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A Method of Piecewise-Smooth Numerical Branching

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A method of numerical branching is proposed for piecewise-smooth steady-state problems when any analytical expressions are not known for the regions of smoothness of the function involved. It is shown in model examples of contact problems with Coulomb friction that the method can reliably discover all solution branches around a known solution if its input parameters are set up properly. It is also suggested how to choose the optimum parameter settings.

1 Introduction

The steady-state bifurcation problem:

$$\left. \begin{array}{l} \text{Find } \mathbf{y} \in U \text{ such that} \\ \mathbf{H}(\mathbf{y}) = \mathbf{0}, \end{array} \right\} \quad (\mathcal{P})$$

where $U \subset \mathbb{R}^{N+1}$ and $\mathbf{H}: U \rightarrow \mathbb{R}^N$, has been the subject of large number of studies in the last decades. A variety of numerical bifurcation methods has been constructed if \mathbf{H} is smooth, say continuously differentiable (see, e.g., [2, 6, 9] and references therein). On the other hand, there are many equilibrium problems in economics and diverse engineering fields whose models lead to a system of non-smooth equations [1, 5, 16]. Let us mention frictional and frictionless contact problems in solid mechanics, which are of our specific interest, for instance. However, methods of numerical branching of solutions of non-smooth problems depending on a parameter are still very little explored to our knowledge: Branching of static equilibrium curves of discretised frictionless contact problems were treated in [3, 17], where the tangential directions of curves emanating from points of non-smoothness were characterised as the solutions of a certain linear complementarity problem, and a method based on resolution of this problem was suggested for branch switching during numerical continuation (see also the theoretical survey [11]). Furthermore, the possible evolutions in quasi-static plane frictional contact problems were determined similarly in [14].

In our recent paper [12], we developed a restarted predictor-corrector method for numerical continuation of solution curves of Problem (\mathcal{P}) provided that \mathbf{H} is piecewise C^1 (PC^1). The method consists in continuing smooth solution branches by a standard predictor-corrector method and joining the smooth branches continuously.

More precisely, our method computes a sequence of points $\{\mathbf{y}_k\}$ lying approximately on a selected solution branch together with a sequence of the corresponding tangent vectors $\{\mathbf{t}_k\}$ with a unit weighted norm $\|\mathbf{t}_k\|_w$. Starting from a couple $(\mathbf{y}_k, \mathbf{t}_k)$, one predictor-corrector step yields a new couple $(\mathbf{y}_{k+1}, \mathbf{t}_{k+1})$ in the following way: In the predictor step, an initial approximation $(\tilde{\mathbf{y}}, \tilde{\mathbf{t}})$ of the new couple is generated in the direction of \mathbf{t}_k as

$$\tilde{\mathbf{y}} := \mathbf{y}_k + h\mathbf{t}_k, \quad \tilde{\mathbf{t}} := \mathbf{t}_k,$$

where h is a step size. Then, the corrector steps, which are iterative steps of Newton's type, are run with the initial approximation $(\tilde{\mathbf{y}}, \tilde{\mathbf{t}})$. If they succeed in bringing the predicted point back to the currently approximated branch, the resulting couple is accepted, the step size h is adapted for the next predictor-corrector step, and the current step is done. Otherwise, h is reduced and the predictor and the corrector steps are repeated. If these predictor-corrector steps fail in computing a new couple, a special procedure is carried out for locating a new smooth branch (the so-called simple tangent switch). Afterwards, the predictor-corrector steps are restarted for continuation of the branch found. Let us emphasise that

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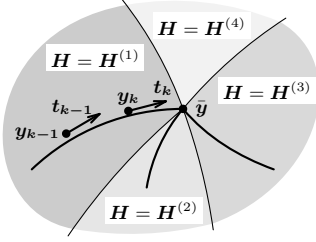


Fig. 1 An example of the considered situation.

the whole method does not obey any analytical expressions of the regions of smoothness of \mathbf{H} , and it is implementable generically.

Similar continuation techniques constructed specially for static plane frictional contact problems can be found in [7, 8]. Nevertheless, all these methods were proposed for finding solely one continuing branch when an end point of the most recently traced smooth branch is encountered. Besides, a bifurcation criterion was proposed merely in [13].

The present paper deals with purely non-smooth numerical branching of solutions of (\mathcal{P}) when \mathbf{H} is PC^1 . We develop an approach that does not rely on any analytical expressions of the regions of smoothness of \mathbf{H} in Section 2, and we demonstrate its performance for model examples of contact problems in Section 3.

Throughout the paper, we employ the following definition of a PC^1 -function [16]:

Definition 1.1 A function $\mathbf{H}: U \rightarrow \mathbb{R}^N$ defined on an open set $U \subset \mathbb{R}^M$ is PC^1 if it is continuous and for every $\bar{\mathbf{y}} \in U$, there exist an open neighbourhood $O \subset U$ of $\bar{\mathbf{y}}$ and a finite family of C^1 -functions $\mathbf{H}^{(i)}: O \rightarrow \mathbb{R}^N$, $i \in \mathcal{I}(\bar{\mathbf{y}})$, such that

$$\forall \mathbf{y} \in O: \mathbf{H}(\mathbf{y}) \in \{\mathbf{H}^{(i)}(\mathbf{y}); i \in \mathcal{I}(\bar{\mathbf{y}})\}.$$

The functions $\mathbf{H}^{(i)}$ are termed *selections* of \mathbf{H} at $\bar{\mathbf{y}}$.

2 Numerical Branching

Consider that an approximation of a smooth solution branch of (\mathcal{P}) has been obtained by the restarted predictor-corrector method from [12], and either the method has failed in locating a new smooth branch or it has located one but we want to find other ones because a bifurcation is expected by virtue of the criterion from [13], for example. Restricting ourselves to purely non-smooth branching (Assumptions (I)–(III) below), we shall describe a method designed for discovering potentially all continuing branches.

