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# WAVELET-GALERKIN METHOD FOR PLANE ELASTOSTATICS

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**ABSTRACT:** *This paper is devoted to a new algorithm to solve elastostatics problems on any shape domain. Our approach is based on a wavelet-Galerkin method. Due to the form of the wavelet bases, the general domain is included in a square fictitious domain, and the initial problem is extended over this square. The boundary conditions are imposed by an augmented Lagrangian method. Numerical examples are presented. Comparisons are done with analytic solutions and with finite element computations.*

**KEY WORDS:** *Elastostatics, wavelet transform, fictitious domains, augmented Lagrangian.*

## 1. INTRODUCTION

Solving hard problems in computational structure mechanics, requires conceiving accurate, robust and high performance algorithms. Classically, this kind of problem is solved by finite element methods, but, it may be very difficult to generate the grid, for example when the geometry of the solid is complex or in the study of localized phenomena. Moreover, the localization of high gradient zones is not easy, and usually needs stress tensor computation.

We propose an original alternative method, based on wavelet transform. The advantages of this wavelet–Galerkin method are, in particular: localization, adaptativity and the fact that there is no creation of a grid. This kind of method has already been used by the authors to solve periodic homogenization problems and permits us studying very inhomogeneous materials [10], [12].

On the other hand, there is no fast wavelet transform on general domains. We have therefore been interested in a fictitious domain method consisting of including the general domain in a larger square or rectangle.

This method is already used with the finite element method or with the finite difference method when the shape of the domain is too complex.

The paper is organized as follows: in section 2, we introduce the method that is used to solve a bidimensional elastostatics problem on any shape domain with the wavelet transform, and we give mathematical results. Section 3 is a brief summary of wavelets’ theory and of the wavelet-Galerkin method. Section 4 is devoted to the boundary conditions and the way we take them into consideration. In section 5, we give the numerical algorithms that we use to solve the linear system. Finally, section 6 contains some numerical examples whose validity is shown by comparing the results with analytical solutions and with solutions given in the literature.

## 2. MATHEMATICAL FORMULATION OF THE PROBLEM

### 2.1. Some Functional Spaces

In this first section, we define some functional spaces which will be used in the following.

We consider an open bounded domain  $\omega$  in  $\mathbb{R}^2$ . We suppose that the boundary  $\partial\omega$  of  $\omega$  can be written as  $\partial\omega = \overline{\partial\omega_D} \cup \overline{\partial\omega_F}$  where  $|\partial\omega_D| > 0$  ( $|\partial\omega_D| = \int_{\partial\omega_D} ds$ ).

The displacements on  $\omega$  are in the space,

$$H = \left( H^1(\omega) \right)^2 = \left( \left\{ v \in L^2(\omega) : \frac{\partial v}{\partial x_1} \in L^2(\omega), \frac{\partial v}{\partial x_2} \in L^2(\omega) \right\} \right)^2 \quad (1)$$

and the associated constraints are in the space,

$$L = \left( L^2_{sym}(\omega) \right)^4 = \left\{ \sigma_{ij}, i, j = 1, 2, 3 : \sigma_{ij} \in L^2(\omega), \sigma_{ij} = \sigma_{ji} \text{ a.e.} \right\} \quad (2)$$

We will also consider the square domain  $\Omega = (0, 1) \times (0, 1)$ , and the functional spaces,

$$L^2_p(\Omega) = \left\{ v \in L^2(\mathbb{R}^2) : v(x_1 + k_1, x_2 + k_2) = v(x_1, x_2) \text{ a.e. } \forall (k_1, k_2) \in \mathbb{Z}^2 \right\} \quad (3)$$

$$V = \left( H_p^1(\Omega) \right)^2 = \left\{ v \in L_p^2(\Omega) : \frac{\partial v}{\partial x_1} \in L^2(\Omega), \frac{\partial v}{\partial x_2} \in L^2(\Omega) \right\}^2 \quad (4)$$

with the scalar product,

$$(u, v)_V = \sum_{i=1}^2 \int_{\Omega} \left[ u_i v_i + \sum_{k=1}^2 \frac{\partial u_i}{\partial x_k} \frac{\partial v_i}{\partial x_k} \right] dx.$$

- On the parts  $\partial\omega_D$  and  $\partial\omega_F$  of the boundary  $\partial\omega$ , we define the trace:

$$\gamma_D : V \mapsto \left( H_{00}^{1/2}(\partial\omega_D) \right)^2 \quad ; \quad \gamma_F : V \mapsto \left( H_{00}^{1/2}(\partial\omega_F) \right)^2. \quad (5)$$

- On  $\partial\omega_D$ , let us define,

$$\mathcal{M}_1 = \left( H_{00}^{1/2}(\partial\omega_D) \right)^2 \quad \text{and its dual:} \quad \mathcal{M}'_1 = \left( H^{-1/2}(\partial\omega_D) \right)^2 \quad (6)$$

for the duality product:  $\langle w, \mu_1 \rangle_1 = \int_{\partial\omega_D} w \mu_1 ds$ .

