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Cédric Bonnafé and Meinolf Geck

To David Vogan on the occasion of his 60th birthday

Abstract. In 1979, Vogan introduced a generalised τ -invariant for characterising primitive ideals in enveloping algebras. Via a known dictionary this translates to an invariant of left cells in the sense of Kazhdan and Lusztig. Although it is not a complete invariant, it is extremely useful in describing left cells. Here, we propose a general framework for defining such invariants which also applies to Hecke algebras with unequal parameters.

Key words: Coxeter groups, Hecke algebras, Kazhdan–Lusztig cells, MSC 2010: Primary 20C08, Secondary 20F55

1. Introduction

Let W be a finite Weyl group. Using the corresponding generic Iwahori– Hecke algebra and the "new" basis of this algebra introduced by Kazhdan and Lusztig [16], we obtain partitions of W into left, right and two-sided cells. Analogous notions originally arose in the theory of primitive ideals in enveloping algebras; see Joseph [15]. This is one of the sources for the interest in knowing the cell partitions of W; there are also deep connections [19]

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with representations of reductive groups, singularities of Schubert cells and the geometry of unipotent classes. Vogan [24], [25] introduced invariants of left cells which are computable in terms of certain combinatorially defined operators $T_{\alpha\beta}$, $S_{\alpha\beta}$ where α, β are adjacent simple roots of W. In the case where W is the symmetric group \mathfrak{S}_n , these invariants completely characterise the left cells; see [16, §5], [24, §6]. Although Vogan's invariants are not complete invariants in general, they have turned out to be extremely useful in describing left cells.

Now, the Kazhdan–Lusztig cell partitions are not only defined and interesting for finite Weyl groups, but also for affine Weyl groups and Coxeter groups in general; see, e.g., Lusztig [18], [20]. Furthermore, the original theory was extended by Lusztig [17] to allow the possibility of attaching weights to the simple reflections. The original setting then corresponds to the case where all weights are equal to 1; we will refer to this case as the "equal parameter case". Our aim here is to propose analogues of Vogan's invariants which work in general, i.e., for arbitrary Coxeter groups and arbitrary (positive) weights.

In Sections 2 and 3 we briefly recall the basic set-up concerning Iwahori– Hecke algebras, cells in the sense of Kazhdan and Lusztig, and the concept of induction of cells. In Section 4 we introduce the notion of *left cellular maps*; a fundamental example is given by the Kazhdan–Lusztig *-operations. In Section 5, we discuss the equal parameter case and Vogan's original definition of a generalised τ -invariant. As this definition relied on the theory of primitive ideals, it only applies to finite Weyl groups. In Theorem 5.2, we show that this works for arbitrary Coxeter groups satisfying a certain boundedness condition. (A similar result has also been proved by Shi [22, 4.2], but he uses a definition slightly different from Vogan's; our argument seems to be more direct.) In Sections 6 and 7, we develop an abstract setting for defining such invariants; this essentially relies on the concept of induction of cells and is inspired by Lusztig's method of *strings* [18, §10]. As a bi-product of our approach, we obtain that the *-operations also work in the unequal parameter case. We conclude by discussing examples and stating open problems.

Remark. In [4, Cor. 6.2], the first author implicitly assumed that the results on the Kazhdan–Lusztig *-operations [16, §4] also hold in the unequal parameter context — which was a serious mistake at the time. Corollary 6.5 below justifies *a posteriori* this assumption.

Notation. We fix a Coxeter system (W, S) and we denote by $\ell : W \to \mathbb{Z}_{\geq 0}$ the associated length function. We also fix a totally ordered abelian group \mathscr{A} . We use an exponential notation for the group algebra $A = \mathbb{Z}[\mathscr{A}]$:

$$A = \bigoplus_{a \in \mathscr{A}} \mathbb{Z} v^a \quad \text{where} \quad v^a v^{a'} = v^{a+a'} \text{ for all } a, a' \in \mathscr{A}.$$

We write $\mathscr{A}_{\leq 0} := \{ \alpha \in \mathscr{A} \mid \alpha \leq 0 \}$ and $A_{\leq 0} := \bigoplus_{a \in \mathscr{A}_{\leq 0}} \mathbb{Z}v^a$; the symbols $\mathscr{A}_{\geq 0}, A_{\geq 0}$ etc. have analogous meanings. We denote by $\overline{} : A \to A$ the involutive automorphism such that $\overline{v^a} = v^{-a}$ for all $a \in \mathscr{A}$.

2. Weight functions and cells

Let $p: W \to \mathscr{A}, w \mapsto p_w$, be a weight function in the sense of Lusztig [20], that is, we have $p_s = p_t$ whenever $s, t \in S$ are conjugate in W, and $p_w = p_{s_1} + \cdots + p_{s_k}$ if $w = s_1 \cdots s_k$ (with $s_i \in S$) is a reduced expression for $w \in W$. The original setup in [16] corresponds to the case where $\mathscr{A} = \mathbb{Z}$ and $p_s = 1$ for all $s \in S$; this will be called the "equal parameter case". We shall assume throughout that $p_s > 0$ for all $s \in S$. (There are standard techniques for reducing the general case to this case [3, §2].)

Let $\mathcal{H} = \mathcal{H}_A(W, S, p)$ be the corresponding generic Iwahori–Hecke algebra. This algebra is free over A with basis $(T_w)_{w \in W}$, and the multiplication is given by the rule

$$T_s T_w = \begin{cases} T_{sw} & \text{if } sw > w, \\ T_{sw} + (v^{p_s} - v^{-p_s})T_w & \text{if } sw < w, \end{cases}$$

where $s \in S$ and $w \in W$; here, \leq denotes the Bruhat–Chevalley order on W.

