C_c^\infty(G, \Omega^{1/2}),$$

hence $\overline{k_P} = i^* k_Q \in \mathcal{E}'_s(G, \Omega^{1/2}) \cap \mathcal{E}'_r(G, \Omega^{1/2})$. This proves that compactly supported adjointable left G -operators are those with convolution distributional kernel belonging to $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$.

There is another way to characterize adjointable G -operators. Note that for G -operators with $k_P \in \mathcal{E}'_{r,s}(G, \Omega^{1/2})$, it follows from Theorem 13 that P can be extended into a continuous linear operator $\mathcal{E}'(G, \Omega^{1/2}) \longrightarrow \mathcal{E}'(G, \Omega^{1/2})$ which leaves stable the subspace $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$, by using the left convolution with $i^* k_P$. We are going to check that the converse is true and we conclude

Proposition 16. *A compactly supported left G -operator P is adjointable if and only if it has a continuous extension to a linear map*

$$\tilde{P} : \mathcal{E}'(G, \Omega^{1/2}) \longrightarrow \mathcal{E}'(G, \Omega^{1/2})$$

such that $\tilde{P}(\mathcal{E}'_{r,s}(G, \Omega^{1/2})) \subset \mathcal{E}'_{r,s}(G, \Omega^{1/2})$.

In that case, \tilde{P} is given by left convolution with $i^*k_P = \tilde{P}(\delta) \in \mathcal{E}'_{r,s}(G, \Omega^{1/2})$.

Proof. For any $u \in \mathcal{E}'(G, \Omega^{1/2})$ and $f \in C_c^\infty(G, \Omega^{1/2})$ we have, using a sequence $(u_n) \subset C_c^\infty(G, \Omega^{1/2})$ converging to u in \mathcal{E}'

$$\tilde{P}(u * f) = \lim P(u_n * f) = \lim P(u_n) * f = \tilde{P}(u) * f.$$

Thus \tilde{P} is automatically $C_c^\infty(G, \Omega^{1/2})$ -right linear. For any $u \in \mathcal{E}'_{r,s}(G, \Omega^{1/2})$, $v \in \mathcal{E}'(G, \Omega^{1/2})$ and using a sequence $(v_n) \subset C_c^\infty(G, \Omega^{1/2})$ converging to v in \mathcal{E}' , we then have

$$\tilde{P}(u * v) = \lim \tilde{P}(u * v_n) = \lim \tilde{P}(u) * v_n = \tilde{P}(u) * v.$$

Now we can write the equalities

$$i^*k_P * f = P(f) = P(\delta * f) = \tilde{P}(\delta) * f$$

for any $f \in C_c^\infty(G, \Omega^{1/2})$, which proves that $i^*k_P = \tilde{P}(\delta) \in \mathcal{E}'_{r,s}(G, \Omega^{1/2})$ and the remaining assertion is obvious. \square

5. WAVE FRONT SETS AND TRANSVERSALITY

We now turn to some microlocal aspects of distributions on groupoids. We are going to compare transversality conditions for distributions with transversality conditions for their wave front sets. We begin with the case of submersions.

Given a submersion $\pi : M \longrightarrow B$, a sufficient condition of π -transversality is provided by the behavior of the wave front set. We denote

$$(53) \quad \mathcal{M}(\pi) = \ker d\pi^\perp = \{(x, \xi) \in T^*M, ; \langle \xi, t \rangle = 0, \forall t \in \ker d\pi_x\}.$$

Observe that $\mathcal{M}(\pi)$ is the union of the conormal bundles of the fibers of π . If $F \xrightarrow{\iota} M$ is an immersion, we denote by $N^*(\iota)$ or $N^*(F)$ the conormal bundle of F in M , which is defined by $N^*(F) = \{(m, \xi) \in T^*M \mid m \in F; \langle \xi, t \rangle = 0, \forall t \in T_m F\}$. Then

$$(54) \quad \mathcal{M}(\pi) = \bigcup_{b \in B} N^*(\pi^{-1}(b)).$$

Proposition 17. *Let $\pi : M \longrightarrow B$ be a C^∞ submersion onto B and $W \subset T^*M \setminus 0$ such that $W \cap \mathcal{M}(\pi) = \emptyset$. Then*

$$\mathcal{D}'_W(M, E) \subset \mathcal{D}'_\pi(M, E).$$

Proof. We apply the formula (3.6) p. 328 of [5] to $u_f = \pi_*(u.f) :$

$$\text{WF}(\pi_*(u.f)) \subset (d\pi)_*(\text{WF}(u.f)) \subset (d\pi)_*(\text{WF}(u)) = \{(x, \xi) ; x = \pi(m), (m, {}^t d\pi_m(\xi)) \in \text{WF}(u)\}$$

Since $\mathcal{M}(\pi) = \{(m, \zeta) ; \zeta \in \text{Im}({}^t d\pi_m)\}$, we obtain $\text{WF}(\pi_*(u.f)) = \emptyset$, that is, u_f is C^∞ . \square

Remark 18. The reverse inclusion does not hold. It is possible to find a distribution in \mathcal{D}'_π the wave front set of which does not avoid $\mathcal{M}(\pi)$. To see this, we suppose $M = \mathbb{R}^p \times \mathbb{R}^q$ and $B = \mathbb{R}^p$, with π equal to the first projection. Let us consider the distribution u on M , with Fourier transform equal to:

$$\widehat{u}(\xi, \eta) = v(\xi, \eta) = \chi(\eta)a(\xi, \eta).$$

Here a is positively homogeneous of degree zero and smooth away from the origin, and χ is a smooth function on \mathbb{R}^q such that $\chi(\eta) = 0$ for $\|\eta\| \leq 1$ and $\chi(\eta) = 1$ for $\|\eta\| \geq 2$. Hence v is well-defined as a smooth function on M . The function v defines a tempered distribution, hence $u = \mathcal{F}^{-1}(v)$ is also tempered. We see that the wave front set $WF(u)$ contains the nonempty subset $\{(0, 0; \xi, 0), \xi \in \mathbb{R}^p\}$ of $\mathcal{M}(\pi) = \{(x, y; \xi, \eta) \in T^*M, \eta = 0\}$. For example, one can choose $a(\xi, \eta) = \exp(-\frac{\|\xi\|^2}{\|\eta\|^2})$, which yields the estimates:

$$(55) \quad |\widehat{u}(\xi, \eta)| \leq C_N \sqrt{1 + \|\xi\|^2 / \|\eta\|^2}^{-N} = C_N \|\eta\|^N \|(\xi, \eta)\|^{-N}, \quad N = 1, 2, 3, \dots$$

We shall suppose from now on that the estimates (55) are verified, and use the shorthand notation $\Lambda(-) := \sqrt{1 + \|\cdot\|^2}$. To prove $u \in \mathcal{D}'_\pi$ we have to show, in view of Definition 2, that the push-forward $\pi_*(fu)$ is a C^∞ function on B for any $f \in C_c^\infty(M)$. This will result from the following computation with a test function $\varphi \in C_c^\infty(B)$:

$$\begin{aligned} \langle \pi_*(fu), \varphi \rangle &= \langle fu, \varphi \circ \pi \rangle = \langle \widehat{fu}, \widehat{\varphi \circ \pi} \rangle \\ &= \iint (\widehat{f} * \widehat{u})(\xi, \eta) \widehat{\varphi \circ \pi}(\xi, \eta) d\xi d\eta \\ &= \iiint \widehat{u}(\xi', \eta') \widehat{f}(\xi - \xi', \eta - \eta') \widehat{\varphi}(\xi) \delta(\eta) d\xi d\xi' d\eta d\eta' \\ &= \iiint \widehat{u}(\xi', \eta') \widehat{f}(\xi - \xi', -\eta') \widehat{\varphi}(\xi) d\xi d\xi' d\eta' \\ &= \int \left(\int \left(\iint \widehat{u}(\xi', \eta') \widehat{f}(\xi - \xi', -\eta') d\eta' d\xi' \right) e^{-i\langle \xi, x \rangle} d\xi \right) \varphi(x) dx. \end{aligned}$$