Let $\bar{\mathbf{y}}$ be an end point of the recovered branch. We shall assume the following:

- (I) The whole solution set of (\mathcal{P}) in a vicinity of $\bar{\mathbf{y}}$ is formed by one-sided smooth solution branches emanating from $\bar{\mathbf{y}}$ into mutually distinct regions of smoothness $\{\mathbf{y} \in O; \mathbf{H}(\mathbf{y}) = \mathbf{H}^{(i)}(\mathbf{y})\}$ for some $i \in \mathcal{I}(\bar{\mathbf{y}})$.
- (II) The gradients $\nabla \mathbf{H}^{(i)}(\bar{\mathbf{y}})$, $i \in \mathcal{I}(\bar{\mathbf{y}})$, have a full rank.
- (III) $\text{Ker } \nabla \mathbf{H}^{(i)}(\bar{\mathbf{y}}) \cap \text{Ker } \nabla \mathbf{H}^{(j)}(\bar{\mathbf{y}}) = \{\mathbf{0}\}$, $\forall i, j \in \mathcal{I}(\bar{\mathbf{y}}), i \neq j$.

Note that these assumptions guarantee particularly that tangent vectors at $\bar{\mathbf{y}}$ to any two different branches are linearly independent (see Fig. 1 for an illustration of branching under our consideration).

Since it seems to be hardly possible to encounter a point of non-smoothness in practical computations, we shall suppose in addition that \mathbf{H} is smooth in any point considered in the procedures hereinafter. Then, a unit tangent vector \mathbf{t} at such a point \mathbf{y} lying in a vicinity of $\bar{\mathbf{y}}$ and belonging to a solution branch from the region $\{z \in U; \mathbf{H}(z) = \mathbf{H}^{(i)}(z)\}$ is uniquely determined up to a direction by the conditions:

$$\nabla \mathbf{H}^{(i)}(\mathbf{y})\mathbf{t} = \mathbf{0}, \quad \|\mathbf{t}\|_w = 1,$$

where $\|\cdot\|_w$ denotes a suitable weighted norm [10, pp. 86 and 87].

Our branching method consists of two steps: an approximation of the end point $\bar{\mathbf{y}}$ and a subsequent location of new branches from a neighbourhood of the approximate end point.

2.1 Approximation of the end point

The end point can be approximated by a bisection-like procedure that is based on the predictor-corrector method used in the process of numerical continuation of smooth branches. The proposed procedure relies on the fact that the individual branches emanating from the end point have numerically distinguishable tangents, which is justified by Assumptions (I)–(III). The algorithm can be sketched as follows:

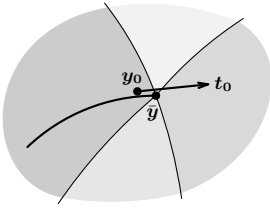


Fig. 2 Output of Algorithm 2.1.

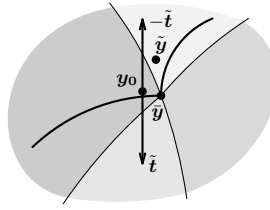


Fig. 3 Prediction direction for finding a new branch.

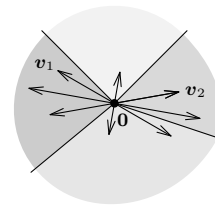


Fig. 4 Generation of directions into all regions.

Algorithm 2.1

Input: $c_{\text{diff}} \in (0, 1)$, $h > h_{\text{min}} > 0$, $\mathbf{y}_0, \mathbf{t}_0 \in \mathbb{R}^{N+1}$ such that

$$\mathbf{H}(\mathbf{y}_0) \approx \mathbf{0}, \quad \nabla \mathbf{H}(\mathbf{y}_0) \mathbf{t}_0 = \mathbf{0}, \quad \|\mathbf{t}_0\|_w = 1,$$

and the corrector steps for the prediction $\tilde{\mathbf{y}} := \mathbf{y}_0 + h\mathbf{t}_0$, $\tilde{\mathbf{t}} := \mathbf{t}_0$ do not converge or they converge to a couple (\mathbf{y}, \mathbf{t}) with $\mathbf{t}^\top \mathbf{t}_0 / (\|\mathbf{t}\| \|\mathbf{t}_0\|) < c_{\text{diff}}$.

Step 1: Set $h := h/2$.

Step 2: Do the prediction $\tilde{\mathbf{y}} := \mathbf{y}_0 + h\mathbf{t}_0$, $\tilde{\mathbf{t}} := \mathbf{t}_0$.

Step 3: Run the correction with $(\tilde{\mathbf{y}}, \tilde{\mathbf{t}})$.

Step 4: If the correction converges to a new couple (\mathbf{y}, \mathbf{t}) with $\mathbf{t}^\top \mathbf{t}_0 / (\|\mathbf{t}\| \|\mathbf{t}_0\|) \geq c_{\text{diff}}$, then set $\mathbf{y}_0 := \mathbf{y}$, $\mathbf{t}_0 := \mathbf{t}$.

Step 5: If $h < h_{\text{min}}$, break. Otherwise, go to Step 1.

Output: The point \mathbf{y}_0 with the corresponding unit tangent \mathbf{t}_0 .

The couple $(\mathbf{y}_0, \mathbf{t}_0)$ in the input is supposed to be the last couple obtained by numerical continuation of the most recent smooth branch (the couple $(\mathbf{y}_k, \mathbf{t}_k)$ in Fig. 1), and h equals typically the minimal step size from the predictor-corrector method employed during the continuation. We denote the Euclidean norm by $\|\cdot\|$ and c_{diff} is the minimal value of the cosine of the angle of two tangent vectors considered to correspond to the same branch. Hence, if the conditions in Step 4 are satisfied, the newly found point \mathbf{y} is supposed to be from the same branch as the one stored in \mathbf{y}_0 at that time, and it is accepted as a better approximation of $\bar{\mathbf{y}}$. The minimal step size h_{min} determines the precision required for the resulting approximation of $\bar{\mathbf{y}}$.

2.2 Location of new branches

Suppose that \mathbf{y}_0 calculated by Algorithm 2.1 is a good approximation of $\bar{\mathbf{y}}$, it belongs to some smooth branch, and \mathbf{t}_0 is the corresponding unit tangent pointing out from the region of smoothness with \mathbf{y}_0 (see Fig. 2). Inspired by the methods for branch switching at a smooth simple bifurcation point [6,9], we shall propose a heuristic technique composed of a sequence of predictor-corrector steps and yielding points on other smooth branches.

