



On the Riesz means of $\frac{n}{\phi(n)}$
A Sankaranarayanan, Saurabh Kumar Singh

► **To cite this version:**

A Sankaranarayanan, Saurabh Kumar Singh. On the Riesz means of $\frac{n}{\phi(n)}$. Hardy-Ramanujan Journal, 2013, Volume 36 - 2013, pp.8 - 20. 10.46298/hrj.2013.179 . hal-01112687

HAL Id: hal-01112687
<https://hal.science/hal-01112687>

Submitted on 3 Feb 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE RIESZ MEANS OF $\frac{n}{\phi(n)}$

A. SANKARANARAYANAN AND SAURABH KUMAR SINGH

In honor of Professor M. Ram Murty on his sixtieth birthday

ABSTRACT. Let $\phi(n)$ denote the Euler-totient function. We study the error term of the general k -th Riesz mean of the arithmetical function $\frac{n}{\phi(n)}$ for any positive integer $k \geq 1$, namely the error term $E_k(x)$ where

$$\frac{1}{k!} \sum_{n \leq x} \frac{n}{\phi(n)} \left(1 - \frac{n}{x}\right)^k = M_k(x) + E_k(x).$$

The upper bound for $|E_k(x)|$ established here thus improves the earlier known upper bound when $k = 1$.

1. INTRODUCTION

Investigating the growth (or decay) of the absolute value of the error term of the summatory function of an arithmetical function is a classical question in number theory. Many results on such interesting questions are available in the literature (for some of them, the readers may refer to chapter 14 of [3]). Let $\phi(n)$ denote the Euler-totient function defined to be the number of positive integers $\leq n$ which are co-prime to n . Let us write

$$(1) \quad \sum_{n \leq x} \frac{1}{\phi(n)} = A (\log x + B) + E_0^*(x)$$

and

$$(2) \quad \sum_{n \leq x} \frac{n}{\phi(n)} = Ax - \log x + E_1^*(x)$$

where

2000 *Mathematics Subject Classification*. Primary 11A25, Secondary 11N37.

Key words and phrases. Euler-totient function, Generating functions, Riemann zeta-function, mean-value theorems.

$$(3) \quad A = \frac{315\zeta(3)}{2\pi^4}, \quad B = \gamma_0 - \sum_p \frac{\log p}{p^2 - p + 1}.$$

Here $\zeta(s)$ and γ_0 denote the Riemann zeta-function and the Euler's constant respectively. The sum defining B extends over all primes p . In [4] (see p.184), E. Landau proved that

$$(4) \quad E_0^*(x) \ll \frac{\log x}{x}$$

as $x \rightarrow \infty$. Using a theorem of Walfisz based on Weyl's inequality, in [6], R. Sitaramachandrarao established (by elementary methods) that

$$(5) \quad E_0^*(x) \ll \frac{(\log x)^{\frac{2}{3}}}{x}$$

as $x \rightarrow \infty$. In an another paper [7], R. Sitaramachandrarao studied the discrete average and integral average of these error terms $E_j^*(x)$ for $j = 0, 1$. In particular, he proved that

$$(6) \quad \int_1^x E_1^*(t) dt = -\frac{D}{2}x + O(x^{\frac{4}{5}}).$$

by elementary methods where

$$(7) \quad D = \gamma_0 + \log(2\pi) + \sum_p \frac{\log p}{p(p-1)}.$$

As a consequence of (2) and (6) (see Remark 4.1 of [7]), he derived that

$$(8) \quad \begin{aligned} \sum_{n \leq x} \frac{n}{\phi(n)} (x - n) &= \int_1^x \left(\sum_{n \leq u} \frac{n}{\phi(n)} \right) du \\ &= \frac{A}{2}x^2 - \frac{1}{2}x \log x + \left(\frac{1-D}{2} \right) x + O\left(x^{\frac{4}{5}}\right) \end{aligned}$$

