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# ON PLANAR RANDOM WALKS IN ENVIRONMENTS INVARIANT BY HORIZONTAL TRANSLATIONS

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## Abstract

We give a recurrence criterion for a model of planar random walk in environment invariant under horizontal translations. Some examples are next developed, for instance when the environment is produced by a dynamical system.

## 1 Introduction

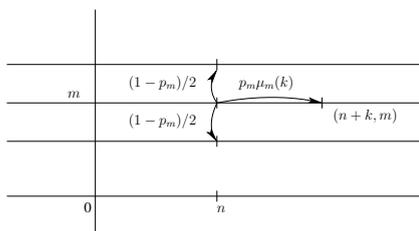
### 1.1 Presentation

We study the qualitative asymptotic behavior of a discrete time Markov chain  $(X_n, Y_n)_{n \geq 0}$  on the lattice  $\mathbb{Z}^2$ . Vertical jumps are nearest neighbors, with equal probability, and the environment is invariant under horizontal translations. To precise the model, introduce parameters  $(p_n)_{n \in \mathbb{Z}}$  and probability measures  $(\mu_n)_{n \in \mathbb{Z}}$  with support on  $\mathbb{Z}$ , such that for some  $\delta > 0$  and all  $n \in \mathbb{Z}$  :

$$\delta \leq p_n \leq 1 - \delta, \quad \mu_n(0) \leq 1 - \delta \quad \text{and} \quad \sum_{k \in \mathbb{Z}} |k|^3 \mu_n(k) \leq 1/\delta.$$

The last condition is introduced for having second order expansion of the characteristic functions uniform in  $n \in \mathbb{Z}$ . Letting  $X_0 = Y_0 = 0$ , the Markovian evolution is given by :

$$\mathbb{P}((n, m) \rightarrow (n+k, m)) = p_m \mu_m(k) \quad \text{and} \quad \mathbb{P}((n, m) \rightarrow (n, m \pm 1)) = (1 - p_m)/2.$$



When  $p_m = 1/2$  and  $\mu_m = (\delta_1 + \delta_{-1})/2$ , we recover simple random walk in  $\mathbb{Z}^2$ . An important submodel, a model with oriented horizontal lines, was introduced by Campanino and Petritis [1]. It corresponds to taking  $p_n = p \in (0, 1)$  and  $\mu_n = \delta_{\varepsilon_n}$ , where  $(\varepsilon_n)_{n \in \mathbb{Z}}$  is a sequence of  $\pm 1$ .

Let us detail known results, principally regarding this last model. For simplicity we say “i.i.d” for “a typical realization of an i.i.d. sequence”. Campanino and Petritis [1] prove recurrence when  $\varepsilon_n = (-1)^n$  and transience for  $\varepsilon_n = 1_{n \geq 0} - 1_{n < 0}$  and when the  $(\varepsilon_n)$  are i.i.d.. Guillotin-Plantard and Le Ny [5] show transience results when the  $(\varepsilon_n)$  form an independent family with marginals described by some dynamical system. Pene [8] also prove transience when the  $(\varepsilon_n)$  are stationary, under a decorrelation condition. Castell, Guillotin-Plantard, Pene and Schapira [3] consider a

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similar model where the  $(\varepsilon_n)$  are i.i.d. and the vertical steps follow some measure  $\mu$  in the domain of attraction of a centered stable distribution. A very precise analysis of the annealed return times to 0 is performed, implying transience in this case. Devulder and Pene [4] consider an extension of the model with oriented lines and establish transience when the  $(p_n)$  form an i.i.d. non-constant sequence and  $\mu_n = \delta_{\varepsilon_n}$ , with arbitrary  $(\varepsilon_n)$ . Recently Campanino and Petritis [2], still considering the oriented lines model, study a random perturbation of the periodic case, where the perturbation decreases via some power law as the vertical coordinate increases. They show the existence of a critical parameter for the question recurrence/transience.

One may observe that most results on the model with oriented lines concern transience. As we shall detail later, the asymptotics of the random walk is related to the growth of the sums  $(\varepsilon_1 + \dots + \varepsilon_n)$  and the frontier between recurrence and transience lies at logarithmic scales. Recurrence is therefore rare when the  $(\varepsilon_n)$  are stationary and a little of stochasticity is assumed.

## 1.2 Results

Let us present our results. Let  $\eta_i = m_i p_i / (1 - p_i)$ , where  $m_i$  is the expectation of  $\mu_i$ . This quantity can be interpreted when grouping in packets the successive horizontal steps of the random walk. Precisely, when arriving at the horizontal line  $i$ , the number  $\Gamma_i$  of steps on this line follows a geometrical law of parameter  $p_i$ , whereas the displacement is  $J_i = \sum_{1 \leq k \leq \Gamma_i} \xi_k$ , where the  $(\xi_k)$  are i.i.d. with law  $\mu_i$  and are independent of  $\Gamma_i$ . Then  $\eta_i = \mathbb{E}(J_i)$ . For instance, in the context of the oriented lines model we have  $\eta_i = (p/(1-p))\varepsilon_i$ . Set next :

$$R_k^l = \sum_{k \leq i \leq l} \eta_i,$$

for  $k \leq l$  and  $R_k^l = 0$  when  $l < k$ . The main quantities of interest for the sequel are the following.

**Definition 1.1** *Introduce the strictly increasing functions :*

$$\varphi(n) = \sqrt{n^2 + \sum_{-n \leq k \leq l \leq n} (R_k^l)^2} \text{ and } \varphi_+(n) = \sqrt{n^2 + \sum_{-n \leq k \leq l \leq n, kl > 0} (R_k^l)^2}.$$

For large  $x > 0$ , let  $\varphi^{-1}(x)$  be the unique integer  $n$  so that  $\varphi(n) \leq x < \varphi(n+1)$ . Idem for  $\varphi_+^{-1}(x)$ .

*Remark.* — In general,  $n \leq \varphi_+(n) \leq \varphi(n) \leq Cn^2$ , so  $c\sqrt{n} \leq \varphi^{-1}(n) \leq \varphi_+^{-1}(n) \leq n$ . An important property, detailed later, is that  $\varphi^{-1}$  and  $\varphi_+^{-1}$  are examples of regularly varying functions  $f$  in the sense that for all  $C \geq 1$  there exists  $K > 0$  so that  $f(Cx) \leq Kf(x)$ .

**Theorem 1.2** *The random walk is recurrent if and only if :*

$$\sum_{n \geq 1} \frac{1}{n^2} \frac{(\varphi^{-1}(n))^2}{\varphi_+^{-1}(n)} = +\infty.$$

*Remark.* — For simple random walk,  $R_k^l = 0$ , giving  $\varphi_+^{-1}(n) = \varphi^{-1}(n) = n$ . Recurrence hence follows from the divergence of the harmonic series. Notice that the generic term in the sum is always less than or equal to  $n^{-2} \varphi_+^{-1}(n) \leq n^{-1}$  and that simple random walk realizes equality. In this rough sense, all other random walks in the class considered are less recurrent.

*Remark.* — It is quite clear from the theorem that the asymptotics is not modified when changing a finite number of lines. Indeed, one gets some new  $(\tilde{R}_k^l)$  which check  $(\tilde{R}_k^l)^2 \leq 2(R_k^l)^2 + C$ . This gives  $\tilde{\varphi}(n) \leq C'\varphi(n)$ , with also a similar symmetric inequality, and the same is true for  $\varphi_+$ . The regular variation properties of the inverse functions allow to conclude.

The difference of order between  $\varphi$  and  $\varphi_+$  is related to symmetry reasons concerning the two half sequences  $(\eta_n)_{n \geq 1}$  and  $(\eta_n)_{n \leq -1}$ . Symmetry gives more transience, whereas antisymmetry implies more recurrence. This appears in the following statement.

**Corollary 1.3**

i) In the general case, a sufficient condition for transience is :

$$\sum_{n \geq 1} \frac{1}{\sqrt{n^2 + \sum_{-n \leq k \leq l \leq n} (R_k^l)^2}} < +\infty.$$

ii) Antisymmetric model :  $\eta_{-n} = -\eta_n$ ,  $n \geq 1$ . The random walk is recurrent if and only if :

$$\sum_{n \geq 1} \frac{1}{\sqrt{n^2 + \sum_{1 \leq k \leq l \leq n} (R_k^l)^2}} = +\infty.$$

iii) Symmetric model :  $\eta_{-n} = \eta_n$ ,  $n \geq 1$ . The random walk is transient if :

$$\sum_{n \geq 1} \frac{1}{\sqrt{n^2 + n \sum_{1 \leq k \leq n} (R_1^k)^2}} < +\infty.$$

*Remark.* — Notice that the symmetric model can be seen as a model on  $\mathbb{Z} \times \mathbb{N}$  with reflection on the horizontal axis  $\mathbb{Z} \times \{0\}$ . Let us point out that the conditions appearing in ii) and iii) are not equivalent due to the following examples.

**Proposition 1.4**

i) In the general case, if  $|R_1^n| + |R_{-n}^{-1}| = O(\log^{1/2} n)$ , then the random walk is recurrent.

ii) In the antisymmetric case, if  $|R_1^n| = O(\log n)$ , then the random walk is recurrent.

iii) Let  $1 < \alpha \leq 2$  and suppose that  $(\eta_n)_{n \geq 0}$  satisfies  $|R_1^n - \log^\alpha n| = O(\log^{\alpha-1} n)$ .

- Let  $(\eta'_n)_{n \in \mathbb{Z}}$  be such that  $\eta'_n = \eta_{|n|}$ . Then the random walk is transient.
- Let  $(\eta''_n)_{n \in \mathbb{Z}}$  be such that  $\eta''_n = \text{sign}(n)\eta_n$ ,  $n \neq 0$ ,  $\eta''_0 = \eta_0$ . Then the random walk is recurrent.

*Remark.* — The second case in item iii) is particularly interesting, because the random walk is recurrent but, roughly speaking, cannot come back when staying only North or only South. This implies an oscillating behavior (perhaps of sinusoidal or spiral type) for the random walk which would be interesting to quantify.

We next investigate the case when the  $(\eta_n)$  are produced by some dynamical system, first of all of quasi-periodic type and next with more randomness.

**Proposition 1.5** Take  $p_n = p \in (0, 1)$  and  $\mu_n = \delta_{\varepsilon_n}$  with  $\varepsilon_n = 1_{[0,1/2)}(n\alpha) - 1_{[1/2,1)}(n\alpha)$ . Suppose that  $\alpha \notin \mathbb{Q}$  has a continued fraction expansion  $[a_1, a_2, \dots]$  verifying :

$$\sum_{n \geq 1} \frac{\log(1 + a_n)}{a_1 + \dots + a_n} = +\infty.$$

Then the random walk is recurrent.

*Remark.* — The proof relies on the antisymmetry of  $x \mapsto 1_{[0,1/2)}(x) - 1_{[1/2,1)}(x)$  and works without change for  $\varepsilon_n = 1_I(n\alpha) - 1_{\mathbb{T} \setminus I}(n\alpha)$ , where  $I$  is a finite union of intervals on the torus  $\mathbb{T}$  with  $\mathbb{T} \setminus I = -I$ . The case of  $\varepsilon_n = 1_{[0,1/2)}(n\alpha + x) - 1_{[1/2,1)}(n\alpha + x)$  for a general  $x \in \mathbb{R} \setminus \mathbb{Z}$ , for example  $x = 1/4$ , is more difficult, because we are in the critical zone. A specific study is a priori required, with finer estimates, especially lower bounds, on the ergodic sums associated to  $(\varepsilon_n)$ . Notice that the condition on  $\alpha$  is verified when the  $(a_n)$  are bounded. Also, defining the type of  $\alpha$  as  $\eta(\alpha) = \sup\{s \geq 1 \mid \liminf n^s \text{dist}(n\alpha, \mathbb{Z}) = 0\}$ , it is possible to have  $\eta(\alpha) = a$  for any  $a \in [1, +\infty]$  together with the condition of the proposition verified (one may simply choose  $a_n$  arbitrary large on a very sparse subsequence).

For the other application, we suppose that the  $(\eta_n)$  are a realization of strictly stationary process (meaning that for any  $b \geq 0$  the law of  $(\eta_n)_{a \leq n \leq a+b}$  is independent on  $a \in \mathbb{Z}$ ). For  $n \geq 1$ , let us introduce the strong mixing coefficient :

$$\alpha(n) = \sup_{\substack{k \geq 1, A \in \sigma(\eta_1, \dots, \eta_k) \\ B \in \sigma(\eta_l, l \geq k+n)}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

The following proposition extends results in [1] and [4] relying on independence.

**Proposition 1.6** *Let  $\eta_n = \eta_n(\omega)$ , where the  $(\eta_n)_{n \in \mathbb{Z}}$  form a strictly stationary process. Suppose that  $\mathbb{E}(\eta_0) = 0$  and that for some  $\beta > 1$  and  $\gamma > 1$  :*

$$\alpha(n) = O(\log^{-\gamma} n) \text{ and } \liminf \frac{1}{N} \sum_{1 \leq n \leq N} \mathbb{P}(|R_1^n| \geq \log^\beta n) > 0.$$

*Then the random walk is transient for almost-every  $\omega$ .*

*Remark.* — Mention that if replacing  $\log^\beta n$  by  $n^\varepsilon$  in the statement of the proposition, then  $\gamma > 0$  is sufficient in the proof. The second condition is satisfied if a central limit theorem is verified.

### 1.3 Notations

- The law of a random variable  $\Gamma$  is  $\mathcal{G}(p)$ ,  $0 < p < 1$ , if  $\mathbb{P}(\Gamma = m) = (1-p)p^m$ ,  $m \geq 0$ .
- Let  $\chi_n(t) = \sum_{k \in \mathbb{Z}} e^{itk} \mu_n(k)$ ,  $m_n = \sum_{k \in \mathbb{Z}} k \mu_n(k)$ ,  $m_{2,n} = \sum_{k \in \mathbb{Z}} k^2 \mu_n(k)$ ,  $\text{var}_n = m_{2,n} - m_n^2$ .
- For  $n \in \mathbb{Z}$ , let  $\varphi_n(t)$  be the characteristic function of  $\sum_{1 \leq k \leq \Gamma} \xi_k$ , where the  $(\xi_k)_{k \geq 1}$ ,  $\Gamma$  are independent with  $\xi_k \sim \mu_n$  and  $\Gamma \sim \mathcal{G}(p_n)$ . We have :

$$\varphi_n(t) = \frac{1 - p_n}{1 - p_n \chi_n(t)} = r_n(t) e^{ia_n(t)},$$

with argument (odd) and modulus (even), with  $O$  uniform in  $n \in \mathbb{Z}$  :

$$\begin{cases} a_n(t) &= \arctan\left(\frac{p_n \text{Im}(\chi_n(t))}{1 - p_n \text{Re}(\chi_n(t))}\right) = t \frac{p_n}{1 - p_n} m_n + O(t^3), \\ r_n(t) &= \frac{1 - p_n}{(1 + p_n^2 |\chi_n|^2 - 2p_n \text{Re}(\chi_n(t)))^{1/2}} = 1 - \frac{t^2}{2} \frac{p_n}{(1 - p_n)^2} (m_{2,n} - p_n \text{var}_n) + O(t^3). \end{cases} \quad (1)$$

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## 2 Reductions

### 2.1 A one-dimensional problem

As the second coordinate  $(Y_n)_{n \geq 0}$  is simple random walk on  $\mathbb{Z}$  (with waiting times at each site), the random walk is vertically recurrent. Introduce  $0 = \sigma_0 < \tau_0 < \sigma_1 < \tau_1 < \dots$ , where:

$$\tau_k = \min\{n > \sigma_k \mid Y_n \neq 0\}, \quad \sigma_{k+1} = \{n > \tau_k \mid Y_n = 0\}.$$

Set  $D_n = X_{\sigma_n} - X_{\sigma_{n-1}}$ . The  $(D_n)_{n \geq 1}$  are independent and identically distributed, because the environment is invariant under horizontal translations.