We split the innermost double integral $I(\xi)$ into two terms, the first one on the domain A defined by $\Lambda(\xi - \xi') \leq \frac{1}{4}\Lambda(\xi)$, the second one on the complementary A' . An elementary straightforward computation shows the following double inequality for any $\xi' \in A$:

$$(56) \quad \frac{\sqrt{2}}{2}\Lambda(\xi) \leq \Lambda(\xi') \leq \frac{5}{4}\Lambda(\xi).$$

The Fourier transform \widehat{f} is rapidly decreasing, which yields the estimates:

$$(57) \quad \|\widehat{f}(\xi - \xi', -\eta')\| \leq D_N \Lambda(\xi' - \xi, \eta')^{-N}, \quad N = 1, 2, 3, \dots$$

We have then, using (55), (56), (57) and the fact that $\|\eta'\| \geq 1$ for any (ξ', η') in the support of \widehat{u} :

$$\begin{aligned} |I(\xi)| &\leq \iint_A C_{2N} D_M \|\eta'\|^{2N} \|(\xi', \eta')\|^{-2N} \Lambda(\xi' - \xi, \eta')^{-M} d\xi' d\eta' + \iint_{A'} C_N D_{2M} \|\eta'\|^N \|(\xi', \eta')\|^{-N} \Lambda(\xi' - \xi, \eta')^{-2M} d\xi' d\eta' \\ &\leq 2^{\frac{N}{2}} C_{2N} D_M \iint \Lambda(\eta')^{2N-M} \Lambda(\xi')^{-N} \Lambda(\xi)^{-N} d\xi' d\eta' + 5^M C_N D_{2M} \iint \Lambda(\eta')^{N-M} \Lambda(\xi')^{-N} \Lambda(\xi)^{-M} d\xi' d\eta'. \end{aligned}$$

As this holds for arbitrary M and N , the innermost double integral $I(\xi)$ converges and defines a rapidly decreasing function of ξ . Hence the inner triple integral defines a smooth function of x , which proves the claim.

It is not obvious to us how to characterize transversal distributions whose wave front set avoids $\mathcal{M}(\pi)$. We give in the following lemma a sufficient condition.

Lemma 19. *Let $v \in \mathcal{D}'_{\pi_Y}(X \times Y)$ and assume that there exists constants $d \in \mathbb{N}$ and $\delta \in [0, 1)$ such that for any compact subset K of X and multi-index $\beta \in \mathbb{N}^{n_Y}$, one can find a constant $C_{K\beta}$ such that*

$$(58) \quad |\langle \partial^\beta v_y, f \rangle| \leq C_{K\beta} \|f\|_{K, d+\delta|\beta|}, \quad \forall f \in C_c^\infty(X), \quad y \in Y.$$

Here, we have set $\|f\|_{K, d+\delta|\beta|} = \sum_{|\alpha| \leq d+\delta|\beta|} \sup_K |\partial^\alpha f|$. Then

$$(59) \quad \text{WF}(v) \subset \mathcal{M}(\pi_X).$$

In particular, $\text{WF}(v) \cap \mathcal{M}(\pi_Y) = \emptyset$.

Proof. Let us fix $(x_0, y_0, \xi_0, \eta_0) \notin \mathcal{M}(\pi_X)$, that is, $\eta_0 \neq 0$ and assume that $|(\xi_0, \eta_0)| = 1$. We work below in a conic neighborhood Γ of $(x_0, y_0, \xi_0, \eta_0)$ such that for all $(x, y, \xi, \eta) \in \Gamma$ with $|(\xi, \eta)| = 1$, we have $|\eta_j| \geq |\eta_{0j}|/2$ for some fixed j such that $\eta_{0j} \neq 0$.

Let $(x, y, \xi, \eta) \in \Gamma$ be such that $|(\xi, \eta)| = 1$ and $\varphi(x, y)$ be supported in a compact neighborhood $K \times L$ of (x_0, y_0) in $X \times Y$. Denoting $\varphi_y = \varphi(\cdot, y)$, we have for any $N > 0$

$$(60) \quad \begin{aligned} |\langle v, \varphi e^{-it\langle(\cdot, \cdot), (\xi, \eta)\rangle} \rangle| &= \left| \int \langle v_y, \varphi_y e^{-it\langle(\cdot, y), (\xi, \eta)\rangle} \rangle dy \right| \\ &= \left| \int \widehat{\varphi_y v_y}(t\xi) e^{-it\langle y, \eta \rangle} dy \right| \\ &\leq C \cdot \left(\sum_{|\alpha| \leq N} \sup_{y \in L} |\partial_y^\alpha \widehat{\varphi_y v_y}(t\xi)| |\eta|^{|\alpha| - 2N} \right) t^{-N} \text{ by [7, Theorem 7.7.1].} \end{aligned}$$

Moreover, since $v : y \mapsto v_y$ is C^∞ , we have

$$\partial_{y_j}^N \widehat{\varphi_y v_y}(t\xi) = \partial_{y_j}^N \langle \varphi_y v_y, e^{-it\langle \cdot, \xi \rangle} \rangle = \langle \partial_{y_j}^N \varphi_y v_y, e^{-it\langle \cdot, \xi \rangle} \rangle = \widehat{\partial_{y_j}^N \varphi_y v_y}(t\xi).$$

We note $K_\epsilon = \{x + y; x \in K, |y| < \epsilon\}$ for any $\epsilon > 0$ and let $\chi_\epsilon \in C_c^\infty(K_\epsilon)$ be such that $\chi_\epsilon = 1$ on $K_{\epsilon/2}$. If $H(\xi)$ denotes the supporting function of K [7, 4.3.1], we get using the assumption (58) and the proof of the Paley-Wiener-Schwartz Theorem in [7, 7.3.1]

$$\begin{aligned} |\widehat{\partial_{y_j}^N \varphi_y v_y}(\xi)| &= |\partial_{y_j}^N \varphi_y v_y(\chi_\epsilon e^{-i\langle \cdot, \xi \rangle})| \\ &\leq C_{K_\epsilon N} \sum_{|\beta| \leq d+\delta N} \sup |\partial^\beta (\chi_\epsilon e^{-i\langle \cdot, \xi \rangle})| \\ &\leq C_{K_\epsilon N} \cdot C \cdot e^{H(0)} \cdot \sum_{|\beta| \leq d+\delta N} \epsilon^{-\beta} (1 + |\xi|)^{d+\delta N - |\beta|}. \end{aligned}$$

With $\epsilon = 1/(1 + |\xi|)$ and using the inequalities $C_{K_\epsilon N} \leq C_{K_{\epsilon'} N}$ if $\epsilon < \epsilon'$, we obtain

$$|\widehat{\partial_{y_j}^N \varphi_y v_y}(\xi)| \leq C_{K_1 N} \cdot C \cdot (1 + |\xi|)^{d+\delta N} \leq C'_{K_1 N} (1 + |\xi|)^{d+\delta N}.$$

Using uniform estimates $|\eta| \geq c_1 > 0$ and $(1 + |t\xi|) \leq c_2 t$ for $(\xi, \eta) \in \Gamma$, $|(\xi, \eta)| = 1$ and the estimate (61) applied to (60), we get

$$|\langle v, \varphi e^{-it\langle(\cdot, \cdot), (\xi, \eta)\rangle} \rangle| \leq C \cdot t^{d+(\delta-1)N}.$$

since $\delta - 1 < 0$, we conclude that $(x_0, y_0, \xi_0, \eta_0) \notin \text{WF}(v)$. \square

In view of Proposition 17, it is natural to set

Definition 5. Let G be a C^∞ groupoid. A subset $W \subset T^*G \setminus 0$ is r -transversal if

$$(61) \quad W \cap N^*(G^x) = \emptyset \text{ for all } x \in G^{(0)},$$

A subset $W \subset T^*G \setminus 0$ is s -transversal if

$$(62) \quad W \cap N^*(G_x) = \emptyset \text{ for all } x \in G^{(0)}.$$

We call bi-transversal a set which is r -transversal and s -transversal. We denote by $\mathcal{E}'_a(G, \Omega^{1/2})$ the subset of $\mathcal{E}'(G, \Omega^{1/2})$ of distributions whose wavefront set $WF(u)$ is bi-transversal.