Equivalently, he established that the first Riesz mean satisfies the asymptotic relation

$$(9) \quad \sum_{n \leq x} \frac{n}{\phi(n)} \left(1 - \frac{n}{x} \right) = \frac{A}{2}x - \frac{1}{2} \log x + \left(\frac{1-D}{2} \right) + O\left(x^{-\frac{1}{5}}\right).$$

If we denote the error term of the first Riesz mean related to the arithmetic function $\frac{n}{\phi(n)}$ in (9) by $E_1(x)$, then a conjecture of Sitaramachandrarao (see Remark 4.1 of [7]) asserts that

$$(10) \quad E_1(x) \ll \frac{1}{x^{\frac{3}{4}-\delta}}$$

for every small fixed positive δ .

The aim of this article is to study the error term of the general k -th Riesz mean related to the arithmetic function $\frac{n}{\phi(n)}$ for any positive integer $k \geq 1$. More precisely, we write

$$(11) \quad \frac{1}{k!} \sum_{n \leq x} \frac{n}{\phi(n)} \left(1 - \frac{n}{x}\right)^k = M_k(x) + E_k(x)$$

where $M_k(x)$ is the main term and $E_k(x)$ is the error term of the sum under investigation. It should be mentioned here that in [1], K. Chandrasekharan and R. Narasimhan have developed a general method to study Omega and O -results for the error term of the general k -th Riesz mean whenever the generating function (i.e the Dirichlet series) corresponding to the coefficients satisfies a functional equation (with multiple gamma factors) analogous to the functional equation of the Riemann zeta-type. Though the Lemma 3.1 in the sequel suggests that the generating function in our case do have some nice factors (essentially $\zeta(s)$ and its translate), in totality nothing can be drawn about its functional equation. Therefore, such problems need to be treated in a different way and of course we can make use of the presence of these nice factors in the generating function.

We prove

Main Theorem. *Let $x \geq x_0$ where x_0 is a sufficiently large positive number. For any integer $k \geq 1$, we have*

$$\frac{1}{k!} \sum_{n \leq x} \frac{n}{\phi(n)} \left(1 - \frac{n}{x}\right)^k = c_1(k)x + c_2(k) \log x + c_3(k) + E_k(x)$$

where $c_1(k)$, $c_2(k)$ and $c_3(k)$ are certain specific constants (depend only on k) and

$$E_k(x) \ll \frac{1}{x^{\frac{1}{2}-\delta}}$$

for any small fixed positive constant δ satisfying $\delta < \frac{1}{100}$ and the implied constant is independent of k .

Remark: One expects the error term of the sum

$$\sum_{n \leq x} \frac{n}{\phi(n)} (x - n)$$

(whose average is the first Riesz mean of $\frac{n}{\phi(n)}$) to behave like the error term of the sum $\sum_{n \leq x} d(n)$ (where $d(n)$ is the number of positive divisors of n) since the generating function related to $\frac{n}{\phi(n)}$ behaves almost like $\zeta(s)\zeta(s+1)$ (see for example Lemma 3.1 in the sequel). This justifies the conjecture of Sitaramachandrarao. We also observe that for every integer $k \geq 1$, we have

$$\frac{1}{k!} \sum_{n \leq x} \frac{n}{\phi(n)} \left(1 - \frac{n}{x}\right)^k = \frac{1}{k!} \sum_{n \leq x} \frac{n}{\phi(n)} \left(1 + O\left(2^k \frac{n}{x}\right)\right)$$

Thus, it is reasonable to expect the error $E_1(x)$ to dominate over all other errors $E_k(x)$ in absolute value. In view of the above main theorem, we propose the following :

Conjecture. *For every integer $k \geq 1$,*

$$E_k(x) \ll \frac{1}{x^{\frac{3}{4}-\delta}}$$

for any small fixed positive constant δ and the implied constant is independent of k .