**Lemma 2.1** *The random walk  $(X_n, Y_n)_{n \geq 0}$  is recurrent if and only  $(\sum_{k=1}^n D_k)_{n \geq 1}$  is recurrent.*

*Proof of the lemma :*

We follow lemma 2.9 in [1]. If  $(\sum_{k=1}^n D_k)_{n \geq 1}$  is recurrent, then so is  $(X_n, Y_n)$ , because  $(X_{\sigma_n}, Y_{\sigma_n}) = (\sum_{k=1}^{\sigma_n} D_k, 0)$ . In case of transience, using the invariance of the environment by horizontal translations, there is a constant  $C$  so that :

$$\sum_{n \geq 1} \mathbb{P}(X_{\sigma_n} = x) \leq C, \quad x \in \mathbb{Z}.$$

Let  $((\xi_k)_{k \geq 1}, \Gamma)$  be independent (and also from the  $(X_{\sigma_n})$ ) such that  $\xi_k \sim \mu_0$  and  $\Gamma \sim \mathcal{G}(p_0)$ . Set  $H = \sum_{1 \leq k \leq \Gamma} \xi_k$ . We have :

$$\begin{aligned} \sum_{n \geq 1} \mathbb{P}(0 \in (X_{\sigma_n}, X_{\tau_n-1})) &= \sum_{n \geq 1} \mathbb{P}(0 \in (X_{\sigma_n}, X_{\sigma_n} + H)) \leq \sum_{x \in \mathbb{Z}} \sum_{n \geq 1} \mathbb{P}(X_{\sigma_n} = x) \mathbb{P}(H \geq |x|) \\ &\leq C \sum_{x \in \mathbb{Z}} \mathbb{P}(H \geq |x|) < +\infty, \end{aligned}$$

because  $H$  is integrable. As  $(X_n, Y_n)$  can be 0 only for  $n$  in some  $[\sigma_k, \tau_k - 1]$ , we conclude that  $(X_n, Y_n)$  is transient and this completes the proof of the lemma.  $\square$

We thus concentrate on the recurrence properties of  $(\sum_{k=1}^n D_k)_{n \geq 1}$ . Set  $D = D_1$  and :

$$\chi_D(t) = \mathbb{E}(e^{itD}), \quad t \in \mathbb{R}.$$

In case of aperiodicity, i.e. when the support of the law of  $D$  generates  $\mathbb{Z}$ , an analytical recurrence theorem of Kesten and Spitzer, cf Spitzer [9], says that  $(\sum_{k=1}^n D_k)_{n \geq 1}$  is transient if and only if :

$$\int_0^\eta \operatorname{Re} \left( \frac{1}{1 - \chi_D(t)} \right) dt < +\infty, \quad \text{for some } \eta > 0. \quad (2)$$

Our hypotheses imply that the subgroup of  $(\mathbb{Z}, +)$  generated by the support of the law of  $D$  is not reduced to  $\{0\}$ . Up to considering some sublattice  $m\mathbb{Z} \times \mathbb{Z}$ ,  $m > 0$ , we assume aperiodicity.

We therefore focus on the estimation of  $\chi_D$  near the origin, which is now the only singularity of  $1/(1 - \chi_D)$ . We fix  $0 < t < \eta$  and omit the dependence in  $t$  to lighten the notations.

## 2.2 Local time and contour of a Galton-Watson tree

For simplicity, we first change notations. Denote now by  $(Y_n)_{n \geq 0}$  simple symmetric random walk on  $\mathbb{Z}$ , starting at 0. Let  $\sigma = \min\{k \geq 1 \mid Y_k = 0\}$  be the return time to 0. As in [1], we later make use of the standard properties  $\mathbb{P}(\sigma > x) = O(1/\sqrt{x})$  and :

$$\mathbb{E}(s^\sigma) = 1 - \sqrt{1 - s^2}, \quad 0 \leq s \leq 1. \quad (3)$$

Grouping the successive horizontal steps of the random walk, observe that  $D$  can be written as :

$$D = \sum_{k=0}^{\sigma-1} \left( \sum_{u=1}^{\Gamma_k} \xi_u^{(k)} \right),$$

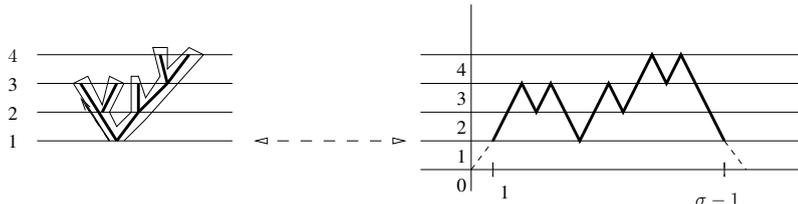
where, conditionally on the  $(Y_l)$ , the  $((\xi_u^{(k)})_{u \geq 1, k \geq 0}, (\Gamma_k)_{k \geq 0})$  are independent with  $\xi_u^{(k)} \sim \mu_{Y_k}$  and  $\Gamma_k \sim (\mathcal{G}(p_{Y_k}))$ . This furnishes :

$$\begin{aligned} \chi_D(t) &= \mathbb{E}(\mathbb{E}(e^{itD} \mid (Y_l))) = \mathbb{E} \left( \prod_{k=0}^{\sigma-1} \mathbb{E} \left( e^{it \sum_{u=1}^{\Gamma_k} \xi_u^{(k)}} \mid (Y_l) \right) \right) = \mathbb{E} \left( \prod_{k=0}^{\sigma-1} \varphi_{Y_k}(t) \right) \\ &= \frac{1}{2} \mathbb{E} \left( \varphi_0(t) \prod_{n \geq 1} (\varphi_n(t))^{N_n} \right) + \frac{1}{2} \mathbb{E} \left( \varphi_0(t) \prod_{n \geq 1} (\varphi_{-n}(t))^{N_n} \right), \end{aligned} \quad (4)$$

where we have distinguished between positive and negative excursions and set :

$$N_n = \#\{0 < k < \sigma, |Y_k| = n\}.$$

To detail  $N_n$ , recall first that positive excursions of simple random walk can be described via the contour process of a critical Galton-Watson tree  $(Z_n)_{n \geq 1}$ , with root  $Z_1 = 1$  and offspring distribution  $\mathcal{G}(1/2)$ . See for instance Le Gall [6]. This is illustrated by the following picture :



As shown in the left-hand side, start from the root of the tree and turn clockwise. We recover a positive excursion of simple random walk on the right-hand side in the time interval  $[1, \sigma - 1]$ , by associating to each ascending/descending movement a  $\pm 1$  step. One easily counts that the total number of visits at level  $n \geq 1$  is  $N_n = Z_n + Z_{n+1}$ . This gives :

$$\varphi_0(t) \prod_{n \geq 1} (\varphi_n(t))^{N_n} = \varphi_0(t) \prod_{n \geq 1} (\varphi_n(t))^{Z_n + Z_{n+1}} = \prod_{n \geq 1} [\varphi_n(t) \varphi_{n-1}(t)]^{Z_n}.$$

We therefore obtain  $\chi_D(t) = (\chi_+(t) + \chi_-(t))/2$ , with :

$$\chi_+(t) = \mathbb{E} \left[ \prod_{n \geq 0} [\varphi_n(t) \varphi_{n+1}(t)]^{Z_{n+1}} \right] \text{ and } \chi_-(t) = \frac{1}{2} \mathbb{E} \left[ \prod_{n \geq 0} [\varphi_{-n}(t) \varphi_{-n-1}(t)]^{Z_{n+1}} \right]. \quad (5)$$

### 3 Continued fraction expansions

The Markovian structure of a Galton-Watson tree allows to develop  $\chi_+$  and  $\chi_-$  in continued fractions. We first recall some classical facts about general continued fractions.

#### 3.1 SP-continued fractions and their convergents

For sequences  $(a_n)_{n \geq 1}$  and  $(b_n)_{n \geq 0}$ , we employ the following notation for a generalized finite continued fraction :

$$[b_0; (a_1, b_1); (a_2, b_2); \dots; (a_n, b_n)] = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{\dots}{b_n + \frac{a_n}{b_n}}}}.$$

In the present article, we use this notation for Sleszynski-Pringsheim (SP) continued fractions. Finite SP-continued fractions are obtained by successively applying to an initial  $z_0 \in \mathbb{C}$  in the unit disc maps of the form  $z \mapsto a/(b+z)$ , with complex numbers such that  $|a| + 1 \leq |b|$ , therefore preserving the unit disc. We write  $[b_0; (a_1, b_1); (a_2, b_2); \dots]$  for infinite SP-continued fractions. The latter are converging, by the Sleszynski-Pringsheim theorem (see [7]). We won't use the result directly, but largely reproduce the ideas from the proof.

For  $n \geq 0$ , the finite SP-continued fraction  $[b_0; (a_1, b_1); (a_2, b_2); \dots; (a_n, b_n)]$  can be written as  $A_n/B_n$ , where the  $(A_n)$  and  $(B_n)$  satisfy the following recursive relations :

$$\begin{cases} A_n = b_n A_{n-1} + a_n A_{n-2}, & A_{-1} = 1, \quad A_0 = b_0, \\ B_n = b_n B_{n-1} + a_n B_{n-2}, & B_{-1} = 0, \quad B_0 = 1. \end{cases}$$

In order to study convergence, we also require the classical determinant, for  $n \geq 1$  :

$$\begin{aligned} A_n B_{n-1} - A_{n-1} B_n &= (-a_n)(A_{n-1} B_{n-2} - A_{n-2} B_{n-1}) = \dots \\ &= (-1)^n a_1 \dots a_n (A_0 B_{-1} - A_{-1} B_0) = (-1)^{n+1} a_1 \dots a_n. \end{aligned}$$

### 3.2 Development of $\chi_+$ and $\chi_-$

Focusing on  $\chi_+$ , cf (5), set :

$$\chi_+^{(N)}(t) = \mathbb{E} \left( \prod_{n=0}^N [\varphi_n(t) \varphi_{n+1}(t)]^{Z_{n+1}} \right).$$

By dominated convergence (by 1, using that  $Z_n = 0$  for large  $n$ , a.s.) we have  $\chi_+^{(N)}(t) \rightarrow \chi_+(t)$ . Recall now a classical way of describing the Galton-Watson tree  $(Z_n)_{n \geq 1}$  with root  $Z_1 = 1$  and offspring distribution  $\mathcal{G}(1/2)$ . Fixing a bi-indexed collection  $(R_{k,l})_{k \geq 1, l \geq 1}$  of independent random variables with distribution  $\mathcal{G}(1/2)$ , one can recursively write :

$$Z_{N+1} = \sum_{k=1}^{Z_N} R_{k,N+1}, \quad N \geq 1.$$

The generating function of  $\mathcal{G}(1/2)$  being  $s \mapsto (2-s)^{-1}$ , omitting the  $t$ , we obtain :

$$\chi_+^{(N)}(t) = \mathbb{E} \left[ \prod_{n=0}^{N-1} [\varphi_n \varphi_{n+1}]^{Z_{n+1}} \left( \frac{1}{2 - \varphi_N \varphi_{N+1}} \right)^{Z_N} \right] = \mathbb{E} \left[ \prod_{n=0}^{N-2} [\varphi_n \varphi_{n+1}]^{Z_{n+1}} \left( \frac{\varphi_{N-1} \varphi_N}{2 - \varphi_N \varphi_{N+1}} \right)^{Z_N} \right].$$

Iterating the procedure, this leads to :

$$\begin{aligned} \chi_+^{(N)}(t) &= \mathbb{E} \left[ \varphi_0 \varphi_1 \left( \frac{\varphi_1 \varphi_2}{2 - \frac{\varphi_2 \varphi_3}{2 - \frac{\varphi_3 \varphi_4}{2 - \dots - \varphi_N \varphi_{N+1}}}} \right)^{Z_2} \right] \\ &= [0; (\varphi_0 \varphi_1, 2); (-\varphi_1 \varphi_2, 2); \dots; (-\varphi_{N-1} \varphi_N, 2 - \varphi_N \varphi_{N+1})] = \varphi_0(t) f_+^{(N)}(t), \end{aligned}$$

with  $f_+^{(N)}(t) = [0; (1, 2/\varphi_1(t)); (-1, 2/\varphi_2(t)); \dots; (-1, 2/\varphi_N(t) - \varphi_{N+1}(t))]$ .

In the framework of the previous section, take  $a_1 = 1$ ,  $a_n = -1$  for  $n \geq 2$  and  $b_0 = 0$ ,  $b_n = 2/\varphi_n$ , for  $n \geq 1$ . Call  $h_n$  the continued fraction of length  $n$ . As  $A_n B_{n-1} - A_{n-1} B_n = 1$ ,  $n \geq 1$ , and  $A_0 = 0$ , we obtain the equality :

$$h_N(t) = \frac{A_0}{B_0} + \sum_{k=1}^N \left( \frac{A_k}{B_k} - \frac{A_{k-1}}{B_{k-1}} \right) = \sum_{k=1}^N \frac{1}{B_k B_{k-1}}.$$

Idem, with the modified term  $\tilde{B}_N = \tilde{b}_N B_{N-1} - B_{N-2}$ , where  $\tilde{b}_N = 2/\varphi_N - \varphi_{N+1}$ , we get :

$$f_+^{(N)}(t) = \sum_{k=1}^{N-1} \frac{1}{B_k B_{k-1}} + \frac{1}{\tilde{B}_N B_{N-1}}. \quad (6)$$

We now show that both  $h_N(t)$  and  $f_+^{(N)}(t)$  converge, as  $N \rightarrow +\infty$ , to the same  $f_+(t)$ , where :

$$f_+(t) = \sum_{k \geq 1} \frac{1}{(B_k B_{k-1})(t)}. \quad (7)$$

This way, from the relation  $B_n = (2/\varphi_n)B_{n-1} - B_{n-2}$ ,  $n \geq 2$ , and  $B_1 = (2/\varphi_1)B_0 + B_{-1}$  we have :

$$|B_n| \geq 2|B_{n-1}| - |B_{n-2}|, \quad n \geq 1.$$

This gives  $|B_n| - |B_{n-1}| \geq |B_{n-1}| - |B_{n-2}| \geq \dots \geq |B_0| - |B_{-1}| = 1$ . Hence  $|B_n| \geq n+1$ . Similarly  $|\tilde{B}_N| \geq (2-1)|B_{N-1}| - |B_{N-2}| \geq 1$ . This implies the desired convergence.

Defining symmetrically some  $f_-$  associated to  $\chi_-$ , we obtain :

$$\chi_D = \frac{\varphi_0}{2}(f_+ + f_-). \quad (8)$$

## 4 First estimates

### 4.1 General lower bound on $\operatorname{Re}(1 - \chi_D)$

**Proposition 4.1** *We have  $\operatorname{Re}(1 - \chi_D(t)) \geq \delta^2 t$ .*

*Proof of the proposition :*

We suppose  $\delta > 0$  small. We have  $\chi_D = (\varphi_0/2)(f_+ + f_-)$ . Hence :

$$\operatorname{Re}(1 - \chi_D(t)) \geq 1 - |\chi_D(t)| \geq 1 - (|f_+(t)| + |f_-(t)|)/2.$$

We prove that  $|f_{\pm}(t)| \leq 1 - \delta^2 t$ . By (1), using  $m_{2,n} \geq \delta$ , for small  $t$  uniformly in  $n$  :

$$r_n(t) \leq 1 - \frac{t^2}{4} \frac{\delta}{(1-\delta)^2} m_{2,n} (1 - (1-\delta)) \leq 1 - \delta^3 t^2 / (4(1-\delta)^2) \leq 1 - \delta^4 t^2.$$

Therefore for  $n \geq 1$  :

$$|B_n| \geq \frac{2}{r_n} |B_{n-1}| - |B_{n-2}| \geq \frac{2}{1 - \delta^4 t^2} |B_{n-1}| - |B_{n-2}|.$$

Introduce the real sequence  $(u_n)$  such that  $u_{-1} = 0$ ,  $u_0 = 1$  and  $u_n = (2/(1 - \delta^4 t^2))u_{n-1} - u_{n-2}$ . Define next  $v_n = |B_n| - u_n$ , which verifies  $v_{-1} = v_0 = 0$  and :

$$v_n \geq \frac{2}{1 - \delta^4 t^2} v_{n-1} - v_{n-2}.$$

As  $v_n - v_{n-1} \geq (2/(1 - \delta^4 t^2) - 2)v_{n-1} + v_{n-1} - v_{n-2}$ , the condition “ $v_{n-1} \geq v_{n-2} \geq 0$ ” is transmitted recursively. Hence  $v_n$  is non-decreasing and checks  $v_n \geq 0$ , i.e.  $|B_n| \geq u_n$ . Also :

$$u_n = \frac{\rho_+^{n+1} - \rho_-^{n+1}}{\rho_+ - \rho_-} = \rho_+^n \sum_{k=0}^n \rho_+^{-2k}, \quad n \geq -1,$$

with  $\rho_{\pm} = (1 - \delta^4 t^2)^{-1} (1 \pm \sqrt{1 - (1 - \delta^4 t^2)^2}) = 1 \pm \delta^2 \sqrt{2} t + O(t^2)$ , satisfying  $\rho_+ \rho_- = 1$ . Then, remarking that  $u_n = \rho_+^n \sum_{0 \leq k \leq n} \rho_+^{-2k}$ , we obtain :

$$|f_+| \leq \sum_{k \geq 1} \frac{1}{|B_k B_{k-1}|} \leq \sum_{k \geq 1} \frac{1}{u_k u_{k-1}} \leq \sum_{k \geq 1} \frac{1}{\rho_+^{2k-1}} \left( \frac{1}{\sum_{j=0}^{k-1} \rho_+^{-2j}} - \frac{1}{\sum_{j=0}^k \rho_+^{-2j}} \right) \frac{1}{\rho_+^{-2k}}.$$

We conclude that  $|f_+| \leq \rho_+ (1 - 1/\sum_{k \geq 0} \rho_+^{-2k}) = \rho_+ (1 - (1 - \rho_+^{-2})) = \rho_- \leq 1 - \delta^2 t$ , for  $t$  small enough. Idem for  $f_-$ . This concludes the proof of the proposition.  $\square$

*Remark.* — This general result is independent from the distribution of the  $(p_n)$  and  $(\mu_n)$ . It gives the upper bound  $\operatorname{Re}(1/(1 - \chi_D(t))) \leq 1/\operatorname{Re}(1 - \chi_D(t)) \leq 2/(\delta t)$ , not sufficient to decide the

question of integrability. This proposition will allow to remove terms of order at most  $t$  in the final part of the proof. One may also notice that the expression of  $u_n$  furnishes :

$$|B_n| \geq \frac{n+1}{(1-\delta^4 t^2)^n} \sum_{k=0}^{n/2} \frac{1}{2k+1} \binom{n}{2k} (1 - (1 - \delta^4 t^2)^2)^k.$$

For small  $t$ , this gives a lower bound for  $|B_n|$  of the form  $(n+1)(1 + \alpha n^2 t^2)$ ,  $\alpha > 0$ .