Example 1. We obviously get that A^*G is a bi-transversal subset of T^*G . Remember that the space $\Psi(G)$ of G -pseudodifferential operators can be defined as the space of distributions on G which are conormal to $G^{(0)}$ [11]. Since such distributions have wave front sets included in $A^*G = N^*(G^{(0)})$, we get

$$(63) \quad \Psi_c(G) \subset \mathcal{E}'_a(G, \Omega^{1/2}).$$

In our previous notation (53), condition (61) reads $W \cap \mathcal{M}(r) = \emptyset$ and condition (62) reads $W \cap \mathcal{M}(s) = \emptyset$. Observe that Proposition 17 gives :

- If W is r -transversal, $\mathcal{E}'_W(G, \Omega^{1/2}) \subset \mathcal{E}'_r(G, \Omega^{1/2})$.
- If W is s -transversal, $\mathcal{E}'_W(G, \Omega^{1/2}) \subset \mathcal{E}'_s(G, \Omega^{1/2})$.
- If W is bi-transversal, $\mathcal{E}'_W(G, \Omega^{1/2}) \subset \mathcal{E}'_{r,s}(G, \Omega^{1/2})$.

In particular, one has $\mathcal{E}'_a(G, \Omega^{1/2}) \subset \mathcal{E}'_{r,s}(G, \Omega^{1/2})$. Using Theorem 13 and Proposition 14, we obtain

Corollary 20. Let W_1 and W_2 be closed conic subsets $T^*G \setminus 0$ which are respectively s -transversal and r -transversal. The convolution product restricts to bilinear maps

$$\begin{aligned} \mathcal{E}'_{W_1}(G, \Omega^{1/2}) \times \mathcal{E}'(G, \Omega^{1/2}) &\xrightarrow{*} \mathcal{E}'(G, \Omega^{1/2}), \\ \mathcal{E}'_{W_2}(G, \Omega^{1/2}) \times C_c^\infty(G, \Omega^{1/2}) &\xrightarrow{*} C_c^\infty(G, \Omega^{1/2}), \\ \mathcal{E}'(G, \Omega^{1/2}) \times \mathcal{E}'_{W_2}(G, \Omega^{1/2}) &\xrightarrow{*} \mathcal{E}'(G, \Omega^{1/2}), \\ C_c^\infty(G, \Omega^{1/2}) \times \mathcal{E}'_{W_1}(G, \Omega^{1/2}) &\xrightarrow{*} C_c^\infty(G, \Omega^{1/2}). \\ \mathcal{E}'_a(G, \Omega^{1/2}) \times \mathcal{E}'_a(G, \Omega^{1/2}) &\xrightarrow{*} \mathcal{E}'_{r,s}(G, \Omega^{1/2}). \end{aligned}$$

The next natural question is to understand the wave front set of the convolution product of two distributions with bi-transversal wave front sets. To do so, we will need a better understanding of the structure of T^*G .

6. THE COTANGENT GROUPOID OF COSTES-DAZORD-WEINSTEIN

We recall the definition of the cotangent groupoid of Costes-Dazord-Weinstein. We have tried to make more concise the construction of the source and target map given in [3] and to enlight the role played by the transpose of the differential of the multiplication map of G . More general and conceptual approaches can also be found in [14, 9]. See also [10].

Let G be a Lie groupoid whose multiplication is denoted by m , source and target by s, r and inversion by i . It is straightforward that, differentiating all the structure maps of G , we get that $TG \rightrightarrows TG^{(0)}$ is a Lie groupoid whose multiplication is given by dm , source and target by ds, dr and inversion by di . Hence, it is natural to try to transpose everything to get a groupoid structure

on T^*G . Following this idea, it is natural to define $(\gamma, \xi) = (\gamma_1, \xi_1) \cdot (\gamma_2, \xi_2) \in T^*G$ by $\gamma = \gamma_1\gamma_2$ and ξ by the equation

$$(64) \quad {}^t dm_{(\gamma_1, \gamma_2)}(\xi) = (\xi_1, \xi_2).$$

The point is that $m : G^{(2)} \rightarrow G$ is a submersion, so that, at any point $(\gamma_1, \gamma_2) \in G^{(2)}$, the linear map ${}^t dm_{(\gamma_1, \gamma_2)}$ is injective. Thus, the solution ξ of (64) exists, and then is unique, if and only if

$$(65) \quad (\xi_1, \xi_2) \in \text{Im } {}^t dm_{(\gamma_1, \gamma_2)}.$$

In that case, we note

$$\xi = \xi_1 \oplus \xi_2 = \mu(\xi_1, \xi_2).$$

Since $\text{Im } {}^t dm_{(\gamma_1, \gamma_2)} = (\ker dm_{(\gamma_1, \gamma_2)})^\perp$, the condition (65) is equivalent to

$$(66) \quad \forall (X_1, X_2) \in T_{(\gamma_1, \gamma_2)}G^{(2)}, \quad dm_{(\gamma_1, \gamma_2)}(X_1, X_2) = 0 \Rightarrow \xi_1(X_1) + \xi_2(X_2) = 0.$$

We are going to reformulate condition (66) in term of composability of (ξ_1, ξ_2) in the expected groupoid structure on T^*G and thus we will clarify what are the source map, target map and unit space of T^*G . For that purpose, we collect useful information on the kernel of dm .

Lemma 21. *Let $(X_1, X_2) \in T_{(\gamma_1, \gamma_2)}G^{(2)}$ and $y = s(\gamma_1) = r(\gamma_2)$, $Y = ds(X_1) = dr(X_2)$. Then the equivalences below hold.*

$$\begin{aligned} & (X_1, X_2) \in \ker dm_{(\gamma_1, \gamma_2)} \\ \Leftrightarrow & X_1 \in \ker dr, X_2 \in \ker ds \text{ and } X_1 = (dL_{\gamma_1})(Y_1), X_2 = (dR_{\gamma_2})(Y_2) \text{ with } (Y_1, Y_2) \in \ker dm_{(y, y)} \\ \Leftrightarrow & X_1 \in \ker dr, X_2 \in \ker ds \text{ and } X_1 = (dL_{\gamma_1})(Y_1), X_2 = (dR_{\gamma_2})(Y_2) \text{ with } Y_1 + Y_2 = Y \\ \Leftrightarrow & X_1 \in \ker dr, X_2 \in \ker ds \text{ and } X_1 = (dL_{\gamma_1})(Y_1), X_2 = (dR_{\gamma_2})(Y_2) \text{ with } di(Y_1) = Y_2, \\ \Leftrightarrow & X_1 \in \ker dr \text{ and } X_2 = dR_{\gamma_2} \circ di \circ dL_{\gamma_1}^{-1}(X_1), \\ \Leftrightarrow & X_2 \in \ker ds \text{ and } X_1 = dL_{\gamma_1} \circ di \circ dR_{\gamma_2}^{-1}(X_2). \end{aligned}$$

are true.

