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Lyapunov function for Hawkes-driven limit order books

Frédéric Abergel

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Abstract

In this note, I provide a detailed construction of a Lyapunov function for the limit order book model of [2], under the classical stability condition for the Hawkes process. In passing, a Lyapunov function for the multivariate Hawkes process with exponential kernels is exhibited - although it is extremely likely that this construction is not new, I couldn't find it explicitly elsewhere.

0.1 A Lyapunov function for markovian Hawkes processes

Consider a D -dimensional Hawkes processes (N^i) , $i = 1, \dots, D$ with intensities

$$\lambda_t^i = \lambda_0^i + \sum_{j=1}^D \int_0^t \alpha_{ij} e^{-\beta_{ij}(t-s)} dN_s^j.$$

Classically, we introduce the

$$\mu_t^{ij} \equiv \int_0^t \alpha_{ij} e^{-\beta_{ij}(t-s)} dN_s^j,$$

so that there holds

$$\lambda_t^i = \lambda_0^i + \sum_{j=1}^D \mu_t^{ij}. \quad (0.1.1)$$

We assume the following

$$\forall i, j, \alpha_{ij} > 0, \beta_{ij} > 0, \quad (0.1.2)$$

and also that the spectral radius of the matrix \mathbf{M} with components $\mathbf{M}_{pq} = \frac{\alpha_{qp}}{\beta_{qp}}$ is smaller than 1:

$$\rho(\mathbf{M}) < 1. \quad (0.1.3)$$

The infinitesimal generator associated to the markovian process (μ^{ij}) , $1 \leq i, j \leq D$ is the operator

$$\mathcal{L}_H(F)(\vec{\mu}) = \sum_j \lambda^j (F(\vec{\mu} + \Delta^j(\vec{\mu})) - F(\vec{\mu})) - \sum_{i,j} \beta_{ij} \mu_{ij} \frac{\partial F}{\partial \mu_{ij}}, \quad (0.1.4)$$

where $\vec{\mu} \in \mathcal{R}^{D^2}$ is the vector with components μ_{ij} and λ_j is as in (0.1.1). The notation $\Delta^j(\vec{\mu})$ characterizes the jumps in those of the entries in $\vec{\mu}$ that are affected by a jump of the process N^j . For a **fixed** index j , it is given by the vector with entries α_{ij} at the corresponding spots for $i = 1, \dots, D$, and zero entries elsewhere.

A Lyapunov function for the associated semi-group is sought under the form

$$V(\vec{\mu}) = \sum_{i,j} \delta_{ij} \mu_{ij} \quad (0.1.5)$$

(since the intensities are always positive, a linear function will be coercive). Assuming (0.1.5), there holds

$$\mathcal{L}_H(V) = \sum_j \lambda^j \left(\sum_i \delta_{ij} \alpha_{ij} \right) - \sum_{i,j} \beta_{ij} \mu_{ij} \delta_{ij} \quad (0.1.6)$$

or

$$\mathcal{L}_H(V) = \sum_{i,j} (\lambda_0^j + \sum_k \mu_{jk}) \delta_{ij} \alpha_{ij} - \beta_{ij} \mu_{ij} \delta_{ij}. \quad (0.1.7)$$

At this stage, it is convenient to introduce $\vec{\epsilon}$ the maximal eigenvector of \mathbf{M} . Denote by κ the associated eigenvalue. By Assumption (0.1.3), one has that $0 < \kappa < 1$ and furthermore, by Perron-Frobenius theorem, there holds: $\forall i, \epsilon_i > 0$.

Assuming that

$$\delta_{ij} \equiv \frac{\epsilon_i}{\beta_{ij}}, \quad (0.1.8)$$

the expression for V becomes

$$V(\vec{\mu}) = \sum_{i,j} \epsilon_i \frac{\mu_{ij}}{\beta_{ij}}. \quad (0.1.9)$$

Plugging (0.1.9) in (0.1.7) yields

$$\begin{aligned} \mathcal{L}_H(V) &= \sum_{i,j} \lambda_0^j \delta_{ij} \alpha_{ij} + \sum_{i,j,k} \mu_{jk} \epsilon_i \frac{\alpha_{ij}}{\beta_{ij}} - \sum_{j,k} \beta_{jk} \mu_{jk} \delta_{jk} \\ &= \sum_{i,j} \lambda_0^j \delta_{ij} \alpha_{ij} + (\kappa - 1) \sum_{j,k} \epsilon_k \mu_{jk}, \end{aligned} \quad (0.1.10)$$

using the identity $\sum_j \mathbf{M}_{ji} \epsilon_i = \kappa \epsilon_j$. A comparison with (0.1.9) easily yields the upper bound

$$\mathcal{L}_H V \leq -\gamma V + C, \quad (0.1.11)$$

with $\gamma = (1 - \kappa) \beta_{\min}$, $\beta_{\min} \equiv \inf_{i,j} (\beta_{ij}) > 0$ by assumption, and $C = \sum_{i,j} \lambda_0^j \delta_{ij} \alpha_{ij} \equiv \kappa \vec{\epsilon} \cdot \vec{\lambda}_0$.