Let $(C'_w)_{w \in W}$ be the "new" basis of \mathscr{H} introduced in [16, (1.1.c)], [17, §2]. (These basis elements are denoted c_w in [20].) For any $x, y \in W$, we write

$$C'_x C'_y = \sum_{z \in W} h_{x,y,z} C'_z$$
 where $h_{x,y,z} \in A$ for all $x, y, z \in W$.

We have the following more explicit formula for $s \in S$, $y \in W$ (see [17, §6], [20, Chap. 6]):

$$C'_{s} C'_{y} = \begin{cases} (v^{p_{s}} + v^{-p_{s}}) C'_{y} & \text{if } sy < y, \\ C'_{sy} + \sum_{z \in W : \, sz < z < y} M^{s}_{z,y} C'_{z} & \text{if } sy > y, \end{cases}$$

where $C'_s = T_s + v^{-p_s}T_1$ and $M^s_{z,y} = \overline{M}^s_{z,y} \in A$ is determined as in [17, §3]. As in [20, §8], we write $x \leftarrow_L y$ if there exists some $s \in S$ such that

As in [20, §8], we write $x \leftarrow_L y$ if there exists some $s \in S$ such that $h_{s,y,x} \neq 0$, that is, C'_x occurs with a non-zero coefficient in the expression of $C'_s C'_y$ in the C'-basis. The Kazhdan–Lusztig left pre-order \leq_L is the transitive closure of \leftarrow_L . The equivalence relation associated with \leq_L will be denoted by \sim_L and the corresponding equivalence classes are called the *left cells* of W. Note that $\mathscr{H}C_w \subseteq \sum_y AC_y$ where the sum runs over all $y \in W$ with $y \leq_L w$.

Similarly, we can define a pre-order \leq_R by considering multiplication by C'_s on the right in the defining relation. The equivalence relation associated with \leq_R will be denoted by \sim_R and the corresponding equivalence classes are called the *right cells* of W. We have

(2.1)
$$x \leqslant_R y \iff x^{-1} \leqslant_L y^{-1};$$

see [20, 5.6, 8.1]. Finally, we define a pre-order \leq_{LR} by the condition that $x \leq_{LR} y$ if there exists a sequence $x = x_0, x_1, \ldots, x_k = y$ such that, for

each $i \in \{1, \ldots, k\}$, we have $x_{i-1} \leq_L x_i$ or $x_{i-1} \leq_R x_i$. The equivalence relation associated with \leq_{LR} will be denoted by \sim_{LR} and the corresponding equivalence classes are called the *two-sided cells* of W.

Definition 2.2. A (non-empty) subset Γ of W is called *left-closed* if, for any $x, y \in \Gamma$, we have $\{z \in W \mid x \leq_L z \leq_L y\} \subseteq \Gamma$.

Note that any such subset is a union of left cells. A left cell itself is clearly left-closed with respect to \leq_L . It immediately follows from these definitions that, given any left-closed subset $\Gamma \subseteq W$, the A-submodules

$$\mathscr{I}_{\Gamma} = \langle C'_{w} \mid w \leqslant_{L} z \text{ for some } z \in \Gamma \rangle_{A},$$
$$\mathscr{I}_{\Gamma} = \langle C'_{w} \mid w \notin \Gamma, w \leqslant_{L} z \text{ for some } z \in \Gamma \rangle_{A}.$$

are left ideals in \mathscr{H} . Hence we obtain an \mathscr{H} -module $[\Gamma]_A := \mathscr{I}_{\Gamma}/\mathscr{I}_{\Gamma}$, which is free over A with basis given by $(e_x)_{x \in \Gamma}$, where e_x denotes the residue class of C'_x in $[\Gamma]_A$. The action of C'_w $(w \in W)$ is given by the formula

$$C'_w \cdot e_x = \sum_{y \in \Gamma} h_{w,x,y} \, e_y.$$

3. Cells and parabolic subgroups

A key tool in this work will be the process of *induction of cells*. Let $I \subseteq S$ and consider the parabolic subgroup $W_I \subseteq W$ generated by I. Then

$$X_I := \{ w \in W \mid ws > w \text{ for all } s \in I \}$$

is the set of distinguished left coset representatives of W_I in W. The map $X_I \times W_I \to W$, $(x, u) \mapsto xu$, is a bijection and we have $\ell(xu) = \ell(x) + \ell(u)$ for all $x \in X_I$ and $u \in W_I$; see [13, §2.1]. Thus, given $w \in W$, we can write uniquely w = xu where $x \in X_I$ and $u \in W_I$. In this case, we denote $\operatorname{pr}_I(w) := u$. Let $\leq_{L,I}$ and $\sim_{L,I}$ be respectively the pre-order and equivalence relations for W_I defined similarly as \leq_L and \sim_L are defined in W.

Theorem 3.1. Let $I \subseteq S$. If $x, y \in W$ are such that $x \leq_L y$ (resp. $x \sim_L y$), then $\operatorname{pr}_I(x) \leq_{L,I} \operatorname{pr}_I(y)$ (resp. $\operatorname{pr}_I(x) \sim_{L,I} \operatorname{pr}_I(y)$). In particular, if Γ is a left cell of W_I , then $X_I \Gamma$ is a union of left cells of W.

This was first proved by Barbasch–Vogan [1, Cor. 3.7] for finite Weyl groups in the equal parameter case (using the theory of primitive ideals); see [9] for the general case.