Proof. Let $(X_1, X_2) \in T_{(\gamma_1, \gamma_2)}G^{(2)}$ such that $dm_{(\gamma_1, \gamma_2)}(X_1, X_2) = 0$. We write $y = s(\gamma_1) = r(\gamma_2)$ and $Y = ds_{\gamma_1}(X_1) = dr_{\gamma_2}(X_2) \in T_yG^{(0)} \subset T_yG$.

Since $r \circ m = r \circ p_1$ and $s \circ m = s \circ p_2$ if $p_i : G^2 \rightarrow G$ denote the canonical projections, we have

$$dr_{\gamma_1}(X_1) = ds_{\gamma_2}(X_2) = 0 \text{ that is } X_1 \in T_{\gamma_1}G^{r(\gamma_1)} \text{ and } X_2 \in T_{\gamma_2}G_{s(\gamma_2)}.$$

We introduce the vectors $Y_1 \in T_yG^y$ and $Y_2 \in T_yG_y$ such that

$$X_1 = d(L_{\gamma_1})_y(Y_1) \text{ and } X_2 = d(R_{\gamma_2})_y(Y_2).$$

The equality $m \circ (L_{\gamma_1}, R_{\gamma_2}) = L_{\gamma_1}R_{\gamma_2} \circ m$ then gives the equivalence

$$dm_{(\gamma_1, \gamma_2)}(X_1, X_2) = 0 \iff dm_{(y, y)}(Y_1, Y_2) = 0.$$

The relations $s \circ L_\gamma = s$ and $r \circ R_\gamma = r$ give

$$ds_y(Y_1) = dr_y(Y_2) = Y.$$

A simple computation shows that $dm_{(y, y)}(Y, Y) = Y$ and since $Y_1 - Y \in \ker ds$, $Y_2 - Y \in \ker dr$, we also have

$$dm_{(y, y)}(Y_1 - Y, Y_2 - Y) = d(R_y)_y(Y_1 - Y) + d(L_y)_y(Y_2 - Y) = Y_1 - Y + Y_2 - Y = Y_1 + Y_2 - 2Y.$$

This implies the equality

$$(67) \quad dm_{(y,y)}(Y_1, Y_2) = dm_{(y,y)}(Y, Y) + dm_{(y,y)}(Y_1 - Y, Y_2 - Y) = Y_1 + Y_2 - Y$$

which proves the second equivalence of the lemma.

Differentiating the relation $\gamma^{-1}\gamma = s(\gamma)$ at $\gamma = y$, we get $di_y + \text{id} = ds_y + dr_y$, hence

$$di_y(Y_1) - Y_2 = ds_y(Y_1) + dr_y(Y_1) - Y_1 - Y_2 = Y - Y_1 - Y_2,$$

which proves the third equivalence. The last ones are obvious. \square

Going back to (66), we have under the assumption $dm(X_1, X_2) = 0$ and using the notation of the lemma

$$\xi_1(X_1) + \xi_2(X_2) = {}^t di_y {}^t d(L_{\gamma_1})_y(\xi_1)(Y_2) + {}^t d(R_{\gamma_2})_y(\xi_2)(Y_2)$$

Thus (66) holds if and only if

$$(68) \quad {}^t di_y {}^t d(L_{\gamma_1})_y(\xi_1) + {}^t d(R_{\gamma_2})_y(\xi_2) = 0.$$

This equality in A_y^*G (viewed as the space of linear forms on T_yG_y that is, $A^*G = (AG)^*$ with $AG = \ker ds|_{G^{(0)}}$) leads to the definition of source and target maps for $\Gamma = T^*G$:

Definition 6. [3]. *Let G be a C^∞ groupoid. We denote by $\Gamma = T^*G$ the cotangent space of G endowed with the C^∞ groupoid structure given by*

- unit space $\Gamma^{(0)} = A^*G$;
- source map

$$(69) \quad s_\Gamma(\gamma, \xi) = (s(\gamma), \tilde{s}(\xi)) = (s(\gamma), -{}^t di_{s(\gamma)} {}^t d(L_\gamma)_{s(\gamma)}(\xi)) \in A^*G;$$

- target map

$$(70) \quad r_\Gamma(\gamma, \xi) = (r(\gamma), \tilde{r}(\xi)) = (r(\gamma), {}^t d(R_\gamma)_{r(\gamma)}(\xi)) \in A^*G;$$

- product map

$$(71) \quad m_\Gamma(\gamma_1, \xi_1, \gamma_2, \xi_2) = (\gamma_1\gamma_2, \mu(\xi_1, \xi_2)), \quad \mu(\xi_1, \xi_2) = ({}^t dm_{(\gamma_1, \gamma_2)})^{-1}(\xi_1, \xi_2);$$

- inversion map

$$(72) \quad i_\Gamma(\gamma, \xi) = (\gamma^{-1}, ({}^t di_\gamma)^{-1}(\xi)).$$

Remark 22. The Lie algebroid of G is defined above by $AG = \ker ds|_{G^{(0)}}$. It is also often defined by $AG = (TG|_{G^{(0)}})/TG^{(0)}$. Both bundles are canonically isomorphic since $\ker ds|_{G^{(0)}} \oplus TG^{(0)} = TG|_{G^{(0)}}$. Looking at AG as the normal bundle of $G^{(0)}$ into G gives an alternative and useful definition of source and target maps of T^*G .

Let $\xi \in T_\gamma^*G$ and $x = s(\gamma)$. The extension of the linear form

$$\xi \circ d(L_\gamma)_x : T_xG^x \longrightarrow \mathbb{R}$$

by 0 on the subspace $T_xG^{(0)}$ of T_xG gives an element of $A_x^*G = (T_xG/T_xG^{(0)})^*$ denoted by $\bar{s}(\xi)$. Similarly, with $y = r(\gamma)$, we denote $\bar{r}(\xi)$ the extension by 0 on the subspace $T_yG^{(0)}$ of T_yG of the linear form

$$\xi \circ d(R_\gamma)_y : T_yG_y \longrightarrow \mathbb{R},$$

If we use again the relation $di_y + \text{id} = ds_y + dr_y$, we obtain that (68) is equivalent to

$$(73) \quad \bar{s}(\xi_1) = \bar{r}(\xi_2),$$

hence we can also set

$$(74) \quad s_\Gamma(\gamma, \xi) = (s(\gamma), \bar{s}(\xi)) ; r_\Gamma(\gamma, \xi) = (r(\gamma), \bar{r}(\xi)).$$

Remark 23. An obvious computation shows that the graph Γ_μ of the map $(\gamma_1, \xi_1, \gamma_2, \xi_2) \mapsto (\gamma_1\gamma_2, \xi_1 \oplus \xi_2)$ is canonically isomorphic to the conormal bundle N^*G_m of the graph of the multiplication of G

$$(\gamma, \xi, \gamma_1, \xi_1, \gamma_2, \xi_2) \in \Gamma_\mu \Leftrightarrow (\gamma, -\xi, \gamma_1, \xi_1, \gamma_2, \xi_2) \in N^*G_m = (TG_m)^\perp.$$

Since N^*G_m is lagrangian in $(T^*G)^3$, this remark proves that the groupoid structure of $\Gamma = T^*G$ which follows from these considerations is automatically symplectic.