Remark: The constants $c_1(k)$, $c_2(k)$ and $c_3(k)$ are determined explicitly in the last section 5. With $k = 1$, we find that

$$c_1(1) = \frac{315\zeta(3)}{2\pi^4}, \quad c_2(1) = -\frac{1}{2}$$

and

$$c_3(1) = \frac{1}{2} - \frac{\gamma_0}{2} - \frac{1}{2} \log(2\pi) - \frac{1}{2} \sum_p \frac{\log p}{p(p-1)} = \frac{1-D}{2}.$$

Thus, the main theorem with $k = 1$ improves Sitaramachandrarao's bound on $|E_1(x)|$ in (9) considerably though his conjecture is still far from being resolved.

2. NOTATIONS AND PRELIMINARIES

Notations: 1. Throughout the paper, $s = \sigma + it$; the parameters T and x are sufficiently large real numbers and k is an integer ≥ 1 .

2. δ , ϵ always denote sufficiently small positive constants.

3. As usual $\zeta(s)$ denotes the Riemann zeta-function and γ_0 is Euler's constant.

3. SOME LEMMAS

Lemma 3.1. *For $\Re s > 1$, we have*

$$F(s) := \sum_{n=1}^{\infty} \frac{n}{\phi(n)n^s} = \zeta(s)\zeta(s+1)g(s)$$

where

$$g(s) := \prod_p \left(1 + \frac{1/p^{s+2} - 1/p^{2s+2}}{(1 - 1/p)} \right)$$

with $g(s)$ is absolutely and uniformly convergent in any compact set in the half-plane $\Re s \geq -\frac{1}{2} + 2\delta$ for any small fixed positive δ satisfying $0 < \delta < \frac{1}{100}$.

Proof. For $\Re s > 1$, we have

$$\begin{aligned} F(s) &:= \sum_{n=1}^{\infty} \frac{n}{\phi(n)n^s} \\ &= \prod_p \left(1 + \frac{p}{\phi(p)p^s} + \frac{p^2}{\phi(p^2)p^{2s}} + \cdots \right) \\ &= \prod_p \left(1 + \frac{p}{(p-1)p^s} \frac{1}{(1 - 1/p^s)} \right) \\ &= \prod_p \left(\frac{(1 - 1/p)(1 - 1/p^s) + 1/p^s}{(1 - 1/p)(1 - 1/p^s)} \right) \\ &= \zeta(s) \prod_p \left(1 + \frac{1}{p^{s+1}(1 - 1/p)} \right) \\ &= \zeta(s)\zeta(s+1) \prod_p \left(1 + \frac{1}{p^{s+1}(1 - 1/p)} \right) \left(1 - \frac{1}{p^{s+1}} \right) \\ &= \zeta(s)\zeta(s+1) \prod_p \left(1 + \frac{1/p^{s+2} - 1/p^{2s+2}}{(1 - 1/p)} \right) \\ F(s) &= \zeta(s)\zeta(s+1)g(s) \end{aligned}$$

where

$$g(s) := \prod_p \left(1 + \frac{1/p^{s+2} - 1/p^{2s+2}}{(1 - 1/p)} \right).$$

We observe that $g(s)$ is an infinite product of the form $\prod_p (1 + a_p)$ which is absolutely convergent if and only if the sum $\sum |a_p|$ is convergent. Thus, the sum

$$\sum_p \left| \frac{1/p^{s+2} - 1/p^{2s+2}}{(1 - 1/p)} \right| \leq 2 \sum_p \frac{1}{p^{\sigma+2}} + 2 \sum_p \frac{1}{p^{2\sigma+2}}$$

is absolutely and uniformly convergent in any compact set in the half-plane $\sigma > -\frac{1}{2} + 2\delta$ for any fixed δ satisfying $0 < \delta < \frac{1}{100}$. \square

We prove the following lemma 3.2 adapting the arguments of A.E. Ingham (see p.31 Theorem B of [2]) with the dependence of the implied constants on k explicit.