## 4.2 Isolating the main term

Let us start from the relation  $\chi_D(t) = \mathbb{E}(\prod_{k=0}^{\sigma-1} \varphi_{Y_k}(t))$ , extracted from (4). Our aim now is to replace the  $(\varphi_n)$  by the  $(\psi_n)$ , where we set :

$$\psi_n(t) = e^{ia_n(t)} w_n(t), \quad w_n(t) = \cos a_n(t) \left( \frac{1 - ip_n \operatorname{Im}(\chi_n(t)) / (1 - p_n \operatorname{Re}(\chi_n(t)))}{1 - it p_n m_n / (1 - p_n)} \right). \quad (9)$$

First of all, recalling the exact value of  $a_n(t)$  given in (1) and that  $\cos(\arctan x) = 1/\sqrt{1+x^2}$ , we observe that :

$$|\psi_n(t)| = \frac{1}{\sqrt{1 + \left( \frac{p_n \operatorname{Im}(\chi_n(t))}{1 - p_n \operatorname{Re}(\chi_n(t))} \right)^2}} \frac{\sqrt{1 + \left( \frac{p_n \operatorname{Im}(\chi_n(t))}{1 - p_n \operatorname{Re}(\chi_n(t))} \right)^2}}{\sqrt{1 + \frac{p_n^2 m_n^2}{(1-p_n)^2} t^2}} = \left( 1 + \frac{p_n^2 m_n^2}{(1-p_n)^2} t^2 \right)^{-1/2} \leq 1.$$

We now set  $g(t) = \mathbb{E}(\prod_{k=0}^{\sigma-1} \psi_{Y_k}(t))$  and estimate the difference  $g(t) - \chi_D(t)$  :

$$\begin{aligned} |g(t) - \chi_D(t)| &= \left| \mathbb{E} \left[ \left( \prod_{k=0}^{\sigma-1} w_{Y_k} - \prod_{k=0}^{\sigma-1} r_{Y_k} \right) \prod_{k=0}^{\sigma-1} e^{ia_{Y_k}} \right] \right| \leq \mathbb{E} \left( \left| \prod_{k=0}^{\sigma-1} w_{Y_k} - \prod_{k=0}^{\sigma-1} r_{Y_k} \right| \right) \\ &\leq \mathbb{E} \left( \left| \prod_{k=0}^{\sigma-1} w_{Y_k} - 1 \right| \right) + \mathbb{E} \left( \left| \prod_{k=0}^{\sigma-1} r_{Y_k} - 1 \right| \right) = (A) + (B). \end{aligned}$$

From (1), for some constant  $c > 0$  we have  $1 - ct^2 \leq r_n(t)$ . Using also  $r_n(t) \leq 1$  and (3), we get :

$$(B) = \mathbb{E} \left( 1 - \prod_{k=0}^{\sigma-1} r_{Y_k}(t) \right) \leq \mathbb{E} (1 - (1 - ct^2)^\sigma) = \sqrt{1 - (1 - ct^2)^2} = \sqrt{2c} t + O(t^2).$$

For (A), denoting by “arg” the principal determination of the argument, write :

$$\begin{aligned} (A) &\leq \mathbb{E} \left( \left| \left( \prod_{k=0}^{\sigma-1} |w_{Y_k}| - 1 \right) \prod_{k=0}^{\sigma-1} e^{i \operatorname{arg}(w_{Y_k})} \right| \right) + \mathbb{E} \left( \left| \prod_{k=0}^{\sigma-1} e^{i \operatorname{arg}(w_{Y_k})} - 1 \right| \right) \\ &\leq \mathbb{E} \left( 1 - \prod_{0 \leq k < \sigma} |w_{Y_k}| \right) + \mathbb{E} \left( \left| \prod_{k=0}^{\sigma-1} e^{i \operatorname{arg}(w_{Y_k})} - 1 \right| \right). \end{aligned}$$

For the first term, for another constant  $c > 0$  we have for small  $t$ , uniformly in  $n$  :

$$|w_n(t)| = \left( 1 + \frac{p_n^2 m_n^2}{(1-p_n)^2} t^2 \right)^{-1/2} \geq (1 + ct^2)^{-1/2} \geq 1 - ct^2,$$

giving that is this term is bounded from above by  $\mathbb{E} (1 - (1 - ct^2)^\sigma) = \sqrt{1 - (1 - ct^2)^2} \leq \sqrt{2c} t + O(t^2)$ . For the second term, observe first that, uniformly in  $n$  :

$$|\arg(w_n(t))| = \left| \arctan \left( \frac{p_n \operatorname{Im}(\chi_n(t))}{1 - p_n \operatorname{Re}(\chi_n(t))} \right) - \arctan \left( \frac{p_n m_n t}{1 - p_n} \right) \right| = O(t^3). \quad (10)$$

From the fact that  $\mathbb{P}(\sigma > 1/t) = O(\sqrt{t})$ , as  $t \rightarrow 0$ , we now get :

$$\mathbb{E} \left( \left| \prod_{k=0}^{\sigma-1} e^{i \arg(w_{Y_k})} - 1 \right| \right) \leq \mathbb{E} \left( \left| \prod_{k=0}^{\sigma-1} e^{i \arg(w_{Y_k})} - 1 \right| 1_{\sigma \leq \frac{1}{t^2}} \right) + \mathbb{E} \left( \left| \prod_{k=0}^{\sigma-1} e^{i \arg(w_{Y_k})} - 1 \right| 1_{\sigma > \frac{1}{t^2}} \right).$$

The second term is thus less than or equal than  $c't + 2\mathbb{P}(\sigma > 1/t^2) \leq c''t$ . The conclusion is that :

$$\chi_D(t) = g(t) + O(t), \quad g(t) = \mathbb{E} \left( \prod_{k=0}^{\sigma-1} \psi_{Y_k}(t) \right). \quad (11)$$

The purpose of the previous modification was to obtain, using (1) :

$$\begin{aligned} \frac{1}{\psi_n(t)} &= \frac{1}{\cos a_n(t)} e^{-ia_n(t)} \frac{1 - itp_n m_n / (1 - p_n)}{1 - ip_n \operatorname{Im}(\chi_n(t)) / (1 - p_n \operatorname{Re}(\chi_n(t)))} \\ &= (1 - i \tan a_n(t)) \frac{1 - itp_n m_n / (1 - p_n)}{1 - ip_n \operatorname{Im}(\chi_n(t)) / (1 - p_n \operatorname{Re}(\chi_n(t)))} \\ &= (1 - ip_n \operatorname{Im}(\chi_n(t)) / (1 - p_n \operatorname{Re}(\chi_n(t)))) \frac{1 - itp_n m_n / (1 - p_n)}{1 - ip_n \operatorname{Im}(\chi_n(t)) / (1 - p_n \operatorname{Re}(\chi_n(t)))} \\ &= 1 - itp_n m_n / (1 - p_n). \end{aligned}$$

Now, since  $|\psi_n| \leq 1$ , in the same way as for  $\chi_D$  (cf (8)), we can write :

$$g(t) = \frac{\psi_0(t)}{2} (g_+(t) + g_-(t)), \quad (12)$$

where both  $g_+$  and  $g_-$  can be developed as infinite SP-continued fractions. As a result, from (11) and (12) and the fact that  $\psi_0(t) = 1 + O(t)$  we finally get :

$$\chi_D(t) = \frac{1}{2} (g_+(t) + g_-(t)) + O(t). \quad (13)$$

Focusing on  $g_+$  (this is symmetric for  $g_-$ ) and setting  $\rho_n = 2p_n m_n / (1 - p_n)$ , we have :

$$g_+(t) = \sum_{n \geq 1} \frac{1}{B_n(t) B_{n-1}(t)},$$

where (using that  $2/\psi_n(t) = 2 - it\rho_n$ ) the  $(B_n)$  satisfy  $B_{-1} = 0$ ,  $B_0 = 1$ ,  $B_1 = (2 - it\rho_1)B_0 + B_{-1}$  and  $B_n = (2 - it\rho_n)B_{n-1} - B_{n-2}$ ,  $n \geq 2$ . We now have the following estimates.

**Proposition 4.2**

- 1)  $|B_n| \geq 2|B_{n-1}| - |B_{n-2}|$ ,  $n \geq 1$ . Also  $n \mapsto |B_n| - |B_{n-1}| \geq 1$  is non-decreasing and  $|B_n| \geq n+1$ .
- 2) We have the upper bound, for  $n \geq 1$  :

$$\left| \sum_{k > n} \frac{1}{B_k B_{k-1}} \right| \leq \frac{n+1}{|B_n|^2}.$$

*Proof of the proposition :*

The first point is treated as before. For the second one, we first prove that  $(n+1)|B_{n+1}| \geq (n+2)|B_n|$ ,  $n \geq 1$ . We will use it in the equivalent form  $|B_n| / (|B_{n+1}| - |B_n|) \leq n+1$ . Write :

$$\begin{aligned} (n+1)|B_{n+1}| - (n+2)|B_n| &\geq (n+1)(2|B_n| - |B_{n-1}|) - (n+2)|B_n| \\ &= n|B_n| - (n+1)|B_{n-1}|. \end{aligned}$$

Recursively,  $(n+1)|B_{n+1}| - (n+2)|B_n| \geq |B_1| - 2|B_0| = 0$ . Now :

$$\begin{aligned} \left| \sum_{k>n} \frac{1}{B_k B_{k-1}} \right| &\leq \sum_{k>n} \frac{1}{|B_k B_{k-1}|} \leq \sum_{k>n} \left( \frac{1}{|B_{k-1}|} - \frac{1}{|B_k|} \right) \frac{1}{|B_k| - |B_{k-1}|} \\ &\leq \sum_{k>n} \left( \frac{1}{|B_{k-1}|} - \frac{1}{|B_k|} \right) \frac{1}{|B_{n+1}| - |B_n|}. \end{aligned}$$

Finally  $|\sum_{k>n} 1/(B_k B_{k-1})| \leq 1/(|B_n|(|B_{n+1}| - |B_n|)) \leq (n+1)/|B_n|^2$ .  $\square$

## 5 Precise study of the convergents of a continued fraction

We now perform exact computations. Recall that  $g_+(t) = \lim_{n \rightarrow +\infty} A_n(t)/B_n(t)$ , where :

$$\begin{cases} A_n = (2 - it\rho_n)A_{n-1} + a_n A_{n-2}, & A_{-1} = 1, \quad A_0 = 0, \\ B_n = (2 - it\rho_n)B_{n-1} + a_n B_{n-2}, & B_{-1} = 0, \quad B_0 = 1, \end{cases}$$

with  $a_1 = 1$  and  $a_n = -1$  for  $n \geq 2$ . Introduce the matrices :

$$M_t = \begin{pmatrix} 2 - it & -1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Notice that  $A^n = \begin{pmatrix} n+1 & -n \\ n & -n+1 \end{pmatrix}$ ,  $n \in \mathbb{Z}$ , and  $B^n = B$ ,  $n \geq 0$ . Rewrite the recursions as :

$$\begin{cases} \begin{pmatrix} B_n \\ B_{n-1} \end{pmatrix} = M_{t\rho_n} \cdots M_{t\rho_2} \begin{pmatrix} 2 - it\rho_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = M_{t\rho_n} \cdots M_{t\rho_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & n \geq 1, \\ \begin{pmatrix} A_n \\ A_{n-1} \end{pmatrix} = M_{t\rho_n} \cdots M_{t\rho_2} \begin{pmatrix} 2 - it\rho_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = M_{t\rho_n} \cdots M_{t\rho_2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & n \geq 1. \end{cases}$$

Using obvious notations and writing  $B_n = B_n(t, \rho_1, \dots, \rho_n)$ , remark that :

$$A_n = B_{n-1}(t, \rho_2, \dots, \rho_n). \quad (14)$$

Hence, we only have to focus on  $(B_n)$ . Developing according to the positions of the  $B$ 's :

$$\begin{aligned} \begin{pmatrix} B_n - (n+1) \\ B_{n-1} - n \end{pmatrix} &= \sum_{r=1}^n (-it)^r \sum_{1 \leq k_1 < \dots < k_r \leq n} \prod_{j=1}^r \rho_{k_j} A^{n-k_r} B A^{k_r - k_{r-1} - 1} \dots B A^{k_1 - 1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \sum_{r=1}^n (-it)^r \sum_{1 \leq k_1 < \dots < k_r \leq n} \prod_{j=1}^r \rho_{k_j} \begin{pmatrix} n - k_r + 1 & 0 \\ n - k_r & 0 \end{pmatrix} \cdots \begin{pmatrix} k_2 - k_1 & 0 \\ k_2 - k_1 - 1 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_1 - 1 \end{pmatrix} \\ &= \sum_{r=1}^n (-it)^r \sum_{1 \leq k_1 < \dots < k_r \leq n} \prod_{j=1}^r \rho_{k_j} \begin{pmatrix} n - k_r + 1 \\ n - k_r \end{pmatrix} (k_r - k_{r-1}) \cdots (k_2 - k_1) k_1. \end{aligned}$$

Consequently :

$$B_n - B_{n-1} = 1 + \sum_{r=1}^n (-it)^r \sum_{1 \leq k_1 < \dots < k_r \leq n} k_1 (k_2 - k_1) \cdots (k_r - k_{r-1}) \rho_{k_1} \cdots \rho_{k_r}. \quad (15)$$

We now perform  $r$  successive Abel transforms on the right-hand side. This way, for  $k \leq l$ , set  $R_k^l = \rho_k + \dots + \rho_l$ , with  $R_k^l = 0$  if  $l < k$ . Fixing  $k_{r-1}$ ,  $n$ , and as  $\rho_{k_r} = R_{k_r}^n - R_{k_r+1}^n$ , we write :

$$\begin{aligned}
\sum_{k_{r-1} < k_r \leq n} \rho_{k_r}(k_r - k_{r-1}) &= \sum_{k_{r-1} < k_r \leq n} R_{k_r}^n(k_r - k_{r-1}) - \sum_{k_{r-1}+1 < k_r \leq n+1} R_{k_r}^n(k_r - 1 - k_{r-1}) \\
&= \sum_{k_{r-1} < k_r \leq n} R_{k_r}^n(k_r - k_{r-1}) - \sum_{k_{r-1} < k_r \leq n} R_{k_r}^n(k_r - 1 - k_{r-1}) \\
&= \sum_{k_{r-1} < k_r \leq n} R_{k_r}^n.
\end{aligned}$$

We make this transformation in the right-hand side of (15). Repeat next this manipulation with :

$$\sum_{k_{r-2} < k_{r-1} \leq k_r} \rho_{k_{r-1}}(k_{r-1} - k_{r-2}),$$

fixing this time all variables except  $k_{r-1}$  and taking  $R_{k_{r-1}}^{k_r-1}$  in place of  $R_{k_r}^n$ . Doing this successively for all variables  $k_r, \dots, k_1$ , we arrive at :

$$B_n - B_{n-1} = 1 + \sum_{r=1}^n (-it)^r \sum_{1 \leq k_1 < \dots < k_r \leq n} R_{k_1}^{k_2-1} R_{k_2}^{k_3-1} \dots R_{k_r}^n. \quad (16)$$

As a result, summing in  $n$ , this furnishes :

$$B_n = n + 1 + \sum_{r=1}^n (-it)^r \sum_{1 \leq k_1 < \dots < k_{r+1} \leq n+1} R_{k_1}^{k_2-1} R_{k_2}^{k_3-1} \dots R_{k_r}^{k_{r+1}-1}. \quad (17)$$

Similarly, via (14) :

$$A_n = n + \sum_{r=1}^{n-1} (-it)^r \sum_{2 \leq k_1 < \dots < k_{r+1} \leq n+1} R_{k_1}^{k_2-1} R_{k_2}^{k_3-1} \dots R_{k_r}^{k_{r+1}-1}.$$

The previous two equalities next furnish :

$$B_n - A_n = 1 + \sum_{r=1}^n (-it)^r \sum_{2 \leq k_2 < \dots < k_{r+1} \leq n+1} R_1^{k_2-1} \dots R_{k_r}^{k_{r+1}-1} = \sum_{0 \leq r \leq n} (-it)^r \Delta_r^n, \quad (18)$$

where we set :

$$\Delta_r^n = \sum_{1 \leq k_1 < \dots < k_r \leq n} R_1^{k_1} R_{k_1+1}^{k_2} \dots R_{k_{r-1}+1}^{k_r}.$$

We convene that  $\Delta_0^n = 1$  and  $\Delta_r^n = 0$  as soon as  $r > n$  or  $r < 0$ .