Remark 24. Observe that $\Gamma = T^*G \xrightarrow{\pi} G$ is a vector bundle and that μ is a vector bundle homomorphism over m . Actually, the diagram

$$(75) \quad \begin{array}{ccc} \Gamma^{(2)} & \xrightarrow{\mu} & \Gamma \\ \downarrow \pi \times \pi & & \downarrow \pi \\ G^{(2)} & \xrightarrow{m} & G \end{array}$$

commutes and μ is linear in the fibers. We observe that

$$(76) \quad N^*G^{(2)} = \ker \mu \subset \Gamma^{(2)}.$$

We finish this review with two basic examples, the second one being certainly the initial motivation of [3].

Example 2. We take $G = X \times X \times Z \rightrightarrows X \times Z$ (cartesian product of the pair groupoid $X \times X$ with the space Z). Here we have

$$\Gamma^{(0)} = A^*G = \{(x, \xi, x, -\xi, 0) ; (x, \xi) \in T^*X\}.$$

Let $\gamma = (x, y, z)$ and $\xi = (\zeta, \eta, \sigma) \in T_\gamma^*G$. Then $\tilde{s}(\xi)$ is the element in $T_{(y,y,z)}^*X \times X \times Z$ obtained from $\eta \in T_y^*X \simeq 0 \times T_y^*X \times 0$ after extension by 0 into the subspace of vectors of the form (u, u, w) . In the same way, $\tilde{r}(\xi) \in T_{(x,x,z)}^*X \times X \times Z$ is obtained from $\zeta \in T_x^*X \simeq T_x^*X \times 0 \times 0$ after extension by 0 into the subspace of vectors of the form (u, u, w) . Writing

$$(u, v, w) = (u - v, 0, 0) + (v, v, w) = (0, v - u, 0) + (u, u, w)$$

we see that $(x_1, y_1, z_1; \zeta_1, \eta_1, \sigma_1)$ and $(x_2, y_2, z_2; \zeta_2, \eta_2, \sigma_2)$ are composable in Γ if and only if

$$y_1 = x_2 ; z_1 = z_2 ; \eta_1 = -\zeta_2 ;$$

and the product is then given by (γ, ξ) with $\gamma = \gamma_1\gamma_2$ and

$$\xi = (\zeta_1, \eta_2, \sigma_1 + \sigma_2).$$

When Z is a point, the groupoid $\Gamma = T^*(X \times X)$ is isomorphic to the pair groupoid of T^*X , with isomorphism given by

$$T^*X \times T^*X \longrightarrow T^*X \times T^*X ; (x, \zeta, y, \eta) \mapsto (x, \zeta, y, -\eta).$$

This “change of sign”, denoted by $'$ in [6] and used there to relate Lagrangian submanifolds and canonical relations, shows that the cotangent groupoid $\Gamma = T^*(X \times X)$ is implicitly used [6] and central in the results about composition of Fourier integral distributions stated in [6].

Example 3. We suppose that the Lie groupoid G is a Lie group with Lie algebra \mathfrak{g} . Two elements (g, ξ) and (h, η) in T^*G are composable if and only if :

- g and h are composable (this condition is void in a Lie group),
- $L_g^*\xi = R_h^*\eta$ (equality in \mathfrak{g}^*).

In this case, the composition is defined by:

$$(g, \xi)(h, \eta) = (gh, \xi \oplus \eta)$$

with:

$$\langle {}^t dm_{(g,h)} \xi \oplus \eta, (X, Y) \rangle = \langle \xi, X \rangle + \langle \eta, Y \rangle$$

for any $X \in T_g G$ and $Y \in T_h G$. This can also be written as:

$$(77) \quad \langle \xi \oplus \eta, (L_g)_* Y + (R_h)_* X \rangle = \langle \xi, X \rangle + \langle \eta, Y \rangle.$$

Let us remark at this stage that we obtain:

$$(78) \quad \xi \oplus \eta = R_{h^{-1}}^* \xi = L_{g^{-1}}^* \eta,$$

as we can see by setting respectively $Y = 0$ and $X = 0$ in (77). Using the compositability condition and the fact that left and right translations commute, we can directly check:

$$\begin{aligned} R_{h^{-1}}^* \xi &= R_{h^{-1}}^* L_{g^{-1}}^* L_g^* \xi \\ &= R_{h^{-1}}^* L_{g^{-1}}^* R_h^* \eta \\ &= L_{g^{-1}}^* \eta. \end{aligned}$$

Now, as the cotangent bundle of a Lie group is trivial, we can identify T^*G with the product $G \times \mathfrak{g}^*$ via the map:

$$\begin{aligned} \Phi : T^*G &\longrightarrow G \times \mathfrak{g}^* \\ (g, \xi) &\longmapsto (g, R_g^* \xi). \end{aligned}$$

We will denote $R_g^* \xi$ by $\bar{\xi}$. The compositability condition can be rewritten as:

$$L_g^* R_{g^{-1}}^* \bar{\xi} = R_h^* R_{h^{-1}}^* \bar{\eta}$$

hence:

$$\text{Ad}^* g \cdot \bar{\xi} = \bar{\eta}.$$

The composition law is then rewritten as:

$$\begin{aligned} (g, \bar{\xi})(h, \bar{\eta}) &= \Phi((g, R_{g^{-1}}^* \bar{\xi})(h, R_{h^{-1}}^* \bar{\eta})) \\ &= \phi(gh, R_{h^{-1}}^* R_{g^{-1}}^* \bar{\xi}) \\ &= (gh, R_{gh}^* R_{h^{-1}}^* R_{g^{-1}}^* \bar{\xi}) \\ &= (gh, \bar{\xi}). \end{aligned}$$

The cotangent groupoid T^*G is thus identified via Φ with the transformation groupoid $G \times \mathfrak{g}^*$, with source, target and composition defined by:

$$\begin{aligned} s(g, \bar{\xi}) &= \text{Ad}^* g \cdot \bar{\xi}, \\ r(g, \bar{\xi}) &= \bar{\xi}, \\ (g, \bar{\xi})(h, \text{Ad}^* g \cdot \bar{\xi}) &= (gh, \bar{\xi}). \end{aligned}$$

7. CONVOLUTION AND WAVE FRONT SETS ON GROUPOIDS

We revisit the convolution of distributions on a groupoid in terms of their wave front sets. The starting point is a beautiful observation of A. Weinstein [17]: the convolution product is a distribution on $G \times G \times G$. More precisely, this distribution

$$\mathcal{C} \in \mathcal{D}'(G^3, \Omega^{1/2} \otimes (\Omega^{-1/2} \otimes \Omega_G) \otimes (\Omega^{-1/2} \otimes \Omega_G)) = C_c^\infty(G^3, (\Omega^{-1/2} \otimes \Omega_G) \otimes \Omega^{1/2} \otimes \Omega^{1/2})'$$

is given by

$$\langle \mathcal{C}, w \otimes u \otimes v \rangle = \int_{G_m} w(\gamma)u(\gamma_1)v(\gamma_2)$$

where $G_m = \{(\gamma, \gamma_1, \gamma_2) ; \gamma = \gamma_1\gamma_2\}$ is the graph of the multiplication map of G .

From this point of view, the wave front set of $*$, that is of \mathcal{C} , is the conormal bundle $N^*(G_m) \subset T^*G^3$. This allows us to recover precise conditions on the wave front sets of compactly supported distributions in order that their convolution product is well defined and then to get information on the wave front set of the resulting distribution.