Lemma 3.2. *Let c and y be any positive real numbers and $T \geq T_0$ where T_0 is sufficiently large. Then we have,*

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s}{s(s+1)\dots(s+k)} ds = \begin{cases} \frac{1}{k!} \left(1 - \frac{1}{y}\right)^k + O\left(\frac{4^k y^c}{T^k}\right) & \text{if } y \geq 1, \\ O\left(\frac{1}{T^k}\right) & \text{if } 0 < y \leq 1. \end{cases}$$

Proof. If $y \geq 1$, we move the line of integration to the far left say to the line $\Re s = -R$ (with $R \geq 10k$, sufficiently large). Then, in the rectangular contour formed by the line segments joining the points $c - iT, c + iT, -R + iT, -R - iT$ and $c - iT$ in this anti-clockwise order, we find that $s = 0, -1, \dots, -k$ are the simple poles. The residue at $s = -r$ is $\frac{(-1)^r}{r!(k-r)!} y^{-r}$ and hence the sum of the residues namely

$$(12) \quad \sum_{r=0}^k \frac{(-1)^r}{r!(k-r)!} y^{-r} = \frac{1}{k!} \left(1 - \frac{1}{y}\right)^k.$$

The sum of the horizontal lines contributions in absolute value is

$$(13) \quad \ll \int_{-R}^c \frac{y^\sigma}{T^{k+1}} d\sigma \ll \frac{(R+c)y^c}{T^{k+1}}.$$

The left vertical line contribution in absolute value is

$$\begin{aligned}
& \ll \int_{-T}^T \frac{y^{-R}}{|(-R+it)(-R+1+it)\dots(-R+k+it)|} dt \\
& \ll \int_{|t|\leq R} \dots + \int_{T\geq|t|>R} \dots \\
& \ll \frac{Ry^{-R}}{R(R-1)(R-2)\dots(R-k)} + \frac{y^{-R}}{kR^k} \\
& \ll \frac{2^k y^{-R}}{R^k} + \frac{y^{-R}}{kR^k} \\
(14) \quad & \ll \frac{2^k y^{-R}}{R^k}.
\end{aligned}$$

With $R = \frac{T}{2}$, we obtain the desired asymptotic when $y \geq 1$.

If $0 < y \leq 1$, then we move the line of integration to the far-right namely to the line $\Re s = R_1$ (say with R_1 sufficiently large). Since there are no poles in the rectangular contour formed by the line segments joining the points $c - iT, c + iT, R_1 + iT, R_1 - iT$ and $c - iT$ in the anti-clockwise order and the integrand is analytic, by Cauchy's theorem for analytic function of a rectangular contour, we obtain the main term to be zero. However, the horizontal lines together contribute an error which is in absolute value

$$(15) \quad \ll \int_c^{R_1} \frac{y^\sigma}{T^{k+1}} d\sigma \ll \frac{R_1 y^{R_1}}{T^{k+1}} \ll \frac{R_1}{T^{k+1}},$$

and the vertical line contributes an error which is in absolute value

$$(16) \quad \ll \int_{-T}^T \frac{y^{R_1}}{|(R_1+it)(R_1+1+it)\dots(R_1+k+it)|} dt \ll \frac{T y^{R_1}}{R_1^{k+1}} \ll \frac{T}{R_1^{k+1}}.$$

We choose $R_1 = T$. This proves the lemma. \square

Lemma 3.3. *The Riemann zeta-function is extended as a meromorphic function in the whole complex plane \mathbb{C} with a simple pole at $s = 1$ and it satisfies a functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ where*

$$\chi(s) = \frac{\pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right)}{\pi^{-s/2} \Gamma\left(\frac{s}{2}\right)}.$$

Also, in any bounded vertical strip, using Stirling's formula, we have

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-1/2} e^{i(t+\frac{\pi}{4})} (1 + O(t^{-1}))$$

as $|t| \rightarrow \infty$. Thus, in any bounded vertical strip,

$$|\chi(s)| \asymp t^{1/2-\sigma} (1 + O(t^{-1}))$$

as $|t| \rightarrow \infty$.