**Proposition 5.1** *i)  $|B_n - A_n|^2 = \sum_{r=0}^n t^{2r} K_r(n)$ , with  $K_0(n) = 1$  and :*

$$K_r(n) = \sum_{1=k_1 \leq l_1 < \dots < k_r \leq l_r \leq n} (R_1^{l_1} R_{k_2}^{l_2} \dots R_{k_r}^{l_r})^2 (1 + 2(n - l_r)) 2^{N((k_i, l_i)_{1 \leq i \leq r})},$$

where  $N((k_i, l_i)_{1 \leq i \leq r}) = \#\{2 \leq i \leq r \mid k_i \geq l_{i-1} + 2\}$ .

*ii)  $|B_n|^2 = \sum_{r=0}^n t^{2r} L_r(n)$ , with  $L_0(n) = (n+1)^2$  and :*

$$L_r(n) = \sum_{1 \leq k_1 \leq l_1 < \dots < k_r \leq l_r \leq n} (R_1^{l_1} R_{k_2}^{l_2} \dots R_{k_r}^{l_r})^2 (2k_1 - 1)(1 + 2(n - l_r)) 2^{N((k_i, l_i)_{1 \leq i \leq r})}.$$

*iii)  $Re((B_n - A_n)\overline{B_n}) = \sum_{r=0}^n t^{2r} M_r(n)$ , with  $M_0(n) = n+1$  and :*

$$M_r(n) = \sum_{1 \leq k_1 \leq l_1 < \dots < k_r \leq l_r \leq n} (R_{k_1}^{l_1} R_{k_2}^{l_2} \dots R_{k_r}^{l_r})^2 (1 + 2(n - l_r)) 2^{N((k_i, l_i)_{1 \leq i \leq r})}.$$

*iv)*  $\text{Im}((B_n - A_n)\overline{B_n}) = -\sum_{r=0}^{n-1} t^{2r+1} N_r(n)$ , with :

$$N_r(n) = \sum_{1 = k_1 \leq l_1 < \dots < k_{r+1} \leq l_{r+1} \leq n} R_1^{l_1} (R_{k_2}^{l_2} \dots R_{k_{r+1}}^{l_{r+1}})^2 (1 + 2(n - l_{r+1})) 2^{N((k_i, l_i)_{1 \leq i \leq r+1})}.$$

*Remark.* — Point *i)* induces a similar formula for  $|B_n - B_{n-1}|^2$ , because of the identity  $(B_n - B_{n-1})(t, \rho_1, \dots, \rho_n) = (B_n - A_n)(t, \rho_n, \dots, \rho_1)$ , resulting from (16) and (18).

*Remark.* — It results from *ii)* that if  $(\Omega, \mathcal{F}, T, \mu)$  is an ergodic dynamical system and  $f \in L^\infty(\mu)$ , then the main Lyapunov exponent of the matrix :

$$M(\omega) = \begin{pmatrix} 2 + if(\omega) & -1 \\ 1 & 0 \end{pmatrix},$$

defined as  $\gamma_{\max}(M) = \lim n^{-1} \int_{\Omega} \|M(T^{n-1}\omega) \dots M(\omega)\| d\mu(\omega)$  equals :

$$\gamma_{\max}(M) = \frac{1}{2} \lim \frac{1}{n} \sup_{1 \leq r \leq n} \log \sum_{1 \leq k_1 \leq l_1 < \dots < k_r \leq l_r \leq n} (R_{k_1}^{l_1})^2 \dots (R_{k_r}^{l_r})^2,$$

with  $R_k^l = \sum_{k \leq j \leq l} f(T^j \omega)$ . It may possible to deduce some variational formula for this exponent.

## 5.1 Proof of *i)*

Starting from  $B_n - A_n = \sum_{0 \leq r \leq n} (-it)^r \Delta_r^n$ , we compute :

$$\begin{aligned} |B_n - A_n|^2 &= \sum_{r=0}^n t^{2r} (\Delta_r^n)^2 + 2 \sum_{0 \leq r < r' \leq n/2} (-1)^{r+r'} t^{2r+2r'} \Delta_{2r}^n \Delta_{2r'}^n \\ &\quad + 2 \sum_{0 \leq r < r' \leq (n-1)/2} (-1)^{r+r'} t^{2r+2r'+2} \Delta_{2r+1}^n \Delta_{2r'+1}^n. \end{aligned}$$

Hence  $|B_n - A_n|^2 = \sum_{r=0}^n t^{2r} (\Delta_r^n)^2 + 2 \sum_{0 \leq r < r' \leq n, r'-r \text{ even}} (-1)^{(r-r')/2} t^{r+r'} \Delta_r^n \Delta_{r'}^n$ . Thus :

$$\begin{aligned} |B_n - A_n|^2 &= \sum_{r=0}^n t^{2r} \left[ (\Delta_r^n)^2 + 2 \sum_{1 \leq p \leq r} (-1)^{(r+p)/2 - (r-p)/2} \Delta_{r+p}^n \Delta_{r-p}^n \right] \\ &= \sum_{r=0}^n t^{2r} \left[ \sum_{-r \leq p \leq r} (-1)^p \Delta_{r+p}^n \Delta_{r-p}^n \right]. \end{aligned} \tag{19}$$

Call  $K_r(n)$  the term between brackets. We check that it equals the formula given in *i)*. The latter is a consequence of the recursion :

$$\begin{cases} K_1(n) = \sum_{1 \leq k \leq n} (R_1^k)^2 (1 + 2(n - k)), \\ K_r(n) = \sum_{1 \leq k \leq n} (R_1^k)^2 \left[ \theta^k K_{r-1}(n - k) + 2 \sum_{k < l \leq n} \theta^l K_{r-1}(n - l) \right], \quad r \geq 2. \end{cases} \tag{20}$$

In the second formula,  $\theta$  denotes the shift  $\theta((\rho_n)_{n \geq 0}) = (\rho_{n+1})_{n \geq 0}$  and for a function  $\psi((\rho_n)_{n \geq 0})$  we write  $\theta\psi$  for  $\psi \circ \theta$ . For the first relation, simply compute :

$$\begin{aligned}
K_1(n) &= (\Delta_1^n)^2 - 2\Delta_0^n \Delta_2^n = \left( \sum_{1 \leq k \leq n} R_1^k \right)^2 - 2 \sum_{1 \leq k < l \leq n} R_1^k R_{k+1}^l \\
&= \sum_{1 \leq k \leq n} (R_1^k)^2 + 2 \sum_{1 \leq k < l \leq n} R_1^k (R_1^l - R_{k+1}^l).
\end{aligned}$$

As  $R_1^l - R_{k+1}^l = R_1^k$  and the number of  $l$  in  $(k, n]$  is  $n - k$ , we get the expression of  $K_1(n)$ .

We turn to the proof of the recursive relation. This needs some preparation. Taking general  $p \geq 1, q \geq 1$ , we have :

$$\Delta_p^n \Delta_q^n = \sum_{\substack{1 \leq k_1 < \dots < k_p \leq n \\ 1 \leq k'_1 < \dots < k'_q \leq n}} (R_1^{k_1} \dots R_{k_{p-1}+1}^{k_p}) (R_1^{k'_1} \dots R_{k'_{q-1}+1}^{k'_q}).$$

Discuss now according to the relative positions of  $k_1$  and  $k'_1$  :

$$\begin{aligned}
\Delta_p^n \Delta_q^n &= \sum_{1 \leq k \leq n} (R_1^k)^2 \theta^k \Delta_{p-1}^{n-k} \theta^k \Delta_{q-1}^{n-k} \\
&+ \sum_{\substack{1 \leq k_1 < \dots < k_p \leq n \\ k_1 < k'_1 < \dots < k'_q \leq n}} R_1^{k_1} (R_{k_1+1}^{k_2} \dots R_{k_{p-1}+1}^{k_p}) (R_1^{k'_1} + R_{k_1+1}^{k'_1}) (R_{k'_1+1}^{k'_2} \dots R_{k'_{q-1}+1}^{k'_q}) \\
&+ \sum_{\substack{1 \leq k'_1 < \dots < k'_q \leq n \\ k'_1 < k_1 < \dots < k_p \leq n}} (R_1^{k'_1} + R_{k'_1+1}^{k'_1}) (R_{k_1+1}^{k_2} \dots R_{k_{p-1}+1}^{k_p}) (R_1^{k'_1} \dots R_{k'_{q-1}+1}^{k'_q}).
\end{aligned}$$

Regrouping terms, this can be rewritten more concisely as :

$$\begin{aligned}
\Delta_p^n \Delta_q^n &= \sum_{1 \leq k \leq n} (R_1^k)^2 \left[ \theta^k \Delta_{p-1}^{n-k} \sum_{k \leq l \leq n} \theta^l \Delta_{q-1}^{n-l} + \theta^k \Delta_{q-1}^{n-k} \sum_{k < l \leq n} \theta^l \Delta_{p-1}^{n-l} \right] \\
&+ \sum_{1 \leq k \leq n} R_1^k [\theta^k \Delta_{p-1}^{n-k} \theta^k \Delta_q^{n-k} + \theta^k \Delta_p^{n-k} \theta^k \Delta_{q-1}^{n-k}].
\end{aligned}$$

Let  $r \geq 2$ . We now plug this expression in  $K_r(n) = \sum_{-r \leq p \leq r} (-1)^p \Delta_{r+p}^n \Delta_{r-p}^n$  :

$$\begin{aligned}
K_r(n) &= \sum_{-r+1 \leq p \leq r-1} (-1)^p \Delta_{r+p}^n \Delta_{r-p}^n + 2(-1)^r \Delta_{2r}^n \\
&= \sum_{1 \leq k \leq n} (R_1^k)^2 \sum_{-r+1 \leq p \leq r-1} (-1)^p \left[ \theta^k \Delta_{r+p-1}^{n-k} \sum_{k \leq l \leq n} \theta^l \Delta_{r-p-1}^{n-l} + \theta^k \Delta_{r-p-1}^{n-k} \sum_{k < l \leq n} \theta^l \Delta_{r+p-1}^{n-l} \right] \\
&+ 2(-1)^r \Delta_{2r}^n + 2 \sum_{1 \leq k \leq n} R_1^k \left[ \sum_{-r+1 \leq p \leq r-1} (-1)^p \theta^k \Delta_{r+p-1}^{n-k} \theta^k \Delta_{r-p}^{n-k} \right].
\end{aligned}$$

The last line is  $2 \sum_{1 \leq k \leq n} R_1^k \left[ \sum_{-r+1 \leq p \leq r} (-1)^p \theta^k \Delta_{r+p-1}^{n-k} \theta^k \Delta_{r-p}^{n-k} \right]$ . The inner brackets are 0, so :

$$K_r(n) = \sum_{1 \leq k \leq n} (R_1^k)^2 \left[ \theta^k K_{r-1}(n-k) + 2 \sum_{-r+1 \leq p \leq r-1} (-1)^p \theta^k \Delta_{r+p-1}^{n-k} \sum_{k < l \leq n} \theta^l \Delta_{r-p-1}^{n-l} \right].$$

Set  $m = n - k$  and  $Z_r(m) = \sum_{-r \leq p \leq r} (-1)^p \Delta_{r+p}^m \sum_{1 \leq l \leq m} \theta^l \Delta_{r-p}^{m-l}$ . We shall show that :

$$Z_r(m) = \sum_{1 \leq k \leq m} \theta^k K_r(m-k), \quad r \geq 1. \quad (21)$$

To complete the proof of (20), we apply this equality to  $Z_{r-1}(m)$ . First of all, with  $0 \leq p \leq r-1$  :

$$\begin{aligned} \Delta_{r+p}^m \sum_{1 \leq l \leq m} \theta^l \Delta_{r-p}^{m-l} &= \sum_{\substack{1 \leq k_1 < \dots < k_{r+p} \leq m \\ 1 \leq l_1 < l_2 < \dots < l_{r-p+1} \leq m}} R_1^{k_1} \dots R_{k_{r+p-1}+1}^{k_{r+p}} R_{l_1+1}^{l_2} \dots R_{l_{r-p+1}}^{l_{r-p+1}} \\ &= \sum_{\substack{1 \leq k_1 < \dots < k_{r+p} \leq m \\ k_1 < l_2 < \dots < l_{r-p+1} \leq m}} R_1^{k_1} R_{k_1+1}^{k_2} \dots R_{k_{r+p-1}+1}^{k_{r+p}} R_{k_1+1}^{l_2} \dots R_{l_{r-p+1}}^{l_{r-p+1}} \\ &+ \sum_{\substack{1 \leq k_1 < \dots < k_{r+p} \leq m \\ k_1 < l_1 < \dots < l_{r-p+1} \leq m}} R_1^{k_1} R_{k_1+1}^{k_2} \dots R_{k_{r+p-1}+1}^{k_{r+p}} R_{l_1+1}^{l_2} \dots R_{l_{r-p+1}}^{l_{r-p+1}} \\ &+ \sum_{\substack{1 \leq l_1 < \dots < l_{r-p+1} \leq m \\ l_1 < k_1 < \dots < k_{r+p} \leq m}} (R_1^{l_1} + R_{l_1+1}^{k_1}) R_{k_1+1}^{k_2} \dots R_{k_{r+p-1}+1}^{k_{r+p}} R_{l_1+1}^{l_2} \dots R_{l_{r-p+1}}^{l_{r-p+1}}. \end{aligned}$$

In short :

$$\begin{aligned} \Delta_{r+p}^m \sum_{1 \leq l \leq m} \theta^l \Delta_{r-p}^{m-l} &= \sum_{1 \leq k \leq m} R_1^k \left[ \theta^k \Delta_{r+p-1}^{m-k} \sum_{k \leq l \leq m} \theta^l \Delta_{r-p}^{m-l} + \theta^k \Delta_{r-p}^{m-k} \sum_{k < l \leq m} \theta^l \Delta_{r+p-1}^{m-l} \right] \\ &+ \sum_{1 \leq k \leq m} \theta^k \Delta_{r+p}^{m-k} \theta^k \Delta_{r-p}^{m-k}. \end{aligned}$$

Consequently :

$$\begin{aligned} Z_r(m) &= (-1)^r \left[ m \Delta_{2r}^m + \sum_{1 \leq l \leq m} \theta^l \Delta_{2r}^{m-l} \right] + \sum_{1 \leq k \leq m} \sum_{-r+1 \leq p \leq r-1} (-1)^p \theta^k \Delta_{r+p}^{m-k} \theta^k \Delta_{r-p}^{m-k} \\ &+ \sum_{1 \leq k \leq m} R_1^k \sum_{-r+1 \leq p \leq r-1} (-1)^p \left[ \theta^k \Delta_{r+p-1}^{m-k} \sum_{k \leq l \leq m} \theta^l \Delta_{r-p}^{m-l} + \theta^k \Delta_{r-p}^{m-k} \sum_{k < l \leq m} \theta^l \Delta_{r+p-1}^{m-l} \right]. \end{aligned}$$

Recognizing some  $\theta^k K_r(m-k)$ , we get :

$$\begin{aligned} Z_r(m) &= (-1)^r \left[ m \Delta_{2r}^m - \sum_{1 \leq l \leq m} \theta^l \Delta_{2r}^{m-l} \right] + \sum_{1 \leq k \leq m} \theta^k K_r(m-k) \\ &+ \sum_{1 \leq k \leq m} R_1^k \sum_{-r+1 \leq p \leq r-1} (-1)^p \left[ \theta^k \Delta_{r+p-1}^{m-k} \sum_{k \leq l \leq m} \theta^l \Delta_{r-p}^{m-l} \right] \\ &+ \sum_{1 \leq k \leq m} R_1^k \sum_{-r+2 \leq p \leq r} (-1)^{p+1} \left[ \theta^k \Delta_{r+p-1}^{m-k} \sum_{k < l \leq m} \theta^l \Delta_{r-p}^{m-l} \right]. \end{aligned}$$

