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Observer Design for Frictionless and Unilaterally Constrained Mechanical Systems: A Passivity-Based Approach

Aneel Tanwani, Bernard Brogliato, and Christophe Prieur

Abstract

This paper addresses the problem of estimating the velocity variables, using the position measurement as output, in nonlinear Lagrangian dynamical systems with perfect unilateral constraints. Using the class of bounded variation functions to model the velocity variables (so that Zeno phenomenon is not ruled out), we represent the derivative of such functions with the Lebesgue-Stieltjes measure, and use the framework of measure differential inclusion (MDI) to describe the dynamics. Under the assumption that the velocity of the system is uniformly bounded, an observer is designed which is also a measure differential inclusion. It is proved that there exists a unique solution to the proposed observer and this solution converges asymptotically to the actual velocity.

I. INTRODUCTION

In this paper, we consider Lagrangian mechanical systems with unilateral constraints (without friction) on the position of a moving point, whose position and velocity we choose to denote by q and \dot{q} , respectively. The unconstrained motion of the system satisfies the equation

$$\ddot{q} = G(t, q, \dot{q}), \quad (1a)$$

and the position q is constrained by:

$$h_i(q) \geq 0, \quad i \in \{1, \dots, m\} \quad (1b)$$

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where $G : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes a vector field, and $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ represent the unilateral constraints imposed on the system's motion. Mechanical systems with impacts, such as robots and colliding rigid bodies could be seen as systems with unilateral constraints. It is seen that the nonsmooth behavior, or discontinuity in state trajectory, appears in such systems when any of the constraint is active, that is, $h_i(q) = 0$, and the velocity points outside the admissible domain, that is, $\nabla h_i^\top(q)\dot{q} < 0$, where we use the notation $\nabla h_i(\cdot)$ to denote the gradient of the function $h_i(\cdot)$. This is because the velocity must change its direction instantaneously to keep the moving point inside the admissible set. In general, it is seen that the trajectories of such systems are algebraically constrained. They exhibit continuous (due to flow within the interior of the set) as well as discrete dynamics (due to impacts with the boundary of the set), and hence form an important class of nonsmooth systems. There are several modeling frameworks for such nonsmooth systems; one such modeling framework, which is used to model the motion of state-constrained trajectories is the so-called *sweeping process* [18], [20], [22], [23]. The term so-coined because it represents the motion of a point inside a closed set. As the the set moves, the point is swept across by the moving set. If for such processes, the constraint set is parameterized by time *only*, then we call it the *first-order* sweeping process. However, for system (1), we first define an admissible set for velocity $\dot{q}(\cdot)$ which is parameterized by the position $q(\cdot)$, and this formulation leads to a *second-order* sweeping process.

This paper is concerned with the design of observers for estimating the velocity $\dot{q}(\cdot)$ using the position $q(\cdot)$ as the output, while using the sweeping process formulation to describe the dynamics of the system and the observer. The construction of observers, or state estimators, is a classical problem in the design of control systems and has found many useful applications such as output feedback control, and fault diagnosis. For smooth systems, there are several standard techniques such as Kalman filter, Luenberger observer, or high-gain approach. Such techniques have also been applied to design of observers for certain classes of smooth and unconstrained Lagrangian systems with application to output feedback control, see for example, [6], [7], [25], [38]. A common element of these designs is to assume that the velocity $\dot{q}(\cdot)$ is uniformly bounded (in time) which is primarily because $G(t, q, \cdot)$ is quadratic in general for mechanical systems.

Lately, the researchers have started looking at the state-estimation problem in nonsmooth systems. In this regard, we mention the recent work on observer design of switched systems with ordinary differential equations [31], [33], [34], switched differential-algebraic equations

[35], [36], certain classes of differential inclusions [10], [27], [32], complementarity systems [16], and the references therein for more details. Classical approaches for observer design are based on constructing an auxiliary dynamical system driven by the error between the measured output and the estimated output, where it is shown that the resulting dynamics of the state estimation error converge to the origin. However, for nonsmooth systems subjected to impacts, such schemes are not easily implementable since the impacts, or discrete dynamics, are not influenced by error injection and hence destroy the integration effect.

For nonsmooth Lagrangian systems with impacts, the problem of state estimation has been considered in [19] under certain restrictive assumptions, and state estimation with tracking control in [15] for motions restricted within a convex polyhedral domain. The work of [5] also deals with the problem of tracking control (without estimation) for similar kind of systems. This article, however, deals with a more general class of nonsmooth Lagrangian dynamics, that allow more general domains, using the formalism of differential inclusions. The approach adopted is closely related to the observer design presented in [10] that is itself strongly inspired from the material in [9]. The authors in [10] work with differential inclusions that represent the first-order sweeping process and the resulting trajectories are absolutely continuous. Since the constraint set is a function of time, this information is passed to the state estimator. The state estimator replicates the system dynamics, and hence is a sweeping process of the first-order. The convexity of the constraint set then generates the passivity relation between the error dynamics and the output estimation error, which is the key component in proving the error convergence.

In this paper, however, the system under consideration is a second-order sweeping process. This way, the state-trajectories of the system are allowed to be of locally bounded variation (BV), and hence discontinuous, which introduces the major difference. Since locally BV functions may admit an infinite number of discontinuities in finite time, the Zeno phenomenon is not excluded in our setup. The proposed observer only describes the dynamics for the velocity estimate (and not the position) in the form of a differential inclusion driven by the measured output (position). It is proved that, for each output, there exists a unique solution to such differential inclusions, and that this solution converges to the actual velocity of the system asymptotically.

The article is organized as follows: in section II some useful mathematical definitions are recalled. The Moreau's sweeping process in which we embed Lagrangian nonsmooth mechanical systems, and the definition of its solutions are described in section III. Section IV is dedicated to

the velocity observer design, the error dynamics stability analysis, and its passivity interpretation. In section V, numerical simulations obtained with the INRIA software package SICONOS are presented for two systems: the bouncing ball and a planar ball bouncing in a parabola. Section VI is dedicated to one of the main results of this article, *i.e.* the proof of the well-posedness (existence and uniqueness of solutions) of the velocity observer. Conclusions end the paper in section VII.

II. PRELIMINARIES

In this section, we collect some basic definitions and notations that will be used later on.

Functions of bounded variation: The total variation of a function $f : [a, b] \rightarrow \mathbb{R}^n$ is defined as $var_f(t) = \sup \sum_{i=1}^k |f(t_i) - f(t_{i-1})|$, $t \in [a, b]$, where the supremum is taken over all integers k , and all possible choices of the sequence $\{t_i\}$ such that $a = t_0 < t_1 < \dots < t_k = b$. The function $f(\cdot)$ is said to be of bounded variation (BV) on $[a, b]$ if $var_f(t) < \infty$. If it is right continuous with bounded variation we denote it with *rcbv*. It is locally *rcbv* if this holds for any bounded interval $[a, b] \subset \mathbb{R}$. If $f(\cdot)$ is BV on $[a, b]$ then it has at most a countable number of jump discontinuities. Moreover, it has right and left limits everywhere. The right and left limits of a function at t are denoted by $f(t^+) := \lim_{s \searrow t} f(s)$ and $f(t^-) := \lim_{s \nearrow t} f(s)$, respectively, provided they exist. In this notation, right continuity of $f(\cdot)$ in t , means that $f(t^+) = f(t)$.

Locally integrable functions: For an interval $I \subseteq \mathbb{R}$, we denote by $\mathcal{L}^1(I, \mathbb{R}^n; d\mu)$ and $\mathcal{L}_{loc}^1(I, \mathbb{R}^n; d\mu)$ the space of integrable and locally integrable functions, respectively, from I to \mathbb{R}^n with respect to the measure $d\mu$. If the measure is not specified then the integration is with respect to the Lebesgue measure. An absolutely continuous (AC) function $f : [a, b] \rightarrow \mathbb{R}^n$ is a function that can be written as $f(x) - f(a) = \int_a^x \dot{f}(s) ds$ for any $x \geq a$ for a function $\dot{f}(\cdot) \in \mathcal{L}^1([a, b], \mathbb{R}^n)$, which is considered as its derivative. The space of continuously differentiable functions is denoted by $\mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$, for $m, n \in \mathbb{N}$.

Stieltjes measure associated with BV functions: If $v : I \rightarrow \mathbb{R}^n$ is a function of bounded variation, then one can associate with it a Lebesgue-Stieltjes measure or the so-called *differential measure* dv on I . Also, if $v(\cdot)$ is *rcbv* on $[a, b]$, then we have the relation that $v(t) = v(a) + \int_{]a, t]} dv$.

The density of the measure dv with respect to a positive Radon measure $d\mu$ over an interval I is defined as:

$$\frac{dv}{d\mu}(t) := \lim_{\varepsilon \rightarrow 0} \frac{dv(I(t, \varepsilon))}{d\mu(I(t, \varepsilon))}, \quad (2)$$

where $I(t, \varepsilon) := I \cap [t - \varepsilon, t + \varepsilon]$. Similarly, one can define the density of the Lebesgue measure dt with respect to the Radon measure $d\mu$. A Radon measure $d\nu$ is absolutely continuous with respect to $d\mu$ if for every measurable set \mathcal{A} , $d\mu(\mathcal{A}) = 0$ implies that $d\nu(\mathcal{A}) = 0$. Further, the measure $d\nu$ is absolutely continuous with respect to $d\mu$ if and only if the density function $\frac{d\nu}{d\mu}(\cdot)$ is well-defined (finite μ -almost everywhere) and is $d\mu$ integrable.

Convex analysis: For a set $V \subset \mathbb{R}^n$, we will denote its interior by $\text{int } V$, and the boundary of this set is denoted by $\text{bd}(V)$. If V is convex, then $\mathcal{N}_V(v)$ denotes the normal cone to V at $v \in V$ and is defined as:

$$\mathcal{N}_V(v) := \{w \in \mathbb{R}^n \mid \langle w, x - v \rangle \leq 0 \ \forall x \in V\}. \quad (3)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in \mathbb{R}^n . Obvious from the definition, the normal cone to a convex set is a monotone operator. When V is a closed convex cone, we denote by $V^\circ(q)$ the closed convex polyhedral cone polar to $V(q)$ with respect to usual inner product on \mathbb{R}^n , which is defined as:

$$V^\circ(q) := \{w \in \mathbb{R}^n : w^\top v \leq 0, \ \forall v \in V(q)\}. \quad (4)$$

III. SOLUTION FRAMEWORK

In this section, we will describe the dynamics of nonsmooth Lagrangian systems using differential inclusions and briefly talk about their solutions. The observer will then be designed using this formalism.

A. Mathematical model

We consider mechanical systems with a finite number of degrees of freedom that are subjected to some unilateral constraints. The position variable $q \in \mathbb{R}^n$ is thus assumed to evolve in a set that admits the following form:

$$\Phi := \{q \in \mathbb{R}^n \mid h_i(q) \geq 0, \ i = 1, 2, \dots, m\}. \quad (5)$$

The geometry of the set Φ is determined by the functions $h_i(\cdot)$, and the only condition we will impose on the functions $h_i(\cdot)$ is that they are continuously differentiable so that $\nabla h_i(\cdot)$ is continuous for each i . This allows us to model a large number of closed domains which may even be nonconvex.

The convex polyhedral tangent cone $V(q)$ to the region Φ at a point q is given by:

$$V(q) := \{v \in \mathbb{R}^n \mid v^\top \nabla h_i(q) \geq 0, \forall i \in \mathcal{J}(q)\} \quad (6)$$

where the set $\mathcal{J}(q)$ denotes the set of active constraints at q , i.e.,

$$\mathcal{J}(q) := \{i \in \{1, \dots, m\} : h_i(q) \leq 0\}.$$

One can think of the set $V(q(t))$ as the set of admissible velocities that keep the position variable $q(t)$ inside the set Φ . It is noted that $V(q) = \mathbb{R}^n$ and consequently $\mathcal{N}_{V(q)}(\cdot) = \{0\}$ if $q \in \text{int } \Phi$.