Proof. See for example p.116 of [8] or p.8-12 of [3]. \square

Lemma 3.4. *For any fixed σ satisfying $\frac{1}{2} < \sigma < 1$, we have*

$$\int_1^T |\zeta(\sigma + it)|^2 dt = \zeta(2\sigma)T + O(T^{2-2\sigma} \log T).$$

Proof. See for example p.151 of [8]. \square

Lemma 3.5. *Let $U \geq U_0$ where U_0 is sufficiently large. Then, unconditionally there exists a point $t^* \in [U, U + U^{1/3}]$ such that the estimate*

$$\max_{1/2 \leq \sigma \leq 2} |\zeta(\sigma + it^*)| \ll \exp(c^*(\log \log U)^2)$$

holds where c^ is an absolute positive constant.*

Proof. This is part of theorem 2 of [5]. See for example Lemma 2 in p.73 of [5]. \square

4. PROOF OF THE MAIN THEOREM

We first choose the free large parameter T such that

$$(17) \quad \max_{1/2 \leq \sigma \leq 2} |\zeta(\sigma + iT)| \ll \exp(c^*(\log \log T)^2).$$

The existence of such a T is ensured by Lemma 3.5.

From Lemma 3.2, with $c = 1 + \frac{1}{\log x}$ and writing $F(s) := \zeta(s)\zeta(s+1)g(s)$, we have

$$\begin{aligned} S &:= \frac{1}{k!} \sum_{n \leq x} \frac{n}{\phi(n)} \left(1 - \frac{n}{x}\right)^k \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s(s+1)(s+2)\cdots(s+k)} ds \\ (18) \quad &= \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s(s+1)\cdots(s+k)} ds + O\left(\frac{4^k x^c \log x}{T^k}\right) \end{aligned}$$

Now move the line of integration in the above integral to $\Re s = -1/2 + 2\delta$ (where δ is any fixed positive constant $< \frac{1}{100}$). In the rectangular contour formed by the line segments joining the points $c - iT$, $c + iT$, $-\frac{1}{2} + 2\delta + iT$, $-\frac{1}{2} + 2\delta - iT$ and $c - iT$ in the anticlockwise order, we observe that $s = 1$ is a simple pole and $s = 0$ is a double

pole of the integrand, thus we get the main term from the sum of the residues coming from the poles $s = 1$ and $s = 0$, namely $c_1(k)x + c_2(k) \log x + c_3(k)$. We note that

$$\begin{aligned}
 & \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s(s+1)\dots(s+k)} ds \\
 (19) \quad &= \int_{-\frac{1}{2}+2\delta+iT}^{c+iT} \dots + \int_{-\frac{1}{2}+2\delta-iT}^{-\frac{1}{2}+2\delta+iT} \dots + \int_{c-iT}^{-\frac{1}{2}+2\delta-iT} \dots + \text{sum of the residues.}
 \end{aligned}$$

The left vertical line segment contributes the quantity:

$$\begin{aligned}
 Q_1 &= \frac{1}{2\pi} \int_{-T}^T F(-\frac{1}{2} + 2\delta + it) \frac{x^{-\frac{1}{2}+2\delta+it}}{(-\frac{1}{2} + 2\delta + it)(\frac{1}{2} + 2\delta + it) \dots (k - \frac{1}{2} + 2\delta + it)} dt \\
 &= \frac{1}{2\pi} \left(\int_{-1}^1 + \int_{1 < |t| \leq T} \right) \frac{x^{-\frac{1}{2}+2\delta+it} \zeta(-\frac{1}{2} + 2\delta + it) \zeta(\frac{1}{2} + 2\delta + it) g(-\frac{1}{2} + 2\delta + it)}{(-\frac{1}{2} + 2\delta + it)(\frac{1}{2} + 2\delta + it) \dots (k - \frac{1}{2} + 2\delta + it)} dt \\
 &\ll \frac{x^{-\frac{1}{2}+2\delta}}{(k-1)!} + x^{-\frac{1}{2}+2\delta} \int_{1 < |t| \leq T} t^{\frac{1}{2} - (-\frac{1}{2}+2\delta)} |\zeta(3/2 + 2\delta + it)| |\zeta(1/2 + 2\delta + it)| \frac{dt}{t^k} \\
 &\ll \frac{x^{-\frac{1}{2}+2\delta}}{(k-1)!} + x^{-\frac{1}{2}+2\delta} \int_{1 < |t| \leq T} \frac{1}{\sqrt{t}} \frac{|\zeta(1/2 + 2\delta + it)|}{t^{k-\frac{1}{2}}} dt \\
 &\ll \frac{x^{-\frac{1}{2}+2\delta}}{(k-1)!} + x^{-\frac{1}{2}+2\delta} \left(\int_1^T \frac{1}{t} dt \right)^{\frac{1}{2}} \left(\int_1^T \frac{|\zeta(1/2 + 2\delta + it)|^2}{t^{2k-1}} dt \right)^{\frac{1}{2}} \\
 (20) \quad &\ll \frac{x^{-\frac{1}{2}+2\delta}}{(k-1)!} + x^{-\frac{1}{2}+2\delta} (\log T)
 \end{aligned}$$

since, by Lemma 3.4, letting

$$(21) \quad v(T) := \int_1^T |\zeta(1/2 + 2\delta + it)|^2 dt (\ll_\delta T)$$

we have using integration by parts

$$\begin{aligned}
\int_1^T \frac{|\zeta(1/2 + 2\delta + it)|^2}{t^{2k-1}} dt &= \int_1^T \frac{1}{t^{2k-1}} dv(T) \\
&= \frac{v(t)}{t^{2k-1}} \Big|_1^T + (2k-1) \int_1^T \frac{v(t)}{t^{2k}} dt \\
&\ll_\delta \frac{1}{T^{2k-2}} + 1 + \max(1, \log T) \\
(22) \quad &\ll \log T
\end{aligned}$$

(by splitting the cases $k = 1$ and $k \geq 2$ separately) where the implied constant in (22) is independent of k and note that δ is any small fixed positive constant.

Now we will estimate the contributions coming from the upper horizontal line (lower horizontal line is similar). Let

$$\begin{aligned}
Q_2 &:= \max_{-1/2 \leq \sigma \leq 1/2} |\zeta(\sigma + iT)| \\
&\ll \max_{-1/2 \leq \sigma \leq 1/2} |T|^{1/2-\sigma} |\zeta(1-\sigma-iT)| \\
&\ll T \max_{1/2 \leq 1-\sigma \leq 3/2} |\zeta(1-\sigma-iT)| \\
(23) \quad &\ll T \exp(c^*(\log \log T)^2)
\end{aligned}$$

Therefore with our choice of T , from (17), we have

$$(24) \quad \max_{-1/2 \leq \sigma \leq 2} |\zeta(\sigma + iT)| \ll T \exp(c^*(\log \log T)^2).$$

Thus the horizontal lines in total contribute a quantity which is in absolute value

$$\begin{aligned}
&\ll \int_{-1/2+2\delta}^c \left| \zeta(\sigma + iT) \zeta(\sigma + 1 + iT) \frac{g(\sigma + iT) x^{\sigma+iT}}{(\sigma + iT)(\sigma + 1 + iT) \cdots (\sigma + k + iT)} \right| d\sigma \\
(25) \quad &\ll_\delta \frac{(x \exp(2c^*(\log \log T)^2))}{T^k}
\end{aligned}$$

Collecting all the estimates, we get

$$(26) \quad E_k(x) \ll \frac{4^k x^c \log x}{T^k} + x^{-1/2+2\delta} (\log T) + \frac{(x \exp(2c^*(\log \log T)^2))}{T^k}.$$