$\mathcal{M}_1$  has the scalar product,

$$(u, v)_{\partial\omega_D} = \int_{\partial\omega_D} uv ds,$$

and the norm:  $\|w\|_{\mathcal{M}_1} = \inf \{ \|v\|_V : v \in H_p^1(\Omega) \quad \gamma_D(v) = w \}$ .

- On  $\partial\omega_F$ , let us define,

$$\mathcal{M}_2 = \left( H_{00}^{1/2}(\partial\omega_F) \right)^2 \quad \text{and its dual:} \quad \mathcal{M}'_2 = \left( H^{-1/2}(\partial\omega_F) \right)^2 \quad (7)$$

for the duality product:  $\langle \mu_2, g \rangle_2 = \int_{\partial\omega_F} \mu_2 g ds$ .

The scalar product on  $\mathcal{M}_2$ , and the norms on  $\mathcal{M}_2$  and  $\mathcal{M}'_2$  are defined as those on  $\mathcal{M}_1$  and  $\mathcal{M}'_1$ .

## 2.2. The Elasticity Problem

In this part, we consider a bidimensional elastic material with rigid tensor  $\mathbb{K}$ , occupying the open bounded domain  $\omega$  in  $\mathbb{R}^2$ . We want to find the displacement  $u$  solution of the equilibrium problem:

**Problem 1.**  $f$ ,  $u_0$ , and  $g$  be given, find  $u$  and  $\sigma$  such that:

$$(P_\omega) \quad \begin{cases} \operatorname{div} \sigma + f = 0 & \text{in } \omega \\ u = u_0 & \text{on } \partial\omega_D \\ \sigma \cdot n = g & \text{on } \partial\omega_F \\ \sigma = \mathbb{K}e(u) & \text{(behaviour law)} \end{cases} \quad (8)$$

where  $n$  is the unit vector of the outward normal to  $\partial\omega$ ,  $e(u)$  is the deformation tensor associated to  $u$ ,  $e(u) = \operatorname{grad}_s(u)$  i.e.  $e_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$ ,  $i, j = 1$  or  $2$ .

Classically, if  $f \in \left( L^2(\omega) \right)^2$ ,  $u_0 \in \mathcal{M}_1$ ,  $g \in \mathcal{M}'_2$  and  $|\partial\omega_D| > 0$ , this elasticity problem has an unique solution  $(\sigma, u)$  with  $\sigma \in L$ ,  $u \in H$  [13].

### 2.3. The Fictitious Domains

We want to solve this problem using the wavelet transform. Due to the form of the wavelets, it is necessary to introduce an associated method. Then, we include the domain  $\omega$  in a larger square domain  $\Omega$ , and we extend the problem  $(P_\omega)$  on  $\Omega$ :

**Problem 2.**  $\tilde{f}$ ,  $u_0$ , and  $g$  be given, find  $\tilde{u}$  and  $\tilde{\sigma}$  such that:

$$(P_\Omega) \quad \begin{cases} \operatorname{div} \tilde{\sigma} + \tilde{f} = 0 & \text{in } \Omega \\ \tilde{u} = u_0 & \text{on } \partial\omega_D \\ \tilde{\sigma} \cdot n = g & \text{on } \partial\omega_F \\ \tilde{\sigma} = \mathbb{K}e(\tilde{u}) & \text{(behaviour law)} \\ \tilde{u} \text{ } \Omega\text{-periodic} \end{cases} \quad (9)$$

where  $\tilde{f}$  is an extension of  $f$  on  $\Omega$ .

**Remark.** The condition  $\tilde{u}$   $\Omega$ -periodic is not necessary but in the following only periodic wavelets are used.

From a given function  $f$ , the computation of the periodic extension  $\tilde{f}$  of  $f$  may be done by different ways according to the properties of  $f$  and  $\omega$  (by periodizing  $f$  or extending  $f$  by a constant, etc.).

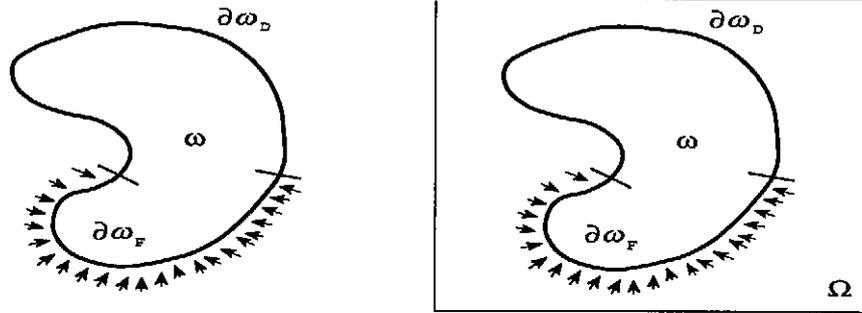


Fig.1: The fictitious domains

If  $\tilde{f} \in \left(L^2(\Omega)\right)^2$ ,  $u_0 \in \mathcal{M}_1$ ,  $g \in \mathcal{M}'_2$  and  $|\partial\omega_D| > 0$ , this problem has an unique solution  $(\tilde{\sigma}, \tilde{u})$  (with  $\tilde{u} \in V$ ) equal to the solution  $(\sigma, u)$  of  $(P_\omega)$  on  $\omega$  [16], [13]. This solution can be written as a problem of minimization under constraint:

**Problem 3.** Let  $\tilde{f}$ ,  $u_0$  and  $g$  be given. Find  $u$  such that,

$$(P_m) \quad \begin{cases} u \in \operatorname{Argmin} \phi \\ u = u_0 & \text{on } \partial\omega_D \\ \sigma \cdot n = g & \text{on } \partial\omega_F \end{cases} \quad (10)$$

where

$$\phi(v) = \frac{1}{2} \int_{\Omega} \sigma(v) : e(v) \, dx - \int_{\Omega} \tilde{f}v \, dx = \frac{1}{2} \int_{\Omega} \mathbb{K}e(v) : e(v) \, dx - \int_{\Omega} \tilde{f}v \, dx \quad (11)$$

is the potential energy of the system.

If  $u_0 \in (L^2(\partial\omega_D))^2$ ,  $g \in (L^2(\partial\omega_F))^2$ ,  $\tilde{f} \in (L^2(\Omega))^2$ , this problem has an unique solution  $(\sigma, u)$  with  $u \in H$  [12], [16].

## 2.4. The Augmented Lagrangian

In order to solve this problem, we have chosen to use the augmented Lagrangian method. The interests of this method are that the displacement solution of problem  $(P_r)$  is exact for all  $r$  (contrary to a penalty method for example), and the term depending on  $r$  acts as a preconditioner of a simply Lagrangian method. This augmented Lagrangian method consists in adding a new variable  $\lambda_i$  for each constraint  $i$ , called the Lagrangian multiplier. With those variables, we build a function called Lagrangian such that the solution is a saddle point of the Lagrangian (minimum for the displacement, maximum for the multipliers).

More precisely, we define the augmented Lagrangian of the problem of minimization under constraint  $(P_m)$  by,

$$L_r(v, \mu_1, \mu_2) = \phi(v) + \langle \mu_1, v - u_0 \rangle_1 + \langle \mu_2, \sigma(v) \cdot n - g \rangle_2 + \frac{r}{2} \|v - u_0\|_{\mathcal{M}_1}^2 \quad (12)$$

and we have the theorem.

**Theorem 1.** [5] *The point  $(u, \lambda_1, \lambda_2)$  in  $V \times \mathcal{M}'_1 \times \mathcal{M}_2$  solution of  $(P_m)$  is a saddle point of  $L_r$ . In other words,  $(u, \lambda_1, \lambda_2)$  is the unique solution of the problem:*

$$(P_r) \quad \min_{v \in V} \max_{\mu_1 \in \mathcal{M}'_1} \max_{\mu_2 \in \mathcal{M}_2} L_r(v, \mu_1, \mu_2). \quad (13)$$

**Remark.** The augmented Lagrangian is a partial augmented Lagrangian, only the first constraint is concerned.

## 2.5. The Variational Problem

To find the saddle point solution of problem  $(P_r)$ , we take the derivative of the augmented Lagrangian equal to zero. In other words, we have,

$$(P_N) \quad L'_r(u, \lambda_1, \lambda_2)(v, \mu_1, \mu_2) = 0 \quad \text{for all } (v, \mu_1, \mu_2) \text{ in } V \times \mathcal{M}'_1 \times \mathcal{M}_2. \quad (14)$$

Then, we have to solve the following linear problem.

**Problem 4.** Let  $f$ ,  $u_0$ , and  $g$  be given. Find  $(u, \lambda_1, \lambda_2)$  in  $V \times \mathcal{M}'_1 \times \mathcal{M}_2$  such that,

$$(P'_N) \quad \begin{cases} a_r(u, v) + b_1(v, \lambda_1) + b_2(v, \lambda_2) = h(v) & \forall v \in V \\ b_1(u, \mu_1) = e_1(\mu_1) & \forall \mu_1 \in \mathcal{M}'_1 \\ b_2(u, \mu_2) = e_2(\mu_2) & \forall \mu_2 \in \mathcal{M}_2 \end{cases} \quad (15)$$

with

$$\begin{cases} a_r(u, v) = \int_{\Omega} \sigma(u) : e(v) \, dx + r(u, v)_{\partial\omega_D}; h(v) = \int_{\Omega} \tilde{f}v \, dx + r(u_0, v)_{\partial\omega_D} \\ b_1(v, \mu_1) = \langle v, \mu_1 \rangle_1; e_1(\mu_1) = \langle u_0, \mu_1 \rangle_1 \\ b_2(v, \mu_2) = \langle \mu_2, \sigma(v) \cdot n \rangle_2; e_2(\mu_2) = \langle \mu_2, g \rangle_2 \end{cases} \quad (16)$$

## 2.6. The Unicity of the Solution

Using the LBB theorem [2], [5], [11], [16], it is easy to show the existence and unicity of the triplet  $(u, \lambda_1, \lambda_2)$  in  $V \times \mathcal{M}'_1 \times \mathcal{M}_2$ .

The problem  $(P'_N)$  is similar to the classical mixed formulations as presented by Babuska in [2].

## 3. THE WAVELET-GALERKIN METHOD

### 3.1. One Dimensional Daubechies Wavelets

In this section, we present a brief summary of one dimensional Daubechies wavelets [8].