0.2 Lyapunov function for a Hawkes-process-driven limit order book

This section addresses the existence of a Lyapunov function for a limit order book model introduced in [2].

0.2.1 The infinitesimal generator

We now address the case of the Hawkes-process-driven limit order book as in [2], and provide a proof of the ergodicity of the limit order book under the less stringent, more natural assumption (0.1.3).

A markovian $D = (2K + 2)$ -dimensional Hawkes process now models the intensities of the arrivals of market and limit orders, whereas those of cancellations orders remain constant. With the same notations as in [1][2], one can work out the infinitesimal generator associated with the process describing the joint evolution of the limit order book, characterized by the $2K$ processes of available quantities on the ask

and bid side: $(\vec{a}; \vec{b})$, and the vector of intensities $\vec{\mu}$. In fact, one can write that

$$\begin{aligned}
\mathcal{L}F(\vec{a}; \vec{b}; \vec{\mu}) &= \lambda^{M^+} \left(F \left([a_i - (q - A(i-1))_+]_+; J^{M^+}(\vec{b}); \vec{\mu} + \Delta^{M^+}(\vec{\mu}) \right) - F \right) \\
&+ \sum_{i=1}^K \lambda_i^{L^+} \left(F \left(a_i + q; J^{L^+}(\vec{b}); \vec{\mu} + \Delta^{L^+}(\vec{\mu}) \right) - F \right) \\
&+ \sum_{i=1}^K \lambda_i^{C^+} a_i \left(F \left(a_i - q; J^{C^+}(\vec{b}) \right) - F \right) \\
&+ \lambda^{M^-} \left(F \left(J^{M^-}(\vec{a}); [b_i + (q - B(i-1))_-]_-; \vec{\mu} + \Delta^{M^-}(\vec{\mu}) \right) - F \right) \\
&+ \sum_{i=1}^K \lambda_i^{L^-} \left(F \left(J^{L^-}(\vec{a}); b_i - q; \vec{\mu} + \Delta^{L^-}(\vec{\mu}) \right) - F \right) \\
&+ \sum_{i=1}^K \lambda_i^{C^-} |b_i| \left(f \left(J^{C^-}(\vec{a}); b_i + q \right) - f \right) \\
&- \sum_{i,j=1}^{2K+2} \beta_{ij} \mu_{ij} \frac{\partial F}{\partial \mu_{ij}}. \tag{0.2.1}
\end{aligned}$$

In order to ease the already cumbersome notations, we note $F(a_i; \vec{b}; \vec{\mu})$ instead of $F(a_1, \dots, a_i, \dots, a_K; \vec{b}; \vec{\mu})$. Moreover, as in Section 0.1, the notations $\Delta^{\dots}(\vec{\mu})$ stands for the jump of the intensity vector $\vec{\mu}$ corresponding to a jump of the process N^{\dots} . As for the J 's, they are *shift operators* corresponding to the renumbering on one side of the limit order book following an event affecting the other side. For instance the shift operator corresponding to the arrival of a sell market order ($dM^-(t) = 1$) of size q is

$$J^{M^-}(\mathbf{a}) = \left(\underbrace{0, 0, \dots, 0}_{k \text{ times}}, a_1, a_2, \dots, a_{K-k} \right), \tag{0.2.2}$$

with

$$k := \inf \left\{ p : \sum_{j=1}^p |b_j| > q \right\} - \inf \{ p : |b_p| > 0 \}. \tag{0.2.3}$$

Similar expressions are derived for the other events affecting the order book. We refer to [1] for a thorough analysis of the operator just introduced, and simply note that it is a combination of

- standard difference operators corresponding to the arrival or cancellation of orders at each limit
- shift operators expressing the moves in the best limits and therefore, in the origins of the frames for the two sides of the order book. These shifts depend on the profile of the order book on the opposite side, namely the cumulative depth up to level i defined by

$$A(i) := \sum_{k=1}^i a_k, \tag{0.2.4}$$

and

$$B(i) := \sum_{k=1}^i |b_k|, \quad (0.2.5)$$

and their generalized inverse functions

$$A^{-1}(q') := \inf\{p : \sum_{j=1}^p a_j > q'\}, \quad (0.2.6)$$

and

$$B^{-1}(q') := \inf\{p : \sum_{j=1}^p |b_j| > q'\}. \quad (0.2.7)$$

- drift terms coming from the mean-reverting behaviour of the intensities between jumps.