Example 3.2. Let Γ be a left-closed subset of W_I . Then the subset $X_I\Gamma$ of W is left-closed (see Theorem 3.1). Let $\mathscr{H}_I \subseteq \mathscr{H}$ be the parabolic subalgebra

spanned by all T_w where $w \in W_I$. Then we obtain the \mathscr{H}_I -module $[\Gamma]_A$, with standard basis $(e_w)_{w \in \Gamma}$, and the \mathscr{H} -module $[X_I \Gamma]_A$, with standard basis $(e_{xw})_{x \in X_I, w \in \Gamma}$. By [10, 3.6], we have an isomorphism of \mathscr{H} -modules

$$[X_I\Gamma]_A \xrightarrow{\sim} \operatorname{Ind}_I^S([\Gamma]_A), \qquad e_{yv} \mapsto \sum_{x \in X_I, w \in \Gamma} p_{xu, yv}^* \left(T_x \otimes e_u \right),$$

where $p_{xu,yv}^* \in A$ are the *relative* Kazhdan–Lusztig polynomials of [9, Prop. 3.3] and, for any \mathscr{H}_I -module V, we denote by $\operatorname{Ind}_I^S(V) := \mathscr{H} \otimes_{\mathscr{H}_I} V$ the *induced module*, with basis $(T_x \otimes e_w)_{x \in X_I, w \in \Gamma}$. (In [10, §3], it is not stated explicitly that $\Gamma = X_I \Gamma$ is left-closed, but this condition is used implicitly in the discussion there.)

A first invariant of left cells is given as follows. For any $w \in W$, we denote by $\mathscr{R}(w) := \{s \in S \mid ws < w\}$ the *right descent set* of w (or τ -invariant of win the language of primitive ideals; see [1]). The next result has been proved in [16, 2.4] (for the equal parameter case) and in [20, 8.6] (for the unequal parameter case).

Proposition 3.3 (Kazhdan-Lusztig, Lusztig). Let $x, y \in W$.

- (a) If $x \leq_L y$ then $\mathscr{R}(y) \subseteq \mathscr{R}(x)$.
- (b) If $x \sim_L y$, then $\mathscr{R}(x) = \mathscr{R}(y)$.
- (c) For any $I \subseteq S$, the set $\{w \in W \mid \mathscr{R}(w) = I\}$ is a union of left cells of W.

We show how this can be deduced from Theorem 3.1. First, note that (b) and (c) easily follow from (a), so we only need to prove (a). Let $x, y \in W$ be such that $x \leq_L y$. Let $s \in \mathscr{R}(y)$ and set $I = \{s\}$. Then $\operatorname{pr}_I(y) = s$ and so $\operatorname{pr}_I(x) \leq_{L,I} \operatorname{pr}_I(y) = s \in W_I = \{1, s\}$. Since $p_s > 0$, the definitions immediately show that $s \leq_{I,L} 1$ but $s \not\sim_{L,I} 1$. Hence, we must have $\operatorname{pr}_I(x) = s$ and so $s \in \mathscr{R}(x)$. Thus, we have $\mathscr{R}(y) \subseteq \mathscr{R}(x)$.

4. Left cellular maps

Definition 4.1. A map $\delta : W \to W$ is called *left cellular* if the following conditions are satisfied for every left cell $\Gamma \subseteq W$ (with respect to the given weights $\{p_s \mid s \in S\}$):

(A1) $\delta(\Gamma)$ also is a left cell.

(A2) The map δ induces an \mathscr{H} -module isomorphism $[\Gamma]_A \cong [\delta(\Gamma)]_A$.

A prototype of such a map is given by the Kazhdan–Lusztig *-operations. We briefly recall how this works. For any $s, t \in S$ such that $st \neq ts$, we set

 $\mathscr{D}_R(s,t) := \{ w \in W \,|\, \mathscr{R}(w) \cap \{s,t\} \text{ has exactly one element} \}$

and, for any $w \in \mathscr{D}_R(s,t)$, we set $\mathfrak{T}_{s,t}(w) := \{ws, wt\} \cap \mathscr{D}_R(s,t)$. (See [16, §4], [24, §3].) Note that $\mathfrak{T}_{s,t}(w)$ consists of one or two elements; in order to have a uniform notation, we consider $\mathfrak{T}_{s,t}(w)$ as a multiset with two identical elements if $\{ws, wt\} \cap \mathscr{D}_R(s,t)$ consists of only one element.

If st has order 3, then the intersection $\{ws, wt\} \cap \mathscr{D}_R(s, t)$ consists of only one element which will be denoted by w^* . Thus, we have $\mathfrak{T}_{s,t}(w) = \{w^*, w^*\}$ in this case. With this notation, we can now state:

Proposition 4.2 (Kazhdan–Lusztig *-operations [16, §4]). Assume that we are in the equal parameter case and that st has order 3. Then we obtain a left cellular map $\delta: W \to W$ by setting

$$\delta(w) = \begin{cases} w^* & \text{if } w \in \mathscr{D}_R(s,t), \\ w & \text{otherwise.} \end{cases}$$

In particular, if $\Gamma \subseteq \mathscr{D}_R(s,t)$ is a left cell, then so is $\Gamma^* := \{w^* \mid w \in \Gamma\}.$

(In Corollary 6.5 below, we extend this to the unequal parameter case.)

If st has order ≥ 4 , then the set $\mathfrak{T}_{s,t}(w)$ may contain two distinct elements. In order to obtain a single-valued operator, Vogan [25, §4] (for the case m = 4; see also McGovern [21, §4]) and Lusztig [18, §10] (for any $m \geq 4$) propose an alternative construction, as follows.