Therefore :

$$\begin{aligned}
Z_r(m) &= (-1)^r \left[ m\Delta_{2r}^m - \sum_{1 \leq l \leq m} \theta^l \Delta_{2r}^{m-l} - \sum_{1 \leq k \leq m} R_1^k \left( \theta^k \Delta_{2r-1}^{m-k} (m-k+1) + \sum_{k < l \leq m} \theta^l \Delta_{2r-1}^{m-l} \right) \right] \\
&+ \sum_{1 \leq k \leq m} \theta^k K_r(m-k) + \sum_{1 \leq k \leq m} R_1^k \sum_{-r+1 \leq p \leq r} (-1)^p \left[ \theta^k \Delta_{r+p-1}^{m-k} \sum_{k \leq l \leq m} \theta^l \Delta_{r-p}^{m-l} \right] \\
&+ \sum_{1 \leq k \leq m} R_1^k \sum_{-r+1 \leq p \leq r} (-1)^{p+1} \left[ \theta^k \Delta_{r+p-1}^{m-k} \sum_{k < l \leq m} \theta^l \Delta_{r-p}^{m-l} \right].
\end{aligned}$$

The last two terms are  $\sum_{1 \leq k \leq m} R_1^k [\sum_{-r+1 \leq p \leq r} (-1)^p \theta^k \Delta_{r+p-1}^{m-k} \theta^k \Delta_{r-p}^{m-k}]$  and the brackets are equal to 0. It remains to show that the first bracket above is also 0. It equals :

$$- \sum_{1 \leq l \leq m} \theta^l \Delta_{2r}^{m-l} + \sum_{1 \leq k \leq m} (k-1) R_1^k \theta^k \Delta_{2r-1}^{m-k} - \sum_{1 \leq k < l \leq m} R_1^k \theta^l \Delta_{2r-1}^{m-l}.$$

In the last term, write  $R_1^k = R_1^l - R_{k+1}^l$ . We conclude from the observation that :

$$\sum_{1 \leq k < l \leq m} R_1^l \theta^l \Delta_{2r-1}^{m-l} = \sum_{1 \leq k \leq m} (k-1) R_1^k \theta^k \Delta_{2r-1}^{m-k} \text{ and } \sum_{1 \leq k < l \leq m} R_{k+1}^l \theta^l \Delta_{2r-1}^{m-l} = \sum_{1 \leq l \leq m} \theta^l \Delta_{2r}^{m-l}.$$

This completes the proof of *i*).

## 5.2 Proof of *ii*)

Set  $\tilde{\Delta}_r^n = \sum_{0 \leq k_0 < k_1 < \dots < k_r \leq n} R_{k_0+1}^{k_1} R_{k_1+1}^{k_2} \dots R_{k_{r-1}+1}^{k_r}$ , with  $\tilde{\Delta}_0^n = n+1$  and  $\tilde{\Delta}_r^n = 0$  whenever  $r > n$  or  $r < 0$ . From (17), we get :

$$B_n = \sum_{0 \leq r \leq n} (-it)^r \tilde{\Delta}_r^n. \quad (22)$$

Exactly in the same way as for deriving relation (19), we have :

$$|B_n|^2 = \sum_{r=0}^n t^{2r} \left[ \sum_{-r \leq p \leq r} (-1)^p \tilde{\Delta}_{r+p}^n \tilde{\Delta}_{r-p}^n \right] =: \sum_{r=0}^n t^{2r} L_r(n).$$

To compute  $L_r(n)$  we first observe that :

$$\tilde{\Delta}_{r+p}^n \tilde{\Delta}_{r-p}^n = \sum_{0 \leq k \leq n} \left[ \theta^k \Delta_{r+p}^{n-k} \sum_{k \leq l \leq n} \theta^l \Delta_{r-p}^{n-l} + \theta^k \Delta_{r-p}^{n-k} \sum_{k < l \leq n} \theta^l \Delta_{r+p}^{n-l} \right].$$

Replacing in  $L_r(n)$ , this allows to write, recognizing  $K_r(n)$  and  $Z_r(n)$  given in (21) :

$$\begin{aligned}
L_r(n) &= \sum_{0 \leq k \leq n} \sum_{-r \leq p \leq r} (-1)^p \left[ \theta^k \Delta_{r+p}^{n-k} \sum_{k \leq l \leq n} \theta^l \Delta_{r-p}^{n-l} + \theta^k \Delta_{r-p}^{n-k} \sum_{k < l \leq n} \theta^l \Delta_{r+p}^{n-l} \right] \\
&= \sum_{0 \leq k \leq n} \theta^k K_r(n-k) + 2 \sum_{0 \leq k \leq n} \theta^k Z_r(n-k) \\
&= \sum_{0 \leq k \leq n} \theta^k K_r(n-k) + 2 \sum_{0 \leq k < l \leq n} \theta^l K_r(n-l) = \sum_{0 \leq k \leq n} (1+2k) \theta^k K_r(n-k).
\end{aligned}$$

This completes the proof of *ii*).

### 5.3 Proof of *iii*)

From (18) and (22), we deduce :

$$(B_n - A_n)\overline{B}_n = \left[ \sum_{0 \leq r \leq n} (-it)^r \Delta_r^n \right] \left[ \sum_{0 \leq r \leq n} (it)^r \tilde{\Delta}_r^n \right]. \quad (23)$$

Therefore :

$$\begin{aligned} \operatorname{Re}((B_n - A_n)\overline{B}_n) &= \sum_{0 \leq r \leq n/2} (-t^2)^r \Delta_{2r}^n \sum_{0 \leq r' \leq n/2} (-t^2)^{r'} \tilde{\Delta}_{2r'}^n \\ &+ \sum_{0 \leq r \leq (n-1)/2} (-1)^r t^{2r+1} \Delta_{2r+1}^n \sum_{0 \leq r' \leq (n-1)/2} (-1)^{r'} t^{2r'+1} \tilde{\Delta}_{2r'+1}^n. \end{aligned}$$

Consequently :

$$\begin{aligned} \operatorname{Re}((B_n - A_n)\overline{B}_n) &= \sum_{0 \leq r \leq n} t^{2r} \Delta_r^n \tilde{\Delta}_r^n + \sum_{0 \leq r < r' \leq n/2} (-1)^{r+r'} t^{2r+2r'} (\Delta_{2r}^n \tilde{\Delta}_{2r'}^n + \tilde{\Delta}_{2r}^n \Delta_{2r'}^n) \\ &+ \sum_{0 \leq r < r' \leq (n-1)/2} (-1)^{r+r'} t^{2r+2r'+2} (\Delta_{2r+1}^n \tilde{\Delta}_{2r'+1}^n + \tilde{\Delta}_{2r+1}^n \Delta_{2r'+1}^n) \\ &= \sum_{0 \leq r \leq n} t^{2r} \Delta_r^n \tilde{\Delta}_r^n + \sum_{\substack{0 \leq r < r' \leq n \\ r'-r \text{ even}}} (-1)^{(r'-r)/2} t^{r+r'} (\Delta_r^n \tilde{\Delta}_{r'}^n + \tilde{\Delta}_r^n \Delta_{r'}^n) \\ &= \sum_{0 \leq r \leq n} t^{2r} \left[ \sum_{-r \leq p \leq r} (-1)^p \Delta_{r+p}^n \tilde{\Delta}_{r-p}^n \right] =: \sum_{0 \leq r \leq n} t^{2r} M_r(n). \end{aligned}$$

As  $\tilde{\Delta}_r^n = \Delta_r^n + \sum_{1 \leq k \leq n} \theta^k \Delta_r^{n-k}$ , using  $K_r(n)$  and the value of  $Z_r(n)$  in (21), we have :

$$M_r(n) = K_r(n) + Z_r(n) = K_r(n) + \sum_{1 \leq k \leq n} \theta^k K_r(n-k) = \sum_{0 \leq k \leq n} \theta^k K_r(n-k).$$

This ends the proof of *iii*).

### 5.4 Proof of *iv*)

Starting from (23) and (18), we have :

$$\begin{aligned} \operatorname{Im}((B_n - A_n)\overline{B}_n) &= \sum_{0 \leq r \leq n/2} (-1)^r t^{2r} \Delta_{2r}^n \sum_{0 \leq r' \leq (n-1)/2} (-1)^{r'} t^{2r'+1} \tilde{\Delta}_{2r'+1}^n \\ &- \sum_{0 \leq r \leq (n-1)/2} (-1)^r t^{2r+1} \Delta_{2r+1}^n \sum_{0 \leq r' \leq n/2} (-1)^{r'} t^{2r'} \tilde{\Delta}_{2r'}^n \\ &= \sum_{\substack{0 \leq r \leq n/2 \\ 0 \leq r' \leq (n-1)/2}} t^{2r+2r'+1} (-1)^{r+r'} (\Delta_{2r}^n \tilde{\Delta}_{2r'+1}^n - \tilde{\Delta}_{2r}^n \Delta_{2r'+1}^n). \\ &= \sum_{\substack{0 \leq r \leq n \\ 0 \leq r' \leq n, r-r' \text{ odd}}} (-1)^r (-1)^{(r+r'-1)/2} t^{r+r'} \Delta_r^n \tilde{\Delta}_{r'}^n \\ &= - \sum_{0 \leq r \leq n-1} t^{2r+1} \left[ \sum_{-r-1 \leq p \leq r} (-1)^p \Delta_{r+p+1}^n \tilde{\Delta}_{r-p}^n \right] =: - \sum_{0 \leq r \leq n-1} t^{2r+1} N_r(n). \end{aligned}$$

Use now that  $\tilde{\Delta}_r^n = \Delta_r^n + \sum_{1 \leq k \leq n} \theta^k \Delta_r^{n-k}$  and  $\sum_{-r-1 \leq p \leq r} (-1)^p \Delta_{r+p+1}^n \Delta_{r-p}^n = 0$  to get :

$$\begin{aligned} N_r(n) &= (-1)^{r+1} \sum_{1 \leq k \leq n} \theta^k \Delta_{2r+1}^{n-k} + \sum_{-r \leq p \leq r} (-1)^p \Delta_{r+p+1}^n \sum_{1 \leq k \leq n} \theta^k \Delta_{r-p}^{n-k} \\ &= (-1)^{r+1} \sum_{1 \leq k \leq n} \theta^k \Delta_{2r+1}^{n-k} + \sum_{1 \leq l \leq n} R_1^l \sum_{-r \leq p \leq r} (-1)^p \theta^l \Delta_{r+p}^{n-l} \sum_{l < k \leq n} \theta^k \Delta_{r-p}^{n-k} \\ &\quad + \sum_{1 \leq k \leq n} \sum_{-r \leq p \leq r} (-1)^p \theta^k \Delta_{r-p}^{n-k} \sum_{k \leq l \leq n} R_1^l \theta^l \Delta_{r+p}^{n-l}. \end{aligned}$$

As a result :

$$\begin{aligned} N_r(n) &= \sum_{1 \leq k \leq n} R_1^k [\theta^k K_r(n-k) + \theta^k Z_r(n-k)] + (-1)^{r+1} \sum_{1 \leq k \leq n} \theta^k \Delta_{2r+1}^{n-k} \\ &\quad + \sum_{1 \leq k \leq n} \sum_{-r \leq p \leq r} (-1)^p \theta^k \Delta_{r-p}^{n-k} \sum_{k < l \leq n} R_1^l \theta^l \Delta_{r+p}^{n-l}. \end{aligned} \quad (24)$$

We next define and focus on :

$$\begin{aligned} O_r(n) &= (-1)^{r+1} \sum_{1 \leq k \leq n} \theta^k \Delta_{2r+1}^{n-k} + \sum_{1 \leq k \leq n} \sum_{-r \leq p \leq r} (-1)^p \theta^k \Delta_{r-p}^{n-k} \sum_{k < l \leq n} (R_1^k + R_{k+1}^l) \theta^l \Delta_{r+p}^{n-l} \\ &= \sum_{1 \leq k \leq n} R_1^k \sum_{-r \leq p \leq r} (-1)^p \theta^k \Delta_{r-p}^{n-k} \sum_{k < l \leq n} \theta^l \Delta_{r+p}^{n-l} + (-1)^{r+1} \sum_{1 \leq k \leq n} \theta^k \Delta_{2r+1}^{n-k} \\ &\quad + \sum_{1 \leq k \leq n} \sum_{-r \leq p \leq r} (-1)^p \theta^k \Delta_{r-p}^{n-k} \theta^k \Delta_{r+p+1}^{n-k} \\ &= \sum_{1 \leq k \leq n} R_1^k \theta^k Z_r(n-k) + \sum_{1 \leq k \leq n} \left[ \sum_{-r \leq p \leq r-1} (-1)^p \theta^k \Delta_{r-p}^{n-k} \theta^k \Delta_{r+p+1}^{n-k} \right]. \end{aligned}$$

The terms between brackets are 0. Together with (24), this gives :

$$N_r(n) = \sum_{1 \leq k \leq n} R_1^k \left[ \theta^k K_r(n-k) + 2 \sum_{k < l \leq n} \theta^l K_r(n-l) \right]. \quad (25)$$

This completes the proof of *iv*).

## 6 End the proof of the theorem

### 6.1 Preliminaries

Let us begin with a lemma on the regular variation of  $\varphi^{-1}$  and  $\varphi_+^{-1}$ .

**Lemma 6.1** *i) For all  $C \geq 1$ , there exists a constant  $K > 0$  so that for large  $x$  :*

$$K\varphi^{-1}(Cx) \leq \varphi^{-1}(x) \leq \varphi^{-1}(Cx) \text{ and } K\varphi_+^{-1}(Cx) \leq \varphi_+^{-1}(x) \leq \varphi_+^{-1}(Cx).$$

*ii) There is a constant  $c > 0$  so that for all  $n$  :  $\varphi(n+1) \leq c\varphi(n)$  and  $\varphi_+(n+1) \leq c\varphi_+(n)$ .*

*Proof of the lemma :*

*i)* Functions are increasing, giving the second inequalities. For the first ones, consider the case of  $\varphi$ , that of  $\varphi_+$  being similar. We take  $C = \sqrt{2}$ . If  $\varphi^{-1}(x) \geq K\varphi^{-1}(\sqrt{2}x)$  and now  $C \geq 1$  is arbitrary, choose  $m$  so that  $(\sqrt{2})^m \geq C$ . Then  $\varphi^{-1}(x) \geq K^m \varphi^{-1}((\sqrt{2})^m x) \geq K^m \varphi^{-1}(Cx)$ . Now :

$$\begin{aligned}
\sum_{1 \leq k \leq l \leq 2n} (R_k^l)^2 &\geq \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 + \sum_{1 \leq k \leq n < l \leq 2n} (R_k^n + R_{n+1}^l)^2 \\
&\geq \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 + n \sum_{1 \leq k \leq n} (R_k^n)^2 + n \sum_{n < l \leq 2n} (R_{n+1}^l)^2 + 2 \sum_{1 \leq k \leq n} R_k^n \sum_{n < l \leq 2n} R_{n+1}^l.
\end{aligned}$$

As a result :

$$\begin{aligned}
\sum_{1 \leq k \leq l \leq 2n} (R_k^l)^2 &\geq \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 + n \sum_{1 \leq k \leq n} (R_k^n)^2 - \left( \sum_{1 \leq k \leq n} R_k^n \right)^2 \\
&\quad + n \sum_{n < l \leq 2n} (R_{n+1}^l)^2 - \left( \sum_{n+1 \leq l \leq 2n} R_{n+1}^l \right)^2 + \left( \sum_{1 \leq k \leq n} R_k^n + \sum_{n < l \leq 2n} R_{n+1}^l \right)^2.
\end{aligned}$$

Let us now compute  $n \sum_{1 \leq k \leq n} (R_k^n)^2 - \left( \sum_{1 \leq k \leq n} R_k^n \right)^2$ . It equals :

$$\begin{aligned}
(n-1) \sum_{1 \leq k \leq n} (R_k^n)^2 - 2 \sum_{1 \leq k < k' \leq n} R_k^n R_{k'}^n &= (n-1) \sum_{1 \leq k \leq n} (R_k^n)^2 - 2 \sum_{1 \leq k < k' \leq n} R_k^n R_{k'}^n \\
&= (n-1) \sum_{1 \leq k \leq n} (R_k^n)^2 + \sum_{1 \leq k < k' \leq n} (R_k^{k'})^2 \\
&\quad - \sum_{1 \leq k \leq n} (R_k^n)^2 (n-k+1) - \sum_{1 \leq k \leq n} (R_{k+1}^n)^2 k.
\end{aligned}$$