We now formulate the dynamics of system (1) as a measure differential inclusion:

$$dq = v dt \quad (7a)$$

$$M(q)dv + F(t, q, v)dt \in -\mathcal{N}_{V(q)}(v_e) \subseteq V^\circ(q) \quad (7b)$$

where

$$v_e(t) := \frac{v(t^+) + ev(t^-)}{1 + e}, \quad (7c)$$

and $e \in [0, 1]$ is the coefficient of restitution. The initial condition is assumed to satisfy $q_0 := q(0) \in \Phi$, and $v_0 := v(0) \in V(q_0)$. Since $V(q)$ is convex, the normal cone $\mathcal{N}_{V(q)}$ is defined as in (3). For a proof of the inclusion in the right-hand side of (7b) see [9].

The formulation for constrained mechanical systems, as in (7), was pioneered by J. J. Moreau [22], and the MDI (7) is called a second order sweeping process (because the constraint set for velocities appearing in (7b) depends on the state variable $q(\cdot)$). The motivation for working with the MDI is that we are seeking a solution to the evolution problem in the space of locally *rcbv* functions to deal with possible collisions with the boundary of the admissible set. Functions which are locally *rcbv* possess generalized derivatives that can be identified with Stieltjes measure and equation (7b) precisely describes the inclusion of the measure dv , associated with $v(\cdot)$, into a normal cone described by the constraint set Φ . It is also noted that the post-impact velocity determined according to Moreau's collision rule (or Newton's impact law) is directly encoded in the MDI (7). Further details on inclusions of type (7) and comparisons with other modeling frameworks could be found in [8, Section 5.3]. For our purpose, it is seen that the observer design given in Section IV is partially aided by this compact formulation. It is noteworthy that there is a close link between the sweeping process in (7) and so-called complementarity Lagrangian systems, see *e.g.* [2, Section 3.6], and the examples in section V.

B. Existence of Solution

The solution of MDI (7) is considered in the following sense:

Definition 1. A solution to the Cauchy problem (7) with initial data $(q_0, v_0) \in \Phi \times V(q_0)$, over an interval $I = [0, T]$, is a pair (q, v) such that $v(\cdot)$ is rcbv on I ; $q(t) = q_0 + \int_0^t v(s)ds$; $q(t) \in \Phi$ and $v(t) \in V(q(t))$ for all $t \geq 0$; and furthermore, there exists a positive measure (represented by) $d\mu$ such that both dt and dv possess densities with respect to $d\mu$, denoted by $dt/d\mu$, and $dv/d\mu$ respectively, such that

$$M(q) \frac{dv}{d\mu}(t) + F(t, q, v) \frac{dt}{d\mu}(t) \in -\mathcal{N}_{V(q(t))}(v_e), \quad d\mu\text{-a.e. on } I. \quad (8)$$

The choice of the measure $d\mu$ is not unique since the right-hand side of (7) is a cone. However, by Lebesgue-Radon-Nikodym theorem, the functions $dt/d\mu(\cdot) \in \mathcal{L}^1(I, \mathbb{R}; d\mu)$ and $dv/d\mu(\cdot) \in \mathcal{L}^1(I, \mathbb{R}^n; d\mu)$ are uniquely determined for a given $d\mu$.

The problem of existence of solutions for evolution problems (1) has been studied for a long time. Earlier results on this problem dealt with the single constraint case ($m = 1$) and one may refer to [20, Chapter 3], [29] for results in this direction. The basic idea in these works is to introduce a time discretization scheme, either at position level [29] or velocity level [20] to construct a sequence of approximate solutions which is shown to converge as the step size converges to zero. For several unilateral constraints ($m \geq 2$), the existence and uniqueness has been proved in [3] under analytic assumptions on the data using the solution theory for differential equations and variational inequalities. The most relaxed conditions, under which the existence of solutions has been proved using discretization at velocity level, have appeared recently in [13] for the inelastic case ($e = 0$), and in [14], [28] for general values of $e \in [0, 1]$. Based on the work of [28], the following regularity assumptions are required on the system data for the existence of solution, and are also needed for the observer design:

- (H1) The function $F(\cdot, \cdot, \cdot)$ is continuous and is continuously differentiable (\mathcal{C}^1) with respect to its second and third arguments.
- (H2) The mapping $M(\cdot)$, from \mathbb{R}^n to the set of symmetric positive definite matrices, belongs to class \mathcal{C}^1 and

$$\underline{\lambda}_M |v|^2 \leq v^\top M(q)v \leq \bar{\lambda}_M |v|^2 \quad \forall (q, v) \in \Phi \times \mathbb{R}^n. \quad (9)$$

- (H3)** For each $i \in \{1, \dots, m\}$, the function $h_i \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$, its Euclidean gradient $\nabla h_i(q)$ is locally Lipschitz continuous and does not vanish in a neighborhood of $\{q \in \mathbb{R}^n \mid h_i(q) = 0\}$.
- (H4)** The active constraints are functionally independent, i.e., $\{\nabla h_i(q)\}_{i \in \mathcal{J}(q)}$ is linearly independent for all $q \in \Phi$.

Without recalling the formal result on existence and assuming that a solution exists in the sense of Definition 1 under hypotheses (H1) – (H4), we only collect the properties of the solutions to system (7) which provide more insight.

C. Solution Characteristics

1) *The post-impact velocity:* The condition $q(t) \in \Phi$ for all t , allows for jumps in the velocity when q hits the boundary of Φ . At that instant, one must have from Moreau's viability Lemma (see e.g. [8, Proposition 5.1] or [23]) that $v(t^-) \in -V(q(t))$, and $v(t^+) \in V(q(t))$. According to (7b), the jumps in $v(\cdot)$ satisfy:

$$M(q(t))(v(t^+) - v(t^-)) \in -\mathcal{N}_{V(q(t))}(v_e(t)).$$

If there is only one active constraint, i.e. $\mathcal{J}(q(t)) = \{i\}$, for some fixed $i \in \{1, \dots, m\}$, then $V(q) = \mathbb{R}_+ \nabla h_i(q(t))$, or equivalently, $\mathcal{N}_{V(q(t))} = \mathbb{R}_- \nabla h_i(q(t))$. These relations imply that

$$v(t^+) = -ev(t^-) + (1 + e) \text{proj}_{M(q(t))}(V(q(t)), v(t^-)), \quad (10)$$

where $\text{proj}_{M(q)}(V(q))$ denotes the projection on $V(q)$ according to the kinetic metric at q , which is defined by the inner product $\langle v, w \rangle_{M(q)} = \langle v, M(q)w \rangle = \langle M(q)v, w \rangle$.

2) *Regularity of state trajectories:* The function q is absolutely continuous, but not necessarily everywhere differentiable. The velocity $v(\cdot)$ is a locally *rcbv* function, for which the left and right limits are defined everywhere. The acceleration is represented by the measure dv and can be decomposed as a sum of two measures: an atomic measure $d\mu_a$ and a Lebesgue-integrable function $\dot{v}(\cdot)$, i.e., $dv = d\mu_a + \dot{v}dt$.

3) *Countably many impacts:* The set of impact times, at which $v(\cdot)$ is discontinuous, is at most countable. One may simply take $d\mu_a = \sum_{k \geq 0} [v(t_k^+) - v(t_k^-)] \delta_{t_k}$, where δ_{t_k} is the Dirac impulse at time t_k and $\{t_k\}_{k \geq 0}$ is an ordered sequence of impact times. Thus, the formulation (7) does not exclude the Zeno phenomenon (with a finite or infinite number of left accumulation points). However, if $e = 1$, then it is shown in [4] that there exists a constant $\rho > 0$ such that $t_{k+1} - t_k > \rho$, for each $k \geq 0$.

4) *Non-uniqueness and Continuity of solutions:* The solution of system (7) is unique if the system data is analytic [3], but in general, it may not be the case. Even under the analyticity assumption, the solutions may not vary continuously with respect to initial conditions under the hypotheses (H1) - (H4). For this to hold, there is an additional condition on the set $\{\nabla h_i(q)\}_{i \in \mathcal{J}(q)}$ given in [28], which states that for $e = 0$, the active constraints must satisfy $\langle \nabla h_i(q), M^{-1}(q) \nabla h_j(q) \rangle \leq 0$, and for $e \in (0, 1]$, $\langle \nabla h_i(q), M^{-1}(q) \nabla h_j(q) \rangle = 0$. The work of [15] assumes this condition because their design is based on closeness of solutions (in graphical sense) with respect to initial conditions. However, our observer design doesn't require this property and hence no such condition is imposed in our results.

IV. OBSERVER DESIGN

We now address the problem of designing observers for the systems considered in Section III. It will be assumed that the position $q(\cdot)$ is the measured variable, and the objective is then to design an estimator for the velocity $v(\cdot)$ of the moving particle. In addition, we suppose that the following assumption holds:

Assumption 1. *The velocity $v(\cdot)$ obtained as a solution to (7) stays bounded, that is:*

$$v(t) \in \mathcal{B}_v := \{v \in \mathbb{R}^n : |v| \leq C_v\} \quad \forall t \geq 0. \quad (11)$$

Before describing the observer dynamics, we let $\bar{F}(t, q, \cdot)$ denote the Lipschitz extension¹ of $F(t, q, \cdot)$ from \mathcal{B}_v such that there exists $C_F(t, q)$ satisfying

$$|F(t, q, v_1) - F(t, q, v_2)| \leq C_F(t, q) \cdot |v_1 - v_2|, \quad \forall v_1, v_2 \in \mathcal{B}_v.$$

The observer we propose is:

$$\boxed{M(q)dz + \bar{F}(t, q, \hat{v})dt - M(q)L\hat{v}dt \in -\mathcal{N}_{V(q)}(\hat{v}_e)} \quad (12a)$$

where

$$\hat{v}(t) := z(t) - Lq(t), \quad (12b)$$

$$\hat{v}_e(t) := \frac{\hat{v}(t^+) + e\hat{v}(t^-)}{1 + e} \quad (12c)$$

¹For a locally Lipschitz function $F(t, q, \cdot) : \mathcal{B} \rightarrow \mathbb{R}^n$, the function $\bar{F}(t, q, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called the Lipschitz extension of $F(t, q, \cdot)$ from $\mathcal{B} \subset \mathbb{R}^n$ if $\bar{F}(t, q, \cdot)$ is globally Lipschitz over \mathbb{R}^n and $\bar{F}(t, q, v) = F(t, q, v)$ for all $(t, q) \in \mathbb{R} \times \mathbb{R}^n$ and $v \in \mathcal{B}$.

and $q(\cdot)$ in (12a) and (12b) is an absolutely continuous function of time which is obtained from (7) as the measured output. The initial condition $\hat{v}_0 := \hat{v}(0) \in \mathcal{B}_v$. The design criteria for the output injection matrix L will appear later in Section IV-B. The solution to system (12) is interpreted in the sense similar to Definition 1. The idea of using Lipschitz extension of the system vector fields for state estimators appeared in [30].

In the sequel, we will show that the proposed observer (12) has the following two properties:

- **Well-posedness:** For each absolutely continuous function $q(\cdot)$, there exists a unique locally *rcbv* function $\hat{v}(\cdot)$ that satisfies (12).
- **Error convergence:** The estimate $\hat{v}(\cdot)$ converges to $v(\cdot)$ asymptotically, i.e., $\lim_{t \rightarrow \infty} |v(t) - \hat{v}(t)| = 0$.

In the following two subsections, it is proved that both these properties indeed hold. Before proceeding, note that the original system may not have unique solutions, but the observer has the property that it generates a unique trajectory corresponding to the each function $q(\cdot)$ observed as an output of system (7); see [10, Remark 3.3] for further explanation along these lines.