Let  $M$  be a positive integer. There exists a sequence of real numbers  $\{h(n)\}_{n=0, \dots, 2M-1}$ , a function  $\varphi$  called the scaled function and a function  $\psi$ , orthogonal to the first one called the associated wavelet, such that,

$$\varphi(x) = \sqrt{2} \sum_{n=0}^{2M-1} h(n) \varphi(2x - n) \quad \forall x \in \mathbb{R}, \quad (17)$$

$$\psi(x) = \sqrt{2} \sum_{n=0}^{2M-1} g(n) \varphi(2x - n) \quad \forall x \in \mathbb{R}, \quad (18)$$

where  $g(n) = (-1)^n h(2M - 1 - n)$ ,  $n = 0, \dots, 2M - 1$ .

The support of  $\varphi$  and  $\psi$  are included in the closed and compact interval  $[0, 2M - 1]$ .

Let

$$\varphi_{j,k}(x) = 2^{j/2} \varphi(2^j x - k) \quad ; \quad \psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k) \quad \forall x \in \mathbb{R}, \forall j, k \in \mathbb{Z} \quad (19)$$

The functions  $\varphi_{j,k}$  and  $\psi_{j,k}$  verify,

$$\int_{\mathbb{R}} \varphi_{j,k}(x) \varphi_{j,\ell}(x) \, dx = \delta_{k\ell} \quad ; \quad \int_{\mathbb{R}} \psi_{j,k}(x) \psi_{j,\ell}(x) \, dx = \delta_{ij} \delta_{k\ell} \quad \forall i, j, k, \ell \in \mathbb{Z}. \quad (20)$$

We define  $V_j$  the closure of the space generated by  $\{\varphi_{jk}, k \in \mathbb{Z}\}$ . For all  $j \in \mathbb{Z}$ , this subspace  $V_j$  verifies the following conditions,

$$V_j \subset V_{j+1} \quad ; \quad \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \quad ; \quad L^2(\mathbb{R}) = \overline{\bigcup_{j \in \mathbb{Z}} V_j}. \quad (21)$$

The set of spaces  $\{V_j, j \in \mathbb{Z}\}$  is called a multiresolution analysis of  $L^2(\mathbb{R})$ .

### 3.2. Periodic Wavelets

Due to the form of the problem, it is necessary to define periodic wavelets and a multiresolution analysis of  $(L_p^2(\Omega))^2$ . First, we define the periodic scaled function  $\phi_{jk}$  by,

$$\phi_{jk}(x) = 2^{j/2} \sum_{n \in \mathbb{Z}} \varphi(2^j(x+n) - k) \quad j \geq 0 ; k = 0, \dots, 2^j - 1, \quad (22)$$

and the same for the periodic associated wavelet  $\psi$ . Then, we can define the periodic scale function on  $\mathbb{R}^2$  by,

$$\Phi_{k_1 k_2}^j(x_1, x_2) = \phi_{jk_1}(x_1) \phi_{jk_2}(x_2). \quad (23)$$

The closure spaces  $V_j^p$  generated by  $\{\Phi_{k_1 k_2}^j ; k_1, k_2 = 0, \dots, 2^j - 1\}$  is a multiresolution analysis of  $L_p^2(\Omega)$  and the dimension of  $V_j^p$  is  $2^{2j}$ . The bidimensional periodic associated wavelets are,

$$\Psi_{k_1 k_2}^{1j}(x_1, x_2) = \phi_{jk_1}(x_1) \psi_{jk_2}(x_2), \quad (24)$$

$$\Psi_{k_1 k_2}^{2j}(x_1, x_2) = \psi_{jk_1}(x_1) \phi_{jk_2}(x_2), \quad (25)$$

$$\Psi_{k_1 k_2}^{3j}(x_1, x_2) = \psi_{jk_1}(x_1) \psi_{jk_2}(x_2). \quad (26)$$

If we denote  $\underline{k}$  the couple  $\underline{k} = (k_1, k_2)$ , and  $\Lambda_j$  the subset of  $\mathbb{N}^2$ ,

$$\Lambda_j = \{\underline{k} \in \mathbb{N} : 0 \leq k_1 \leq 2^j - 1 ; 0 \leq k_2 \leq 2^j - 1\}, \quad (27)$$

then, if we denote  $\{\mathcal{V}_j, j \geq 0\}$  the multiresolution analysis of  $(L_p^2(\Omega))^2$ , an element  $u$  of  $\mathcal{V}_j$  is written,

$$u(x_1, x_2) = \begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \end{pmatrix} \quad \text{with} \quad u_d(x_1, x_2) = \sum_{\underline{k} \in \Lambda_j} u_d^{\underline{k}} \Phi_{\underline{k}}^j(x_1, x_2) \quad (28)$$

where  $d$  is the direction ( $d = 1, 2$ ). The dimension of a space  $\mathcal{V}_j$  is equal to  $2^{2j+1}$ .