Our heuristic stems from the idea used in the simple tangent switch [12]: Let $\tilde{\mathbf{t}}$ be a vector satisfying

$$\nabla \mathbf{H}(\tilde{\mathbf{y}}) \tilde{\mathbf{t}} = \mathbf{0}, \quad \|\tilde{\mathbf{t}}\|_w = 1, \tag{1}$$

where $\tilde{\mathbf{y}}$ is a point close to $\bar{\mathbf{y}}$ and lies in the region of smoothness containing a new smooth branch (Fig. 3). Then, either $\tilde{\mathbf{t}}$ or $-\tilde{\mathbf{t}}$ should be a suitable prediction direction for a predictor-corrector step giving a point on this new branch.

Imagine for a while that you have a sequence of points $\tilde{\mathbf{y}}$ that pass through all regions of smoothness intersecting at $\bar{\mathbf{y}}$ at your disposal. Under Assumptions (I)–(III), you could discover all solution branches by computing the sequence of vectors $\tilde{\mathbf{t}}$ from (1) and trying the corresponding predictor-corrector steps with $\tilde{\mathbf{t}}$ and $-\tilde{\mathbf{t}}$ successively. Hence, we would like to *guess* such a sequence of points $\tilde{\mathbf{y}}$ from all the regions intersecting at $\bar{\mathbf{y}}$.

For this purpose, consider the most probable branching scenarios described in [13]: By linearisation and projection, the corresponding regions of smoothness can be represented by cones in a two-dimensional vector space with vertices at $\mathbf{0}$. This representative space can be spanned by any two linearly independent vectors, say \mathbf{v}_1 and \mathbf{v}_2 . Thus, a sequence of linear combinations of \mathbf{v}_1 and \mathbf{v}_2 whose directions are deployed densely enough in the space comprises the directions pointing into the representatives of all regions of smoothness as indicated in Fig. 4. From here, one can conclude that if

\mathbf{v}_1 and \mathbf{v}_2 represent some vectors $\tilde{\mathbf{t}}_1$ and $\tilde{\mathbf{t}}_2$, then the corresponding sequence of linear combinations of $\tilde{\mathbf{t}}_1$ and $\tilde{\mathbf{t}}_2$ contains directions pointing into all regions and it can be used for calculating the sequence of points as desired.

In the general setting with $(\mathbf{y}_0, \mathbf{t}_0)$ from the output of Algorithm 2.1, let us choose a step length h sufficiently large in comparison with the precision of the approximation of $\bar{\mathbf{y}}$ by \mathbf{y}_0 , in particular so large that $\mathbf{y}_0 + h\mathbf{t}_0$ leaves the region with \mathbf{y}_0 . Next, let us take $\tilde{\mathbf{t}}_1 := -\mathbf{t}_0$ and calculate a vector $\tilde{\mathbf{t}}_2$ satisfying

$$\nabla H(\mathbf{y}_0 + h\mathbf{t}_0)\tilde{\mathbf{t}}_2 = \mathbf{0}, \quad \|\tilde{\mathbf{t}}_2\|_w = 1.$$

According to the imposed assumptions, $\tilde{\mathbf{t}}_1$ and $\tilde{\mathbf{t}}_2$ are linearly independent. One can choose a set $\mathcal{B} \subset \mathbb{R}^2$ so that $\{\beta_1\tilde{\mathbf{t}}_1 + \beta_2\tilde{\mathbf{t}}_2\}_{(\beta_1, \beta_2) \in \mathcal{B}}$ is a sequence of densely deployed directions. With regard to the argumentation for the most probable branching scenarios, one can expect that the sequence $\{\mathbf{y}_0 + h(\beta_1\tilde{\mathbf{t}}_1 + \beta_2\tilde{\mathbf{t}}_2)\}_{(\beta_1, \beta_2) \in \mathcal{B}}$ might be the desired one containing points from all regions intersecting at $\bar{\mathbf{y}}$.

These considerations lead us to the following algorithm:

Algorithm 2.2

Input: $c_{\text{diff}} \in (0, 1)$, $n_{\text{dir}} \in \mathbb{N}$, $h, h_{\text{init}} > 0$, $\mathbf{y}_0, \mathbf{t}_0 \in \mathbb{R}^{N+1}$ such that

$$H(\mathbf{y}_0) \approx \mathbf{0}, \quad \nabla H(\mathbf{y}_0)\mathbf{t}_0 = \mathbf{0}, \quad \|\mathbf{t}_0\|_w = 1,$$

and $\mathbf{y}_0 + h\mathbf{t}_0$ leaves the region of smoothness with \mathbf{y}_0 .

Step 1: Take $\tilde{\mathbf{t}}_1 := -\mathbf{t}_0$ and compute $\tilde{\mathbf{t}}_2$ such that

$$\nabla H(\mathbf{y}_0 + h\mathbf{t}_0)\tilde{\mathbf{t}}_2 = \mathbf{0}, \quad \|\tilde{\mathbf{t}}_2\|_w = 1.$$

Set $n_{\text{br}} := 1$, $\mathbf{y}_{n_{\text{br}}} := \mathbf{y}_0$, $\mathbf{t}_{n_{\text{br}}} := \tilde{\mathbf{t}}_1$.

Step 2: Set $i := 0$.

Step 3: Set

$$\alpha := \frac{2\pi i}{n_{\text{dir}}}, \quad \mathbf{v} := (\sin \alpha)\tilde{\mathbf{t}}_1 + (\cos \alpha)\tilde{\mathbf{t}}_2, \quad \mathbf{v} := \frac{\mathbf{v}}{\|\mathbf{v}\|_w},$$

and compute $\tilde{\mathbf{t}}$ such that

$$\nabla H(\mathbf{y}_0 + h\mathbf{v})\tilde{\mathbf{t}} = \mathbf{0}, \quad \|\tilde{\mathbf{t}}\|_w = 1.$$

Step 4: If $i = 0$ or $|\tilde{\mathbf{t}}^\top \mathbf{t}_0| / (\|\tilde{\mathbf{t}}\| \|\mathbf{t}_0\|) < c_{\text{diff}}$, set $\mathbf{t}_0 := \tilde{\mathbf{t}}$. Otherwise, go to Step 7.