A. Observer Well-posedness

It is noted that the function $q(\cdot)$ is seen as an external “input” by the observer and hence $V(q(\cdot))$ is seen as a time-parameterized multi-valued function that does not depend on any of the internal states of the estimator. This makes the observer (12) a sweeping process of first order. Letting $g(t, \hat{v}) := \bar{F}(t, q(t), \hat{v}) - M(q(t))L(v(t) - \hat{v})$, equation (12a) is rewritten as:

$$M(q(t))d\hat{v} + g(t, \hat{v})dt \in -\mathcal{N}_{V(q(t))}(\hat{v}_e) \quad (13)$$

where, by definition, $g(\cdot, \hat{v})$ is measurable and $g(t, \cdot)$ is globally Lipschitz. Specifically, there exist some constants $C_{g,l}$ and $C_{g,b}$ such that

$$|g(t, \hat{v}_1) - g(t, \hat{v}_2)| \leq C_{g,l}|\hat{v}_1 - \hat{v}_2|, \quad \forall \hat{v}_1, \hat{v}_2 \in \mathbb{R}^n, \quad \forall t \in [0, T] \quad (14)$$

$$|g(t, \hat{v})| \leq C_{g,l}|\hat{v}| + |g(t, 0)| \leq C_{g,b}(1 + |\hat{v}|), \quad \forall t \in [0, T] \quad (15)$$

where, in arriving at (15), we used the fact that $|g(t, 0)| = |M(q(t))Lv(t)| + |\bar{F}(t, q, 0)|$ is bounded by some constant under Assumption 1. Moreover, the set-valued map $t \mapsto V(q(t))$ is lower semicontinuous (because $t \mapsto q(t)$ is absolutely continuous and $q \mapsto V(q)$ is lower semicontinuous).

Let us now state the following result on existence and uniqueness of solution to (13). This is a fundamental step since existence of solutions secures that the error stability analysis is meaningful, while uniqueness property secures that the observer output is unique for a given plant trajectory, as reminded above.

Theorem 1. *Assume that hypotheses (H1) - (H4) and Assumption 1 hold, then the system (13) is well-posed, that is, there exists a unique solution $\hat{v} \in BV([0, T]; \mathbb{R}^n)$ for any initial condition $\hat{v}(0) \in V(q(0))$. Moreover, it holds that*

$$\hat{v}_e(t) \in V(q(t)) \quad \forall t \geq 0. \quad (16)$$

The result on existence and uniqueness of solutions for MDI (13), stated in Theorem 1, is important in several respects: Firstly, the multivalued operator on the right-hand side is non-compact, time-varying and the variation of this set-valued map (measured using Hausdorff-distance) is not of bounded variation because of which we cannot use the existing results on solutions of first-order sweeping processes; Secondly, the numerical implementation of the examples considered in this paper (see Section V) is based on a time-discretization procedure and the proof of Theorem 1 shows that the proposed sequence of discretized solutions indeed converges to a unique solution of system (13). With this motivation, we work out a formal proof of Theorem 1 in this paper. In this section, we will only develop an outline which shows all the steps involved in the proof and for some of these steps, detailed calculations are given in Section VI.

Proof Outline for Theorem 1: Consider a partition \mathcal{P} of the interval $[0, T]$ given by:

$$\mathcal{P} := \{t_{\mathcal{P},i}, 0 \leq i \leq N_{\mathcal{P}}\}, \quad 0 = t_{\mathcal{P},0} < t_{\mathcal{P},1} < t_{\mathcal{P},2} < \dots < t_{\mathcal{P},N_{\mathcal{P}}} = T$$

and let

$$\hat{v}_{\mathcal{P},0} = \hat{v}_0 \quad (17a)$$

$$\hat{v}_{\mathcal{P},i} = -e\hat{v}_{\mathcal{P},i-1} + (1+e) \text{proj}_{M_{\mathcal{P},i}} \left[\hat{v}_{\mathcal{P},i-1} - \frac{1}{1+e} M_{\mathcal{P},i}^{-1} G_{\mathcal{P},i}, V_{\mathcal{P},i} \right] \quad (17b)$$

where $G_{\mathcal{P},i} := \int_{t_{\mathcal{P},i-1}}^{t_{\mathcal{P},i}} g(s, \hat{v}_{\mathcal{P},i-1}) ds$ and $M_{\mathcal{P},i} := M(q(t_{\mathcal{P},i}))$. One can then define a piecewise constant solution $\hat{v}_{\mathcal{P}}(\cdot)$ for each partition \mathcal{P} as follows:

$$\hat{v}_{\mathcal{P}}(t) := \begin{cases} \hat{v}_{\mathcal{P},i} & t \in [t_{\mathcal{P},i}, t_{\mathcal{P},i+1}) \\ \hat{v}_{\mathcal{P},N_{\mathcal{P}}} & t = t_{N_{\mathcal{P}}}. \end{cases} \quad (18)$$

The motivation behind defining the successive elements of a piecewise constant solution using (17b) is that² :

$$\begin{aligned} \frac{\hat{v}_{\mathcal{P},i} + e\hat{v}_{\mathcal{P},i-1}}{1+e} &= \text{proj}_{M_{\mathcal{P},i}} \left[\hat{v}_{\mathcal{P},i-1} - \frac{1}{1+e} M_{\mathcal{P},i}^{-1} G_{\mathcal{P},i}, V_{\mathcal{P},i} \right] \\ \iff M_{\mathcal{P},i}(\hat{v}_{\mathcal{P},i} - \hat{v}_{\mathcal{P},i-1}) + G_{\mathcal{P},i} &= -\mathcal{N}_{V_{\mathcal{P},i}} \left(\frac{\hat{v}_{\mathcal{P},i} + e\hat{v}_{\mathcal{P},i-1}}{1+e} \right) \end{aligned}$$

which is a quite natural discretization of (13).

In the sequel,

- a uniform bound (with respect to \mathcal{P}) is derived on $|\hat{v}_{\mathcal{P}}|$ in Section VI-A, and
- an estimate of the total variation of $\hat{v}_{\mathcal{P}}$ over a compact interval is computed in Section VI-B.

Using these bounds to invoke a generalized version of Helly's first theorem (see Theorem A.2 in Appendix A), there exists a filter \mathcal{F} finer than the filter of sections of \mathcal{P} , and a function of bounded variation $\hat{v} : [0, T] \rightarrow \mathbb{R}^n$ which is the weak pointwise limit of $\hat{v}_{\mathcal{P}}(\cdot)$ with respect to \mathcal{F} . Since we are working in the finite-dimensional setup, $\hat{v}(\cdot)$ is a strong pointwise generalized sublimit of $\hat{v}_{\mathcal{P}}$:

$$\lim_{\mathcal{F}} |\hat{v}(t) - \hat{v}_{\mathcal{P}}(t)| = 0 \quad \forall t \in [0, T]. \quad (19)$$

The next step is to show that $\hat{v}(\cdot)$ obtained above is indeed a solution to system (13). We demonstrate it by showing that

- the differential inclusion (13) holds at continuity points of \hat{v} (Section VI-C), and
- the inclusion (13) is satisfied at discontinuity points of \hat{v} (Section VI-D).

The fact that $\hat{v}_e(t) \in V(q(t))$ follows due to closedness of $V(q(t))$. To complete the proof, it remains to show that the solution to (13) is unique, which basically follows due to convexity of $V(q(t))$ and Lipschitz continuity of $g(t, \cdot)$. To see that, let $\hat{v}^1(\cdot), \hat{v}^2(\cdot)$ be two solutions to (13) with $\hat{v}^1(0) = \hat{v}^2(0)$, then there exists a measure $d\hat{\mu}$ such that

$$M(q(t)) \frac{d\hat{v}^i}{d\hat{\mu}}(t) + g(t, \hat{v}^i) \frac{dt}{d\hat{\mu}}(t) \in -\mathcal{N}_{V(q(t))}(\hat{v}^i), \quad i = 1, 2.$$

²We use the fact that for a convex set V , it holds that $x = \arg \min_{y \in V} |z - y|_M$, that is, x is the projection of z onto V with respect to the norm induced by a positive definite matrix M , if and only if $\langle M(z - x), y - x \rangle \leq 0, \forall y \in V \iff M(z - x) \in \mathcal{N}_V(x)$.

Using the monotone property of the normal cone, we get

$$M(q(t)) \left\langle \frac{d\hat{v}^1}{d\hat{\mu}}(t) - \frac{d\hat{v}^2}{d\hat{\mu}}(t), \hat{v}^1(t) - \hat{v}^2(t) \right\rangle \leq |g(t, \hat{v}^2(t)) - g(t, \hat{v}^1(t))| \cdot |\hat{v}^1(t) - \hat{v}^2(t)|.$$

Since $g(t, \cdot)$ is Lipschitz, and $\hat{v}^1(0) = \hat{v}^2(0)$, the above inequality becomes

$$|\hat{v}_1 - \hat{v}_2|^2 \leq \frac{2C_{g,t}}{\lambda_M} \int_{]0,t]} |\hat{v}_1(s) - \hat{v}_2(s)|^2 d\hat{\mu}(s).$$

One can now invoke the Gronwall-Bellman like lemma for functions of bounded variation [17, Lemma 4], to get

$$|\hat{v}_1 - \hat{v}_2|^2 \leq 0,$$

whence it follows that $\hat{v}^1(t) = \hat{v}^2(t)$, for $t \in [0, T]$. \square

In the remainder of the section, we address the convergence of the estimation error to zero.

B. Error Stability Analysis

In order to arrive at a result on convergence of velocity estimation error, we introduce additional structure on the nonlinear term $F(t, q, v)$ which are natural for Lagrangian dynamical systems. It is assumed that $F(t, q, v) = C(q, v)v + f(t, q, v)$ where $C(q, v)v$ is defined using Christofel symbols and denotes the Coriolis and centrifugal torques. The following properties are satisfied by such systems [26]:

(P1) If $\dot{M}(q, v)$ denotes the derivative of the mass matrix, then $\dot{M}(q, v) - 2C(q, v)$ is a skew-symmetric operator, that is, $\tilde{v}^\top (\dot{M}(q, v) - 2C(q, v)) \tilde{v} = 0$, $\forall \tilde{v} \in \mathbb{R}^n$.

(P2) There exists a constant $C_M(q) > 0$ s.t.

$$\|C(q, v)\| \leq C_M(q)|v|, \quad \forall v \in \mathcal{B}_v. \quad (20)$$

The main result on convergence of error now follows:

Theorem 2. Consider system (7) under hypotheses (H1) - (H4) and assume that the properties **(P1)**, **(P2)**, and Assumption 1 hold. If there exist a constant $\beta > 0$ and a matrix L such that, for all $(t, q) \in \mathbb{R}_+ \times \Phi$,

$$M(q)L + L^\top M(q) \leq -(2C_v C_M(q) + 2C_F(t, q) + \beta)I_n, \quad (21)$$

where $I_n \in \mathbb{R}^{n \times n}$ denotes the identity matrix, then the velocity estimate $\hat{v}(\cdot)$ given by (12) converges to $v(\cdot)$ exponentially, that is, for some $c > 0$,

$$|v(t) - \hat{v}(t)| \leq c e^{-\beta t} |v(0) - \hat{v}(0)|. \quad (22)$$

Proof. Define the state estimation error as follows:

$$\tilde{v} := v - \hat{v}.$$

Then the dynamics for the velocity estimation error are:

$$M(q)d\tilde{v} = M(q)dv - M(q)d\hat{v} \quad (23a)$$

$$= M(q)dv - M(q)dz + M(q)Ldq \quad (23b)$$

$$= M(q)L\tilde{v}dt - (F(t, q, v) - \bar{F}(t, q, \hat{v}))dt - (\eta - \hat{\eta}) \quad (23c)$$

where

$$\eta \in \mathcal{N}_{V(q)}(v_e) \quad \text{and} \quad \hat{\eta} \in \mathcal{N}_{V(q)}(\hat{v}_e)$$

and v_e, \hat{v}_e are defined as in (7c) and (12c), respectively. In what follows, we fix $d\mu = dt + d\mu_a + d\hat{\mu}_a$, where the atomic measure $d\mu_a$ (respectively $d\hat{\mu}_a$) is supported by the time instants at which $v(\cdot)$ (respectively $\hat{v}(\cdot)$) is discontinuous. It is seen that $dt + d\mu_a$, and $dt + d\hat{\mu}_a$ are absolutely continuous with respect to $d\mu$ and hence the densities $\frac{dv}{d\mu}(\cdot)$ and $\frac{d\hat{v}}{d\mu}(\cdot)$ are well-defined on the complement of a $d\mu$ -null set.