**Remark.** If we denote by  $\{\mathcal{W}_j, j \geq 0\}$  the subspace of  $(L_p^2(\Omega))^2$  generated by,

$$\mathcal{W}_j = \{\Psi_{\underline{k}}^{\ell j} ; \ell = 1, 2, 3 ; \underline{k} \in \Lambda_j\}, \quad (29)$$

then, for all  $j \geq 0$ , we have,

$$\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j \quad ; \quad (L_p^2(\Omega))^2 = \mathcal{V}_{j_0} \overline{\bigoplus_{j \geq j_0} \mathcal{W}_j} \quad (30)$$

### 3.3. Elastostatics Discretization

In this section, we discretize the problem  $(P'_N)$  using a Galerkin method written in the wavelet basis defined above. We first need to compute the projection into a space  $\mathcal{V}_j$  of the plane elasticity operator,

$$a(u, v) = \int_{\Omega} \mathbb{K} e(u) : e(v) dx \quad \text{for all } u, v \in V \quad (31)$$

Then we obtain the proposition below.

**Proposition 1.** *The projection of the plane elasticity operator into the wavelet basis with scale equal to  $j$  is given by a matrix  $\mathcal{K}$  of order  $2^{2j+1}$  defined by  $[\mathcal{K}_{(i_1, i_2)(\ell_1, \ell_2)}] = [\mathcal{K}_{\underline{i}, \underline{\ell}}]_{\underline{i}, \underline{\ell} \in \Lambda_j}$ , where  $\mathcal{K}_{\underline{i}, \underline{\ell}}$  is an “elementary matrix” of order 2,*

$$\mathcal{K}_{\underline{i}, \underline{\ell}} = \begin{bmatrix} F_{11}^{11} + F_{33}^{22} & F_{12}^{12} + F_{33}^{21} \\ F_{21}^{21} + F_{33}^{12} & F_{22}^{22} + F_{33}^{11} \end{bmatrix}, \quad (32)$$

where

$$F_{\underline{K}}^{\underline{\Delta}} = F_{\underline{K}, \underline{i}, \underline{\ell}}^{\underline{\Delta}, \underline{J}} = \int_{\Omega} \mathbb{K}_{pq}(x) \frac{\partial \Psi_{\underline{i}}^{j_1}}{\partial x_{\alpha}}(x) \frac{\partial \Psi_{\underline{\ell}}^{j_2}}{\partial x_{\beta}}(x) dx \quad (33)$$

with

$$\begin{aligned} \underline{K} &= (p, q) & p, q &= 1, 2 \text{ or } 3; \\ \underline{\Delta} &= (\alpha, \beta) & \alpha, \beta &= 1 \text{ or } 2 && \text{(direction of the derivative);} \\ \underline{J} &= (j_1, j_2) & j_0 &\leq j_1, j_2 \leq j && \text{scale of each wavelet (or scale function).} \end{aligned}$$

For the proof, see [9].

To compute the terms of the matrix  $\mathcal{K}$ , if each  $\mathbb{K}_{pq}$  is a constant function (as in the case of only one material), equation (33) gives us

$$F_{\underline{K}, \underline{i}, \underline{\ell}}^{\underline{\Delta}, \underline{J}} = \mathbb{K}_{pq} \mathcal{A}_{i_1, \ell_1}^{\underline{\Delta}_1, \underline{J}_1} \mathcal{A}_{i_2, \ell_2}^{\underline{\Delta}_2, \underline{J}_2} \quad (34)$$

where

$$\mathcal{A}_{i_1, \ell_1}^{\underline{\Delta}_1, \underline{J}_1} = \int_0^1 \psi_{j_1 i_1}^{(n_1)}(x_1) \psi_{j_1 \ell_1}^{(m_1)}(x_1) dx_1 \quad (\text{ditto for } \mathcal{A}_{i_2, \ell_2}^{\underline{\Delta}_2, \underline{J}_2}) \quad (35)$$

with

$$\begin{aligned}
&\text{if } \alpha = 1 \text{ then } (n_1, n_2) = (1, 0); \\
&\text{if } \alpha = 2 \text{ then } (n_1, n_2) = (0, 1); \\
&\text{if } \beta = 1 \text{ then } (m_1, m_2) = (1, 0); \\
&\text{if } \beta = 2 \text{ then } (m_1, m_2) = (0, 1),
\end{aligned} \tag{36}$$

$\underline{\Delta}_1 = (n_1, m_1)$ ;  $\underline{\Delta}_2 = (n_2, m_2)$ , and  $f^{(n)}$  is the derivative of order  $n$ .

Thus, we have to compute the scalar product of two wavelets and their derivatives. The algorithm to compute these terms is based on theorems given by G. Beylkin [3],[4] and Dahmen-Micchelli [6]. These terms are computed via filter banks as eigenvectors of a low order matrix.

In the case of several materials, see [9] for the computation of  $F_{\underline{\Delta}_i, \underline{\Delta}_j}^{\underline{\Delta}_i, \underline{\Delta}_j}$ . As before, these terms are computed via filter banks as eigenmatrices of a low order tensor.

#### 4. WAVELETS AND BOUNDARY CONDITIONS

We now have to compute the function and operators defined in (16) and especially the boundary terms. This computation is based on the works of Wells-Zhou [17]. The main idea here will be the following: We can represent the function and the measure defined on the boundary in terms of wavelet series defined in a square region and the coding is *independent of the geometry* of the domain  $\omega$ .