Remark 4.3. Let $s, t \in S$ be such that st has any finite order $m \ge 3$. Let $W_{s,t} = \langle s, t \rangle$, a dihedral group of order 2m. For any $w \in W$, the coset $wW_{s,t}$ can be partitioned into four subsets: one consists of the unique element x of minimal length, one consists of the unique element of maximal length, one consists of the (m-1) elements $xs, xst, xsts, \ldots$ and one consists of the (m-1) elements $xt, xts, xtst, \ldots$. Following Lusztig [18, 10.2], the last two subsets (ordered as above) are called *strings*. (Note that Lusztig considers the coset $W_{s,t}w$ but, by taking inverses, the two versions are clearly equivalent.) Thus, if $w \in \mathscr{D}_R(s,t)$, then w belongs to a unique string which we denote by λ_w ; we certainly have $\mathfrak{T}_{s,t}(w) \subseteq \lambda_w \subseteq \mathscr{D}_R(s,t)$ for all $w \in \mathscr{D}_R(s,t)$.

We define an involution $\mathscr{D}_R(s,t) \to \mathscr{D}_R(s,t), w \mapsto \tilde{w}$, as follows. Let $w \in \mathscr{D}_R(s,t)$ and $i \in \{1,\ldots,m-1\}$ be the index such that w is the *i*th element of the string λ_w . Then \tilde{w} is defined to be the (m-i)th element of λ_w . Note that, if m = 3, then $\tilde{w} = w^*$, with w^* as in Proposition 4.2.

Let us write $T_xT_y = \sum_{z \in W} f_{x,y,z}T_z$ where $f_{x,y,z} \in A$ for all $x, y, z \in W$. Following [20, 13.2], we say that \mathscr{H} is **bounded** if there exists some positive $N \in \mathscr{A}$ such that $v^{-N}f_{x,y,z} \in A_{\leq 0}$ for all $x, y, z \in W$. We can now state:

Proposition 4.4 (Lusztig [18, 10.7]). Assume that we are in the equal parameter case and that \mathscr{H} is bounded. If $\Gamma \subseteq \mathscr{D}_R(s,t)$ is a left cell, then so is $\tilde{\Gamma} := \{\tilde{w} \mid w \in \Gamma\}.$

(It is assumed in [*loc. cit.*] that W is crystallographic, but this assumption is now superfluous thanks to Elias–Williamson [6]. The boundedness

assumption is obviously satisfied for all finite Coxeter groups. It also holds, for example, for affine Weyl groups; see the remarks following [20, 13.4].)

In Corollary 6.5 below, we shall show that $w \mapsto \tilde{w}$ also gives rise to a left cellular map and that this works without any assumption, as long as $p_s = p_t$.

5. Vogan's invariants

Hypothesis. Throughout this section, and only in this section, we assume that we are in the equal parameter case.

We recall the following definition.

Definition 5.1 (Vogan [24, 3.10, 3.12]). For $n \ge 0$, we define an equivalence relation \approx_n on W inductively as follows. Let $x, y \in W$.

- For n = 0, we write $x \approx_0 y$ if $\mathscr{R}(x) = \mathscr{R}(y)$.
- For $n \ge 1$, we write $x \approx_n y$ if $x \approx_{n-1} y$ and if, for any $s, t \in S$ such that $x, y \in \mathscr{D}_R(s,t)$ (where st has order 3 or 4), the following holds: if $\mathfrak{T}_{s,t}(x) = \{x_1, x_2\}$ and $\mathfrak{T}_{s,t}(y) = \{y_1, y_2\}$, then either $x_1 \approx_{n-1} y_1$, $x_2 \approx_{n-1} y_2$ or $x_1 \approx_{n-1} y_2$, $x_2 \approx_{n-1} y_1$.

If $x \approx_n y$ for all $n \ge 0$, then x, y are said to have the same *generalised* τ -invariant.

The following result was originally formulated and proved for finite Weyl groups by Vogan [24, §3], in the language of primitive ideals in enveloping algebras. It then follows for cells as defined in Section 2 using a known dictionary (see, e.g., Barbasch–Vogan [1, §2]). The proof in general relies on Proposition 4.2 and results on *strings* as defined in Remark 4.3.

Theorem 5.2 (Kazhdan–Lusztig [16, §4], Lusztig [18, §10], Vogan [24, §3]). Assume that \mathcal{H} is bounded and recall that we are in the equal parameter case. Let Γ be a left cell of W. Then all elements in Γ have the same generalised τ -invariant.

Proof. We prove by induction on n that, if $y, w \in W$ are such that $y \sim_L w$, then $y \approx_n w$. For n = 0, this holds by Propositon 3.3. Now let n > 0. By induction, we already know that $y \approx_{n-1} w$. Then it remains to consider $s, t \in S$ such that $st \neq ts$ and $y, w \in \mathscr{D}_R(s, t)$. If st has order 3, then $\mathfrak{T}_{s,t}(y) = \{y^*, y^*\}$ and $\mathfrak{T}_{s,t}(w) = \{w^*, w^*\}$; furthermore, by Proposition 4.2, we have $y^* \sim_L w^*$ and so $y^* \approx_{n-1} w^*$, by induction. Now assume that sthas order 4. In this case, the argument is more complicated (as it is also in the setting of [24, §3].) Let $I = \{s, t\}$ and Γ be the left cell containing y, w. Since all elements in Γ have the same right descent set (by Proposition 3.3), we can choose the notation such that xs < x and xt > x for all $x \in \Gamma$. Then, for $x \in \Gamma$, we have x = x's, x = x'ts or x = x'sts where $x' \in X_I$. This yields that

(†)
$$\mathfrak{T}_{s,t}(x) = \begin{cases} \{x'st, x'st\} & \text{if } x = x's, \\ \{x't, x'tst\} & \text{if } x = x'ts, \\ \{x'st, x'st\} & \text{if } x = x'sts. \end{cases}$$

We now consider the string λ_x and distinguish two cases.