Remark that  $\sum_{1 \leq k \leq n} (R_k^n)^2 (n-k+1) + \sum_{1 \leq k \leq n} (R_{k+1}^n)^2 k = n \sum_{1 \leq k \leq n} (R_k^n)^2$ . Hence :

$$n \sum_{1 \leq k \leq n} (R_k^n)^2 - \left( \sum_{1 \leq k \leq n} R_k^n \right)^2 = \sum_{1 \leq k \leq l \leq n-1} (R_k^l)^2.$$

Similarly :

$$n \sum_{n < l \leq 2n} (R_{n+1}^l)^2 - \left( \sum_{n+1 \leq l \leq 2n} R_{n+1}^l \right)^2 = \sum_{n+2 \leq k \leq l \leq 2n} (R_k^l)^2 \geq 0.$$

Finally, this furnishes :

$$\sum_{1 \leq k \leq l \leq 2n} (R_k^l)^2 \geq \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 + \sum_{1 \leq k \leq l \leq n-1} (R_k^l)^2.$$

As a result :

$$\varphi(4n)^2 = 16n^2 + \sum_{1 \leq k \leq l \leq 4n} (R_k^l)^2 \geq 2 \left( n^2 + \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 \right) = 2\varphi(n)^2.$$

This concludes the proof of this item.

ii) Simply remark that :

$$\begin{aligned}
\sum_{1 \leq k \leq l \leq n+1} (R_k^l)^2 &\leq \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 + 2 \sum_{1 \leq k \leq n} (R_k^n)^2 + 2 \sum_{1 \leq k \leq n+1} \rho_{n+1}^2 \\
&\leq 3 \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 + 2(n+1) \|\rho\|_\infty \leq C\varphi_+^2(n).
\end{aligned}$$

The case of  $\sum_{1 \leq k \leq l \leq n+1} (R_{-l}^{-k})^2$  is treated similarly, furnishing the result for  $\varphi_+$ . The case of  $\varphi$  is similar, remarking that  $(R_{-n-1}^{n+1})^2 \leq cn^2$ .  $\square$

Recall from (13) that  $\chi_D(t) = (g_+(t) + g_-(t))/2 + O(t)$ . We have  $g_+(t) = \lim_{n \rightarrow +\infty} A_n^+(t)/B_n^+(t)$  and  $g_-(t) = \lim_{n \rightarrow +\infty} A_n^-(t)/B_n^-(t)$ , where we restore the dependency with respect to  $g_+$  and  $g_-$ . We will apply the Kesten-Spitzer theorem, recalled in (2), using the equality :

$$\operatorname{Re} \left( \frac{1}{1 - \chi_D} \right) = \frac{\operatorname{Re}(1 - \chi_D)}{|1 - \chi_D|^2}.$$

It will also be important to remember proposition 4.1, furnishing  $\operatorname{Re}(1 - \chi_D)(t) \geq \delta^2 t$ .

**Proposition 6.2** *For some constant  $K \geq 1$ , which can be made explicit in terms of  $\|\rho\|_\infty$  :*

$$\frac{1}{K\varphi_+^{-1}(1/t)} \leq \operatorname{Re}(1 - \chi_D(t)) \leq \frac{K}{\varphi_+^{-1}(1/t)} \quad \text{and} \quad \frac{1}{K\varphi^{-1}(1/t)} \leq |1 - \chi_D(t)| \leq \frac{K}{\varphi^{-1}(1/t)}.$$

The theorem now follows from the proposition. Indeed, using the Kesten-Spitzer theorem, the random walk is recurrent if and only if :

$$\int_0^n \frac{(\varphi^{-1}(1/t))^2}{\varphi_+^{-1}(1/t)} dt = +\infty.$$

Cutting the above integral with respect to the intervals  $[1/(n+1), 1/n]$  of length  $1/(n(n+1)) \in (n^{-2}/2, n^{-2})$  and using that  $\varphi^{-1}(1/t)$  (resp.  $\varphi_+^{-1}(1/t)$ ) has order exactly  $\varphi^{-1}(n)$  (resp.  $\varphi_+^{-1}(n)$ ) on such an interval, by lemma 6.1, we obtain the formulation given in the statement of the theorem.

## 6.2 Control of the numerator

Fix  $t > 0$  and  $n$ . We have  $g_+(t) = A_n^+(t)/B_n^+(t) + \sum_{k>n} (1/(B_k^+ B_{k-1}^+))$  and similarly for  $g_-$ . This furnishes the equalities :

$$\begin{aligned} 2\operatorname{Re}(1 - \chi_D) &= \operatorname{Re} \left( 1 - \frac{A_n^+}{B_n^+} + 1 - \frac{A_n^-}{B_n^-} - \sum_{k>n} \left( \frac{1}{B_k^+ B_{k-1}^+} + \frac{1}{B_k^- B_{k-1}^-} \right) \right) + O(t) \\ &= \frac{\operatorname{Re}[(B_n^+ - A_n^+) \overline{B_n^+}]}{|B_n^+|^2} + \frac{\operatorname{Re}[(B_n^- - A_n^-) \overline{B_n^-}]}{|B_n^-|^2} \\ &\quad - \operatorname{Re} \sum_{k>n} \left( \frac{1}{B_k^+ B_{k-1}^+} + \frac{1}{B_k^- B_{k-1}^-} \right) + O(t). \end{aligned}$$

1) Towards proving a lower-bound, we use proposition (4.2) and get :

$$\begin{aligned} 2\operatorname{Re}(1 - \chi_D) &\geq \frac{\operatorname{Re}[(B_n^+ - A_n^+) \overline{B_n^+}] - (n+1)}{|B_n^+|^2} + \frac{\operatorname{Re}[(B_n^- - A_n^-) \overline{B_n^-}] - (n+1)}{|B_n^-|^2} + O(t) \\ &\geq \frac{1}{n+1} \left( \frac{\sum_{1 \leq r \leq n} t^{2r} \frac{M_r^+(n)}{n+1}}{1 + \sum_{1 \leq r \leq n} t^{2r} \frac{L_r^+(n)}{(n+1)^2}} + \frac{\sum_{1 \leq r \leq n} t^{2r} \frac{M_r^-(n)}{n+1}}{1 + \sum_{1 \leq r \leq n} t^{2r} \frac{L_r^-(n)}{(n+1)^2}} \right) + O(t). \end{aligned}$$

Remark that  $L_r^\pm(n) \leq 2(n+1)M_r^\pm(n)$  and next use that  $x \mapsto x/(1+2x)$  is increasing to obtain, for some constant  $C > 0$  :

$$\begin{aligned}
2\operatorname{Re}(1 - \chi_D) &\geq \frac{1}{n+1} \left( \frac{\sum_{1 \leq r \leq n} t^{2r} \frac{M_r^+(n)}{n+1}}{1 + 2 \sum_{1 \leq r \leq n} t^{2r} \frac{M_r^+(n)}{(n+1)^2}} + \frac{\sum_{1 \leq r \leq n} t^{2r} \frac{M_r^-(n)}{n+1}}{1 + 2 \sum_{1 \leq r \leq n} t^{2r} \frac{M_r^-(n)}{(n+1)^2}} \right) - Ct. \\
&\geq \frac{1}{n+1} \left( \frac{t^2 \frac{M_1^+(n)}{n+1}}{1 + 2t^2 \frac{M_1^+(n)}{(n+1)^2}} + \frac{t^2 \frac{M_1^-(n)}{n+1}}{1 + 2t^2 \frac{M_1^-(n)}{(n+1)^2}} \right) - Ct.
\end{aligned}$$

Choose now  $n$  so that  $n/2 = \varphi_+^{-1}(1/t)$ . For some constant  $c_0 > 0$  :

$$t^2 \geq \frac{c_0}{\varphi_+^2(n/2)} = \frac{c_0}{n^2/4 + \sum_{1 \leq k \leq l \leq n/2} (R_k^l)^2 + \sum_{1 \leq k \leq l \leq n/2} (R_{-l}^{-k})^2}.$$

We may assume that  $\sum_{1 \leq k \leq l \leq n/2} (R_k^l)^2 \geq \sum_{1 \leq k \leq l \leq n/2} (R_{-l}^{-k})^2$ . Next :

$$\frac{M_1^+(n)}{n+1} = 2 \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 \left(1 - \frac{l+1/2}{n+1}\right) \geq \sum_{1 \leq k \leq l \leq n/2} (R_k^l)^2.$$

As a result :

$$t^2 \frac{M_1^+(n)}{n+1} \geq c_0 \frac{\sum_{1 \leq k \leq l \leq n/2} (R_k^l)^2}{n^2/4 + 2 \sum_{1 \leq k \leq l \leq n/2} (R_k^l)^2}.$$

– Case 1 :  $\sum_{1 \leq k \leq l \leq n/2} (R_k^l)^2 \geq n^2$ . Then  $t^2 M_1^+(n)/(n+1) \geq c_0/4$  and this implies that :

$$2\operatorname{Re}(1 - \chi_D(t)) \geq \frac{c_0/4}{4(n+1)(1+c_0/2)} - Ct = \frac{c_1}{n+1} - Ct.$$

If  $t < c_1/(2C(n+1))$ , then  $2\operatorname{Re}(1 - \chi_D(t)) \geq c_1/(2(n+1))$ . On the contrary, if  $t \geq c_1/(2C(n+1))$ , then recall that  $\operatorname{Re}(1 - \chi_D(t)) \geq \delta^2 t$ . This gives  $\operatorname{Re}(1 - \chi_D(t)) \geq \delta^2 c_1/(2(n+1))$ .

– Case 2 :  $\sum_{1 \leq k \leq l \leq n/2} (R_k^l)^2 < n^2$ . We obtain  $t^2 \geq c_0/(4n^2)$  and conclude that  $\operatorname{Re}(1 - \chi_D(t)) \geq \delta^2 \sqrt{c_0}/(2n)$ . This completes the proof of the lower bound.

2) We now prove the other direction. For some constant  $C > 0$ , using proposition (4.2) and that  $|B_n^\pm| \geq n+1$  we have :

$$\begin{aligned}
2\operatorname{Re}(1 - \chi_D(t)) &\leq \frac{\operatorname{Re}[(B_n^+ - A_n^+) \overline{B_n^+}]}{|B_n^+|^2} + \frac{\operatorname{Re}[(B_n^- - A_n^-) \overline{B_n^-}]}{|B_n^-|^2} + \frac{1}{n+1} + \frac{1}{n+1} + Ct \\
&\leq \frac{1}{n+1} \left( 1 + \sum_{1 \leq r \leq n} t^{2r} \frac{M_r^+(n)}{n+1} + 1 + \sum_{1 \leq r \leq n} t^{2r} \frac{M_r^-(n)}{n+1} + 2 \right) + Ct.
\end{aligned}$$

Using lemma 6.1, choose now  $n$  so that  $n = \varphi_+^{-1}(1/(\kappa t))$ , for some constant  $\kappa \geq 1$  so that :

$$t^2 \leq \left(\frac{1}{4}\right) \frac{1}{n^2 + \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 + \sum_{1 \leq k \leq l \leq n} (R_{-l}^{-k})^2}.$$

From  $\frac{M_r^+(n)}{n+1} \leq 2 \left(2 \sum_{1 \leq k \leq l \leq n} (R_k^l)^2\right)^r$  and the symmetric inequality for  $\frac{M_r^-(n)}{n+1}$  we obtain :

$$2\operatorname{Re}(1 - \chi_D(t)) \leq \frac{1}{n+1} \left( 1 + \sum_{1 \leq r \leq n} 2^{-r} + 1 + \sum_{1 \leq r \leq n} 2^{-r} + 2 \right) + Ct \leq \frac{6}{n+1} + Ct.$$

As  $t \leq 1/(\kappa n)$ , for some constant  $K > 0$ , we have  $\operatorname{Re}(1 - \chi_D(t)) \leq K/n$ . We conclude by lemma 6.1, saying that  $n$  has the same order as  $\varphi_+^{-1}(t)$ .

### 6.3 Denominator

In a first step, let us compute :

$$\left|1 - \frac{A_n^+}{B_n^+} + 1 - \frac{A_n^-}{B_n^-}\right|^2 = \frac{|B_n^+ - A_n^+|^2}{|B_n^+|^2} + \frac{|B_n^- - A_n^-|^2}{|B_n^-|^2} + 2 \frac{\operatorname{Re}[(B_n^+ - A_n^+) \overline{B_n^+}] \operatorname{Re}[(B_n^- - A_n^-) \overline{B_n^-}]}{|B_n^+ B_n^-|^2} + 2 \frac{\operatorname{Im}[(B_n^+ - A_n^+) \overline{B_n^+}] \operatorname{Im}[(B_n^- - A_n^-) \overline{B_n^-}]}{|B_n^+ B_n^-|^2}.$$

Putting all terms with  $|B_n^+|^2 |B_n^-|^2$  as denominator, call  $W_n$  the numerator :

$$W_n = |B_n^+ - A_n^+|^2 |B_n^-|^2 + |B_n^- - A_n^-|^2 |B_n^+|^2 + 2 \operatorname{Re}[(B_n^+ - A_n^+) \overline{B_n^+}] \operatorname{Re}[(B_n^- - A_n^-) \overline{B_n^-}] + 2 \operatorname{Im}[(B_n^+ - A_n^+) \overline{B_n^+}] \operatorname{Im}[(B_n^- - A_n^-) \overline{B_n^-}].$$

By proposition 5.1 we get :

$$W_n = \sum_{\substack{0 \leq r \leq n \\ 0 \leq r' \leq n}} t^{2r+2r'} (K_r^+(n) L_{r'}^-(n) + L_r^+(n) K_{r'}^-(n) + 2M_r^+(n) M_{r'}^-(n)) + 2 \sum_{\substack{0 \leq r \leq n-1 \\ 0 \leq r' \leq n-1}} t^{2(r+r'+1)} N_r^+(n) N_{r'}^-(n).$$

Consequently :

$$W_n = \sum_{0 \leq m \leq 2n} t^{2m} \sum_{0 \leq r \leq m} (K_r^+(n) L_{m-r}^-(n) + L_r^+(n) K_{m-r}^-(n) + 2M_r^+(n) M_{m-r}^-(n)) + 2 \sum_{1 \leq m \leq 2n-1} t^{2m} \sum_{0 \leq r \leq m-1} N_r^+(n) N_{m-1-r}^-(n).$$

We now focus on  $2N_r^+(n) N_{m-1-r}^-(n)$ . Using (25) and setting  $(k, k) = 0$  and  $(k, l) = 1$  for  $k < l$  :

$$2N_r^+(n) N_{m-1-r}^-(n) = \sum_{\substack{1 \leq k \leq l \leq n \\ 1 \leq k' \leq l' \leq n}} 2R_1^k R_{-k'}^{-1} 2^{(k,l)+(k',l')} \theta^l K_r^+(n-l) \theta^{-l'} K_{m-1-r}^-(n-l').$$

Write  $2R_1^k R_{-k'}^{-1} = (R_{-k'}^{-1} + R_1^k)^2 - (R_1^k)^2 - (R_{-k'}^{-1})^2$ . Then  $2N_r^+(n) N_{m-1-r}^-(n)$  equals :

$$\begin{aligned} & \sum_{\substack{1 \leq k \leq l \leq n \\ 1 \leq k' \leq l' \leq n}} (R_1^k + R_{-k'}^{-1})^2 2^{(k,l)+(k',l')} \theta^l K_r^+(n-l) \theta^{-l'} K_{m-1-r}^-(n-l') \\ & - \left( \sum_{1 \leq k \leq l \leq n} (R_1^k)^2 2^{(k,l)} \theta^l K_r^+(n-l) \right) \left( \sum_{1 \leq k' \leq l' \leq n} 2^{(k',l')} \theta^{-l'} K_{m-1-r}^-(n-l') \right) \\ & - \left( \sum_{1 \leq k \leq l \leq n} 2^{(k,l)} \theta^l K_r^+(n-l) \right) \left( \sum_{1 \leq k' \leq l' \leq n} (R_{-k'}^{-1})^2 2^{(k',l')} \theta^{-l'} K_{m-1-r}^-(n-l') \right). \end{aligned}$$

The latter can be rewritten as :

$$\begin{aligned} & \sum_{\substack{1 \leq k \leq l \leq n \\ 1 \leq k' \leq l' \leq n}} (R_1^k + R_{-k'}^{-1})^2 2^{(k,l)+(k',l')} \theta^l K_r^+(n-l) \theta^{-l'} K_{m-1-r}^-(n-l') \\ & - K_{r+1}^+(n) \left( \sum_{1 \leq l \leq n} (2l-1) \theta^{-l} K_{m-1-r}^-(n-l) \right) - K_{m-r}^-(n) \left( \sum_{1 \leq k \leq n} (2k-1) \theta^k K_r^+(n-k) \right). \end{aligned}$$