Pick $W(q, \tilde{v}) = \tilde{v}^\top M(q)v$, then $W(\cdot)$ is locally *rcbv* using the chain rule [24, Theorem 3], and its differential is computed as follows:

$$\frac{dW}{d\mu}(t) = (\tilde{v}(t^+) + \tilde{v}(t^-))^\top M(q) \frac{d\tilde{v}}{d\mu}(t) + \frac{\partial}{\partial q}(\tilde{v}(t^+)^\top M(q)\tilde{v}(t^+)) \frac{dq}{d\mu}(t). \quad (24)$$

At time t_k , if there is a jump in $v(\cdot)$ or $\hat{v}(\cdot)$, then $\frac{dt}{d\mu}(t_k) = 0$, which also implies that $\frac{dq}{d\mu} = \frac{dq}{dt} \cdot \frac{dt}{d\mu} = 0$, and from (23c), we thus obtain:

$$\frac{dW}{d\mu}(t_k) = (\tilde{v}^+ + \tilde{v}^-)^\top M_k \frac{d\tilde{v}}{d\mu}(t_k) = (\tilde{v}^+ + \tilde{v}^-)^\top M_k (\tilde{v}^+ - \tilde{v}^-),$$

where we used the notation $\tilde{v}^+ := \tilde{v}(t_k^+)$, $\tilde{v}^- := \tilde{v}(t_k^-)$, and $M_k := M(q(t_k))$. We can rewrite the above expression as:

$$\begin{aligned}
\frac{dW}{d\mu}(t_k) &= \tilde{v}^{+\top} M_k \tilde{v}^+ - \tilde{v}^{-\top} M_k \tilde{v}^- + \frac{(1-e)}{(1+e)} (\tilde{v}^+ - \tilde{v}^-)^\top M_k (\tilde{v}^+ - \tilde{v}^-) - \frac{(1-e)}{(1+e)} (\tilde{v}^+ - \tilde{v}^-)^\top M_k (\tilde{v}^+ - \tilde{v}^-) \\
&= \frac{1}{(1+e)} \left[(1+e) (\tilde{v}^{+\top} M_k \tilde{v}^+ - \tilde{v}^{-\top} M_k \tilde{v}^-) + (1-e) (\tilde{v}^+ - \tilde{v}^-)^\top M_k (\tilde{v}^+ - \tilde{v}^-) \right] \\
&\quad - \frac{(1-e)}{(1+e)} (\tilde{v}^+ - \tilde{v}^-)^\top M_k (\tilde{v}^+ - \tilde{v}^-) \\
&= \frac{2}{(1+e)} \left[\tilde{v}^{+\top} M_k \tilde{v}^+ - e \tilde{v}^{-\top} M_k \tilde{v}^- - (1-e) \tilde{v}^{+\top} M_k \tilde{v}^- \right] - \frac{(1-e)}{(1+e)} (\tilde{v}^+ - \tilde{v}^-)^\top M_k (\tilde{v}^+ - \tilde{v}^-) \\
&= \frac{2}{(1+e)} \langle M_k (\tilde{v}^+ - \tilde{v}^-), \tilde{v}^+ + e \tilde{v}^- \rangle - \frac{(1-e)}{(1+e)} (\tilde{v}^+ - \tilde{v}^-)^\top M_k (\tilde{v}^+ - \tilde{v}^-).
\end{aligned}$$

Substituting $M(q(t_k))(\tilde{v}^+ - \tilde{v}^-) = -(\eta - \hat{\eta})$, the above equation becomes

$$\frac{dW}{d\mu}(t_k) = -\frac{2}{(1+e)} \left\langle \eta - \hat{\eta}, \frac{\tilde{v}^+ + e \tilde{v}^-}{1+e} \right\rangle - \frac{(1-e)}{(1+e)} (\tilde{v}^+ - \tilde{v}^-)^\top M_k (\tilde{v}^+ - \tilde{v}^-). \quad (25)$$

By definition (see (3)), it follows that

$$\eta \in -\mathcal{N}_{V(q(t))}(v_e) \iff \langle \eta, v_e - \hat{v}_e \rangle \geq 0 \quad (26)$$

$$\hat{\eta} \in -\mathcal{N}_{V(q(t))}(\hat{v}_e) \iff \langle \hat{\eta}, v_e - \hat{v}_e \rangle \leq 0 \quad (27)$$

which in turn implies that

$$\langle \eta - \hat{\eta}, v_e - \hat{v}_e \rangle \geq 0$$

or equivalently,

$$\left\langle \eta - \hat{\eta}, \frac{\tilde{v}^+ + e \tilde{v}^-}{1+e} \right\rangle \geq 0. \quad (28)$$

Using the inequality (28) in equation (25), we get

$$\frac{dW}{d\mu}(t_k) \leq -\frac{(1-e)}{(1+e)} (\tilde{v}^+ - \tilde{v}^-)^\top M_k (\tilde{v}^+ - \tilde{v}^-) \leq 0. \quad (29)$$

Thus, when $0 \leq e < 1$, we have a strict decrease in the value of Lyapunov function $W(\cdot)$ at jump instants, and $W(\cdot)$ at most remains constant for the case $e = 1$.

If $t \neq t_k$, then $\frac{dt}{d\mu}(t) = 1$, and we let

$$\frac{\partial}{\partial q} (\tilde{v}(t) M(q(t)) \tilde{v}(t)) \frac{dq}{dt}(t) = \tilde{v}(t) \dot{M}(q(t), v(t)) \tilde{v}(t)$$

which results in

$$\begin{aligned} \frac{dW}{d\mu}(t) &= \frac{dW}{dt}(t) = \tilde{v}^\top (M(q)L + L^\top M(q))\tilde{v} - \tilde{v}^\top (\eta - \hat{\eta}) \\ &\quad - 2\tilde{v}^\top (F(t, q, v) - \bar{F}(t, q, \hat{v})) + \tilde{v}^\top \dot{M}(q(t), v(t))\tilde{v} \end{aligned}$$

where \tilde{v} and q are evaluated at time t and the argument has been suppressed for conciseness. In the above expression, $\tilde{v}^\top (\eta - \hat{\eta}) \geq 0$ because $v(t), \hat{v}(t) \in V(q(t))$ for all t , due to which $\langle \eta, v - \hat{v} \rangle \geq 0$, and $\langle \hat{\eta}, v - \hat{v} \rangle \leq 0$. Adding and subtracting the term $2\tilde{v}^\top C(q, v)\tilde{v}$ from the right-hand side, and using the fact that $\dot{M}(q, v) - 2C(q, v)$ is a skew symmetric operator, we obtain

$$\frac{dW}{d\mu}(t) = \tilde{v}^\top (M(q)L + L^\top M(q))\tilde{v} - 2\tilde{v}^\top (F(t, q, v) - \bar{F}(t, q, \hat{v}) - C(q, v)\tilde{v})$$

where we can bound the norm of the last term as follows:

$$\begin{aligned} |F(t, q, v) - \bar{F}(t, q, \hat{v}) - C(q, v)\tilde{v}| &= |F(t, q, v) - \bar{F}(t, q, \hat{v})| + |C(q, v)\tilde{v}| \\ &= C_{F(t, q)}(q)|\tilde{v}| + C_M(q)C_v|\tilde{v}| \end{aligned}$$

It now follows under condition (21) that

$$\frac{dW}{d\mu}(t) \leq -\beta W(t), \quad t \neq t_k. \quad (30)$$

Since we had fixed $d\mu = dt + d\mu_a$, and W is non-increasing at the atoms of $d\mu_a$ because of (29), and decreasing exponential with respect to Lebesgue measure due to (30). This results in

$$W(t) \leq e^{-\beta t} W(0) \Rightarrow |\tilde{v}(t)| \leq \sqrt{\frac{\bar{\lambda}_M}{\underline{\lambda}_M}} e^{-\beta t} |\tilde{v}(0)|.$$

for all $t \geq 0$. □

Remark 1. In the proof of Theorem 2, we analyze the velocity estimation error \tilde{v} using a Lyapunov function $W(q, \tilde{v})$. Computing the density of this function with respect to certain measure, we basically capture its evolution at time instants where \tilde{v} is continuous, and at the jump instants of \tilde{v} . In the literature on hybrid systems, we also find conditions on exponential stability under certain conditions on flow and jump dynamics, such as [37]. Although, our approach is compact, it is also limited in the sense that we have to choose a function (in our case quadratic and hence C^1 differentiable) whose composition with *rcbv* functions is also *rcbv*, so that the chain rule could be applied and the density with respect to the measure under consideration

could be computed. However, the approaches adopted in the literature on hybrid systems often allow relaxed assumptions on the regularity of the Lyapunov function.

C. Passivity Interpretation

Lagrangian systems are basically modeled such that the total energy, that is, the sum of kinetic and potential energy, of the system decreases with the passage of time. The kinetic energy is obtained by the quadratic form of v induced by the symmetric positive definite mass matrix $M(q)$. When dealing with impacts, the kinetic energy actually dissipates at each impact. This allows one to state the dissipativity of Lagrangian systems subjected to unilateral constraints and impacts, see [8, Sections 6.8.2 and 7.2.4].

Inspired by these preliminary results, we picked the quadratic form of the velocity estimation error induced by the mass matrix, and in equation (25), it is shown that at any instant t_k where \tilde{v} jumps, we have

$$\frac{dW}{d\mu}(t_k) \leq -\frac{2}{(1+e)} \langle \eta - \hat{\eta}, \tilde{v}_e(t_k) \rangle.$$

During the continuous flow, we pick the gain L such that the unconstrained smooth part of the error dynamics is actually decreasing, so that

$$\frac{dW}{d\mu}(t) \leq -\langle \eta - \hat{\eta}, \tilde{v}_e(t) \rangle.$$

Next, we invoke the monotone property of the normal cone to convex sets which results in $\langle \eta - \hat{\eta}, \tilde{v}_e(t) \rangle \geq 0$ and asymptotic stability of the error dynamics follows. This last argument could also be interpreted as saying that we have passivity from \tilde{v} to $-(\eta - \hat{\eta})$ with respect to the

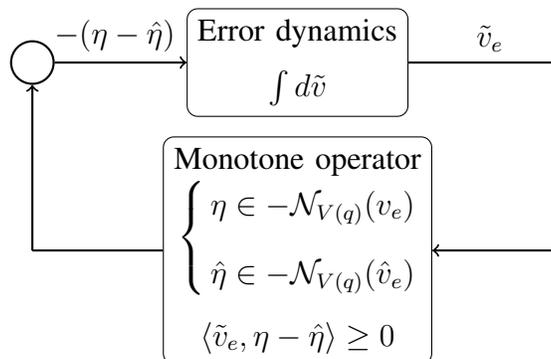


Fig. 1. Interpretation of error dynamics in terms of passivity.

storage function $W(\cdot)$. It is worth mentioning that, in order to deal with discontinuities of v_e , we don't just consider the classical derivative of the storage function, but instead compute the density of dW with respect to $d\mu$. The passive interconnection of the error dynamics is depicted in Figure 1.

We also remark that the condition (21) was introduced explicitly to obtain dissipation of smooth part of the error dynamics with respect to kinetic metric. If there is any other observer for smooth Lagrangian systems for which the error dynamics admit $\tilde{v}^\top M(q)\tilde{v}$ as the Lyapunov function, e.g. in [6], [25] then those designs could be embedded into the formalism of (12) to arrive at a different criteria for error convergence.

V. NUMERICAL IMPLEMENTATION

We now give some remarks on how to execute the measure differential inclusion (12). Since the solutions are allowed to accumulate in finite time, the classical event-driven schemes are not suitable for simulating such systems. We therefore propose a time-stepping scheme to simulate (12), inspired by the Moreau-Jean algorithm [2]. It works as follows: Consider the time interval $[0, T]$ and a sampling time τ_s small enough. With $\hat{v}(0)$ arbitrarily chosen, we want to solve for $\hat{v}(t_{k+1})$, $t_k = k\tau_s$, $k \in \mathbb{N}$, and on the interval $[t_k, t_{k+1})$, \hat{v} is obtained through some interpolation techniques. In the sequel, we discuss how to compute $\hat{v}(t_{k+1})$ by addressing two cases.