Step 5: Try one predictor-corrector step with the couple $(\mathbf{y}_0, \tilde{\mathbf{t}})$ and the initial step size h_{init} . If it converges to a new couple (\mathbf{y}, \mathbf{t}) , then

- (i) if $\mathbf{t}^\top (\mathbf{y} - \mathbf{y}_0) < 0$, take $\mathbf{t} := -\mathbf{t}$;
- (ii) if $\mathbf{t}^\top \mathbf{t}_j / (\|\mathbf{t}\| \|\mathbf{t}_j\|) < c_{\text{diff}}$, $j = 1, \dots, n_{\text{br}}$, then set $n_{\text{br}} := n_{\text{br}} + 1$, $\mathbf{y}_{n_{\text{br}}} := \mathbf{y}$, $\mathbf{t}_{n_{\text{br}}} := \mathbf{t}$.

Step 6: Repeat the procedure from Step 5 with $\tilde{\mathbf{t}} := -\tilde{\mathbf{t}}$.

Step 7: Set $i := i + 1$. If $i < n_{\text{dir}}$, go to Step 3. Otherwise, break.

Output: Points $\mathbf{y}_1, \dots, \mathbf{y}_{n_{\text{br}}}$ with the corresponding unit tangents $\mathbf{t}_1, \dots, \mathbf{t}_{n_{\text{br}}}$.

Here, c_{diff} is the minimal absolute value of the cosine of the angle of two vectors from the kernels of gradients considered to correspond to the same selection similarly as in Algorithm 2.1. Further, n_{dir} is the total number of linear combinations of $\tilde{\mathbf{t}}_1$ and $\tilde{\mathbf{t}}_2$ for seeking points from distinct regions. The step sizes h and h_{init} should be appropriately large in comparison with the expected precision of the approximation of $\bar{\mathbf{y}}$ by \mathbf{y}_0 so that the points determined with them fall into the regions in the respective directions.

The conditions in Step 4 serve for determining whether $\tilde{\mathbf{t}}$ has been computed from a gradient from the same region as previously. If it is so, it is superfluous, hence the corresponding predictor-corrector steps are skipped. In Step 5(ii), one verifies whether (\mathbf{y}, \mathbf{t}) does not belong to any of the branches already found before. This ensures that the points in the output are from different smooth branches. The corresponding tangents are oriented in the directions of branching from $\bar{\mathbf{y}}$, which is guaranteed by Step 5(i). Afterwards, any of the branches located in this way can be traced by predictor-corrector steps.

Remark 2.3 To increase the probability of recovering all branches, one can restart Algorithm 2.2 several times from Step 2 with other choices of $\tilde{\mathbf{t}}_1$ and $\tilde{\mathbf{t}}_2$. We propose the following possibilities:

- (i) Take $\tilde{\mathbf{t}}_i$ from the set $\{\mathbf{t}_1, \dots, \mathbf{t}_{n_{br}}\}$ at your current disposal.
- (ii) Take $\tilde{\mathbf{t}}_i := \tilde{\mathbf{t}}_i^+$ with $\tilde{\mathbf{t}}_i^+$ satisfying

$$\nabla H\left(\mathbf{y}_0 + h\left(\tilde{\mathbf{t}}_i^- + 0.1 \frac{\tilde{\mathbf{t}}}{\|\tilde{\mathbf{t}}\|_w}\right)\right)\tilde{\mathbf{t}}_i^+ = \mathbf{0}, \quad \|\tilde{\mathbf{t}}_i^+\|_w = 1,$$

where the value of h is the same as in Algorithm 2.2, $\tilde{\mathbf{t}}_i^-$ is equal to the vector $\tilde{\mathbf{t}}_i$ employed in the previous run of Algorithm 2.2 and $\tilde{\mathbf{t}}$ is chosen randomly.

3 Numerical Tests

We have tested our technique on two examples of bifurcations in discretised plane contact problems with Coulomb friction from [13]. We have chosen the following strategy based on Remark 2.3 for restarting Algorithm 2.2:

- (1) Pass with $\tilde{\mathbf{t}}_1$ and $\tilde{\mathbf{t}}_2$ successively through the combinations from $\{\mathbf{t}_1, \dots, \mathbf{t}_{n_{br}}\}$ at your disposal.
- (2) If all combinations which are available so far have already been employed, try to let $\tilde{\mathbf{t}}_1$ be the same and choose $\tilde{\mathbf{t}}_2$ according to Remark 2.3(ii).

In our tests, we denote the total number of selections of $\tilde{\mathbf{t}}_1$ and $\tilde{\mathbf{t}}_2$ by n_{span} , that is, the total number of restarts of Algorithm 2.2 equals $n_{\text{span}} - 1$. Further, n_{pc} stands for the total number of the predictor-corrector steps, which constitute the most expensive part of our technique. This number characterises the computational cost of each test.

To start with, we have tested the performance of our method for different values of n_{dir} and n_{span} . Subsequently, we have investigated numerically how the chosen model bifurcations behave for different meshes. Our calculations have been done with the finite-element library GetFEM++ [15].

3.1 Contact problems with Coulomb friction

Let us consider static deformation of an elastic body whose reference configuration is the closure of a bounded domain $\Omega \subset \mathbb{R}^2$. Let the boundary $\partial\Omega$ be Lipschitz-continuous and split into three disjoint relatively open subsets Γ_D , Γ_N and Γ_c (see Fig. 5). The body is fixed along Γ_D whereas a surface force of density \mathbf{h} is prescribed on Γ_N . A flat rigid foundation supports the body along Γ_c , and the contact is modelled by unilateral conditions and the Coulomb friction law. We suppose that there is no initial gap between the body and the foundation. We consider that the surface force is parametrised by a real parameter γ , that is, $\mathbf{h} = \mathbf{h}(\gamma)$ in our model, and we seek equilibrium states of the body for the values of γ from an interval I of our interest.