Let $u_1, u_2 \in \mathcal{E}'(G, \Omega^{1/2})$ be two distributions. By Theorem 2.5.10 of [6], the product $\mathcal{C} \cdot (1 \otimes u_1 \otimes u_2)$ is well defined as a distribution on G^3 if

$$(79) \quad N^*(G_m) + G \times \text{WF}(u_1 \otimes u_2) \subset T^*G^3 \setminus 0.$$

That $(\xi, \xi_1, \xi_2) \in N^*(G_m)$ with base point $(\gamma, \gamma_1, \gamma_2) \in G_m$ and $(0, \eta_1, \eta_2) \in G \times \text{WF}(u_1 \otimes u_2)$ over the same base point satisfy

$$(\xi, \xi_1, \xi_2) + (0, \eta_1, \eta_2) = 0$$

is equivalent to $\xi = 0$ and $-(\xi_1, \xi_2) = (\eta_1, \eta_2)$. Thus (79) is equivalent to

$$(80) \quad \text{WF}(u_1 \otimes u_2) \cap \ker \mu = \emptyset.$$

Let $(\gamma_1, \xi_1, \gamma_2, 0) \in \text{WF}(u_1) \times G \cap \Gamma^{(2)}$. Since $\text{WF}(u_1) \subset T^*G \setminus 0$, we can choose $X_1 \in T_{\gamma_1}G$ such that $\xi_1(X_1) \neq 0$. Using a local section β of r such that $\beta(s(\gamma_1)) = \gamma_2$, we set $X_2 = d\beta ds(X_1) \in T_{\gamma_2}G$. Then $(X_1, X_2) \in T_{(\gamma_1, \gamma_2)}G^{(2)}$ and $\xi_1(X_1) + 0(X_2) \neq 0$, which proves that $(\xi_1, 0) \notin \ker \mu$. This can be rephrased in a more conceptual way. The map

$$\phi : \gamma \longmapsto \gamma \cdot \beta(s(\gamma))$$

is defined in a neighborhood of γ_1 , maps γ_1 to $\gamma_1\gamma_2$ and is a local diffeomorphism at γ_1 . We then check easily the equality

$$\xi_1 \oplus 0 = {}^t(d\phi_{\gamma_1})(\xi_1).$$

From this, we recover that $\xi_1 \neq 0$ implies $(\xi_1, 0) \notin \ker \mu$. The arguments are similar for $(\gamma_1, 0, \gamma_2, \xi_2) \in G \times \text{WF}(u_2) \cap \Gamma^{(2)}$. Since in all cases

$$\text{WF}(u_1 \otimes u_2) \subset \text{WF}(u_1) \times \text{WF}(u_2) \cup \text{WF}(u_1) \times G \cup G \times \text{WF}(u_2),$$

we obtain that (80) is equivalent to

$$(81) \quad \text{WF}(u_1) \times \text{WF}(u_2) \cap \ker \mu = \emptyset.$$

Thus, when (81) holds, the convolution product $u_1 * u_2$ is well defined as a distribution on G by the formula

$$(82) \quad \langle u_1 * u_2, v \rangle = \langle \mathcal{C} \cdot (1 \otimes u_1 \otimes u_2), v \otimes 1 \otimes 1 \rangle, \quad v \in C_c^\infty(G, \Omega^{1/2} \otimes \Omega_G),$$

and it is possible to estimate its wave front set. We shall use the labelling $G^3 = G_3 \times G_1 \times G_2$ and we denote shortly by K the distribution $\mathcal{C}.(1 \otimes u_1 \otimes u_2)$ as well as the operator $C^\infty(G_1 \times G_2, \Omega^{1/2}) \rightarrow \mathcal{E}'(G_3, \Omega^{1/2})$ associated to it via the Schwartz kernel theorem. Theorem 2.5.12 of [6] gives

$$(83) \quad \text{WF}(u_1 * u_2) = \text{WF}(K(1_1 \otimes 1_2)) \subset \text{pr}_{T^*G_3}(\text{WF}(K) \cap (T^*G_3) \times G_1 \times G_2).$$

Using again Theorem 2.5.10 of [6] and $\text{WF}(\mathcal{C}) = N^*G_m \setminus 0$, we have

$$\text{WF}(K) \subset (N^*G_m \setminus 0) + \text{WF}(1 \otimes u_1 \otimes u_2) \cup (N^*G_m \setminus 0) \cup \text{WF}(1 \otimes u_1 \otimes u_2).$$

The second and third term never meet $(T^*G_3) \times G_1 \times G_2$, and

$$(\xi, \xi_1, \xi_2) + (0, \eta_1, \eta_2) \in (N^*(G_m) + \text{WF}(1 \otimes u_1 \otimes u_2)) \cap (T^*G_3) \times G_1 \times G_2$$

means $(\eta_1, \eta_2) = -(\xi_1, \xi_2) \in \Gamma^{(2)} \cap \text{WF}(u_1 \otimes u_2)$ and $\xi = \eta_1 \oplus \eta_2$.

We note for any subsets $W_1, W_2 \in T^*G \setminus 0$:

$$W_1 \bar{\times} W_2 = W_1 \times W_2 \cup W_1 \times 0 \cup 0 \times W_2,$$

$$W_1 \circ W_2 = \mu(W_1 \times W_2 \cap (T^*G)^{(2)}),$$

$$W_1 \bar{\circ} W_2 = \mu(W_1 \bar{\times} W_2 \cap (T^*G)^{(2)}).$$

We have proved

Theorem 25. *Let $u_1, u_2 \in \mathcal{E}'(G, \Omega^{1/2})$ be two compactly supported distributions. If*

$$(84) \quad \text{WF}(u_1) \times \text{WF}(u_2) \cap \ker \mu = \emptyset$$

*then the convolution product $u_1 * u_2 \in \mathcal{E}'(G, \Omega^{1/2})$ is well defined by (82). If W_1, W_2 are closed cones in $T^*G \setminus 0$ such that $W_1 \times W_2 \cap \ker \mu = \emptyset$, then the map*

$$(85) \quad \mathcal{E}'_{W_1}(G, \Omega^{1/2}) \times \mathcal{E}'_{W_2}(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{E}'_{W_1 \bar{\circ} W_2}(G, \Omega^{1/2})$$

is well defined and extends the convolution product on $C_c^\infty(G, \Omega^{1/2})$.

If u_1 or u_2 is smooth then $\text{WF}(u_1) \times \text{WF}(u_2)$ is empty and (84) is trivially satisfied, thus

Corollary 26. *The convolution product of Theorem 25 gives by restriction the maps*

$$(86) \quad \mathcal{E}'(G, \Omega^{1/2}) \times C_c^\infty(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{E}'_{s_\Gamma^{-1}(0)}(G, \Omega^{1/2})$$

and

$$(87) \quad C_c^\infty(G, \Omega^{1/2}) \times \mathcal{E}'(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{E}'_{r_\Gamma^{-1}(0)}(G, \Omega^{1/2})$$

*where $s_\Gamma^{-1}(0)$ and $r_\Gamma^{-1}(0)$ denote the preimages of the zero section in A^*G under the source and target maps of $\Gamma = T^*G$.*

Now we take into account assumptions of transversality on the wave front sets of distributions. Firstly, we can use the groupoid structure of T^*G to rephrase r and s transversality of subsets of T^*G .

Proposition 27. *Let W be a subset of $T^*G \setminus 0$. Then W is r -transversal if and only if*

$$s_\Gamma(W) \subset A^*G \setminus 0.$$

Similarly, W is s -transversal if and only if

$$r_\Gamma(W) \subset A^*G \setminus 0.$$

Proof. Let $W \subset T^*G \setminus 0$ be a set. For any $(\gamma, \xi) \in W$, the non vanishing of

$$\xi \circ d(L_\gamma \circ i)_{s(\gamma)} : T_{s(\gamma)}G_{s(\gamma)} \rightarrow \mathbb{R}$$

is equivalent to the non vanishing of $\xi \circ d(L_\gamma)_{s(\gamma)} : T_{s(\gamma)}G^{s(\gamma)} \rightarrow \mathbb{R}$ which is equivalent to

$$\xi|_{T_\gamma G^{r(\gamma)}} \neq 0.$$

Thus r -transversality is equivalent to (61). The other case is similar. \square

Using the equalities $s_\Gamma \circ \mu_\Gamma = s_\Gamma \circ p_2$ and $r_\Gamma \circ \mu_\Gamma = r_\Gamma \circ p_1$, we get a class of examples for which condition (84) is automatically satisfied.