Sampling interval without impact: The easier case is when it is known that q does not have any contact with the boundary of the set Φ over the entire interval $(t_k, t_{k+1}]$. In that case, $V(q(t)) = \mathbb{R}^n$, for each $t \in (t_k, t_{k+1}]$ and the multivalued part on the right-hand side of (12) is reduced to $\{0\}$. We then obtain $v(t_{k+1})$ by classical numerical integration algorithms. For example, using a semi-explicit Euler's method, we obtain:

$$\begin{aligned} v(t_{k+1}) &= z(t_{k+1}) - Lq(t_{k+1}) \\ z(t_{k+1}) &= z(t_k) + \tau_s \left(L\hat{v}(t_k) - M(q(t_k))^{-1}\bar{F}(t_k, q(t_k), \hat{v}(t_k)) \right). \end{aligned}$$

Sampling intervals with impact: If, however, the contact between q and Φ is detected during the interval $(t_k, t_{k+1}]$ then we basically reformulate the inclusion as a complementarity relation to determine the post impact value of \hat{v} . In order to state the desired complementarity relation, suppose that the contact happens at some time $\bar{t} \in (t_k, t_{k+1}]$, that is, $h_\alpha(\bar{q}) \leq 0$, for some $\alpha = 1, \dots, m$, where $\bar{q} := q(\bar{t})$. It follows from the inclusion (12) that:

$$\hat{v}(\bar{t}^+) - \hat{v}(\bar{t}^-) \in -M(\bar{q})^{-1}\mathcal{N}_{V(\bar{q})}(\hat{v}_e)$$

and from the definition of $V(\bar{q})$, it follows that

$$\hat{v}(\bar{t}^+) - \hat{v}(\bar{t}^-) = - \sum_{\alpha \in \mathcal{J}(\bar{q})} \lambda_\alpha \frac{M^{-1}(\bar{q}) \nabla h_\alpha(\bar{q})}{|M^{-1/2}(\bar{q}) \nabla h_\alpha(\bar{q})|}, \quad \lambda_\alpha > 0, \alpha \in \mathcal{J}(\bar{q})$$

We are thus interested in finding the value of λ_α , $\alpha \in \mathcal{J}(\bar{q})$. Also, using the relation:

$$M(\bar{q})(\hat{v}(\bar{t}^+) - \hat{v}(\bar{t}^-)) \in -\mathcal{N}_{V(\bar{q})}(\hat{v}_e(\bar{t})) \Leftrightarrow \hat{v}_e(\bar{t}) = \text{proj}_{M(\bar{q})} [\hat{v}(\bar{t}^-); V(\bar{q})],$$

it follows that

$$\hat{v}_e(\bar{t}) \in V(\bar{q}) \quad \text{and} \quad \langle \hat{v}(\bar{t}^+) - \hat{v}(\bar{t}^-), \hat{v}(\bar{t}^+) + e\hat{v}(\bar{t}^-) \rangle_{M(\bar{q})} = 0.$$

The preceding relations are in turn equivalent to solving the following complementarity problem:

$$0 \leq \lambda_\alpha \perp \left\langle \frac{M^{-1}(\bar{q}) \nabla h_\alpha(\bar{q})}{|M^{-1/2}(\bar{q}) \nabla h_\alpha(\bar{q})|}, M(\bar{q})(\hat{v}(\bar{t}^+) + e\hat{v}(\bar{t}^-)) \right\rangle \geq 0, \quad \alpha \in \mathcal{J}(\bar{q}). \quad (31)$$

In theory, solving for λ_α allows us to compute $\hat{v}(\bar{t}^+)$ whenever $h_\alpha(q(\bar{t})) = 0$. Knowing that there might be infinitely many such instances in finite interval, running such an algorithm in practice is not feasible on machines with finite precisions. However, note that the second term in (31) is continuous with respect to the variable q and since q is an absolutely continuous function of time, the value of λ_α will not vary a lot with respect to small variations in the value of $q(\bar{t})$. Hence, when implementing this algorithm on the computer, we replace $q(\bar{t})$ with $q(t_{k+1})$ and compute $\hat{v}(t_{k+1})$. With small enough sampling time, the resulting values of \hat{v} are still accurate enough (see [2, Chapter 14] for numerical tests with various step sizes, in particular [2, Table 14.2], see also [21]). The important thing is to know which constraints were active during a certain sampling interval. Thus, *we don't count the number of times q makes contact with the boundary of Φ during a particular sampling interval, we just count the number of constraints that are active during a particular sampling interval*. Since, there are at most m constraints, the above algorithm is feasible for implementation as shown in the following examples.

A. Simulation Results

As an illustration of our results, we consider two examples.

Example 1. We now apply our result to a bouncing ball which is a one degree of freedom system with a single constraint. We let $m(q) = 1$, $h(q) = q \geq 0$, so that $V(q) = \mathbb{R}$ if $q > 0$, and

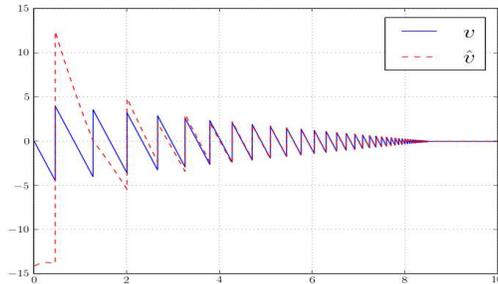


Fig. 2. Simulation of velocity estimator for bouncing ball with $e = 0.9$.

$V(q) = \mathbb{R}_+$ if $q \leq 0$. This leads to

$$\mathcal{N}_{V(q)}(v) = \begin{cases} \{0\}, & \text{if } q > 0, \text{ or } v > 0, \\ \mathbb{R}_-, & \text{if } q \leq 0, \text{ and } v \leq 0. \end{cases}$$

In the form of (7), the dynamics could be described by $dq = v$ and

$$dv + gdt \in -\mathcal{N}_{V(q)}(v_e).$$

If there is a discontinuity at t_k , then $v(t_k^-) < 0$, and the impact occurs when $q = 0$, and at that instant

$$v(t_k^+) - v(t_k^-) \in -\mathcal{N}_{\mathbb{R}_+} \left(\frac{v(t_k^+) + ev(t_k^-)}{1 + e} \right).$$

This is an implicit relation for $v(t_k^+)$, for which the only solution is $v(t_k^+) = -ev(t_k^-)$. The estimated velocity for this system is given by $\hat{v}(t) = z(t) - lq(t)$, where $z(\cdot)$ is obtained from

$$dz + gdt - lzdt + l^2qdt = -\lambda$$

with $l < 0$, and λ computed as follows:

$$\begin{aligned} \lambda &= 0, & \text{if } q > 0 \\ 0 \leq \lambda \perp \hat{v}_e &\geq 0, & \text{if } q = 0. \end{aligned}$$

The proposed observer has been simulated in SICONOS [1] and the simulation results are given in Figure 2. It is seen that the estimated velocity respects the same constraints as imposed on the actual velocity of the system. However, not every impact of the ball with the ground causes a discontinuity in $\hat{v}(\cdot)$.

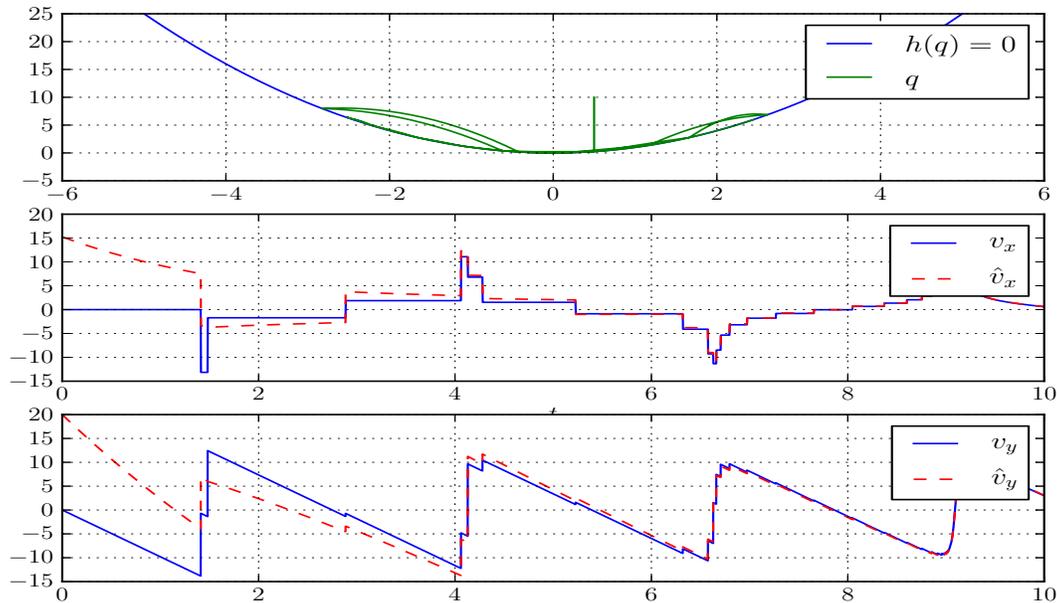


Fig. 3. Velocity estimation for a 2-degree of freedom mechanical system: A ball bouncing inside a parabola. The top plot shows the position of the ball which initially bounces off the boundary and then continues to slide along the boundary of the constraint set $h(q) = 0$. The middle plot and the bottom plot show that the estimate (in red, dashed lines) converges to the actual velocity (blue, solid line) of the system for both coordinates.

Example 2. As a second example, we consider a ball of unit mass bouncing inside a parabola in two-dimensional plane. We denote the position variable by $\begin{pmatrix} q_x \\ q_y \end{pmatrix}$ and the corresponding velocity vector by $\begin{pmatrix} v_x \\ v_y \end{pmatrix}$. The constraint $h(q)$ is defined as:

$$h(q) = q_y - q_x^2 \geq 0.$$

For this constraint, the cone $V(q) = \mathbb{R}^2$, if $h(q) > 0$, else

$$V(q) := \{(v_x, v_y) \in \mathbb{R}^2 \mid v_y - 2q_x v_x \geq 0\}.$$

As an external force, we only consider the gravitation pull in the vertical component of the acceleration, so that the dynamics are:

$$d \begin{pmatrix} v_x \\ v_y \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix} dt = -\mathcal{N}_{V(q)}(v_e).$$

The observer equations for this system are:

$$d \begin{pmatrix} z_x \\ z_y \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix} dt - \begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix} \begin{pmatrix} z_x \\ z_y \end{pmatrix} + \begin{bmatrix} l_1^2 & 0 \\ 0 & l_2^2 \end{bmatrix} \begin{pmatrix} q_x \\ q_y \end{pmatrix} = -\lambda \frac{\nabla h(q)}{|\nabla h(q)|},$$

for some $l_1, l_2 < 0$ appropriately chosen, and λ is computed as follows:

$$\begin{aligned} \lambda &= 0, & \text{if } h(q) > 0 \\ 0 \leq \lambda \perp \left\langle \frac{\nabla h(q)}{|\nabla h(q)|}, \hat{v}_e \right\rangle &\geq 0, & \text{if } h(q) = 0. \end{aligned}$$

The simulation results for that case are shown in Figure 3. It is seen that there is an accumulation of discontinuities in the velocity vector and after that the ball continues to slide along the boundary of the constraint set.

VI. CALCULATIONS FOR THEOREM 1

In this section, we show calculations for the claims made in the proof of Theorem 1.