Combining the approach in [1] and that of Section 0.1, a Lyapunov function can be built, as we now show.

0.2.2 The Lyapunov function

The Lyapunov function is sought under the simple, linear form

$$V(\vec{a}; \vec{b}; \vec{\mu}) = \sum_{i=1}^K a_i + \sum_{i=1}^K |b_i| + \frac{1}{\eta} \sum_{i,j=1}^{2K+2} \delta_{ij} \mu_{ij} = V_1 + \frac{1}{\eta} V_2, \quad (0.2.8)$$

where V_1 (resp. V_2) corresponds to the part that depends only on $\vec{a}; \vec{b}$ (resp. $\vec{\mu}$). We specialize V_2 to be identical - up to a change in the indices - to the function introduced in Section 0.1. The "small" parameter $\eta > 0$ will become handy as a penalization parameter, as we shall see below. Thanks to the linearity of \mathcal{L} and that of V itself¹, there holds

$$\mathcal{L}V = \mathcal{L}V_1 + \frac{1}{\eta} \mathcal{L}V_2. \quad (0.2.9)$$

The first term $\mathcal{L}V_1$ is dealt with exactly as in [1]:

$$\begin{aligned} \mathcal{L}V_1(\mathbf{x}) &\leq -(\lambda^{M^+} + \lambda^{M^-})q + \sum_{i=1}^K (\lambda_i^{L^+} + \lambda_i^{L^-})q - \sum_{i=1}^K (\lambda_i^{C^+} a_i + \lambda_i^{C^-} |b_i|)q \\ &\quad + \sum_{i=1}^K \lambda_i^{L^+} (i_S - i)_+ a_\infty + \sum_{i=1}^K \lambda_i^{L^+} (i_S - i)_+ |b_\infty| \end{aligned} \quad (0.2.10)$$

$$\begin{aligned} &\leq -(\lambda^{M^+} + \lambda^{M^-})q + (\Lambda^{L^-} + \Lambda^{L^+})q - \underline{\lambda}^C q V_1(\mathbf{x}) \\ &\quad + K(\Lambda^{L^-} a_\infty + \Lambda^{L^+} |b_\infty|), \end{aligned} \quad (0.2.11)$$

¹ $|b_i|$ is simply $(-b_i)$, as we are conventionally using negative quantities for the bid side of the limit order book.

where

$$\Lambda^{L^\pm} := \sum_{i=1}^K \lambda_i^{L^\pm} \text{ and } \underline{\lambda}^C := \min_{1 \leq i \leq K} \{\lambda_i^{C^\pm}\} > 0. \quad (0.2.12)$$

At this stage, the dependence on the intensities in the expression above does not allow one to conclude that a geometric drift condition $\mathcal{L}V_1 \leq -\gamma V_1 + C$ is satisfied. Computing $\mathcal{L}V_2$ yields an expression identical to that already obtained in Section 0.1

$$\mathcal{L}(V_2) = \sum_{i,j} \lambda_0^j \delta_{ij} \alpha_{ij} + (\kappa - 1) \sum_{j,k} \epsilon_k \mu_{jk}, \quad (0.2.13)$$

so that there holds

$$\mathcal{L}V \equiv \mathcal{L}V_1 + \frac{1}{\eta} \mathcal{L}V_2 \leq -\underline{\lambda}^C q V_1 - \frac{\gamma}{\eta} V_2 - \vec{G} \cdot \vec{\mu} + C, \quad (0.2.14)$$

where γ is as in Equation (0.1.11), $\vec{G} \cdot \vec{\mu}$ is a compact notation for the linear form in the μ_{ij} s obtained in the inequality (0.2.11), and C is some constant. Now, thanks to the positivity of the coefficients in V_2 and of the μ_{ij} s, one can choose η small enough that there holds

$$\forall \vec{\mu}, |\vec{G} \cdot \vec{\mu}| \leq \frac{\gamma}{2\eta} V_2(\vec{\mu}), \quad (0.2.15)$$

which yields

$$\mathcal{L}V \equiv \mathcal{L}V_1 + \frac{1}{\eta} \mathcal{L}V_2 \leq -\underline{\lambda}^C q V_1 - \frac{\gamma}{2\eta} V_2 + C, \quad (0.2.16)$$

and finally

$$\mathcal{L}V \leq \zeta V + C, \quad (0.2.17)$$

with $\zeta = \text{Min}(\underline{\lambda}^C q, \frac{\gamma}{2\eta})$ and C is some constant.

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