Corollary 28. *Let W_1, W_2 be any subsets of T^*G .*

If W_1 is s -transversal (resp. W_2 is r -transversal) then so is $W_1 \circ W_2$.

In both cases, we have in particular

$$W_1 \times W_2 \cap \ker \mu = \emptyset.$$

We conclude this section by proving, as announced, that $\mathcal{E}'_a(G, \Omega^{1/2})$ is a subalgebra of $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$.

Let $u_1, u_2 \in \mathcal{E}'_a(G, \Omega^{1/2})$. We have by corollary 28 that $\text{WF}(u_1) \times \text{WF}(u_2) \cap \ker \mu = \emptyset$ so the convolution product of Theorem 25 can be computed for u_1 and u_2 . By r -transversality of $\text{WF}(u_1)$, we have $s_\Gamma(\text{WF}(u_1)) \subset A^*G \setminus 0$, so

$$\text{WF}(u_1) \times G \cap \Gamma^2 = \emptyset.$$

Similarly, s -transversality of $\text{WF}(u_2)$ gives $G \times \text{WF}(u_2) \cap \Gamma^2 = \emptyset$. We obtain

$$\text{WF}(u_1) \circ \text{WF}(u_2) = [\text{WF}(u_1) \circ \text{WF}(u_2)] \cup [\text{WF}(u_1) \circ G \times \{0\}] \cup [G \times \{0\} \circ \text{WF}(u_2)] = \text{WF}(u_1) \circ \text{WF}(u_2)$$

Moreover, corollary 28 also gives the bi-transversality of $\text{WF}(u_1) \circ \text{WF}(u_2)$.

Thus, the convolution product provided by Theorem 25 is well defined on $\mathcal{E}'_a(G, \Omega^{1/2})$ and gives a map

$$\mathcal{E}'_a(G, \Omega^{1/2}) \times \mathcal{E}'_a(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{E}'_a(G, \Omega^{1/2}).$$

Since $\text{WF}(\delta) = A^*G \setminus 0$ is bi-transversal, we get $\delta \in \mathcal{E}'_a(G, \Omega^{1/2})$. Also, since $\text{WF}(i^*u) = {}^t(di)(\text{WF}(u))$ and ${}^t(di)$ is the inversion in the groupoid T^*G , we get that the involution of $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ preserves the bi-transversality of the wave front sets. In conclusion,

Theorem 29. *The space $\mathcal{E}'_a(G, \Omega^{1/2})$ is a unital involutive subalgebra of $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$.*

Remark 30. The convolution products defined by Theorems 13 and 25 coincide when both make sense. Indeed, remember that $\ker \mu = N^*G^{(2)}$ and that

$$\text{WF}(u_1 \otimes u_2) \cap \ker \mu = (\text{WF}(u_1) \times \text{WF}(u_2)) \cap \ker \mu = \emptyset.$$

This proves that the restriction $u_1 \otimes u_2|_{G^{(2)}} \in \mathcal{D}'(G^{(2)})$, that is, the pull-back of $u_1 \otimes u_2$ by the map $G^{(2)} \hookrightarrow G^2$ exists, ([7, Theorem 8.2.4]) and it follows that the product given by Theorem 25 is given by

$$(88) \quad \langle u_1 * u_2, \phi \rangle = \langle u_1 \otimes u_2|_{G^{(2)}}, \phi \circ m \rangle.$$

Since u_1 is s -transversal and u_1 is r -transversal, the distribution $u_1 \otimes u_2|_{G^{(2)}}$ is also given by the family

$$u_{1x} \otimes u_2^x \quad x \in G^{(0)}.$$

and we then see that the pairing (88) coincides with the one provided by Theorem 13.

8. G -OPERATORS AND BI-TRANSVERSALITY

In view of the classical result on the composition of operators [7, Theorem 8.2.13] on a manifold M , one may say that the wave front set of the Schwarz kernel K_P of an operator P acting on $C_c^\infty(M)$ is *bi-transversal* whenever one has $\text{WF}(K_P) \in (T^*M \setminus 0) \times (T^*M \setminus 0)$. Indeed, we know that the composition of two such operators is again an operator with $\text{WF}(K_{P \circ Q}) \in (T^*M \setminus 0) \times (T^*M \setminus 0)$. We have seen in section 4 that adjointable compactly supported G -operators are in 1 to 1 correspondence with kernels in $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ and hence the composition of two such operators, P and Q is still an adjointable compactly supported G -operator $P \circ Q$. Using the fact that for any $x \in G^{(0)}$ one has $(P \circ Q)_x = P_x \circ Q_x$, we can thus compose operators acting on $C_c^\infty(G_x)$. However, the conditions of bi-transversality we assume do not imply that the induced operators on the fibers G_x have bi-transversal wave front sets, as seen in remark 18.

In fact, we will see right now that the bi-transversality of the wave front set of the distributional kernel k_P of an adjointable compactly supported G -operator is equivalent to the bi-transversality of the wave front sets of all operators G_x .

First we translate the conditions of r -transversality or s -transversality of a set in $T^*G \setminus 0$ with corresponding transversality properties in $T^*G_x \times T^*G_x$

Proposition 31. *For any $x \in G^{(0)}$, we note $j_x : G_x \times G_x \longrightarrow G$ the map $(\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2^{-1}$. Let $W \subset T^*G \setminus 0$ be any set.*

- (1) *W is r -transversal if and only if $j_x^*(W)$ is r -transversal as a subset of $T^*G_x \times T^*G_x$, that is $j_x^*(W) \subset T^*G_x \times (T^*G_x \setminus 0)$, for all $x \in G^{(0)}$.*
- (2) *W is s -transversal if and only if $j_x^*(W)$ is s -transversal, that is $j_x^*(W) \subset (T^*G_x \setminus 0) \times T^*G_x$, for all $x \in G^{(0)}$.*

Proof. Recall that if $A \subset T^*G$ then

$$j_x^*(A) = \{(\gamma_1, \gamma_2, {}^t d(j_x)_{(\gamma_1, \gamma_2)}(\eta)) ; (\gamma_1, \gamma_2) \in G_x \times G_x (\gamma_1 \gamma_2^{-1}, \eta) \in A\}$$

We decompose ${}^t d(j_x)_{(\gamma_1, \gamma_2)}(\eta) = (\xi_1, \xi_2)$ in the product space $T_{\gamma_1}^* G_x \times T_{\gamma_2}^* G_x$. Observe that

$$T_{\gamma_1}^* G_x \times T_{\gamma_2}^* G_x \longrightarrow T_{\gamma_1}^* G_x, (\xi_1, \xi_2) \longmapsto \xi_1$$

is the transpose of the differential of the map

$$\ell_{\gamma_2} : G_x \longrightarrow G_x \times G_x, \gamma \longmapsto (\gamma, \gamma_2)$$

Since $j_x \circ \ell_{\gamma_2}(\gamma) = R_{\gamma_2^{-1}}(\gamma)$ we get

$$\xi_1 = {}^t d(\ell_{\gamma_2})_{\gamma_1} \circ {}^t d(j_x)_{(\gamma_1, \gamma_2)}(\eta) = {}^t d(R_{\gamma_2^{-1}})_{\gamma_1}(\eta).$$