Discretisation of this problem is done by applying a Lagrange finite-element method to a mixed variational formulation of the problem with Lagrange multipliers enforcing the Dirichlet and the contact boundary conditions. Specifically, the contact conditions are approximated nodally and written in terms of projections. This gives the following discrete problem in the form of (\mathcal{P}) :

$$\left. \begin{aligned} \text{Find } \mathbf{y} &:= (\gamma, \mathbf{u}, \boldsymbol{\lambda}_D, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau) \in I \times \mathbb{R}^{2(n_\Omega + n_D + n_c)} \text{ such that} \\ \mathbf{H}(\mathbf{y}) &= \mathbf{0}, \end{aligned} \right\}$$

where $\mathbf{H}: I \times \mathbb{R}^{2(n_\Omega + n_D + n_c)} \rightarrow \mathbb{R}^{2(n_\Omega + n_D + n_c)}$ is defined by

$$\mathbf{H}(\mathbf{y}) := \begin{pmatrix} \mathbf{A}(\mathbf{u}) - \mathbf{L}(\gamma, \mathbf{u}) - \mathbf{B}_D^\top \boldsymbol{\lambda}_D - \mathbf{B}_\nu^\top \boldsymbol{\lambda}_\nu - \mathbf{B}_\tau^\top \boldsymbol{\lambda}_\tau \\ \mathbf{B}_D \mathbf{u} \\ \lambda_{\nu,j} - ((\lambda_\nu - r \mathbf{B}_\nu \mathbf{u})_j)_-, \quad j = 1, \dots, n_c \\ \lambda_{\tau,j} - P_{[\mathcal{F}((\lambda_\nu - r \mathbf{B}_\nu \mathbf{u})_j)_-, -\mathcal{F}((\lambda_\nu - r \mathbf{B}_\nu \mathbf{u})_j)_-]}((\lambda_\tau - r \mathbf{B}_\tau \mathbf{u})_j), \quad j = 1, \dots, n_c \end{pmatrix}. \quad (2)$$

Here, $\mathbf{u} \in \mathbb{R}^{2n_\Omega}$ is the vector of nodal displacements, $\boldsymbol{\lambda}_D \in \mathbb{R}^{2n_D}$ is the Lagrange multiplier corresponding to the Dirichlet condition, and $\boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau \in \mathbb{R}^{n_c}$ are the normal and tangential Lagrange multipliers on the contact zone, respectively. Further, $\mathbf{A}(\mathbf{u})$ and $\mathbf{L}(\gamma, \mathbf{u})$ are the vectors of internal elastic forces and external applied ones, respectively, and \mathbf{B}_D is the kinematic transformation matrix linking \mathbf{u} with the Lagrange multiplier $\boldsymbol{\lambda}_D$. The matrices \mathbf{B}_ν and \mathbf{B}_τ associate the vector of nodal displacements with the vectors of contact nodal displacements in the directions of the unit inward normal $\boldsymbol{\nu}$ and unit tangent $\boldsymbol{\tau}$ to the foundation, respectively. The notation $(\cdot)_-$ means the negative part (or equivalently, the projection onto $(-\infty, 0]$),

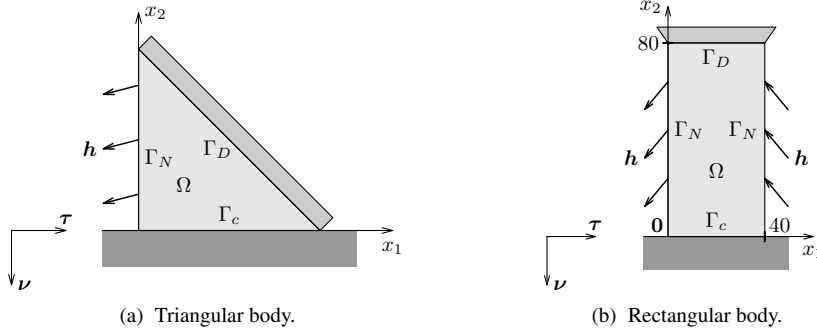


Fig. 5 Geometries of the model problems.

and $P_{[a,b]}(\cdot)$ stands for the projection onto $[a, b]$, $a \leq b$. The friction coefficient is represented by a non-negative constant \mathcal{F} here and $r > 0$ is a fixed augmentation parameter. Since both the negative part and the projection onto an interval are PC^1 -functions, \mathbf{H} is also PC^1 provided that \mathbf{A} and \mathbf{L} are continuously differentiable.

3.2 Problem with a triangular body

Firstly, consider deformation of an isosceles triangle with legs 1 m long shown in Fig. 5(a) in the framework of small-deformation elasticity with the Lamé constants $\lambda = 100 \text{ GN/m}^2$ and $\mu = 82 \text{ GN/m}^2$. We have tested our method for the following two discrete problems obtained by approximating both the displacement and the Lagrange multiplier for the Dirichlet condition with continuous piecewise linear functions:

1. We have chosen $\mathbf{h}(\gamma) = \gamma(-14.5 \text{ GN/m}^2, -7.5 \text{ GN/m}^2)$, $r = 10$ and $\mathcal{F} = 2$, and we have discretised the problem by using a single triangular finite element and by considering the Dirichlet condition in both vertices on the hypotenuse and the contact conditions in the lower left vertex of the triangle.

There are four solution branches intersecting at $\gamma = 0$ in this case. They correspond to loss of contact, contact-slip to the right, contact-stick and contact-slip to the left of the lower left vertex of the triangle. These branches will be denoted respectively by numbers 1 to 4 in what follows.

2. We have set $\mathbf{h}(\gamma) = \gamma(-26 \text{ GN/m}^2, -7.5 \text{ GN/m}^2)$, $r = 10$ and $\mathcal{F} = 1.7$, and we have done the discretisation with a uniform mesh with 4096 triangles and 64 contact nodes.

There are four solution branches intersecting at $\gamma = 0$, again. In this case, they correspond to a partial contact and slip of the triangle to the right, and to no contact, contact-stick and contact-slip to the left of the lower left vertex of the triangle with pulling the whole triangle to the left. In analogy with the previous case, these branches will be referred to as Branch 2, Branch 1, Branch 3 and Branch 4, respectively.

Setting $c_{\text{diff}} = 0.99999$, $h = 5e-8$, $h_{\text{min}} = h/2^{14} \doteq 3.1e-12$ and $h_{\text{init}} = 5e-4$, we have investigated sufficient values of n_{span} and the corresponding values of n_{pc} for finding all four branches for various n_{dir} in both problems. One gets by with selecting $\tilde{\mathbf{t}}_1$ and $\tilde{\mathbf{t}}_2$ solely according to Remark 2.3(i), no random generations are necessary in any of the tests performed. Our computations are summarised in Tables 1 and 2. In the tables, the entry Input means the branch from which

Table 1 Single triangle.