A. Estimate of a uniform bound on $\hat{v}_{\mathcal{P}}$

It is assumed that $|v(t)| \leq C_v$ for all $t \in [0, T]$, so that $q(t) \in \mathcal{B}(q_0, C_v T)$. Let $C_{M^{1/2}}$ be the Lipschitz constant associated with the mapping $q \mapsto M^{1/2}(q)$ on $\mathcal{B}(q_0, C_v T)$. The projection, with respect to the norm induced by $M(q(t_{\mathcal{P},i}))$, on the set $V(q(t_{\mathcal{P},i}))$ is denoted by $P_{\mathcal{P},i}$ and on the set $M^{-1}(q(t_{\mathcal{P},i}))V^\circ(q(t_{\mathcal{P},i}))$ by $Q_{\mathcal{P},i}$. We denote by $\bar{\lambda}_M, \underline{\lambda}_M$ the constants introduced in (9) for the compact set $\mathcal{B}(q_0, C_v T)$. Let $u_{\mathcal{P},i}$ be defined as:

$$u_{\mathcal{P},i} := \hat{v}_{\mathcal{P},i-1} - \frac{1}{1+e} M_{\mathcal{P},i}^{-1} G_{\mathcal{P},i}. \quad (32)$$

These notations are now used in deriving a bound on $\hat{v}_{\mathcal{P}}$. Using Moreau's two-cone lemma (see Lemma A.1 in Appendix A), we first get:

$$\begin{aligned} |\hat{v}_{\mathcal{P},i}|_{M_{\mathcal{P},i}} &= |P_{\mathcal{P},i}(u_{\mathcal{P},i}) - eQ_{\mathcal{P},i}(u_{\mathcal{P},i}) - \frac{e}{1+e} M_{\mathcal{P},i}^{-1} G_{\mathcal{P},i}|_{M_{\mathcal{P},i}} \\ &\leq |u_{\mathcal{P},i}|_{M_{\mathcal{P},i}} + \frac{e}{1+e} \|M_{\mathcal{P},i}^{-1/2}\| \cdot |G_{\mathcal{P},i}| \\ &\leq |\hat{v}_{\mathcal{P},i-1}|_{M_{\mathcal{P},i}} + \frac{1}{\sqrt{\underline{\lambda}_M}} |G_{\mathcal{P},i}| \\ &\leq |\hat{v}_{\mathcal{P},i-1}|_{M_{\mathcal{P},i-1}} + \|M_{\mathcal{P},i}^{1/2} - M_{\mathcal{P},i-1}^{1/2}\| \cdot |\hat{v}_{\mathcal{P},i-1}| + \frac{1}{\sqrt{\underline{\lambda}_M}} |G_{\mathcal{P},i}| \\ &\leq |\hat{v}_{\mathcal{P},i-1}|_{M_{\mathcal{P},i-1}} + C_{M^{1/2}} \int_{t_{\mathcal{P},i-1}}^{t_{\mathcal{P},i}} |v(s)| ds \cdot |\hat{v}_{\mathcal{P},i-1}| + \frac{1}{\sqrt{\underline{\lambda}_M}} |G_{\mathcal{P},i}| \end{aligned}$$

Since $\hat{v}_{\mathcal{P}}(\cdot)$ has the constant value $\hat{v}_{\mathcal{P},i}$ on the interval $[t_{\mathcal{P},i}, t_{\mathcal{P},i+1})$, the above inequality results in:

$$\begin{aligned} \sqrt{\underline{\lambda}_M} |\hat{v}_{\mathcal{P}}(t_{\mathcal{P},i})| - \sqrt{\underline{\lambda}_M} |\hat{v}_{\mathcal{P}}(0)| &\leq \sum_{j=1}^i |\hat{v}_{\mathcal{P},i}|_{M_{\mathcal{P},i}} - |\hat{v}_{\mathcal{P},i-1}|_{M_{\mathcal{P},i-1}} \\ &\leq C_{M^{1/2}} C_v \int_0^{t_{\mathcal{P},i}} |\hat{v}_{\mathcal{P}}(s)| ds + \frac{C_{g,b}}{\sqrt{\underline{\lambda}_M}} \int_0^{t_{\mathcal{P},i}} (1 + |\hat{v}_{\mathcal{P}}(s)|) ds \end{aligned}$$

that is, for each $t \in [0, T]$,

$$|\hat{v}_{\mathcal{P}}(t)| \leq \sqrt{\frac{\bar{\lambda}_M}{\underline{\lambda}_M}} |\hat{v}_0| + \frac{C_{g,b}}{\underline{\lambda}_M} t + \left(\frac{C_{M^{1/2}} C_v}{\sqrt{\underline{\lambda}_M}} + \frac{C_{g,b}}{\underline{\lambda}_M} \right) \int_0^t |\hat{v}_{\mathcal{P}}(s)| ds.$$

Using the Gronwall-Bellman inequality for functions of bounded variation [17, Lemma 4], we get the following bound on $|\hat{v}_{\mathcal{P}}(t)|$, for each $t \in [0, T]$, which is independent of the partition \mathcal{P} :

$$|\hat{v}_{\mathcal{P}}(t)| \leq \sqrt{\frac{\bar{\lambda}_M}{\underline{\lambda}_M}} |\hat{v}_0| + \frac{C_{g,b}}{\underline{\lambda}_M} t + \left(\sqrt{\frac{\bar{\lambda}_M}{\underline{\lambda}_M}} |\hat{v}_0| + \frac{C_{g,b}}{C_{M^{1/2}} C_v \sqrt{\underline{\lambda}_M} + C_{g,b}} \right) \exp \left(\left(\frac{C_{M^{1/2}} C_v}{\sqrt{\underline{\lambda}_M}} + \frac{C_{g,b}}{\underline{\lambda}_M} \right) t \right) \leq C_{sup} \quad (33)$$

where C_{sup} is obtained by evaluating the right-hand side of the first inequality at $t = T$.

B. Estimates on the variation

For a fixed partition \mathcal{P} of the interval $[0, T]$, we now compute the total variation of $\hat{v}_{\mathcal{P}}(\cdot)$. For conciseness, we drop the subscript \mathcal{P} in the quantities appearing in (17) and (18). By definition, we have

$$\hat{v}_i = -e\hat{v}_{i-1} + (1+e)P_i(u_i).$$

Using Moreau's two cone lemma, we can write $u_i := P_i(u_i) + Q_i(u_i)$, so that

$$\begin{aligned} \hat{v}_i - \hat{v}_{i-1} &= -(1+e)u_i + (1+e)P_i(u_i) - M_i^{-1}G_i \\ &= -(1+e)Q_i(u_i) - M_i^{-1}G_i \end{aligned} \quad (34)$$

Since $Q_i(\cdot)$ denotes the projection on $V_i^* := M_i^{-1}V^\circ(q_i)$ with respect to kinetic metric, we take $Q_i(u) = 0$ if $\mathcal{J}(q_i) = \emptyset$, in which case

$$|\hat{v}_i - \hat{v}_{i-1}| = |M_i^{-1}G_i| \leq \frac{|G_i|}{\underline{\lambda}_M}$$

Otherwise, if $\mathcal{J}(q_i) \neq \emptyset$, we have

$$Q_i(u) := \sum_{\alpha \in \mathcal{J}(q_i)} \langle u, \nabla h_\alpha(q_i) \rangle^- M_i^{-1} H_\alpha(q_i) \quad \text{where} \quad H_\alpha(q_i) := \frac{\nabla h_\alpha(q_i)}{\nabla h_\alpha^\top(q_i) M_i^{-1} \nabla h_\alpha(q_i)}$$

and $\langle u, \nabla h_\alpha(q) \rangle^- := \min\{\langle u, \nabla h_\alpha(q) \rangle, 0\}$. One may rewrite (34) as

$$\hat{v}_i - \hat{v}_{i-1} = -(1+e)(Q_i(u_i) - Q_i(\hat{v}_{i-1})) - (1+e)\left(Q_i(\hat{v}_{i-1}) - \tilde{Q}_{i-1}(\hat{v}_{i-1})\right) - (1+e)\tilde{Q}_{i-1}(\hat{v}_{i-1}) - M_i^{-1}G_i \quad (35)$$

where

$$\tilde{Q}_{i-1}(u) := \sum_{\alpha \in \mathcal{J}(q_i)} \langle u, \nabla h_\alpha(q_{i-1}) \rangle^- M_{i-1}^{-1} H_\alpha(q_{i-1})$$

We now compute an upper bound on the norm of the right-hand side of (35).

First term: It is noted using the contraction property of the projection map and the bound derived in (33) that

$$|Q_i(u_i) - Q_i(\hat{v}_{i-1})| \leq \frac{1}{\sqrt{\lambda_M}} |u_i - \hat{v}_{i-1}|_i \leq \frac{1}{(1+e)\sqrt{\lambda_M}} |M_i^{-1}G_i|_i \leq \frac{1}{1+e} \frac{|G_i|}{\lambda_M} \quad (36)$$

Second term: Under the hypothesis that $\nabla h_\alpha(\cdot)$ is locally Lipschitz continuous, for each $\alpha = 1, \dots, m$, and that $q(\cdot)$ evolves within a compact set over the interval $[0, T]$, there exists a constant C_h such that

$$|\nabla h_\alpha(q_i) - \nabla h_\alpha(q_{i-1})| \leq C_h |q_i - q_{i-1}| \leq C_h C_v |t_i - t_{i-1}|, \quad \forall t_i, t_{i-1} \in [0, T]. \quad (37)$$

Similarly, since $M^{-1/2}$ is locally Lipschitz continuous, there exists $C_H > 0$ such that

$$|H_\alpha(q_i) - H_\alpha(q_{i-1})| \leq C_H |q_i - q_{i-1}| \leq C_H C_v (t_i - t_{i-1}), \quad \forall t_i, t_{i-1} \in [0, T]. \quad (38)$$

Thus, we get

$$\begin{aligned} Q_i(\hat{v}_{i-1}) - \tilde{Q}_{i-1}(\hat{v}_{i-1}) &= \sum_{\alpha \in \mathcal{J}(q_i)} \langle \hat{v}_{i-1}, \nabla h_\alpha(q_i) \rangle^- M_i^{-1} H_\alpha(q_i) - \sum_{\alpha \in \mathcal{J}(q_i)} \langle \hat{v}_{i-1}, \nabla h_\alpha(q_{i-1}) \rangle^- M_{i-1}^{-1} H_\alpha(q_{i-1}) \\ &= \sum_{\alpha \in \mathcal{J}(q_i)} \left(\langle \hat{v}_{i-1}, \nabla h_\alpha(q_i) \rangle^- - \langle \hat{v}_{i-1}, \nabla h_\alpha(q_{i-1}) \rangle^- \right) M_i^{-1} H_\alpha(q_i) \\ &\quad + \sum_{\alpha \in \mathcal{J}(q_i)} \langle \hat{v}_{i-1}, \nabla h_\alpha(q_{i-1}) \rangle^- \left[(M_i^{-1} - M_{i-1}^{-1}) H_\alpha(q_i) + M_{i-1}^{-1} (H_\alpha(q_i) - H_\alpha(q_{i-1})) \right] \end{aligned}$$

This further leads to

$$\begin{aligned} |Q_i(\hat{v}_{i-1}) - \tilde{Q}_{i-1}(\hat{v}_{i-1})| &\leq \sum_{\alpha \in \mathcal{J}(q_i)} \left| \langle \hat{v}_{i-1}, \nabla h_\alpha(q_i) \rangle^- - \langle \hat{v}_{i-1}, \nabla h_\alpha(q_{i-1}) \rangle^- \right| \cdot |M_i^{-1} H_\alpha(q_i)| \\ &\quad + \sum_{\alpha \in \mathcal{J}(q_i)} \left| \langle \hat{v}_{i-1}, \nabla h_\alpha(q_{i-1}) \rangle^- \right| \cdot \left(\|M_i^{-1} - M_{i-1}^{-1}\| \cdot |H_\alpha(q_i)| + \|M_{i-1}^{-1}\| \cdot |H_\alpha(q_i) - H_\alpha(q_{i-1})| \right) \\ &\leq \frac{m}{\lambda_M} C_{sup} C_h C_v \tilde{C}_{H_\alpha} (t_i - t_{i-1}) + m C_{sup} \tilde{C}_{h_\alpha} (C_{M^{-1}} \tilde{C}_{H_\alpha} + C_H) C_v (t_i - t_{i-1}) \\ &=: \tilde{C}_{proj} (t_i - t_{i-1}) \end{aligned} \quad (39)$$

where $\tilde{C}_{h_\alpha} = \sup_{q \in \mathcal{B}(q_0, C_v T)} |\nabla h_\alpha(q)|$ and $\tilde{C}_{H_\alpha} := \sqrt{\frac{\lambda_M}{\lambda_M}} C_{h_\alpha} \geq \sup_{q \in \mathcal{B}(q_0, C_v T)} |H_\alpha|$. The constants C_v, C_{sup}, C_h , and C_H were introduced in (11), (33), (37), and (38), respectively.