As  $L_r^+(n) = \sum_{0 \leq k \leq n} (2k+1) \theta^k K_r^+(n-k) = 2M_r^+(n) - K_r^+(n) + \sum_{1 \leq k \leq n} (2k-1) \theta^k K_r^+(n-k)$ , using also a similar expression for  $L_{m-r-1}^-$ , we get :

$$\begin{aligned} 2N_r^+(n)N_{m-1-r}^-(n) &= \sum_{\substack{1 \leq k \leq l \leq n \\ 1 \leq k' \leq l' \leq n}} (R_1^k + R_{-k'}^{-1})^2 2^{(k,l)+(k',l')} \theta^l K_r^+(n-l) \theta^{-l'} K_{m-1-r}^-(n-l') \\ &- K_{r+1}^+(n) (L_{m-r-1}^-(n) - 2M_{m-r-1}^-(n) + K_{m-r-1}^-(n)) \\ &- K_{m-r}^-(n) (L_r^+(n) - 2M_r^+(n) + K_r^+(n)). \end{aligned}$$

Denote by  $Q_{r,m-r-1}(n)$  the first quantity on the right-hand side. Finally :

$$\begin{aligned} W_n &= \sum_{0 \leq m \leq 2n} t^{2m} \sum_{0 \leq r \leq m} (K_r^+(n)L_{m-r}^-(n) + L_r^+(n)K_{m-r}^-(n) + 2M_r^+(n)M_{m-r}^-(n)) \\ &+ \sum_{1 \leq m \leq 2n-1} t^{2m} \sum_{0 \leq r \leq m-1} Q_{r,m-r-1}(n) \\ &- \sum_{1 \leq m \leq 2n-1} t^{2m} \sum_{0 \leq r \leq m-1} K_{r+1}^+(n) (L_{m-r-1}^-(n) - 2M_{m-r-1}^-(n) + K_{m-r-1}^-(n)) \\ &- \sum_{1 \leq m \leq 2n-1} t^{2m} \sum_{0 \leq r \leq m-1} K_{m-r}^-(n) (L_r^+(n) - 2M_r^+(n) + K_r^+(n)). \end{aligned}$$

As a result, separating quantities corresponding to  $m = 0$  and  $m = 2n$  in the first term in the right-hand side :

$$\begin{aligned} W_n &= 4(n+1)^2 + t^{4n} (K_n^+(n)L_n^-(n) + K_n^-(n)L_n^+(n) + 2M_n^+(n)M_n^-(n)) \\ &+ \sum_{1 \leq m \leq 2n-1} t^{2m} (L_m^-(n) + L_m^+(n)) + 2 \sum_{1 \leq m \leq 2n-1} t^{2m} \sum_{0 \leq r \leq m} M_r^+(n)M_{m-r}^-(n) \\ &+ \sum_{1 \leq m \leq 2n-1} t^{2m} \sum_{0 \leq r \leq m-1} Q_{r,m-r-1}(n) \\ &+ \sum_{1 \leq m \leq 2n-1} t^{2m} \sum_{0 \leq r \leq m-1} K_{r+1}^+(n) (2M_{m-r-1}^-(n) - K_{m-r-1}^-(n)) \\ &+ \sum_{1 \leq m \leq 2n-1} t^{2m} \sum_{0 \leq r \leq m-1} K_{m-r}^-(n) (2M_r^+(n) - K_r^+(n)). \end{aligned}$$

One notices that  $(n+1)^2 + \sum_{1 \leq m \leq 2n-1} t^{2m} L_m^+(n) = |B_n^+|^2$  and similarly for  $|B_n^-|^2$ . Also :

$$2M_r^+(n) - K_r^+(n) = \sum_{0 \leq k \leq n} 2^{(0,k)} \theta^k K_r^+(n-k)$$

and similarly  $2M_{m-r-1}^-(n) - K_{m-r-1}^-(n) = \sum_{0 \leq k \leq n} 2^{(0,k)} \theta^{-k} K_{m-r-1}^-(n-k)$ . We deduce :

$$\begin{aligned}
W_n &= 2(n+1)^2 + 4t^{4n} \prod_{1 \leq k \leq n} (\rho_k \rho_{-k})^2 + |B_n^+|^2 + |B_n^-|^2 \\
&+ 2 \sum_{1 \leq m \leq 2n-1} t^{2m} \sum_{0 \leq r \leq m} M_r^+(n) M_{m-r}^-(n) \\
&+ \sum_{1 \leq m \leq 2n-1} t^{2m} \sum_{0 \leq r \leq m-1} \left( Q_{r, m-r-1}(n) + K_{m-r}^-(n) \sum_{0 \leq k \leq n} 2^{(0,k)} \theta^k K_r^+(n-k) \right) \\
&+ \sum_{1 \leq m \leq 2n-1} t^{2m} \sum_{1 \leq r \leq m} K_r^+(n) \sum_{0 \leq k \leq n} 2^{(0,k)} \theta^{-k} K_{m-r}^-(n-k).
\end{aligned}$$

In order to regroup term of line two and the first term of line three, introduce :

$$S_m(n) = \sum_{\substack{-n \leq k_1 \leq l_1 < \dots < k_m \leq l_m \leq n \\ k_i \neq 0, l_i \neq 0}}^* (R_{k_1}^{l_1} \dots R_{k_m}^{l_m})^2 2^{N((k_i, l_i)_{1 \leq i \leq m})} \alpha_n(k_1, l_m),$$

where  $\sum^*$  means that if for some  $i$  we have  $k_i l_i < 0$ , then we remove the term  $\rho_0$  in  $R_{k_i}^{l_i}$  and  $\alpha_n(k_1, l_m) = (1 + 2(n - l_m))(1 + 2(n + k_1))$  if  $k_1 < 0, l_m > 0$ ,  $\alpha_n(k_1, l_m) = (1 + 2(n - l_m))(n + 1)$  if  $k_1 > 0, l_m > 0$ , and  $\alpha_n(k_1, l_m) = (1 + 2(n + k_1))(n + 1)$  if  $k_1 < 0, l_m < 0$ .

Observing that  $S_{2n}(n) = 2 \prod_{1 \leq k \leq n} (\rho_k \rho_{-k})^2$ , we conclude this preliminary computation with :

$$\left| 1 - \frac{A_n^+}{B_n^+} + 1 - \frac{A_n^-}{B_n^-} \right|^2 = \frac{W_n}{|B_n^+|^2 |B_n^-|^2}, \quad (26)$$

where :

$$\begin{aligned}
W_n &= 2(n+1)^2 + |B_n^+|^2 + |B_n^-|^2 + 2t^{4n} \prod_{1 \leq k \leq n} (\rho_k \rho_{-k})^2 + \sum_{1 \leq m \leq 2n} t^{2m} S_m(n) \\
&+ \sum_{1 \leq m \leq 2n-1} t^{2m} \sum_{1 \leq r \leq m} \sum_{0 \leq k \leq n} 2^{(0,k)} (K_r^+(n) \theta^{-k} K_{m-r}^-(n-k) + K_r^-(n) \theta^k K_{m-r}^+(n-k)).
\end{aligned}$$

## 6.4 Control of the denominator

1) For the moment, let  $t > 0$  and  $n$  be arbitrary. Towards proving a lower bound, for some constant  $C > 0$ , we have, using proposition 4.2 :

$$\begin{aligned}
2|1 - \chi_D| &\geq \left| 1 - \frac{A_n^+}{B_n^+} + 1 - \frac{A_n^-}{B_n^-} \right| - \sum_{k > n} \left( \frac{1}{|B_k^+ B_{k-1}^+|} + \frac{1}{|B_k^- B_{k-1}^-|} \right) - Ct \\
&\geq \left| 1 - \frac{A_n^+}{B_n^+} + 1 - \frac{A_n^-}{B_n^-} \right| - \frac{(n+1)(|B_n^+|/|B_n^-| + |B_n^-|/|B_n^+|)}{|B_n^+| |B_n^-|} - Ct.
\end{aligned}$$

Hence, by (26) :

$$2|1 - \chi_D| \geq \frac{\sqrt{W_n} - (n+1) \left( \frac{|B_n^+|}{|B_n^-|} + \frac{|B_n^-|}{|B_n^+|} \right)}{|B_n^+| |B_n^-|} - Ct.$$

Remark now that, still using proposition 4.2 :

$$\begin{aligned}
(n+1)^2 \left( \frac{|B_n^+|}{|B_n^-|} + \frac{|B_n^-|}{|B_n^+|} \right)^2 &= (n+1)^2 \frac{|B_n^-|^2}{|B_n^+|^2} + (n+1)^2 \frac{|B_n^+|^2}{|B_n^-|^2} + 2(n+1)^2 \\
&\leq |B_n^+|^2 + |B_n^-|^2 + 2(n+1)^2 \leq W_n.
\end{aligned}$$

This gives, via (26) :

$$\begin{aligned} 2|1 - \chi_D| &\geq \frac{\sqrt{W_n} - \sqrt{|B_n^+|^2 + |B_n^-|^2 + 2(n+1)^2}}{|B_n^+||B_n^-|} - Ct \\ &\geq \frac{W_n - (|B_n^+|^2 + |B_n^-|^2 + 2(n+1)^2)}{2\sqrt{W_n}|B_n^+||B_n^-|} - Ct \geq \frac{\sum_{1 \leq m \leq 2n} t^{2m} S_m(n)}{2\sqrt{W_n}|B_n^+||B_n^-|} - Ct. \end{aligned}$$

We obtain, observing that  $W_n \leq 16(n+1)^2 + 16 \sum_{1 \leq m \leq 2n} t^{2m} S_m(n)$  :

$$\begin{aligned} 2|1 - \chi_D| &\geq \left( \frac{1}{n+1} \right) \frac{\sum_{1 \leq m \leq 2n} t^{2m} \frac{S_m(n)}{(n+1)^2}}{8\sqrt{1 + \sum_{1 \leq m \leq 2n} t^{2m} \frac{S_m(n)}{(n+1)^2}}} \\ &\quad \times \frac{1}{\sqrt{1 + \sum_{1 \leq r \leq n} t^{2r} \frac{L_r^+(n)}{(n+1)^2}} \sqrt{1 + \sum_{1 \leq r \leq n} t^{2r} \frac{L_r^-(n)}{(n+1)^2}}} - Ct. \end{aligned}$$

Remark next that :

$$\begin{aligned} &\left( 1 + \sum_{1 \leq r \leq n} t^{2r} \frac{L_r^+(n)}{(n+1)^2} \right) \left( 1 + \sum_{1 \leq r \leq n} t^{2r} \frac{L_r^-(n)}{(n+1)^2} \right) \\ &= 1 + \sum_{1 \leq m \leq 2n} t^{2m} \left( \sum_{1 \leq r \leq m-r} \frac{L_r^+(n)L_{m-r}^-(n)}{(n+1)^4} \right) \leq 4 \left( 1 + \sum_{1 \leq m \leq 2n} t^{2m} \frac{S_m(n)}{(n+1)^2} \right). \end{aligned}$$

As a result we obtain :

$$2|1 - \chi_D| \geq \left( \frac{1}{16(n+1)} \right) \frac{\sum_{1 \leq m \leq 2n} t^{2m} S_m(n)/(n+1)^2}{1 + \sum_{1 \leq m \leq 2n} t^{2m} S_m(n)/(n+1)^2} - Ct.$$

As the map  $x \mapsto x/(1+x)$  is increasing, this gives :

$$2|1 - \chi_D| \geq \left( \frac{1}{16(n+1)} \right) \frac{t^2 S_1(n)/(n+1)^2}{1 + t^2 S_1(n)/(n+1)^2} - Ct.$$

Choose now  $n$  so that  $n/2 = \varphi^{-1}(1/t)$ . As a result, for some constant  $c_0 > 0$  :

$$t^2 \geq \frac{c_0}{\varphi^2(n/2)} = \frac{c_0}{n^2/4 + \sum_{-n/2 \leq k \leq l \leq n/2} (R_k^l)^2}. \quad (27)$$

Remark also that :

$$\frac{S_1(n)}{(n+1)^2} \geq \sum_{-n/2 \leq k \leq l \leq n/2, kl \neq 0}^* (R_k^l)^2.$$

– Case 1 :  $\sum_{-n/2 \leq k \leq l \leq n/2, kl \neq 0}^* (R_k^l)^2 \geq n^2$ . Observing that  $(R_0^l)^2 \leq 2(R_1^l)^2 + C$ , one easily gets that  $\sum_{-n/2 \leq k \leq l \leq n/2} (R_k^l)^2 \leq c_1 \sum_{-n/2 \leq k \leq l \leq n/2, kl \neq 0}^* (R_k^l)^2 + c_2 n^2$ . For generic constants  $c_i > 0$  :

$$t^2 \frac{S_1(n)}{(n+1)^2} \geq c_0 \frac{\sum_{-n/2 \leq k \leq l \leq n/2, kl \neq 0}^* (R_k^l)^2}{n^2/4 + \sum_{-n/2 \leq k \leq l \leq n/2} (R_k^l)^2} \geq c_0 \frac{\sum_{-n/2 \leq k \leq l \leq n/2}^* (R_k^l)^2}{c_3 n^2 + c_4 \sum_{-n/2 \leq k \leq l \leq n/2}^* (R_k^l)^2} \geq c_5 > 0.$$

Hence :

$$2|1 - \chi_D| \geq \left( \frac{1}{16(n+1)} \right) \frac{c_5}{1+c_5} - Ct \geq c_6/n - Ct.$$

As a result, if  $t \leq c_6/(2Cn)$  we get  $2|1 - \chi_D| \geq c_6/(2n)$ . If  $t > c_6/(2Cn)$ , then as before, using proposition 4.1,  $|1 - \chi_D| \geq \delta^2 t \geq \delta^2 c_6/(2Cn)$ .

– Case 2 :  $\sum_{-n/2 \leq k \leq l \leq n/2, kl \neq 0} (R_k^l)^2 < n^2$ . This furnishes  $\sum_{-n/2 \leq k \leq l \leq n/2} (R_k^l)^2 < c_7 n^2$  and next from (27),  $t^2 \geq c_8/n^2$ . Hence  $|1 - \chi_D| \geq \delta^2 t \geq \delta^2 \sqrt{c_8}/n$ . This ends the proof of the lower bound.

2) We consider the other direction. Let  $t > 0$  and  $n$  be arbitrary. For some constant  $C > 0$ , using proposition 4.2 :

$$\begin{aligned} 2|1 - \chi_D| &\leq \left| 1 - \frac{A_n^+}{B_n^+} + 1 - \frac{A_n^-}{B_n^-} \right| + \frac{1}{n+1} + \frac{1}{n+1} + Ct \\ &\leq \frac{\sqrt{W_n}}{|B_n^+||B_n^-|} + \frac{2}{n+1} + Ct \leq \frac{\sqrt{W_n}}{(n+1)^2} + \frac{2}{n+1} + Ct. \end{aligned}$$

Remark also that :

$$\begin{aligned} W_n &\leq 16(n+1)^2 \left( 1 + \sum_{1 \leq m \leq 2n} t^{2m} S_m(n)/(n+1)^2 \right) \\ &\leq 32(n+1)^2 \left( 1 + \sum_{1 \leq m \leq 2n} t^{2m} 4^m \left( \sum_{-n \leq k \leq l \leq n} (R_k^l)^2 \right)^m \right). \end{aligned}$$

Using lemma 6.1, choose now  $n$  so that  $n = \varphi^{-1}(1/(\kappa t))$ , for some constant  $\kappa \geq 1$  so that :

$$t^2 \leq \left( \frac{1}{8} \right) \frac{1}{n^2 + \sum_{-n \leq k \leq l \leq n} (R_k^l)^2}.$$

This gives  $W_n \leq 32(n+1)^2 \left( 1 + \sum_{1 \leq m \leq 2n} 2^{-m} \right)$  and therefore :

$$2|1 - \chi_D| \leq \frac{8}{n+1} + \frac{2}{n+1} + Ct.$$

As  $t \leq 1/(\kappa n)$ , using lemma 6.1,  $|1 - \chi_D| \leq c_9/\varphi^{-1}(1/t)$ . This ends the proof of proposition (6.2).