Input	n_{dir}	n_{span}	n_{pc}
	10	2	8
Branch 1	20	2	8
	30	1	8
	10	2	8
	20	2	8
Branch 2	30	2	8
	40	2	8
	50	1	8
Branch 3	10	1	8
Branch 4	10	1	8

Table 2 Refined triangle.

Input	n_{dir}	n_{span}	n_{pc}
	10	2	36
Branch 1	20	1	32
Branch 2	10	1	18
	10	2	34
	20	2	52
Branch 3	30	2	72
	40	1	56
Branch 4	10	1	20

\mathbf{y}_0 is chosen in the input of Algorithm 2.1. One can conclude that the setting with $n_{\text{dir}} = 10$ and $n_{\text{span}} = 2$ seems to be the most suitable in these examples with four branches.

3.3 Problem with a rectangular body

Secondly, consider deformation of a rectangular block that is 40 mm wide and 80 mm high from Fig. 5(b). A plane-strain approximation of the nonlinear Ciarlet-Geymonat constitutive law [4, Chapter 4] is used for the material of the block. Namely, the response function $\hat{\sigma}$ of the first Piola-Kirchhoff stress tensor is defined by

$$\hat{\sigma}(\mathbf{x}, \mathbf{F}) = (\tilde{\sigma}(\tilde{\mathbf{F}}))_{1 \leq i, j \leq 2}, \quad \tilde{\mathbf{F}} = \begin{pmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \quad \mathbf{F} \in \mathbb{R}^{2 \times 2},$$

$$\tilde{\sigma}(\tilde{\mathbf{F}}) = 2b(\text{tr}(\tilde{\mathbf{F}}^\top \tilde{\mathbf{F}}))\mathbf{I} + 2(a - b\tilde{\mathbf{F}}\tilde{\mathbf{F}}^\top)\tilde{\mathbf{F}} + (2c \det(\tilde{\mathbf{F}}^\top \tilde{\mathbf{F}}) - d)\tilde{\mathbf{F}}^{-\top}, \quad \tilde{\mathbf{F}} \in \mathbb{R}^{3 \times 3},$$

where

$$\lambda = 4000 \text{ N/mm}^2, \quad \mu = 120 \text{ N/mm}^2, \quad a = 30 \text{ N/mm}^2$$

and

$$b = \frac{\mu}{2} - a, \quad c = \frac{\lambda}{4} - \frac{\mu}{2} + a, \quad d = \frac{\lambda}{2} + \mu.$$

We have prescribed $\mathbf{h} = \mathbf{h}(\mathbf{x}, \gamma) = \gamma(-2, 0.12(x_1 - 20))$ (in N/mm²) on both parts of Γ_N , $r = 10$ and $\mathcal{F} = 1$. Discretisation is done by approximating the displacement and the Lagrange multiplier for the Dirichlet condition with continuous piecewise bilinear functions on a uniform mesh with 800 squares and 21 contact nodes.

There are six solution branches intersecting at $\gamma = 0$. Branches 1, 2 and 3 correspond to forcing the block to the right with no contact, contact-stick and contact-slip to the right of the lower right vertex of the block. Branches 4, 5 and 6, which are symmetric with respect to the axis $x_1 = 20$ of the block, correspond to forcing the block to the left with no contact, contact-stick and contact-slip to the left of the lower left vertex.

We have set $c_{\text{diff}} = 0.99999$, $h = 5e-7$, $h_{\text{min}} = h/2^{14} \doteq 3.1e-11$, $h_{\text{init}} = 5e-4$ in Algorithms 2.1 and 2.2. As previously, we have started our testing by looking into sufficient values of n_{span} and the corresponding values of n_{pc} for finding all six branches for n_{dir} fixed. In this problem, random generations according to Remark 2.3(ii) are needed for some initial branches and some values of n_{dir} . In such cases, we have repeated the corresponding test two times, and the behaviour of the whole computational process has varied from one test to another; see Table 3 for the results for the first three branches (the behaviour for the second three branches is symmetric). We denote the total number of restarts of Algorithm 2.2 following a random generation by n_{rand} here.

Afterwards, we have prescribed n_{dir} and n_{span} , and we have compared the obtained values of n_{pc} to get an idea about the cheapest but quiet reliable strategy. The summary of the performance of our method is shown in Table 4. All six

Table 3 Rectangular body, n_{dir} given a priori.

Input	n_{dir}	n_{span}	n_{pc}	n_{rand}	Input	n_{dir}	n_{span}	n_{pc}	n_{rand}
	20	8	168	1		20	4	136	1
	20	10	214	3		20	7	220	3
	20	13	266	6		20	8	240	4
Branch 1	40	7	292	2		40	4	260	1
	40	8	330	3		40	5	294	2
	40	14	478	7		40	5	316	2
	50	3	162	0		50	4	274	1
	60	5	232	0	Branch 2	50	5	342	2
	100	3	206	0		50	7	490	4
Branch 3	20	5	128	0		60	5	374	2
	40	3	134	0		60	8	620	5
	50	1	72	0		60	9	654	6
	60	4	202	0		100	4	372	1
	100	1	102	0		100	4	406	1
							100	7	662

Table 4 Rectangular body, n_{dir} and n_{span} given a priori.

Input	n_{dir}	n_{span}	n_{pc}	n_{rand}
Branch 1	20	15	310	3
	20	15	316	5
	20	15	322	1
	20	16	326	3
	20	16	344	5
	20	16	346	1
	50	10	400	0
	60	10	416	0
	100	8	474	0
	Branch 3	20	15	318
20		15	318	0
20		16	334	1
20		16	334	1
50		10	396	0
60		10	414	0
100	8	456	0	