Third term: We have

$$\hat{v}_i = -e\hat{v}_{i-1} + (1+e)P_i(u_i) = P_i(u_i) - eQ_i(u_i) - \frac{e}{1+e}M_i^{-1}G_i$$

which gives

$$\hat{v}_i + \frac{e}{1+e}M_i^{-1}G_i = P_i(u_i) - eQ_i(u_i) \in V(q_i). \quad (40)$$

This further leads to

$$|\tilde{Q}_{i-1}(\hat{v}_{i-1})| \leq \sum_{\alpha \in \mathcal{J}(q_i)} \left| \left\langle \hat{v}_{i-1} + \frac{e}{1+e}M_{i-1}^{-1}G_{i-1} - \frac{e}{1+e}M_{i-1}^{-1}G_{i-1}, \nabla h_\alpha(q_{i-1}) \right\rangle \right| |M_{i-1}^{-1}H_\alpha(q_{i-1})|.$$

From (40), we have

$$\left\langle \hat{v}_{i-1} + \frac{e}{1+e}M_{i-1}^{-1}G_{i-1}, \nabla h_\alpha(q_{i-1}) \right\rangle \geq 0, \quad \forall j = 1, \dots, m.$$

and hence³

$$\begin{aligned} |\tilde{Q}_{i-1}(\hat{v}_{i-1})| &\leq \sum_{\alpha \in \mathcal{J}(q_i)} \frac{e}{1+e} |\langle M_{i-1}^{-1}G_{i-1}, \nabla h_\alpha(q_{i-1}) \rangle| \cdot \frac{|M_{i-1}^{-1}\nabla h_\alpha(q_{i-1})|_{i-1}}{\sqrt{\lambda_M} \nabla h_\alpha^\top(q_{i-1}) M_{i-1}^{-1} \nabla h_\alpha(q_{i-1})} \\ &\leq \frac{e}{1+e} \sum_{\alpha \in \mathcal{J}(q_i)} |M_{i-1}^{-1}G_{i-1}|_{i-1} \cdot |M_{i-1}^{-1}\nabla h_\alpha(q_{i-1})|_{i-1} \cdot \frac{1}{\sqrt{\lambda_M} |M_{i-1}^{-1}\nabla h_\alpha(q_{i-1})|_{i-1}} \\ &\leq \frac{me}{1+e} \frac{|G_{i-1}|}{\lambda_M} \end{aligned} \quad (41)$$

Plugging the bounds from (36), (39), and (41) into (35), we obtain

$$|\hat{v}_i - \hat{v}_{i-1}| \leq \frac{1}{\lambda_M} |G_i| + C_{proj}(t_i - t_{i-1}) + \frac{1}{\lambda_M} |G_i| + \frac{m}{\lambda_M} |G_{i-1}|$$

Using the norm estimate on $\hat{v}_{\mathcal{P}}(\cdot)$, we have $|G_i| \leq (1 + C_{sup})(t_i - t_{i-1})$ and thus for $0 \leq s < t \leq T$, it follows that

$$\text{Var}(\hat{v}_{\mathcal{P}}; [s, t]) \leq C_{var}(t - s) \quad (42)$$

where $C_{var} := \frac{1}{\lambda_M} ((m+2)(1 + C_{sup}) + C_{proj})$.

³We use the fact that for $a, b, c \in \mathbb{R}^n$, satisfying $\langle a, c \rangle \geq 0$, we have $|\langle a+b, c \rangle^-| = |\min\{0, \langle a+b, c \rangle\}| \leq |\min\{0, \langle a, c \rangle\}| + |\min\{0, \langle b, c \rangle\}| = |\langle b, c \rangle|$.

C. Continuity points of the limit function

Assume that $y \in V(q(\tau))$ for all $\tau \in [s, t] \subseteq [0, T]$, then it is claimed that

$$\int_s^t \langle g(\sigma, \hat{v}), (y - \hat{v}) \rangle + \left\langle \dot{M}(\sigma) \hat{v}, \left(y - \frac{\hat{v}}{2} \right) \right\rangle d\sigma \leq \langle M(q(t)) \hat{v}(t) - M(q(s)) \hat{v}(s), y \rangle - \frac{1}{2} (|\hat{v}(t)|_{M(q(t))}^2 - |\hat{v}(s)|_{M(q(s))}^2) \quad (43)$$

In the sequel, we proceed to prove this claim:

Consider a partition \mathcal{P} of the interval $[0, T]$ that contains the nodes $t_{\mathcal{P},j} = s$ and $t_{\mathcal{P},k} = t$ for some $j, k \in \mathbb{N}$. From the discretization scheme (17), for $j+1 \leq i \leq k$, we have

$$\frac{\hat{v}_{\mathcal{P},i} + e\hat{v}_{\mathcal{P},i-1}}{1+e} = P_{\mathcal{P},i} \left(\hat{v}_{\mathcal{P},i-1} - \frac{1}{1+e} M_{\mathcal{P},i}^{-1} G_{\mathcal{P},i} \right).$$

Using the definition of the projection operator, we get

$$\langle -M_{\mathcal{P},i}^{-1} G_{\mathcal{P},i} + \hat{v}_{\mathcal{P},i-1} - \hat{v}_{\mathcal{P},i}, (1+e)y - (\hat{v}_{\mathcal{P},i} + e\hat{v}_{\mathcal{P},i-1}) \rangle_{M_{\mathcal{P},i}} \leq 0, \quad \forall y \in V_{\mathcal{P},i}.$$

The above inequality is equivalently written as:

$$\begin{aligned} & \langle -M_{\mathcal{P},i}^{-1} G_{\mathcal{P},i}, (1+e)y \rangle_{M_{\mathcal{P},i}} + \langle M_{\mathcal{P},i}^{-1} G_{\mathcal{P},i}, \hat{v}_{\mathcal{P},i} + e\hat{v}_{\mathcal{P},i-1} \rangle_{M_{\mathcal{P},i}} \\ & \leq (1+e) \langle \hat{v}_{\mathcal{P},i} - \hat{v}_{\mathcal{P},i-1}, y \rangle_{M_{\mathcal{P},i}} - \langle \hat{v}_{\mathcal{P},i} - \hat{v}_{\mathcal{P},i-1}, \hat{v}_{\mathcal{P},i} + e\hat{v}_{\mathcal{P},i-1} \rangle_{M_{\mathcal{P},i}} \end{aligned} \quad (44a)$$

$$\begin{aligned} & \leq (1+e) \left[\langle M_{\mathcal{P},i} \hat{v}_{\mathcal{P},i} - M_{\mathcal{P},i-1} \hat{v}_{\mathcal{P},i-1}, y \rangle - \frac{1}{2} (|\hat{v}_{\mathcal{P},i}|_{M_{\mathcal{P},i}}^2 - |\hat{v}_{\mathcal{P},i-1}|_{M_{\mathcal{P},i-1}}^2) \right] \\ & \quad - (1+e) \left\langle (M_{\mathcal{P},i} - M_{\mathcal{P},i-1}) \hat{v}_{\mathcal{P},i-1}, y - \frac{\hat{v}_{\mathcal{P},i-1}}{2} \right\rangle. \end{aligned} \quad (44b)$$

To arrive at (44b), the last term in (44a) is rewritten as:

$$\begin{aligned} \langle \hat{v}_{\mathcal{P},i} - \hat{v}_{\mathcal{P},i-1}, \hat{v}_{\mathcal{P},i} \rangle_{M_{\mathcal{P},i}} &= \frac{1}{2} \left[|\hat{v}_{\mathcal{P},i} - \hat{v}_{\mathcal{P},i-1}|_{M_{\mathcal{P},i}}^2 + |\hat{v}_{\mathcal{P},i}|_{M_{\mathcal{P},i}}^2 - |\hat{v}_{\mathcal{P},i-1}|_{M_{\mathcal{P},i-1}}^2 \right. \\ & \quad \left. - \hat{v}_{\mathcal{P},i-1}^\top (M_{\mathcal{P},i} - M_{\mathcal{P},i-1}) \hat{v}_{\mathcal{P},i-1} \right] \end{aligned} \quad (45)$$

and

$$\begin{aligned} \langle \hat{v}_{\mathcal{P},i} - \hat{v}_{\mathcal{P},i-1}, e\hat{v}_{\mathcal{P},i-1} \rangle_{M_{\mathcal{P},i}} &= \frac{e}{2} \left[-|\hat{v}_{\mathcal{P},i} - \hat{v}_{\mathcal{P},i-1}|_{M_{\mathcal{P},i}}^2 + |\hat{v}_{\mathcal{P},i}|_{M_{\mathcal{P},i}}^2 - |\hat{v}_{\mathcal{P},i-1}|_{M_{\mathcal{P},i-1}}^2 \right. \\ & \quad \left. - \hat{v}_{\mathcal{P},i-1}^\top (M_{\mathcal{P},i} - M_{\mathcal{P},i-1}) \hat{v}_{\mathcal{P},i-1} \right]. \end{aligned}$$

Inequality (44b) now leads to:

$$\begin{aligned} \left\langle -G_{\mathcal{P},i}, y - \frac{\hat{v}_{\mathcal{P},i} + e\hat{v}_{\mathcal{P},i-1}}{1+e} \right\rangle + \left\langle (M_{\mathcal{P},i} - M_{\mathcal{P},i-1})\hat{v}_{\mathcal{P},i-1}, y - \frac{\hat{v}_{\mathcal{P},i-1}}{2} \right\rangle \leq \\ \langle M_{\mathcal{P},i}\hat{v}_{\mathcal{P},i} - M_{\mathcal{P},i-1}\hat{v}_{\mathcal{P},i-1}, y \rangle - \frac{1}{2}(|\hat{v}_{\mathcal{P},i}|_{M_{\mathcal{P},i}}^2 - |\hat{v}_{\mathcal{P},i-1}|_{M_{\mathcal{P},i-1}}^2) \end{aligned}$$

which further yields

$$\begin{aligned} \sum_{i=j+1}^k \left\langle -G_{\mathcal{P},i}, y - \frac{\hat{v}_{\mathcal{P},i} + e\hat{v}_{\mathcal{P},i-1}}{1+e} \right\rangle + \left\langle (M_{\mathcal{P},i} - M_{\mathcal{P},i-1})\hat{v}_{\mathcal{P},i-1}, y - \frac{\hat{v}_{\mathcal{P},i-1}}{2} \right\rangle \leq \\ \langle M_{\mathcal{P},k}\hat{v}_{\mathcal{P},k} - M_{\mathcal{P},j}\hat{v}_{\mathcal{P},j}, y \rangle - \frac{1}{2}(|\hat{v}_{\mathcal{P},k}|_{M_{\mathcal{P},k}}^2 - |\hat{v}_{\mathcal{P},j}|_{M_{\mathcal{P},j}}^2). \quad (46) \end{aligned}$$

Using the fact that $\frac{\hat{v}_{\mathcal{P}}(\tau) + e\hat{v}_{\mathcal{P}}(\tau)}{1+e}$ converges to $\hat{v}(\tau)$ for Lebesgue almost all $\tau \in [0, T]$, it follows that

$$\sum_{i=j+1}^k \left\langle G_{\mathcal{P},i}, y - \frac{\hat{v}_{\mathcal{P},i} + e\hat{v}_{\mathcal{P},i-1}}{1+e} \right\rangle \rightarrow \int_s^t \langle g(\tau, \hat{v}(\tau)), y - \hat{v}(\tau) \rangle d\tau. \quad (47)$$

Since $t \mapsto M(q(t))$ is an absolutely continuous function, $\dot{M}(\tau) := \frac{d}{dt}M(q(t))|_{t=\tau}$ exists for Lebesgue almost-all τ , and we have

$$\left\langle (M_{\mathcal{P},i} - M_{\mathcal{P},i-1})\hat{v}_{\mathcal{P},i-1}, y - \frac{\hat{v}_{\mathcal{P},i-1}}{2} \right\rangle = \int_{t_{\mathcal{P},i-1}}^{t_{\mathcal{P},i}} \left\langle \dot{M}(\tau)\hat{v}_{\mathcal{P}}(\tau), y - \frac{\hat{v}_{\mathcal{P}}(\tau)}{2} \right\rangle d\tau,$$

and

$$\sum_{i=j+1}^k \left\langle (M_{\mathcal{P},i} - M_{\mathcal{P},i-1})\hat{v}_{\mathcal{P},i-1}, y - \frac{\hat{v}_{\mathcal{P},i-1}}{2} \right\rangle = \int_{t_{\mathcal{P},j}}^{t_{\mathcal{P},k}} \left\langle \dot{M}(\tau)\hat{v}_{\mathcal{P}}(\tau), y - \frac{\hat{v}_{\mathcal{P}}(\tau)}{2} \right\rangle d\tau. \quad (48)$$

For the terms on the right-hand side, we have the following convergence:

$$\begin{aligned} \langle M_{\mathcal{P},k}\hat{v}_{\mathcal{P},k} - M_{\mathcal{P},j}\hat{v}_{\mathcal{P},j}, y \rangle - \frac{1}{2}(|\hat{v}_{\mathcal{P},k}|_{M_{\mathcal{P},k}}^2 - |\hat{v}_{\mathcal{P},j}|_{M_{\mathcal{P},j}}^2) \rightarrow \langle M(q(t))\hat{v}(t) - M(q(s))\hat{v}(s), y \rangle \\ - \frac{1}{2}(|\hat{v}(t)|_{M(q(t))}^2 - |\hat{v}(s)|_{M(q(s))}^2). \quad (49) \end{aligned}$$

The desired inequality (43) now follows by taking the limit in (46) along all partitions finer than \mathcal{P} and using (47), (48) and (49).