Note that

$$\frac{(x \exp(2c^*(\log \log T)^2))}{T^k} \ll \frac{x}{T^{k-\delta}}$$

for any fixed small positive constant δ . Now we choose $T := 4x^{\frac{3}{2k}}$ so that from (26), we obtain

$$\begin{aligned}
 E_k(x) &\ll \frac{1}{x^{\frac{1}{2}-3\delta}} + \frac{1}{4^{k-\delta}} \cdot \frac{x}{x^{\frac{3}{2}-\frac{3\delta}{2k}}} \\
 &\ll \frac{1}{x^{\frac{1}{2}-3\delta}} + \frac{x}{x^{\frac{3}{2}-2\delta}} \\
 &\ll \frac{1}{x^{\frac{1}{2}-3\delta}}.
 \end{aligned}
 \tag{27}$$

Note that the implied constant in (27) is independent of k . This proves the theorem provided $c_1(k)$, $c_2(k)$ and $c_3(k)$ are precisely determined. This is done in the following section.

5. EVALUATION OF THE CONSTANTS $c_1(k)$, $c_2(k)$ AND $c_3(k)$

We recall that

$$g(s) := \prod_p \left(1 + \frac{1/p^{s+2} - 1/p^{2s+2}}{(1 - 1/p)} \right)$$

is absolutely and uniformly convergent in any compact set contained in the half-plane $\Re s \geq -\frac{1}{2} + 2\delta$ for any fixed small positive δ and thus, we observe that, in any compact region in the half plane $\sigma \geq -\frac{1}{2} + 2\delta$ (taking the logarithmic derivative of $g(s)$) we have

$$g'(s) = g(s) \left(\sum_p \frac{1}{1 + \frac{1/p^{s+2}(1-1/p^s)}{1-1/p}} \frac{1}{1-1/p} \left(-\frac{\log p}{p^{s+2}}(1-1/p^s) + \frac{1}{p^{s+2}} \frac{\log p}{p^s} \right) \right).$$

We note that $g(0) = 1$ and

$$g(1) = \prod_p \left(1 + \frac{1/p^3 - 1/p^4}{1 - 1/p} \right) = \prod_p \left(1 + \frac{1}{p^3} \right) = \prod_p \left(\frac{(1 - \frac{1}{p^6})}{(1 - \frac{1}{p^3})} \right) = \frac{\zeta(3)}{\zeta(6)}$$

and $g'(0) = \sum_p \frac{\log p}{p(p-1)}$.

The Bernoulli numbers B_n are defined to be the coefficients of the exponential generating function, precisely by the relation

$$\frac{t}{e^t - 1} := \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

We note that $\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}$ for any integer $n \geq 1$, $B_2 = \frac{1}{6}$ and $B_6 = \frac{1}{42}$ so that $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(6) = \frac{(2\pi)^6}{2 \cdot 6! \cdot 42}$. Since $s = 1$ is a simple pole inside the rectangular

contour, we obtain

$$(28) \quad \text{Res}_{s=1} \left(F(s) \frac{x^s}{s(s+1)(s+2)\cdots(s+k)} \right) = \frac{\zeta(2)\zeta(3)}{k!\zeta(6)} x = \frac{315\zeta(3)}{2\pi^4 k!} x.$$

Hence,

$$(29) \quad c_1(k) = \frac{315\zeta(3)}{2\pi^4 k!}.$$

We observe that in the neighbourhood of $s = 0$, we have

$$\zeta(s+1) - \frac{1}{s} = \gamma_0 + \gamma_1 s + \gamma_2 s^2 + \cdots$$

and hence

$$\lim_{s \rightarrow 0} \left(\zeta(s+1) - \frac{1}{s} \right) = \gamma_0.$$

If we write

$$h(s) := \prod_{j=1}^k \frac{1}{(s+j)},$$

then

$$h'(s) = h(s) \left(- \sum_{j=1}^k \frac{1}{(s+j)} \right)$$

and hence

$$h'(0) = h(0) \left(- \sum_{j=1}^k \frac{1}{j} \right) = \frac{1}{k!} \left(- \sum_{j=1}^k \frac{1}{j} \right).$$