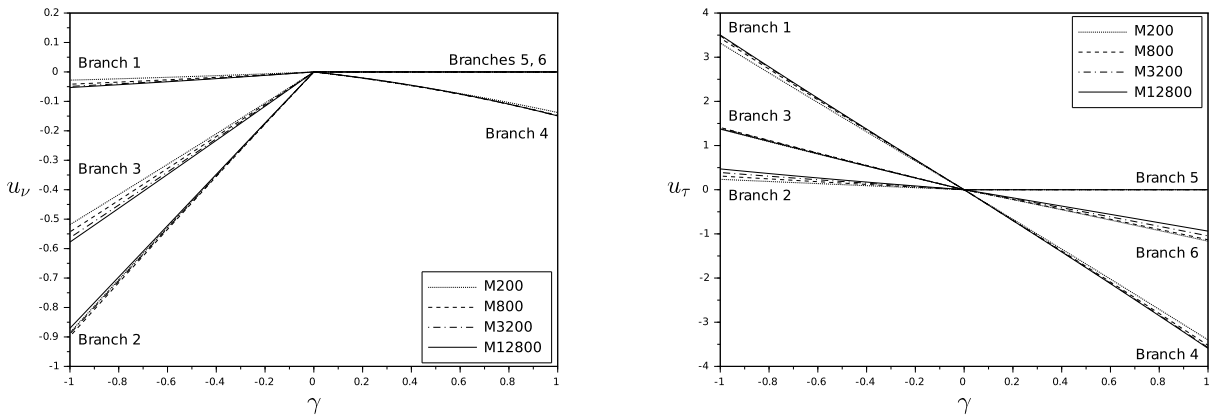
Input	n_{dir}	n_{span}	n_{pc}	n_{rand}
Branch 2	20	15	350	2
	20	15	360	4
	20	15	416	8
	20	16	374	2
	20	16	386	4
	20	16	434	8
	50	10	518	1
	50	10	532	1
	50	10	562	2
	60	10	618	2
60	10	642	3	
60	10	734	6	
100	8	642	1	
100	8	684	2	
100	8	720	3	

branches have been recovered in all the tests. According to the results, we can propose to set n_{dir} to about 20 and n_{span} to about 15 in this example with six branches. In addition, we can suggest to increase rather n_{dir} than n_{span} to increase the probability of finding all branches at the lowest price, in general.

3.4 Behaviour of the bifurcations for different meshes

To investigate the bifurcation in the problem with the rectangular body, we have performed the method of numerical branching for discretisations on uniform meshes with 200, 800, 3200 and 12800 bilinear squares (M200, M800, M3200 and M12800). We have set $n_{\text{dir}} = 30$ and $n_{\text{span}} = 15$ in these computations, letting the other parameters be the same as in Subsection 3.3. We have found six branches distinctly separated for all meshes. The branches seem to be stable and to converge with the meshes, see the bifurcation diagrams in Fig. 6 obtained by the method of piecewise-smooth numerical continuation from [12].

Next, we have looked into the branching in the second case of the problem with the triangular body for discretisations on uniform meshes with 4096, 16384, 65536 and 262144 linear triangles (M4096, M16384, M65536 and M262144). We have used $n_{\text{dir}} = 30$, $n_{\text{span}} = 10$ and we have set $c_{\text{diff}} = 1 - 1e-10$ unlike Subsection 3.2. The bifurcation behaviour is

**Fig. 6** The normal and tangential displacements u_ν and u_τ at the most left contact node of the rectangular body.

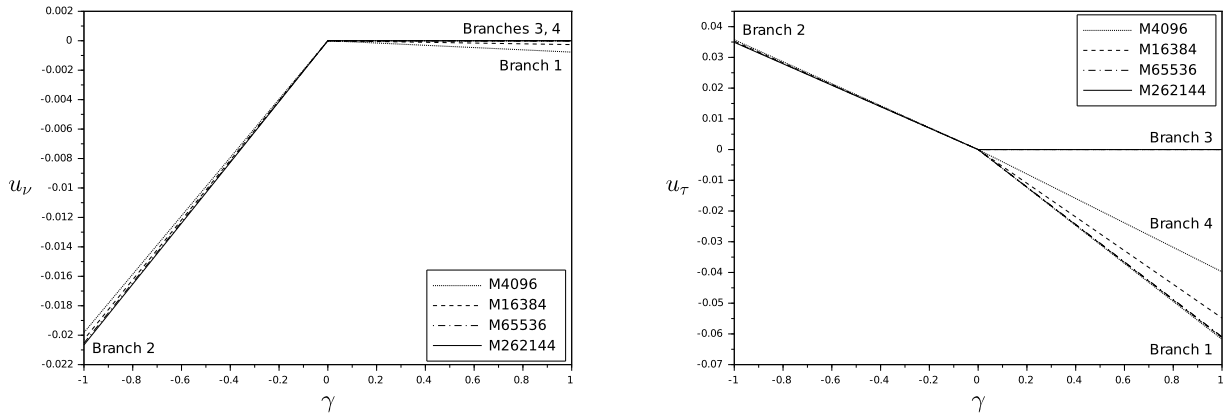


Fig. 7 The normal and tangential displacements u_ν and u_τ at the most left contact node of the triangular body for $\mathbf{h}(\gamma) = \gamma(-26, -7.5)$ (in GN/m^2).

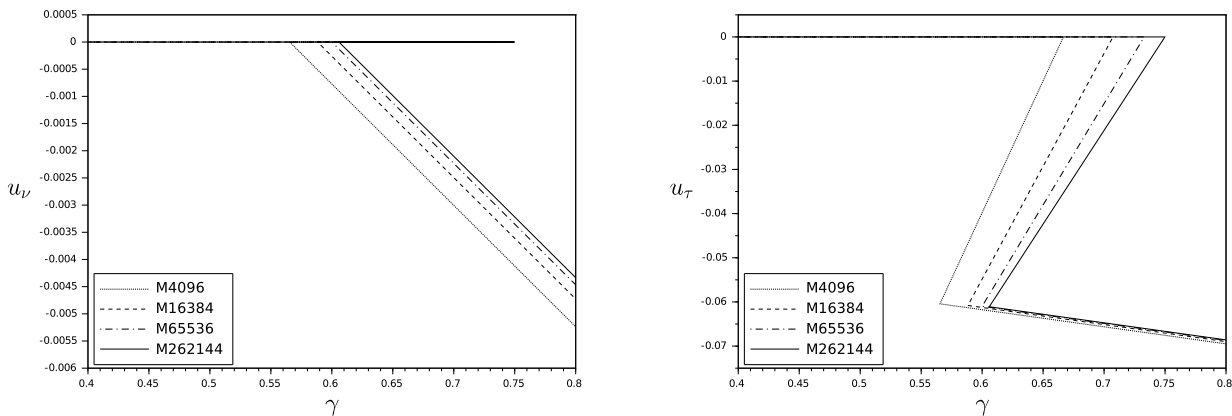


Fig. 8 The normal and tangential displacements u_ν and u_τ at the most left contact node of the triangular body for $\mathbf{h}(\gamma) = (-20 - 10\gamma, -7.5)$ (in GN/m^2).

more complex in this problem. Branches 1 and 4 approach one another for finer meshes and they disappear both for the finest mesh (for the chosen parametrised force $\mathbf{h}(\gamma) = \gamma(-26, -7.5)$ GN/m^2), see Fig. 7.