Let μ be the measure defined by $d\mu = |d\hat{v}| + dt$. Since $d\hat{v}$ and dt are absolutely continuous with respect to $d\mu$ there exists a $d\mu$ negligible set A such that, for all $t \in [0, T] \setminus A$:

$$\frac{dt}{d\mu}(t) = \lim_{\varepsilon \rightarrow 0^+} \frac{dt([t, t + \varepsilon])}{d\mu([t, t + \varepsilon])} \quad \text{and} \quad \frac{d\hat{v}}{d\mu}(t) = \lim_{\varepsilon \rightarrow 0^+} \frac{d\hat{v}([t, t + \varepsilon])}{d\mu([t, t + \varepsilon])}.$$

Assume that \hat{v} is continuous at t and let $y \in \text{int } V(q(t))$. Then due to lower semicontinuity of $q \mapsto V(q)$ and absolute continuity of $t \mapsto q(t)$, $y \in V(q(\tau))$ for all $\tau \in I_\varepsilon := [t, t + \varepsilon]$. Due to the variational inequality (43), we get

$$\int_t^{t+\varepsilon} \langle g(\sigma, \hat{v}), (y - \hat{v}) \rangle + \left\langle \dot{M}(\sigma) \hat{v}(\sigma), \left(y - \frac{\hat{v}}{2} \right) \right\rangle d\sigma \leq \\ \langle M(q(t + \varepsilon)) \hat{v}(t + \varepsilon) - M(q(t)) \hat{v}(t), y \rangle - \frac{1}{2} \left(|\hat{v}(t + \varepsilon)|_{M(q(t+\varepsilon))}^2 - |\hat{v}(t)|_{M(q(t))}^2 \right) \quad (50)$$

Divide both sides by $d\mu([t, t + \varepsilon])$. When $\varepsilon \rightarrow 0$, the left-hand side of (50) converges to

$$\frac{dt}{d\mu} \left[\langle g(t, \hat{v}(t)), y - \hat{v}(t) \rangle + \left\langle \dot{M}(t) \hat{v}(t), y - \frac{1}{2} \hat{v}(t) \right\rangle \right].$$

The first term on the right-hand side of (50) becomes

$$\frac{1}{d\mu(I_\varepsilon)} \langle M(q(t + \varepsilon)) \hat{v}(t + \varepsilon) - M(q(t)) \hat{v}(t), y \rangle = \frac{\langle M(q(t)) d\hat{v}(I_\varepsilon), y \rangle}{d\mu(I_\varepsilon)} + \frac{1}{d\mu(I_\varepsilon)} \left\langle \int_t^{t+\varepsilon} \dot{M}(s) ds \hat{v}(t + \varepsilon), y \right\rangle \\ \xrightarrow{\varepsilon \rightarrow 0} \left\langle M(q(t)) \frac{d\hat{v}}{d\mu}(t), y \right\rangle + \left\langle \dot{M}(t) \hat{v}(t), y \right\rangle \frac{dt}{d\mu},$$

and the second term on the right-hand side of (50) becomes

$$\frac{1}{2d\mu(I_\varepsilon)} \left[\langle M(q(t + \varepsilon)) \hat{v}(t + \varepsilon), d\hat{v}(I_\varepsilon) \rangle + \langle M(q(t)) d\hat{v}(I_\varepsilon), \hat{v}(t) \rangle + \left\langle \int_t^{t+\varepsilon} \dot{M}(s) ds \hat{v}(t + \varepsilon), \hat{v}(t) \right\rangle \right] \\ \xrightarrow{\varepsilon \rightarrow 0} \left\langle M(q(t)) \frac{d\hat{v}}{d\mu}(t), \hat{v}(t) \right\rangle + \left\langle \dot{M}(t) \hat{v}(t), \frac{\hat{v}(t)}{2} \right\rangle \frac{dt}{d\mu}.$$

Thus, in the limit (50) leads to

$$\left\langle M(q(t)) \frac{d\hat{v}}{d\mu}(t) + g(t, \hat{v}(t)) \frac{dt}{d\mu}(t), y - \hat{v}(t) \right\rangle \leq 0, \quad \forall y \in \text{int } V(q(t)).$$

From the definition of $\mathcal{N}_{V(q(t))}(\cdot)$, and using the density argument, it follows that $\hat{v}(\cdot)$ satisfies the differential inclusion (13) at the continuity points of $\hat{v}(\cdot)$.

D. Impact characterization of the limit solution

Let $t_k \in [0, T]$ be the time instant at which $V(q(t_k)) \neq \mathbb{R}^n$. If $\hat{v}(t_k^-) \in V(q(t_k))$, then $\hat{v}(t_k^+) = \hat{v}(t_k^-)$. This is a straightforward consequence of the variational inequality (43). Indeed, let $y \in \text{int } V(q(t_k))$ then $y \in V(q(t))$ for all $t \in [t_k - \delta, t_k + \delta]$ and some $\delta > 0$. Applying the inequality (43) with $s = t_k - \delta$, $t = t_k + \delta$, and letting $\delta \rightarrow 0$, we get

$$\langle M(q(t_k)) (\hat{v}(t_k^+) - \hat{v}(t_k^-)), y \rangle - \frac{1}{2} \left(|\hat{v}(t_k^+)|_{M(q(t_k))} - |\hat{v}(t_k^-)|_{M(q(t_k))} \right) \geq 0.$$

By density, the same inequality holds for all $y \in \text{int } V(q(t_k))$. Picking $y = \hat{v}(t_k^-)$ gives

$$|\hat{v}(t_k^+) - \hat{v}(t_k^-)|^2 \leq 0 \quad \Rightarrow \quad \hat{v}(t_k^+) = \hat{v}(t_k^-).$$

Next, consider the case where $\hat{v}(t_k^-) \notin V(q(t_k))$. We use the shorthand notation $\hat{v}^+ := \hat{v}(t_k^+)$ and $\hat{v}^- := \hat{v}(t_k^-)$, and show that the following impact law holds:

$$\hat{v}^+ = -e\hat{v}^- + (1+e) \text{proj}_{M(q(t_k))}(\hat{v}^-, V(q(t_k))).$$

Define $\tilde{u}_k := -e\hat{v}(t_k^-) + (1+e) \text{proj}_{M(q(t_k))}(\hat{v}(t_k^-), V(q(t_k)))$. Consider the partition \mathcal{P} that contains the node $t_{\mathcal{P}, k_d} = t_k$ for some $k_d \in \mathbb{R}^n$. By definition,

$$\begin{aligned} |\hat{v}_{\mathcal{P}}(t_k) - \tilde{u}| &= e|\hat{v}_{\mathcal{P}, k_d-1} - v^-| \\ &\quad + (1+e) \left| \text{proj}_{M_{\mathcal{P}, k_d}} \left[\hat{v}_{\mathcal{P}, k_d-1} - \frac{1}{1+e} M_{\mathcal{P}, k_d}^{-1} G_{\mathcal{P}, k_d}; V(q(t_k)) \right] - \text{proj}_{M_{\mathcal{P}, k_d}} [\hat{v}^-; V(q(t_k))] \right| \\ &\leq e|\hat{v}_{\mathcal{P}, k_d-1} - v^-| + (1+e) \sqrt{\lambda_M} \left| \hat{v}_{\mathcal{P}, k_d-1} - \frac{1}{1+e} M_{\mathcal{P}, k_d}^{-1} G_{\mathcal{P}, k_d} - v^- \right| \\ &\leq (\sqrt{\lambda_M} + e) |\hat{v}_{\mathcal{P}, k_d-1} - v^-| + \frac{\sqrt{\lambda_M}}{\lambda_M} |G_{\mathcal{P}, k_d}| \end{aligned} \quad (51)$$

The pointwise convergence (19) implies the existence of some filter \mathcal{F} such that, for all $\mathcal{P} \in \mathcal{F}$, we have $t', t_k \in \mathcal{P}$ where t' is such that $|\hat{v}(t') - \hat{v}(t_k^-)| < \frac{\varepsilon}{3}$, $|\hat{v}_{\mathcal{P}}(t') - \hat{v}(t')| < \frac{\varepsilon}{3}$ and $t_k - t' < \frac{\varepsilon}{3C_{var}}$; then

$$\begin{aligned} |\hat{v}_{\mathcal{P}, k_d-1} - \hat{v}(t_k^-)| &\leq |\hat{v}_{\mathcal{P}, k_d-1} - \hat{v}_{\mathcal{P}}(t')| + |\hat{v}_{\mathcal{P}}(t') - \hat{v}(t')| + |\hat{v}(t') - \hat{v}(t_k^-)| \\ &\leq \text{Var}(\hat{v}_{\mathcal{P}}; [t', t_k]) + \frac{2}{3}\varepsilon \leq C_{var}(t_k - t') + \frac{2}{3}\varepsilon < \varepsilon. \end{aligned} \quad (52)$$

Substituting (52) in (51), and taking the limit, we obtain

$$|\hat{v}(t_k) - \tilde{u}_k| < \varepsilon$$

for every $\varepsilon > 0$, whence the desired result follows.

VII. CONCLUSIONS

The problem of designing asymptotically convergent state estimators for nonsmooth mechanical systems was considered in this paper. The formalism of differential inclusions was used for both the system and observer dynamics. For the later, we proved the existence and uniqueness of

solutions. The error analysis (for the convergence of velocity estimate) was based on generalizing the Lyapunov techniques to functions of locally bounded variation.

Several directions of research could stem from the current work. As a first extension, we would like to generalize our result for more complex domains which are not necessarily characterized by the sublevel sets of continuously differentiable functions. Afterwards, we would also like to study the performance of the proposed state estimate in designing controllers that only use the information of the output and not the full state.

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APPENDIX

Lemma A.1 (Moreau's Two-Cone Lemma). *If V and V° denote a pair of mutually polar closed convex cones of a Euclidean linear space \mathbb{R}^n , then the following statements are equivalent for $x, y, z \in \mathbb{R}^n$,*

- $x = \text{proj}(z; V)$ and $y = \text{proj}(z, V^\circ)$
- $z = x + y$, $x \in V$, $y \in V^\circ$, and $\langle x, y \rangle = 0$.

Theorem A.2 (Generalization of Helly's first theorem [20, Theorem 0.2.2]). *Let (u_α) be a generalized sequence or net of functions of bounded variation from the interval $[0, T]$ to a Hilbert space H . Assume that the norm and the variation of u_α are uniformly bounded, that is, there exist C_{\max} and C_{var} such that*

$$\|u_\alpha\| \leq C_{\max} \quad \text{and} \quad \text{Var}(u_\alpha; [0, T]) \leq C_{\text{var}}$$

then there is a filter \mathcal{F} finer than the filter of the sections of the index set (that is, there exists a subnet extracted from the given net) and there exists a function of bounded variation $u : [0, T] \rightarrow H$ that satisfies

$$\text{weak-}\lim_{\mathcal{F}} u_\alpha = u \quad \text{and} \quad \text{Var}(u; [0, T]) \leq C_{\text{var}}.$$

In the foregoing result if the Hilbert space H into consideration is finite dimensional then the convergence is uniform, that is,

$$\lim \|u_\alpha - u\| = 0.$$

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