Case 1. Assume that there exists some $x \in \Gamma$ such that x = x's or x = x'sts. Then $\lambda_x = (x's, x'st, x'sts)$ and so the set $\Gamma^* := (\bigcup_{w \in \Gamma} \lambda_w) \setminus \Gamma$ contains elements with different right descent sets. On the other hand, by [18, Prop. 10.7], Γ^* is the union of at most two left cells. (Again, the assumption in [*loc. cit.*] that W is crystallographic is now superfluous thanks to [6].) We conclude that $\Gamma^* = \Gamma_1 \cup \Gamma_2$ where Γ_1, Γ_2 are left cells such that:

- all elements in Γ_1 have s in their right descent set, but not t;
- all elements in Γ_2 have t in their right descent set, but not s.

Now consider $y, w \in \Gamma$; we write $\mathfrak{T}_{s,t}(y) = \{y_1, y_2\} \subseteq \Gamma^*$ and $\mathfrak{T}_{s,t}(w) = \{w_1, w_2\} \subseteq \Gamma^*$. By (†), all the elements y_1, y_1, w_1, w_2 belong to Γ_2 . In particular, $y_1 \sim_L w_1, y_2 \sim_L w_2$ and so, by induction, $y_1 \approx_{n-1} w_1, y_2 \approx_{n-1} w_2$.

Case 2. We are not in Case 1, that is, all elements $x \in \Gamma$ have the form x = x'ts where $x' \in X_I$. Then $\lambda_x = (x't, x'ts, x'tst)$ for each $x \in \Gamma$. Let us label the elements in such a string as x_1, x_2, x_3 . Then $x = x_2$ and $\mathfrak{T}_{s,t}(x) = \{x't, x'tst\} = \{x_1, x_3\}$.

Now consider $y, w \in \Gamma$. There is a chain of elements which connect y to w via the elementary relations \leftarrow_L , and vice versa. Assume first that y, w are directly connected as $y \leftarrow_L w$. Using the labelling $y = y_2, w = w_2$ and the notation of [18, 10.4], this means that $a_{22} \neq 0$. Hence, the identities " $a_{11} = a_{33}$ ", " $a_{13} = a_{31}$ ", " $a_{22} = a_{11} + a_{13}$ " in [18, 10.4.2] imply that

$$(y_1 \leftarrow_L w_1 \text{ and } y_3 \leftarrow_L w_3)$$
 or $(y_1 \leftarrow_L w_3 \text{ and } y_3 \leftarrow_L w_1)$.

Now, in general, there is a sequence of elements $y = y^{(0)}, y^{(1)}, \ldots, y^{(k)} = w$ in Γ such that $y^{(i-1)} \leftarrow_L y^{(i)}$ for $1 \leq i \leq k$. At each step, the elements in the strings corresponding to these elements are related as above. Combining these steps, one easily sees that

 $(y_1 \leq_L w_1 \text{ and } y_3 \leq_L w_3)$ or $(y_1 \leq_L w_3 \text{ and } y_3 \leq_L w_1)$.

(See also [22, Prop. 4.6].) Now, all elements in a string belong to the same right cell (see [18, 10.5]); in particular, all the elements y_i, w_j belong to the same two-sided cell. Hence, [18, Cor. 6.3] implies that either $y_1 \sim_L w_1$, $y_3 \sim_L w_3$ or $y_1 \sim_L w_3$, $y_3 \sim_L w_1$. (Again, the assumption in [loc. cit.] that W is crystallographic is now superfluous thanks to [6].) Consequently, by induction, we have either $y_1 \approx_{n-1} w_1, y_3 \approx_{n-1} w_3$ or $y_1 \approx_{n-1} w_3, y_3 \approx_{n-1} w_1$.

One of the most striking results about this invariant has been obtained by Garfinkle [8, Theorem 3.5.9]: two elements of a Weyl group of type B_n belong to the same left cell (equal parameter case) if and only if the elements have the same generalised τ -invariant. This fails in general; a counter-example is given by W of type D_n for $n \ge 6$ (as mentioned in the introduction of [7]).

Remark 5.3. Vogan [25, §4] also proposed the following modification of the above invariant. Let $s, t \in S$ be such that st has finite order $m \ge 3$. Let us set $\tilde{\mathfrak{T}}_{s,t}(w) := {\tilde{w}}$ for any $w \in \mathscr{D}_R(s,t)$, with \tilde{w} as in Remark 4.3. Then we obtain a new invariant by exactly the same procedure as in Definition 5.1, but using $\tilde{\mathfrak{T}}_{s,t}$ instead of $\mathfrak{T}_{s,t}$ and allowing any $s, t \in S$ such that st has finite order ≥ 3 . (Note that Vogan only considered the case where m = 4, but then Lusztig's method of strings shows how to deal with the general case.) In any case, this is the model for our more general construction of invariants below.

6. Induction of left cellular maps

We return to the general setting of Section 2, where $\{p_s \mid s \in S\}$ are any positive weights for W.

Definition 6.1. A pair (I, δ) consisting of a subset $I \subseteq S$ and a left cellular map $\delta \colon W_I \to W_I$ is called *KL-admissible*. We recall that this means that the following conditions are satisfied for every left cell $\Gamma \subseteq W_I$ (with respect to the weights $\{p_s \mid s \in I\}$):

(A1) $\delta(\Gamma)$ also is a left cell.

(A2) The map δ induces an \mathscr{H}_I -module isomorphism $[\Gamma]_A \cong [\delta(\Gamma)]_A$.