To examine this behaviour more precisely, we have done an auxiliary numerical continuation of the same parametrised problem except the force $\mathbf{h}(\gamma) = (-20 - 10\gamma, -7.5)$. As one can observe in Fig. 8, there are values of γ for which the contact problem has multiple solutions for this parametrisation, but these values differ a bit for each mesh. Namely, the problem has three solutions for γ approximately from $(0.56541, 0.66665)$, $(0.5883, 0.70731)$, $(0.59996, 0.73289)$ and $(0.60585, 0.7499)$ for M4096, M16384, M65536 and M262144, respectively. Moreover, two solutions approach one another when γ approaches a bound of the interval corresponding to a fixed mesh.

Besides, note that \mathbf{A} from (2) is linear and \mathbf{L} does not depend on \mathbf{u} in this problem. If \mathbf{h} is considered to be linear with respect to γ , then \mathbf{L} is linear, and \mathbf{H} given by (2) is piecewise linear and especially positively homogeneous. This yields that the solutions obtained by the auxiliary continuation characterise the solution behaviour of the corresponding unparametrised contact problem with forces $\mathbf{h} = (h_1, h_2)$ for any $h_2 < 0$.

In particular, we have discovered three solutions of the unparametrised contact problem with forces $\mathbf{h} = (h_1, -7.5)$ for h_1 approximately from $(25.6541, 26.6665)$, $(25.883, 27.0731)$, $(25.9996, 27.3289)$ and $(26.0585, 27.499)$ for the meshes M4096, M16384, M65536 and M262144, respectively. This and the positive homogeneity give us cones in the h_1 - h_2 plane, which are specified in Fig. 9, such that the unparametrised problem has three solutions for the forces from their interiors for each mesh. Furthermore, if a force approaches a half-line from the boundary of the cone for a given mesh, two of the three corresponding solutions approach one another.

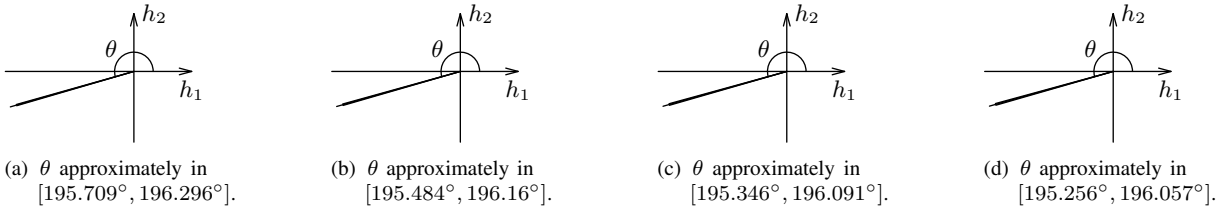


Fig. 9 Cones of forces with three solutions in the interior: (a) M4096; (b) M16384; (c) M65536; (d) M262144.

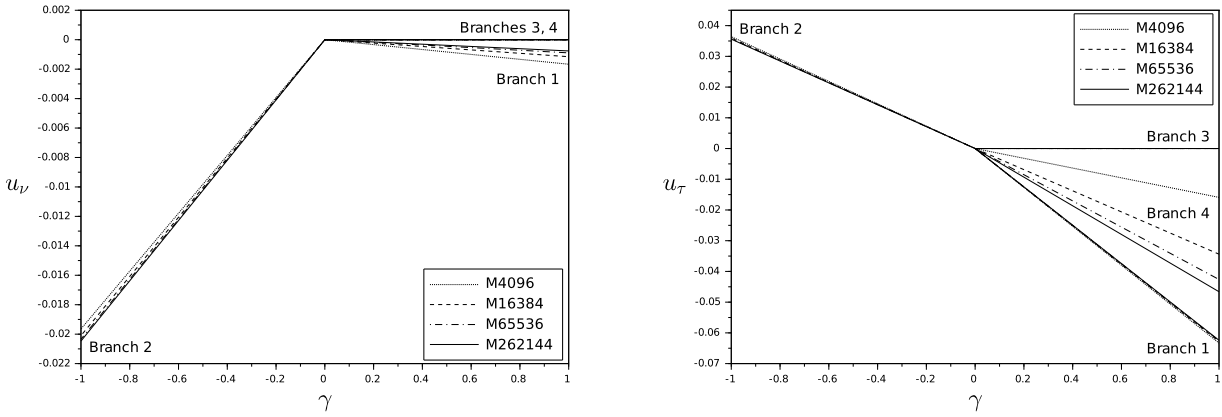


Fig. 10 The normal and tangential displacements u_ν and u_τ at the most left contact node of the triangular body for $\mathbf{h}(\gamma) = \gamma(-26.4, -7.5)$ (in GN/m^2).

Since the line $\mathbf{h}(\gamma) = \gamma(-26, -7.5)$ is closer and closer to a half-line from the boundaries of the cones for finer and finer meshes and out of the cone for the finest mesh, this explains the behaviour of the bifurcation in the presented example. Nevertheless, regarding the branching in the whole h_1 - h_2 plane, one can find it stable and convergent, again. This is illustrated by the bifurcation diagrams in Fig. 10 computed for a bit modified parametrised force $\mathbf{h}(\gamma) = \gamma(-26.4, -7.5)$, for which there are four branches distinctly separated for all meshes.

4 Conclusion

We have developed a method of numerical branching for piecewise-smooth problems, and we have demonstrated that our method can reliably discover all solution branches in two model situations if the parameters of the method are set up properly. We have suggested how to choose the optimum parameter settings, as well. As far as we know, this is the first attempt to devise a numerical method for piecewise-smooth branching that does not rely on any analytical expressions of the regions of smoothness of the function involved.

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