## 7 Proof of the applications

### 7.1 Proof of corollary 1.3

*i*) As  $\varphi^{-1}(1/t) \leq \varphi_+^{-1}(1/t)$  we get  $(\varphi^{-1}(1/t))^2/\varphi_+^{-1}(1/t) \leq \varphi^{-1}(1/t)$ . A sufficient condition for transience is therefore :

$$\int_0^n \varphi^{-1}(1/t) dt < +\infty.$$

On the interval  $[1/\varphi(n+1), 1/\varphi(n)]$ ,  $\varphi^{-1}(1/t)$  has exactly order  $n$ , because  $\varphi(n)$  and  $\varphi(n+1)$  have the same order, by lemma (6.1), *ii*). The last condition is therefore equivalent to the finiteness of :

$$\sum_{n \geq 1} \int_{1/\varphi(n+1)}^{1/\varphi(n)} n dt = \sum_{n \geq 1} n \left( \frac{1}{\varphi(n)} - \frac{1}{\varphi(n+1)} \right) = \lim_N \sum_{1 \leq n \leq N} \frac{1}{\varphi(n)} - \frac{N}{\varphi(N+1)}.$$

As  $N \leq \varphi(N+1)$ , finiteness is equivalent to  $\sum_{n \geq 1} (1/\varphi(n)) < +\infty$ .

ii), iii) We now consider the symmetric case  $\rho_{-n} = \rho_n$ ,  $n \geq 1$ , and the antisymmetric case  $\rho_{-n} = -\rho_n$ ,  $n \geq 1$ . In both situations :

$$\varphi_+^2(n) = n^2 + 2 \sum_{1 \leq k \leq l \leq n} (R_k^l)^2.$$

Next, for some  $C > 0$ , we have  $C^{-1}\psi(n) \leq \varphi(n) \leq C\psi(n)$ , where :

$$\begin{aligned} \psi^2(n) &= n^2 + \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 + \sum_{1 \leq k, l \leq n} (R_{-k}^{-1} + R_1^l)^2 \\ &= n^2 + \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 + n \sum_{1 \leq k \leq n} [(R_{-k}^{-1})^2 + (R_1^k)^2] + 2 \left( \sum_{1 \leq k \leq n} R_{-k}^{-1} \right) \left( \sum_{1 \leq k \leq n} R_1^k \right). \end{aligned}$$

Therefore :

$$\begin{aligned} \psi^2(n) &= n^2 + \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 + n \sum_{1 \leq k \leq n} [(R_{-k}^{-1})^2 + (R_1^k)^2] + \left( \sum_{1 \leq k \leq n} (R_{-k}^{-1} + R_1^k) \right)^2 \\ &\quad - \left( \sum_{1 \leq k \leq n} R_{-k}^{-1} \right)^2 - \left( \sum_{1 \leq k \leq n} R_1^k \right)^2. \\ &= n^2 + \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 + (n-1) \sum_{1 \leq k \leq n} [(R_{-k}^{-1})^2 + (R_1^k)^2] + \left( \sum_{1 \leq k \leq n} (R_{-k}^{-1} + R_1^k) \right)^2 \\ &\quad - 2 \sum_{1 \leq k < l \leq n} R_{-k}^{-1} R_{-l}^{-1} - 2 \sum_{1 \leq k < l \leq n} R_1^k R_1^l. \end{aligned}$$

Observe that :

$$\begin{aligned} - \sum_{1 \leq k < l \leq n} 2R_1^k R_1^l &= - \sum_{2 \leq k \leq l \leq n} 2R_1^{k-1} R_1^l = \sum_{2 \leq k \leq l \leq n} [(R_k^l)^2 - (R_1^{k-1})^2 - (R_1^l)^2] \\ &= \sum_{2 \leq k \leq l \leq n} (R_k^l)^2 - (n-1) \sum_{1 \leq k \leq n} (R_1^k)^2. \end{aligned}$$

As a result :

$$\psi^2(n) = n^2 + \sum_{1 \leq k \leq l \leq n} 2^{(1,k)} [(R_k^l)^2 + (R_{-l}^{-k})^2] + \left( \sum_{1 \leq k \leq n} (R_{-k}^{-1} + R_1^k) \right)^2.$$

- In the antisymmetric case, we have  $\varphi(n) \leq C\psi(n) \leq C\sqrt{2}\varphi_+(n)$ . Hence the random walk is transient if and only if :

$$\int_0^n \varphi_+^{-1}(1/t) dt < +\infty.$$

Proceeding as for i), this last property is equivalent to  $\sum_{n \geq 1} (1/\varphi_+(n)) < +\infty$ .

- In the symmetric case, starting from the beginning of the computation of  $\psi^2(n)$  we have :

$$\psi(n) = n^2 + \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 + 2n \sum_{1 \leq k \leq n} (R_1^k)^2 + 2 \left( \sum_{1 \leq k \leq n} R_1^k \right)^2.$$

As  $(n+1) \sum_{1 \leq k \leq n} (R_1^k)^2 = \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 + \left( \sum_{1 \leq k \leq n} R_1^k \right)^2$ , we get  $\psi(n)$  and thus  $\varphi(n)$  has exact order :

$$n^2 + n \sum_{1 \leq k \leq n} (R_1^k)^2.$$

From point *i*), the random walk is therefore transient if :

$$\sum_{n \geq 1} \frac{1}{\sqrt{n^2 + n \sum_{1 \leq k \leq n} (R_1^k)^2}} < +\infty.$$

## 7.2 Proof of proposition 1.4

*i*) It follows from the hypothesis that both  $\varphi(n)$  and  $\varphi_+(n)$  are  $O(n\sqrt{\log n})$ . Hence, for some constant  $c > 0$  we have the lower bound :

$$\frac{1}{n^2} \frac{(\varphi^{-1}(n))^2}{\varphi_+^{-1}(n)} \geq c \frac{1}{n^2} \frac{(n/\sqrt{\log n})^2}{n} \geq \frac{c}{n \log n}.$$

Using theorem 1.2, we deduce recurrence.

*ii*) We have  $\varphi_+(n) = O(n \log n)$ . We next use corollary 1.3, *ii*).

*iii*) Using corollary 1.3, *ii*) and *iii*), it is enough to show that :

$$\sum_{n \geq 1} \frac{1}{\sqrt{n^2 + n \sum_{1 \leq k \leq n} (R_1^k)^2}} < +\infty, \text{ but } \sum_{n \geq 1} \frac{1}{\sqrt{n^2 + \sum_{1 \leq k \leq l \leq n} (R_k^l)^2}} = +\infty.$$

First of all, making successive integration by parts :

$$\int_1^t \log^\alpha s \, ds = t \log^\alpha t - \alpha t \log^{\alpha-1} t + \alpha(\alpha-1)t(\log^{\alpha-2} t)(1+o(1)).$$

Write now, for some  $C > 0$  :

$$\sum_{1 \leq k \leq n} (R_1^k)^2 \geq \frac{1}{2} \sum_{1 \leq k \leq n} \log^{2\alpha} k - Cn \log^{2\alpha-2} n \geq \frac{1}{2} n(\log^{2\alpha} n)(1+o(1)).$$

The first sum thus converges. On the other hand :

$$\begin{aligned} \frac{1}{2} \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 &\leq \sum_{1 \leq k < l \leq n} (\log^\alpha l - \log^\alpha k)^2 + O(n^2 \log^{2\alpha-2} n) \\ &\leq n \sum_{1 \leq k \leq n} \log^{2\alpha} k - 2 \sum_{1 \leq k < l \leq n} \log^\alpha k \log^\alpha l + O(n^2 \log^{2\alpha-2} n) \\ &\leq n \sum_{1 \leq k \leq n} \log^{2\alpha} k - \left( \sum_{1 \leq k \leq n} \log^\alpha k \right)^2 + O(n^2 \log^{2\alpha-2} n). \end{aligned}$$

As a result :

$$\frac{1}{2} \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 \leq n \int_2^{n+1} \log^{2\alpha} t \, dt - \left( \int_1^n \log^\alpha t \, dt \right)^2 + O(n^2 \log^{2\alpha-2} n).$$

This finally furnishes :

$$\begin{aligned}
\frac{1}{2} \sum_{1 \leq k \leq l \leq n} (R_k^l)^2 &\leq n \int_1^n \log^{2\alpha} t \, dt - \left( \int_1^n \log^\alpha t \, dt \right)^2 + O(n^2 \log^{2\alpha-2} n). \\
&\leq n^2 \log^{2\alpha} n \left( 1 - \frac{2\alpha}{\log n} + \frac{2\alpha(2\alpha-1)}{\log^2 n} (1 + o(1)) \right) \\
&\quad - n^2 \log^{2\alpha} n \left( 1 - \frac{\alpha}{\log n} + \frac{\alpha(\alpha-1)}{\log^2 n} (1 + o(1)) \right)^2 + O(n^2 \log^{2\alpha-2} n) \\
&\leq \alpha^2 n^2 (\log^{2\alpha-2} n) (1 + o(1)) + O(n^2 \log^{2\alpha-2} n).
\end{aligned}$$

Consequently the second series diverges. This completes the proof of the proposition.

### 7.3 Proof of proposition 1.5

We have  $\varepsilon_{-n} = -\varepsilon_n$ ,  $n \geq 1$ . Let  $(q_p)$  be the denominators of the convergence of  $\alpha$ . Every integer  $n < q_{p+1}$  can be decomposed as  $n = b_0(n) + b_1(n)q_1 + \dots + b_p(n)q_p$ , with  $0 \leq b_k(n) \leq a_{k+1} - 1$ . As  $x \mapsto 1_{[0,1/2)}(x) - 1_{[1/2,1)}(x)$  has bounded variation, the Denjoy-Koksma inequality implies that for any  $u$  and any  $k$  that  $|R_u^{u+q_k}| \leq C$ . As a result,  $|R_1^n| \leq C(a_1 + \dots + a_{p+1})$ . Hence, for  $n < q_{p+1}$ :

$$\sum_{1 \leq k \leq l \leq n} (R_k^l)^2 \leq 2n \sum_{1 \leq k \leq n} (R_k^1)^2 \leq 2n^2 (a_1 + \dots + a_{p+1})^2.$$

In the perspective of using corollary 1.3 *ii*), we have for some  $C > 0$ :

$$\sum_{n \geq 1} \frac{1}{\sqrt{\sum_{1 \leq k \leq l \leq n} (R_k^l)^2}} \geq C \sum_{p \geq 1} \sum_{q_p \leq n < q_{p+1}} \frac{1}{n(a_1 + \dots + a_{p+1})} \geq C \sum_{p \geq 1} \frac{\log(q_{p+1}/q_p)}{(a_1 + \dots + a_{p+1})}.$$

Write then :

$$\sum_{p \geq 1} \frac{\log(q_{p+1}/q_p)}{(a_1 + \dots + a_{p+1})} \geq \frac{1}{2} \sum_{p \geq 2} \left[ \frac{\log(q_{p+1}/q_p)}{(a_1 + \dots + a_{p+1})} + \frac{\log(q_p/q_{p-1})}{(a_1 + \dots + a_p)} \right].$$

If  $a_{p+1} \geq 2$ , then  $\log(q_{p+1}/q_p) \geq \log a_{p+1} \geq \frac{\log 2}{\log 3} \log(1 + a_{p+1})$ . If  $a_{p+1} = 1$ , then :

$$\frac{\log(q_{p+1}/q_p)}{(a_1 + \dots + a_{p+1})} + \frac{\log(q_p/q_{p-1})}{(a_1 + \dots + a_p)} \geq \frac{\log(q_{p+1}/q_{p-1})}{(a_1 + \dots + a_{p+1})} \geq \frac{\log 2}{(a_1 + \dots + a_{p+1})},$$

because  $q_{p+1} \geq (1 + a_{p+1}a_p)q_{p-1} \geq 2q_{p-1}$ . As  $\log 2 = \log(1 + a_{p+1})$ , this concludes the proof of the proposition.  $\square$

### 7.4 Proof of proposition 1.6

*Step 1.* One can always decrease  $\gamma > 1$  to get  $\beta > \gamma > 1$ . For  $k \geq 1$ , let  $D_k = [u(k), v(k)]$ , where  $u(k) = k \log^\beta k$  and  $v(k) = k \log^\beta k + k$ , of length  $k$ . Introduce :

$$A_n = \frac{1}{n \log^{-\beta} n} \sum_{k=1}^{n \log^{-\beta} n} 1_{|R_{u(k)}^{v(k)}| \geq \log^\beta k}.$$

There exists  $\alpha > 0$  such that for  $n$  large enough,  $\mathbb{E}(A_n) \geq \alpha$ . Next :

$$\text{var}(A_n) \leq \frac{2}{n^2 \log^{-2\beta} n} \sum_{0 \leq k \leq l \leq n \log^{-\beta} n} \left| \text{cov} \left( 1_{|R_{u(k)}^{v(k)}| \geq \log^\beta k}, 1_{|R_{u(l)}^{v(l)}| \geq \log^\beta l} \right) \right|.$$

– The previous sum restricted to indices  $k \leq l \leq k + k \log^{-\gamma} k$  is less than or equal to :

$$\sum_{0 \leq k \leq n \log^{-\beta} n} k \log^{-\gamma} k \leq cn^2 \log^{-2\beta-\gamma} n.$$

– For indices such that  $l \geq k + k \log^{-\gamma} k$ , the distance between  $D_k$  and  $D_l$  is at least :

$$l \log^\beta l - k \log^\beta k - k \geq k \log^\beta k + k \log^{\beta-\gamma} k - k \log^\beta k - k \geq \frac{1}{2} k \log^{\beta-\gamma} k.$$

It results that the corresponding sum is less than or equal to :

$$\sum_{0 \leq k \leq n \log^{-\beta} n} n \log^{-\beta} n \alpha((k/2) \log^{\beta-\gamma} k) = O\left(n^2 \log^{-2\beta-\gamma} n\right).$$

This thus gives  $\text{var}(A_n) = O(\log^{-\gamma} n)$ . Hence  $\sum_{n \geq 1} \text{var}(A_{2^n}) = O(\sum_{n \geq 1} n^{-\gamma}) < +\infty$ . Therefore, almost surely,  $A_{2^n} - \mathbb{E}(A_{2^n}) \rightarrow 0$  and  $A_{2^n} \geq \alpha/2 > 0$  for  $n$  large enough.

*Step 2.* Denote by  $k_1, \dots, k_N$  the  $k$ 's in  $[1, n \log^{-\beta} n]$  so that  $|R_{u(k)}^{v(k)}| \geq \log^\beta k$ . We have :

$$N \geq \frac{\alpha}{2} n \log^{-\beta} n,$$

for large  $n$  along the sequence  $(2^m)$ . For each  $1 \leq r \leq N$ , introduce two intervals  $I_r$  and  $J_r$  of length  $\log^\beta k_r / (10(1 + \|\rho\|_\infty))$  centered in  $u(k_r)$  and  $v(k_r)$  respectively. The length of  $D_{k_r} \cup I_r \cup J_r$  is equivalent to  $k_r \leq n \log^{-\beta} n$ . Choose an interval  $K_r \subset [1, n]$  of length  $\geq n/4$  in the complementary of the latter union.

Observe now that  $J_r$  can intersect at most three  $I_{r'}$  with  $r' > r$ . Indeed, as  $r'$  increases from the value  $r$ , the center of  $I_{r'}$  makes steps of size  $\geq \log^\beta k_r$ . At most  $k_r \log^{-\beta} k_r$  steps are necessary to cross  $I_r \cup D_{k_r} \cup J_r$ . At the end of the crossing, the length of the  $I_{r'}$  are at most :

$$\log^\beta(k_r + \log^\beta k_r + k_r \log^{-\beta} k_r) = \log^\beta k_r + o(1).$$

In a similar way,  $I_r$  intersects at most three  $J_{r'}$  with  $r' < r$ . Taking into account these overlaps and which  $k$  or  $l$  is in some  $I_r$  or  $J_r$ , we have :

$$\sum_{1 \leq r \leq N} \left( \sum_{k \in I_r, l \in K_r} (R_k^l)^2 + \sum_{k \in J_r, l \in K_r} (R_k^l)^2 \right) \leq 6 \sum_{1 \leq k \leq l \leq n} (R_k^l)^2.$$

In this notation,  $K_r$  is supposed for simplicity to be on the right side of  $I_r \cup D_{k_r} \cup J_r$ . For each  $l \in K_r$ , either  $|R_{u(r)}^l| \geq (1/2) \log^\beta k_r$  or  $|R_{v(r)}^l| \geq (1/2) \log^\beta k_r$  and when one inequality is true, one has  $|R_w^l| \geq (1/4) \log^\beta k_r$ ,  $w \in I_r$ , in the first case and  $|R_w^l| \geq (1/4) \log^\beta k_r$ ,  $w \in J_r$ , in the second case. As a result, the left-hand side of the previous inequality checks, for constants  $c_i > 0$  :

$$\geq c_1 \sum_{r=1}^N n \log^\beta k_r \log^{2\beta} k_r \geq c_2 n N \log^{3\beta} N \geq c_3 n^2 \log^{2\beta} n.$$

We now conclude. Cutting in dyadic blocks, for  $M$  large enough :

$$\sum_{k \geq M} \frac{1}{\varphi_+(k)} \leq \sum_{n \geq n_0} \sum_{2^n \leq k < 2^{n+1}} \frac{1}{\varphi_+(2^n)} \leq C \sum_{n \geq n_0} \frac{2^n}{2^n n^\beta} < +\infty.$$

Applying corollary 1.3 i), we get the result.

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