We say that (I, δ) is *strongly KL-admissible* if, in addition to (A1) and (A2), the following condition is satisfied:

(A3) We have $u \sim_{R,I} \delta(u)$ for all $u \in W_I$.

If $I \subseteq S$ and if $\delta: W_I \to W_I$ is a map, we obtain a map $\delta^L: W \to W$ by

 $\delta^L(xw) = x\delta(w)$ for all $x \in X_I$ and $w \in W_I$.

The map δ^L will be called the *left extension* of δ to W.

Theorem 6.2. Let (I, δ) be a KL-admissible pair. Then the following hold.

(a) The left extension $\delta^L : W \to W$ is a left cellular map for W.

(b) If (I, δ) is strongly admissible, then we have $w \sim_R \delta^L(w)$ for all $w \in W$.

Proof. (a) By Theorem 3.1, there is a left cell Γ' of W_I such that $\Gamma \subseteq X_I \Gamma'$. By condition (A1) in Definition 6.1, the set $\Gamma'_1 := \delta(\Gamma')$ also is a left cell of W_I and, by condition (A2), the map δ induces an \mathscr{H}_I -module isomorphism $[\Gamma']_A \cong [\Gamma'_1]_A$. By Example 3.2, the subsets $X_I \Gamma'$ and $X_I \Gamma'_1$ of W are leftclosed and, hence, we have corresponding \mathscr{H} -modules $[X_I \Gamma']_A$ and $[X_I \Gamma'_1]_A$. These two \mathscr{H} -modules are isomorphic to the induced modules $\operatorname{Ind}_{I}^{S}([\Gamma'])$ and $\operatorname{Ind}_{I}^{S}([\Gamma'_{1}])$, respectively, where explicit isomorphisms are given by the formula in Example 3.2. Now, by [10, Lemma 3.8], we have

$$p_{xu,yv}^* = p_{xu,yv}^*$$
 for all $x, y \in X_I$ and $u, v \in \Gamma'$,

where we set $u_1 = \delta(u)$ and $v_1 = \delta(v)$ for $u, v \in \Gamma'$. By [10, Prop. 3.9], this implies that δ^L maps the partition of $X_I \Gamma'$ into left cells of W onto the analogous partition of $X_I \Gamma'_1$. In particular, since $\Gamma \subseteq X_I \Gamma'$, the set $\delta^L(\Gamma) \subseteq X_I \Gamma'_1$ is a left cell of W; furthermore, [10, Prop. 3.9] also shows that δ^L induces an \mathscr{H} -module isomorphism $[\Gamma]_A \cong [\delta^L(\Gamma)]_A$.

(b) Since condition (A3) in Definition 6.1 is assumed to hold, this is just a restatement of [20, Prop. 9.11(b)]. \square

We will now give examples in which |I| = 2. Let us first fix some notation. If $s, t \in S$ are such that $s \neq t$ and st has finite order, let $w_{s,t}$ denote the longest element of $W_{s,t} = \langle s, t \rangle$ and let

$$\Gamma_{s}^{s,t} = \{ w \in W_{s,t} \mid \ell(ws) < \ell(w) \text{ and } \ell(wt) > \ell(w) \}, \\
\Gamma_{t}^{s,t} = \{ w \in W_{s,t} \mid \ell(ws) > \ell(w) \text{ and } \ell(wt) < \ell(w) \}.$$

Example 6.3 (Dihedral groups with equal parameters). Let $s, t \in S$ be such that $p_s = p_t$ and $s \neq t$. It follows from [20, §8.7] that $\{1\}, \{w_{s,t}\}, \Gamma_s^{s,t}\}$ and $\Gamma_t^{s,t}$ are the left cells of $W_{s,t}$. Let $\sigma_{s,t}$ be the unique group automorphism of $W_{s,t}$ which exchanges s and t. Now, let $\delta_{s,t}$ denote the map $W_{s,t} \mapsto W_{s,t}$ defined by

$$\delta_{s,t}(w) = \begin{cases} w & \text{if } w \in \{1, w_{s,t}\},\\ \sigma_{s,t}(w)w_{s,t} & \text{otherwise.} \end{cases}$$

Then, by [20, Lemma 7.2 and Prop. 7.3], the pair $(\{s,t\}, \delta_{s,t})$ is strongly KL-admissible. Therefore, by Theorem 6.2,

$$\delta^L_{s,t} \colon W \to W$$
 is a left cellular map.

In particular, this means:

(6.4)
$$x \sim_L y \text{ if and only if } \delta^L_{s,t}(x) \sim_L \delta^L_{s,t}(y)$$

for all $x, y \in W$. Note also the following facts:

- If st has odd order, then $\delta_{s,t}$ exchanges the left cells $\Gamma_s^{s,t}$ and $\Gamma_t^{s,t}$. If st has even order, then $\delta_{s,t}$ stabilizes the left cells $\Gamma_s^{s,t}$ and $\Gamma_t^{s,t}$.

For example, if st has order 3, then $\Gamma_s^{s,t} = \{s,ts\}$ and $\Gamma_t^{s,t} = \{t,st\}$; furthermore, $\delta_{s,t}(s) = st$ and $\delta_{s,t}(ts) = t$. The matrix representation afforded by $[\Gamma_s^{s,t}]_A$ with respect to the basis (e_s, e_{ts}) is given by:

$$C'_{s} \mapsto \begin{bmatrix} v^{p_{s}} + v^{-p_{s}} & 1\\ 0 & 0 \end{bmatrix}, \qquad C'_{t} \mapsto \begin{bmatrix} 0 & 0\\ 1 & v^{p_{t}} + v^{-p_{t}} \end{bmatrix} \qquad (p_{s} = p_{t}).$$

The fact that $\delta_{s,t}$ is left cellular just means that we obtain exactly the same matrices when we consider the matrix representation afforded by $[\Gamma_{s,t}^t]_A$ with respect to the basis (e_{st}, e_t) .