First, assume that  $\omega$  is a bounded domain in  $\mathbb{R}^2$  whose boundary  $\partial\omega$  can be written as,

$$\partial\omega = \{x \in \mathbb{R}^2 : F(x) = 0\} \tag{37}$$

for some Lipschitz function  $F$ . Then, the unit normal vector  $n$  along the boundary  $\partial\omega$  can be written as,

$$n = \frac{\nabla F}{|\nabla F|}. \tag{38}$$

The idea is now to compute an integral defined on  $\partial\omega$  from an integral defined on  $\mathbb{R}^2$ . Let assume we can extend  $f$  (defined on  $\partial\omega$ ) to  $\mathbb{R}^2$ , then we have,

$$\int_{\partial\omega} f ds = - \int_{\mathbb{R}^2} f \|\partial\omega\| \tag{39}$$

where  $\|\partial\omega\|$  is the boundary measure defined by,

$$\|\partial\omega\| = -\nabla\chi_\omega \cdot n \tag{40}$$

( $\chi_\omega$  is the characteristic function of  $\omega$ :  $\chi_\omega(x) = 1$  if  $x \in \omega$ ,  $= 0$  otherwise). For more details, see [15].

The functions  $\chi_\omega$  are decomposed in the wavelet basis and so it is not difficult to compute  $\nabla\chi_\omega$  and  $n$  with the coefficients  $\mathcal{A}$  defined in (35), then  $\nabla\chi_\omega \cdot n$ . Finally, when the wavelet

coefficients of the measure  $\|\partial\omega\|$  are obtained, the computation of the terms  $\int_{\mathbb{R}^2} f \nabla \chi_\omega \cdot n$  are similar to the computation of the terms of the stiffness matrix [17].

**Remark.** The wavelets used in the discretizations of the measure  $\|\partial\omega\|$ ,  $u_0$  and  $g$  are the same of those used in the discretization of  $f$  and the matrix  $\mathcal{K}$ .

## 5. NUMERICAL ALGORITHMS

### 5.1. The Uzawa Method

The Uzawa algorithm is an iterative method to solve a linear system like (15). It consists of: first, finding the wavelet coefficients of the displacement associated with given multipliers coefficients by solving a linear system; then, reactualizing the multipliers coefficients with those of the displacement; and finally, going to the first step while the residuous is greater than a chosen threshold  $\varepsilon$ . In our example, this algorithm can be summarized by the following diagram:

(1)  $\gamma^0 \in \mathbb{R}^{L_j^\lambda}$  be given;

(2) minimization on  $v$ , which leads to solve ( $k \geq 1$ ):

*Find  $u^k \in \mathbb{R}^{N_j}$  such that:*

$$A_r u^k = F_r - B\gamma^k ;$$

(3) Actualization of the multiplier:

$$\gamma^{k+1} = \gamma^k + \rho_k (B u^k - U_0)$$

(4) If  $\|u^{k+1} - u^k\|_{L^2(\mathbb{R}^{N_j})} \geq \varepsilon$ , go to (2).

**Remark.** The parameter  $\rho_k$  will be constant equal to  $\rho$ , and we can show that if  $0 < \rho \leq 2r/\|P_2^\dagger\|^2$  where  $P_2$  is the projection from  $H$  into  $\mathcal{M}'_1$ , the method converges to the solution [14].

On each step, we solve the linear system with the conjugate gradient method, described below.

### 5.2. The Conjugate Gradient

The conjugate gradient method is an iterative method to solve symmetric linear system. This method can be described with the following schema:

We initiate with,

$$\begin{cases} x_0 \text{ be chosen;} \\ p_0 = r_0 = b - Ax_0 \end{cases} \quad (41)$$

For  $k = 0, 1, \dots$

$$\begin{aligned} \alpha_k &= \frac{\|r_k\|^2}{(Ap_k, p_k)}; & x_{k+1} &= x_k + \alpha_k p_k & r_{k+1} &= r_k - \alpha_k Ap_k \\ \beta_{k+1} &= \frac{\|r_{k+1}\|^2}{\|r_k\|^2}; & p_{k+1} &= r_{k+1} + \beta_{k+1} p_k \end{aligned} \quad (42)$$

**Remark.** The algorithm needs  $2cN$  operations for each step, where  $N$  is the order of the matrix  $A$ , and  $c$  the average number of not vanishing coefficients by line.

The conjugate gradient converges in at most  $N$  iterations if the matrix's order is equal to  $N$ , and the rate of the convergence depends on the condition number of the matrix  $A$ .

From the second step onward of the Uzawa method, the conjugate gradient initialization is made with the displacement coefficients solution of the previous step.

In order to evaluate the condition number of matrix  $\mathcal{K}$  and the convergence of the conjugate gradient method, we have chosen not to use a preconditioner (diagonal for example, as it is done in works of Dahmen [7]).

## 6. NUMERICAL RESULTS

### 6.1. Boundary Integrals

We present in this section the computation of perimeters for different geometries: a circle, a diamond, and a square. These results enable us to validate the computations of the terms on the boundary given by equations (38), (39), (40).