Similarly we get

$$\xi_2 = {}^t d(L_{\gamma_1} \circ i)_{\gamma_2}(\eta).$$

Thus we get

$$j_x^*(A) = \{(\gamma_1, {}^t d(R_{\gamma_2^{-1}})_{\gamma_1}(\eta), \gamma_2, {}^t d(L_{\gamma_1} \circ i)_{\gamma_2}(\eta)) ; (\gamma_1, \gamma_2) \in G_x \times G_x (\gamma_1 \gamma_2^{-1}, \eta) \in A\}$$

Since $\ker {}^t d(R_{\gamma_2^{-1}})_{\gamma_1} = (\text{Im } d(R_{\gamma_2^{-1}})_{\gamma_1})^\perp = (T_{\gamma_1 \gamma_2^{-1}} G_{r(\gamma_2)})^\perp$, we have for instance the equivalences

$$\begin{aligned} & A \text{ is s-transversal} \\ \Leftrightarrow & \forall (\gamma, \eta) \in A, \eta \notin (T_\gamma G_{s(\gamma)})^\perp \\ \Leftrightarrow & \forall x \in G^{(0)}, \forall (\gamma_1, \gamma_2) \in G_x \times G_x \text{ such that } (\gamma_1 \gamma_2^{-1}, \eta) \in A, \eta \notin (T_{\gamma_1 \gamma_2^{-1}} G_{r(\gamma_2)})^\perp \\ \Leftrightarrow & \forall x \in G^{(0)}, \forall (\gamma_1, \gamma_2) \in G_x \times G_x \text{ such that } (\gamma_1 \gamma_2^{-1}, \eta) \in A, {}^t d(R_{\gamma_2^{-1}})_{\gamma_1}(\eta) \neq 0 \\ \Leftrightarrow & \forall x \in G^{(0)}, j_x^*(A) \subset (T^* G_x \setminus 0) \times T^* G_x. \end{aligned}$$

The other case is identical. \square

Using this, we have the following

Corollary 32. *Let P be a compactly supported G -operator. Then the wave front set of the distributional kernel k_P of P is bi-transversal in T^*G if and only if for any $x \in G^{(0)}$, the wave front set of the Schwartz kernel of P_x is bi-transversal in $T^*(G_x \times G_x)$.*

Remark 33. Using this approach, one could give another proof for Theorem 29, via families of operators by means of the following lemma.

Lemma 34. *Let A_1, A_2 be two subsets of T^*G . Then for any $x \in G^{(0)}$ we have*

$$(89) \quad j_x^*(A_1 \circ A_2) = j_x^*(A_1) \circ j_x^*(A_2)$$

where the composition in the right hand side is the one of $T^*(G_x \times G_x)$.

Suppose now we are given two compactly supported G -operators P and Q such that for any $x \in G^{(0)}$, the wave front sets of the Schwartz kernels of P_x and Q_x are both bi-transversal in $T^*(G_x \times G_x)$. From the classical result [7, Theorem 8.2.14], we then know that the wave front set of the Schwartz kernel of the composition $P_x \circ Q_x$ is again bi-transversal in $T^*(G_x \times G_x)$. Thus, using the lemma $j_x^*(W_P) \circ j_x^*(W_Q) = j_x^*(W_P \circ W_Q)$ and the corollary 32, we recover the result of Theorem 29.

Proof of the lemma. Let A be any subset of T^*G . For any $x \in G^{(0)}$ we have seen in the proof of the Proposition 31 that

$$(90) \quad j_x^*(A) = \{(\gamma_1, {}^t d(R_{\gamma_2^{-1}})_{\gamma_1}(\eta), \gamma_2, {}^t d(L_{\gamma_1} \circ i)_{\gamma_2}(\eta)) ; (\gamma_1, \gamma_2) \in G_x \times G_x, (\gamma_1 \gamma_2^{-1}, \eta) \in A\}$$

Identifying A^*G with $\ker ds|_{G^{(0)}}$, we have for any $\eta \in T_{\gamma_1 \gamma_2^{-1}}^* G$

$$\begin{aligned} \tilde{s}(\eta) &= {}^t d(L_{\gamma_1 \gamma_2^{-1}} \circ i)_{r(\gamma_2)}(\eta) = {}^t d(L_{\gamma_1} \circ L_{\gamma_2^{-1}} \circ i)_{r(\gamma_2)}(\eta) \\ &= {}^t d(L_{\gamma_1} \circ i \circ R_{\gamma_2})_{r(\gamma_2)}(\eta) = {}^t d(R_{\gamma_2})_{r(\gamma_2)} \circ {}^t d(L_{\gamma_1} \circ i)_{\gamma_2}(\eta), \end{aligned}$$

which gives $\xi_2 = ({}^t d(R_{\gamma_2})_{r(\gamma_2)})^{-1}(\tilde{s}(\eta)) = {}^t d(R_{\gamma_2^{-1}})_{\gamma_2}(\tilde{s}(\eta))$. We also have

$$\xi_1 = ({}^t d(R_{\gamma_1})_{r(\gamma_1)})^{-1}(\tilde{r}(\eta)) = {}^t d(R_{\gamma_1^{-1}})_{\gamma_1}(\tilde{r}(\eta)).$$

We thus have

$$(91) \quad j_x^*(A) = \{(\gamma_1, {}^t d(R_{\gamma_1^{-1}})_{\gamma_1}(\tilde{r}(\eta)), \gamma_2, {}^t d(R_{\gamma_2^{-1}})_{\gamma_2}(\tilde{s}(\eta))) ; \gamma_i \in G_x, (\gamma_1 \gamma_2^{-1}, \eta) \in A\}.$$

Next, we apply this formula to $A_1 \circ A_2$:

$$(92) \quad j_x^*(A_1 \circ A_2) = \{(\gamma_1, {}^t d(R_{\gamma_1^{-1}})_{\gamma_1}(\tilde{r}(\eta_1)), \gamma_2, {}^t d(R_{\gamma_2^{-1}})_{\gamma_2}(\tilde{s}(\eta_2))) ;$$

$$(93) \quad \forall \gamma_i \in G_x, (\delta_1, \eta_1, \delta_2, \eta_2) \in (A_1 \times A_2)^{(2)}, \text{ s.t. } \gamma_1 \gamma_2^{-1} = \delta_1 \delta_2\}.$$

On the other hand, the composition of $j_x^*(A_1)$ and $j_x^*(A_2)$ gives

$$(94) \quad j_x^*(A_1) \circ j_x^*(A_2) = \{(\gamma_1, {}^t d(R_{\gamma_1^{-1}})_{\gamma_1}(\tilde{r}(\eta_1)), \gamma_2, {}^t d(R_{\gamma_2^{-1}})_{\gamma_2}(\tilde{s}(\eta_2)))\};$$

$$(95) \quad \forall \gamma_i \in G_x, \gamma_3 \in G_x, \text{ s.t. } (\gamma_1 \gamma_3^{-1}, \eta_1, \gamma_3 \gamma_2^{-1}, \eta_2) \in (A_1 \times A_2)^{(2)}\}.$$

The inclusion $j_x^*(A_1) \circ j_x^*(A_2) \subset j_x^*(A_1 \circ A_2)$ follows from $\gamma_1 \gamma_3^{-1} \gamma_3 \gamma_2^{-1} = \gamma_1 \gamma_2^{-1}$ and conversely observe that if $(\delta_1, \eta_1, \delta_2, \eta_2) \in (A_1 \times A_2)^{(2)}$ satisfies $\gamma_1 \gamma_2^{-1} = \delta_1 \delta_2$, we get using $\gamma_3 = \delta_2 \gamma_2 \in G_x$:

$$\delta_1 = \gamma_1 \gamma_3^{-1}, \quad \delta_2 = \gamma_3 \gamma_2^{-1}.$$

□

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