Let us explicitly relate the above discussion to the *-operations in Proposition 4.2 and the extension in Proposition 4.4. In particular, this yields new proofs of these two propositions and shows that they also hold in the unequal parameter case, without any further assumptions, as long as $p_s = p_t$. (Partial results in this direction are obtained in [23, Cor. 3.5(4)].)

Corollary 6.5. Let $s, t \in S$ be such that st has finite order ≥ 3 and assume that $p_s = p_t$. Then, with the notation in Remark 4.3, we obtain a left cellular map $\delta \colon W \to W$ by setting

$$\delta(w) = \begin{cases} \tilde{w} & \text{if } w \in \mathscr{D}_R(s,t), \\ w & \text{otherwise.} \end{cases}$$

(If st has order 3, then this coincides with the map defined in Proposition 4.2.)

Proof. Just note that, if $w \in \mathscr{D}_R(s,t)$, then $\delta_{s,t}^L(w) = \tilde{w}$. Thus, the assertion simply is a restatement of the results in Example 6.3. Furthermore, if st has order 3, then $\tilde{w} = w^*$ for all $w \in \mathscr{D}_R(s,t)$, as noted in Remark 4.3.

Example 6.6 (Dihedral groups with unequal parameters). Let $s, t \in S$ be such that st has even order ≥ 4 and that $p_s < p_t$. Then it follows from [20, §8.7] that $\{1\}, \{w_{s,t}\}, \{s\}, \{w_{s,t}s\}, \Gamma_s^{s,t} \setminus \{s\}$ and $\Gamma_t^{s,t} \setminus \{w_{s,t}s\}$ are the left cells of $W_{s,t}$. Now, let $\delta_{s,t}$ denote the map $W_{s,t} \mapsto W_{s,t}$ defined by

$$\delta_{s,t}^{<}(w) = \begin{cases} w & \text{if } w \in \{1, w_{s,t}, s, w_{s,t}s\}, \\ ws & \text{otherwise.} \end{cases}$$

Then, again by [20, Lemma 7.5 and Prop. 7.6] (or [13, Exc. 11.4]), the pair $(\{s,t\}, \delta_{s,t}^{<})$ is strongly KL-admissible. Therefore, again by Theorem 6.2, $\delta_{s,t}^{<,L}: W \to W$ is a left cellular map. In particular, this means:

(6.7)
$$x \sim_L y \text{ if and only if } \delta_{s,t}^{<,L}(x) \sim_L \delta_{s,t}^{<,L}(y)$$

for all $x, y \in W$. Note also that $\delta_{s,t}^{\leq}$ exchanges the left cells $\Gamma_s^{s,t} \setminus \{s\}$ and $\Gamma_t^{s,t} \setminus \{w_{s,t}s\}$ while it stabilizes all other left cells in $W_{s,t}$.

For example, if st has order 4, then $\Gamma_1 := \Gamma_s^{s,t} \setminus \{s\} = \{ts, sts\}$ and $\Gamma_2 := \Gamma_t^{s,t} \setminus \{w_{s,t}s\} = \{t, st\}$; furthermore, $\delta_{s,t}^<(ts) = t$ and $\delta_{s,t}^<(sts) = st$. As before, the fact that $\delta_{s,t}^<$ is left cellular just means that the matrix representation afforded by $[\Gamma_1]_A$ with respect to the basis (e_{ts}, e_{sts}) is exactly the same as the matrix representation afforded by $[\Gamma_2]_A$ with respect to the basis (e_t, e_{st}) .

The next example shows that left extensions from dihedral subgroups are, in general, not enough to describe all left cellular maps.

Example 6.8. Let W be a Coxeter group of type B_r $(r \ge 2)$, with diagram and weight function as follows:

 $B_r \qquad \stackrel{b}{\bullet} \stackrel{a}{\bullet} \stackrel{a}{\bullet} \stackrel{a}{\bullet} \stackrel{a}{\bullet} \dots \quad \stackrel{a}{\bullet} \qquad b > (r-1)a > 0$

This is the *asymptotic case* studied by Iancu and the first-named author [2], [5]. In this case, the left, right and two-sided cells are described in terms of a Robinson–Schensted correspondence for bi-tableaux. Using results from [*loc. cit.*], it is shown in [10, Theorem 6.3] that the following hold:

- (a) If Γ_1 and Γ_2 are two left cells contained in the same two-sided cell, then there exists a bijection $\delta : \Gamma_1 \xrightarrow{\sim} \Gamma_2$ which induces an isomorphism of \mathscr{H} -modules $[\Gamma_1]_A \xrightarrow{\sim} [\Gamma_2]_A$.
- (b) The bijection δ in (a) is uniquely determined by the condition that $w, \delta(w)$ lie in the same right cell.

However, one can check that, for $r \in \{3, 4, 5\}$, the map δ does not always arise from a left extension of a suitable left cellular map of a dihedral subgroup of W. It is probable that this observation holds for any $r \ge 3$.

Example 6.9. Let W be an affine Weyl group and W_0 be the finite Weyl group associated with W. Then there is a well-defined "lowest" two-sided cell, which consists of precicely $|W_0|$ left cells; see Guilhot [14] and the references there. It is likely that these $|W_0|$ left cells are all related by suitable left cellular maps.

7. An extension of the generalised τ -invariant

Notation. We fix in this section a set Δ of KL-admissible pairs, as well as a surjective map $\rho: W \to E$ (where E is a fixed set) such that the fibers of ρ are unions of left cells. We then denote by \mathscr{V}_{Δ} the group of permutations of W generated by the family $(\delta^L)_{(I,\delta)\in\Delta}$.