The geometries are,

- a circle defined by  $(x - 0.5)^2 + (y - 0.5)^2 < R$ ;
- a diamond defined by  $|x - 0.5| - |y - 0.5| < R$ ;
- a square defined by  $Sup(|x - 0.5|, |y - 0.5|) < R$ .

with the value of the radius ranging from  $R = 0.2$  to  $R = 0.45$ .

For the computations, the different geometries are included in a squared box with the edge equal to 1. In all the following, we will use the Daubechies wavelets of order 3 at the level  $j$  equal to 5 (2048 degrees of freedom). The results are given in table 1.

The results obtained for the circle are quite precise. We observe on the one hand that we need to choose a step sufficiently small to correctly describe the circle and on the other hand it is necessary to chose a sufficiently large box to avoid problems on the edges of the circle.

Table 1  
Relative errors on the computed perimeters

Circle		Diamond		Square		
Ray	Perimeter	Error	Perimeter	Error	Perimeter	Error
R=0.20	1.2567	$6.4 \cdot 10^{-5}$	1.0493	$7.2 \cdot 10^{-2}$	1.5758	$1.5 \cdot 10^{-2}$
R=0.25	1.5708	$1.2 \cdot 10^{-5}$	1.2258	$1.3 \cdot 10^{-1}$	1.9868	$6.5 \cdot 10^{-3}$
R=0.30	1.8850	$6.3 \cdot 10^{-5}$	1.5837	$6.6 \cdot 10^{-1}$	2.3667	$1.3 \cdot 10^{-2}$
R=0.40	2.5135	$1.1 \cdot 10^{-4}$	2.1137	$6.5 \cdot 10^{-1}$	3.1747	$7.8 \cdot 10^{-3}$
R=0.45	2.8300	$9.3 \cdot 10^{-4}$	2.4609	$3.3 \cdot 10^{-1}$	3.6943	$2.6 \cdot 10^{-2}$

Results are less precise for the two other cases due to the nondifferentiability of the boundary. These results are similar to those obtained in [17].

## 6.2. An Analytical Example

We present the following example,

$$(P_\omega) \quad \begin{cases} -\operatorname{div}(\mathbb{K}e(u)) = f & \text{in } \omega \\ u = u_0 & \text{on } \partial\omega \end{cases}$$

with

$$f(x, y) = \begin{cases} f_1(x, y) = 4\pi^2 [\mathbb{K}_{11} + \mathbb{K}_{12} + 2 \mathbb{K}_{33}] \sin(2\pi x) \cos(2\pi x) \\ f_2(x, y) = 4\pi^2 [\mathbb{K}_{12} + \mathbb{K}_{22} + 2 \mathbb{K}_{33}] \cos(2\pi x) \sin(2\pi y) \end{cases} \quad (43)$$

and

$$u_0(x, y) = \begin{cases} u_1^0(x, y) = \sin(2\pi x) \cos(2\pi y) \\ u_2^0(x, y) = \cos(2\pi x) \sin(2\pi y) \end{cases} \quad (44)$$

where  $\omega$  is a disc with the radius ranging from  $R = 0.2$  to  $R = 0.4$ , included in a squared box with the edge equal to 1. The solution  $u$  (in  $\Omega$ ) of this problem is,

$$u(x, y) = \begin{cases} u_1(x, y) = \sin(2\pi x) \cos(2\pi y) \\ u_2(x, y) = \cos(2\pi x) \sin(2\pi y) \end{cases} \quad (45)$$

Table 2 and figure 2 show the efficiency and the precision of the method.

Table 2  
 Number of iterations and error ( $r = 10^6, \rho = 5.10^4, \varepsilon_{Uzawa} = 10^{-4}$ )

Ray	Uzawa	C.G. iterations	Error iterations
R=0.2	31	241	$1.10^{-2}$
R=0.3	29	152	$5.10^{-3}$
R=0.4	20	197	$4.10^{-3}$

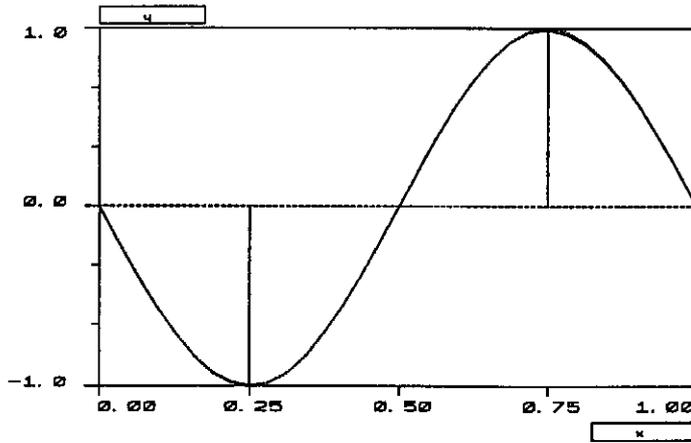


Fig. 2: Section of the computed displacement  $u_1(x_1, 0.5)$

The maximum of the precision is obtained for a ratio disc radius-square edge equal to 0.4.