Now $s = 0$ is a double pole in the rectangular contour and hence the residue is

$$\begin{aligned} Q_3 &:= \text{Res}_{s=0} \left[\frac{F(s)x^s}{s(s+1)(s+2)\cdots(s+k)} \right] \\ &= \lim_{s \rightarrow 0} \frac{d}{ds} \left(s^2 \frac{F(s)x^s}{s(s+1)(s+2)\cdots(s+k)} \right) \\ &= \lim_{s \rightarrow 0} \frac{d}{ds} \left(s\zeta(s)\zeta(s+1)g(s) \frac{x^s}{(s+1)(s+2)\cdots(s+k)} \right) \\ &= \lim_{s \rightarrow 0} \frac{d}{ds} \left[s\zeta(s) \left(\zeta(s+1) - \frac{1}{s} \right) g(s) \frac{x^s}{(s+1)(s+2)\cdots(s+k)} \right] \\ &\quad + \lim_{s \rightarrow 0} \frac{d}{ds} \left[\zeta(s)g(s) \frac{x^s}{(s+1)(s+2)\cdots(s+k)} \right] \\ &= \frac{1}{k!} \{ \zeta(0)\gamma_0 g(0) + \zeta'(0)g(0) + \zeta(0)g'(0) + \zeta(0)g(0) \log x \} + \zeta(0)g(0)h'(0) \\ &= \frac{1}{k!} \left\{ -\frac{\gamma_0}{2} - \frac{1}{2} \log(2\pi) - \frac{1}{2} \sum_p \frac{\log p}{p(p-1)} - \frac{1}{2} \log x + \frac{1}{2} \left(\sum_{j=1}^k \frac{1}{j} \right) \right\}. \end{aligned}$$

Therefore,

$$(30) \quad c_2(k) = -\frac{1}{2(k!)}$$

and

$$(31) \quad c_3(k) = \frac{1}{k!} \left\{ \frac{1}{2} \left(\sum_{j=1}^k \frac{1}{j} \right) - \frac{\gamma_0}{2} - \frac{1}{2} \log(2\pi) - \frac{1}{2} \sum_p \frac{\log p}{p(p-1)} \right\}.$$

This completes the proof of the main theorem.

Acknowledgement: The authors wish to thank the referee for some fruitful comments.

REFERENCES

- [1] K. Chandrashekhara, R. Narasimhan: *Functional equations with multiple gamma factors and average order of an arithmetical functions*, Ann. Math., **61(2)**, (1962) 93-136.
- [2] A.E. Ingham, *The distribution of prime numbers*, Cambridge University Press (1995).
- [3] A. Ivić, *The Riemann Zeta-Function: Theory and Applications*, Dover Publications, Inc, New York.
- [4] E. Landau: *Über die zahlentheoretische Funktion $\phi(n)$ und ihre Beziehung zum Goldbachschen Satz*, Nachr. königlichen Gesellschaft wiss, Göttingen Math. Phys. klasse., 1900, 177-186. Collected works, Vol **1**, Ed.by L. Mirsky et al, Thales Verlag, 106-115.
- [5] K. Ramachandra and A. Sankaranarayanan: *Notes on the Riemann zeta-function*, J. Indian Math. Soc.(N.S.), **57**, no. 1-4 (1991) 67-77 .
- [6] R. Sitaramachandrarao: *On an error term of Landau*, Indian Jour. Pure and Appl. Math., **13**, (1982) 882-885.
- [7] R. Sitaramachandrarao: *On an error term of Landau- II*, Rocky Mountain Jour. of Math., **15**, (1985) 579-588.
- [8] E.C. Titchmarsh: *The Theory of the Riemann Zeta function*, (revised by Dr. Heath-Brown), Clarendon Press, Oxford (1986).

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH,
HOMI BHABHA ROAD, MUMBAI- 400 005, INDIA.

E-mail address: sank@math.tifr.res.in

E-mail address: skumar@math.tifr.res.in