Note that giving a surjective map ρ as above is equivalent to giving an equivalence relation on W which is coarser than \sim_L .

Then, each $w \in W$ defines a map $\tau_w^{\Delta,\rho} : \mathscr{V}_\Delta \longrightarrow E$ as follows:

 $\tau_w^{\Delta,\rho}(\sigma) = \rho(\sigma(w)) \quad \text{for all } \sigma \in \mathscr{V}_{\Delta}.$

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Definition 7.1. Let $x, y \in W$. We say that x and y have the same $\tau^{\Delta,\rho}$ *invariant* if $\tau_x^{\Delta,\rho} = \tau_y^{\Delta,\rho}$ (as maps from \mathscr{V}_{Δ} to E). The equivalence classes for this relation are called the *left Vogan* (Δ, ρ) -*classes*.

An immediate consequence of Theorem 6.2 is the following:

Theorem 7.2. Let $x, y \in W$ be such that $x \sim_L y$. Then x and y have the same $\tau^{\Delta,\rho}$ -invariant.

Remark 7.3. There is an equivalent formulation of Definition 7.1 which is more in the spirit of Vogan's Definition 5.1. We define by induction on n a family of equivalence relations $\approx_n^{\Delta,\rho}$ on W as follows. Let $x, y \in W$.

- For n = 0, we write x ≈^{Δ,ρ}₀ y if ρ(x) = ρ(y).
 For n ≥ 1, we write x ≈^{Δ,ρ}_n y if x ≈^{Δ,ρ}_{n-1} y and δ^L(x) ≈^{Δ,ρ}_{n-1} δ^L(y) for all $(I,\delta) \in \Delta.$

Note that the relation $\approx_{n}^{\Delta,\rho}$ is finer than $\approx_{n-1}^{\Delta,\rho}$. It follows from the definition that x, y have the same $\tau^{\Delta,\rho}$ -invariant if and only if $x \approx_{n}^{\Delta,\rho} y$ for all $n \ge 0$.

This inductive definition is less easy to write than Definition 7.1, but it is more efficient for computational purpose. Indeed, if one finds an n_0 such that the relations $\approx_{n_0}^{\Delta,\rho}$ and $\approx_{n_0+1}^{\Delta,\rho}$ coincide, then x and y have the same $\tau^{\Delta,\rho}$ -invariant precisely when $x \approx_{n_0}^{\Delta,\rho} y$. Note that such an n_0 always exists if W is finite. Also, even in small Coxeter groups, the group \mathscr{V}_{Δ} can become enormous (see Example 7.6 below) while n_0 is reasonably small and the relation $\approx_{n_0}^{\Delta,\rho}$ can be computed quickly.

Example 7.4 (Enhanced right descent set). One could take for ρ the map $\mathscr{R}: W \to \mathscr{P}(S)$ (power set of S); see Proposition 3.3. Assuming that we are in the equal parameter case, we then obtain exactly the invariant in Remark 5.3. In the unequal parameter case, we can somewhat refine this, as follows. Let

$$S^p = S \cup \{sts \mid s, t \in S \text{ such that } p_s < p_t\}$$

and, for $w \in W$, let

$$\mathscr{R}^p(w) = \{ s \in S^p \mid \ell(ws) < \ell(w) \} \subseteq S^p.$$

Then it follows from the description of left cells of $W_{s,t}$ in Example 6.6 and from Theorem 3.1 (by using the same argument as for the proof of Proposition 3.3 given in $\S3$) that

if
$$x \leq_L y$$
, then $\mathscr{R}^p(y) \subseteq \mathscr{R}^p(x)$.

In particular,

if $x \sim_L y$, then $\mathscr{R}^p(x) = \mathscr{R}^p(y)$.

So one could take for ρ the map $\mathscr{R}^p: W \to \mathscr{P}(S^p)$.

Let Δ_2 be the set of all pairs (I, δ) such that $I = \{s, t\}$ with $s \neq t$ and $p_s \leq p_t$; furthermore, if $p_s = p_t$, then $\delta = \delta_{s,t}$ (as defined in Example 6.3) while, if $p_s < p_t$, then $\delta = \delta_{s,t}^<$ (as defined in Example 6.6). Then the pairs in Δ_2 are all strongly KL-admissible. With the notation in Example 7.4, we propose the following conjecture:

Conjecture 7.5. Let $x, y \in W$. Then $x \sim_L y$ if and only if $x \sim_{LR} y$ and x, y have the same $\tau^{\Delta_2, \mathscr{R}^p}$ -invariant.

If W is finite and we are in the equal parameter case, then Conjecture 7.5 is known to hold except possibly in type B_n, D_n ; see the remarks at the end of [12, §6]. We have checked that the conjecture also holds for F_4, B_n $(n \leq 7)$ and all possible weights, using PyCox [11].

By considering collections Δ with subsets $I \subseteq S$ of size bigger than 2, one can obtain further refinements of the above invariants. In particular, it is likely that the results of [2], [5] can be interpreted in terms of generalised $\tau^{\Delta,\rho}$ -invariants for suitable Δ, ρ . This will be discussed elsewhere.

Example 7.6. Let (W, S) be of type H_4 . Then it can be checked by using computer computations in GAP that

$$|\mathscr{V}_{\Delta_2}| = 2^{40} \cdot 3^{20} \cdot 5^8 \cdot 7^4 \cdot 11^2.$$

On the other hand, the computation of left Vogan $(\Delta_2, \mathscr{R}^p)$ -classes using the alternative definition given in Remark 7.3 takes only a few minutes on a standard computer.

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