### 6.3. A Numerical Example

We present the example of a circle with radius equal to 0.25 included in a squared box with edge equal to 1. The problem is given by the following equations,

$$(P_\omega) \quad \begin{cases} -\operatorname{div}(\mathbb{K}e(u)) = 0 & \text{in } \omega \\ u = \alpha n & \text{on } \partial\omega \end{cases} \quad \text{with } \alpha = 0.1 \quad (46)$$

$E = 72GPa, \nu = 0.32$  (Aluminium).

We present in figures 2 to 4 the numerical results.

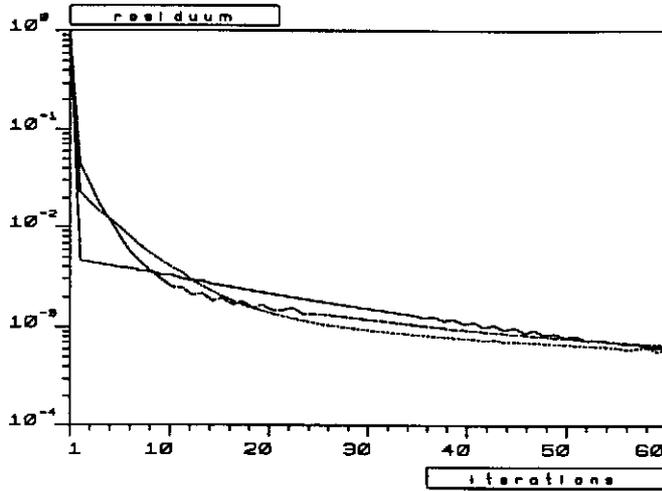


Fig. 3: Residuum along the Uzawa iterations

Figure 3 shows the fluctuations of the residuum during the iterations for  $\rho$  equal to  $10^4$ ,  $5 \cdot 10^4$  and  $10^6$  ( $r = 10^7$ ). It is well known that the Uzawa algorithm has only a linear convergence [14].

The linear system is solved by a preconditioned (SSOR) conjugate gradient method. We show in the figure 3 the computed displacement with Daubechies wavelets of order 3.

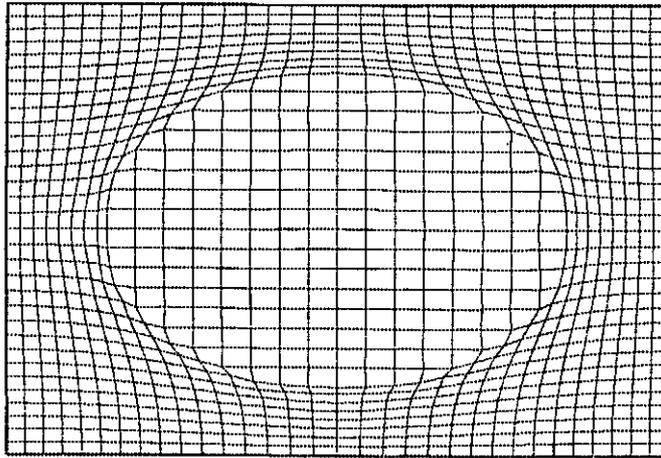


Fig. 4: Computed displacement

Figure 4 shows a comparison between the exact solution and the computed displacement  $u_1$  for  $y = 0.5$ . A good precision is obtained.

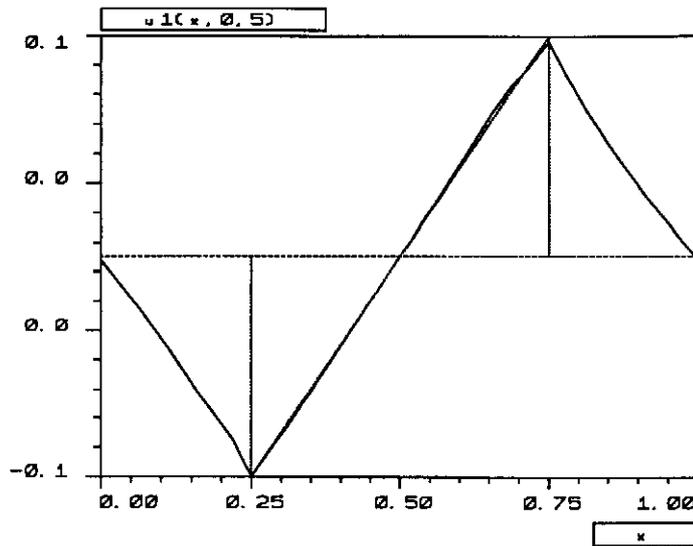


Fig. 5: Section of the displacement

## 7. CONCLUSION

We have developed an original algorithm to solve elastostatics problems: a wavelet-Galerkin method coupled with augmented Lagrangian.

Numerical results show the efficiency of the method and the difficulty in optimizing the different parameters. The method seems to be less efficient near the boundary than a finite elements computation, but adaptativity inside the domain  $\omega$  can be obtained very easily. Other advantages of the wavelets are the lack of mesh and that the structure has only to be defined by a function characteristic of the boundary. This algorithm is promising and will permit us to treat more complex problems (real structures, non linear problems, ...).

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