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ON THE RIESZ MEANS OF $\frac{n}{\phi(n)}$

A. SANKARANARAYANAN AND SAURABH KUMAR SINGH

In honor of Professor M. Ram Murty on his sixtieth birthday

ABSTRACT. Let $\phi(n)$ denote the Euler-totient function. We study the error term of the general k -th Riesz mean of the arithmetical function $\frac{n}{\phi(n)}$ for any positive integer $k \geq 1$, namely the error term $E_k(x)$ where

$$\frac{1}{k!} \sum_{n \leq x} \frac{n}{\phi(n)} \left(1 - \frac{n}{x}\right)^k = M_k(x) + E_k(x).$$

The upper bound for $|E_k(x)|$ established here thus improves the earlier known upper bound when $k = 1$.

1. INTRODUCTION

Investigating the growth (or decay) of the absolute value of the error term of the summatory function of an arithmetical function is a classical question in number theory. Many results on such interesting questions are available in the literature (for some of them, the readers may refer to chapter 14 of [3]). Let $\phi(n)$ denote the Euler-totient function defined to be the number of positive integers $\leq n$ which are co-prime to n . Let us write

$$(1) \quad \sum_{n \leq x} \frac{1}{\phi(n)} = A(\log x + B) + E_0^*(x)$$

and

$$(2) \quad \sum_{n \leq x} \frac{n}{\phi(n)} = Ax - \log x + E_1^*(x)$$

where

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$$(3) \quad A = \frac{315\zeta(3)}{2\pi^4}, \quad B = \gamma_0 - \sum_p \frac{\log p}{p^2 - p + 1}.$$

Here $\zeta(s)$ and γ_0 denote the Riemann zeta-function and the Euler's constant respectively. The sum defining B extends over all primes p . In [4] (see p.184), E. Landau proved that

$$(4) \quad E_0^*(x) \ll \frac{\log x}{x}$$

as $x \rightarrow \infty$. Using a theorem of Walfisz based on Weyl's inequality, in [6], R. Sitaramachandrarao established (by elementary methods) that

$$(5) \quad E_0^*(x) \ll \frac{(\log x)^{\frac{2}{3}}}{x}$$

as $x \rightarrow \infty$. In an another paper [7], R. Sitaramachandrarao studied the discrete average and integral average of these error terms $E_j^*(x)$ for $j = 0, 1$. In particular, he proved that

$$(6) \quad \int_1^x E_1^*(t) dt = -\frac{D}{2}x + O(x^{\frac{4}{5}}).$$

by elementary methods where

$$(7) \quad D = \gamma_0 + \log(2\pi) + \sum_p \frac{\log p}{p(p-1)}.$$

As a consequence of (2) and (6) (see Remark 4.1 of [7]), he derived that

$$(8) \quad \begin{aligned} \sum_{n \leq x} \frac{n}{\phi(n)}(x-n) &= \int_1^x \left(\sum_{n \leq u} \frac{n}{\phi(n)} \right) du \\ &= \frac{A}{2}x^2 - \frac{1}{2}x \log x + \left(\frac{1-D}{2} \right) x + O\left(x^{\frac{4}{5}}\right) \end{aligned}$$

Equivalently, he established that the first Riesz mean satisfies the asymptotic relation

$$(9) \quad \sum_{n \leq x} \frac{n}{\phi(n)} \left(1 - \frac{n}{x}\right) = \frac{A}{2}x - \frac{1}{2} \log x + \left(\frac{1-D}{2} \right) + O\left(x^{-\frac{1}{5}}\right).$$

If we denote the error term of the first Riesz mean related to the arithmetic function $\frac{n}{\phi(n)}$ in (9) by $E_1(x)$, then a conjecture of Sitaramachandrarao (see Remark 4.1 of [7]) asserts that

$$(10) \quad E_1(x) \ll \frac{1}{x^{\frac{3}{4}-\delta}}$$

for every small fixed positive δ .

The aim of this article is to study the error term of the general k -th Riesz mean related to the arithmetic function $\frac{n}{\phi(n)}$ for any positive integer $k \geq 1$. More precisely, we write

$$(11) \quad \frac{1}{k!} \sum_{n \leq x} \frac{n}{\phi(n)} \left(1 - \frac{n}{x}\right)^k = M_k(x) + E_k(x)$$

where $M_k(x)$ is the main term and $E_k(x)$ is the error term of the sum under investigation. It should be mentioned here that in [1], K. Chandrasekharan and R. Narasimhan have developed a general method to study Omega and O -results for the error term of the general k -th Riesz mean whenever the generating function (i.e the Dirichlet series) corresponding to the coefficients satisfies a functional equation (with multiple gamma factors) analogous to the functional equation of the Riemann zeta-type. Though the Lemma 3.1 in the sequel suggests that the generating function in our case do have some nice factors (essentially $\zeta(s)$ and its translate), in totality nothing can be drawn about its functional equation. Therefore, such problems need to be treated in a different way and of course we can make use of the presence of these nice factors in the generating function.

We prove

Main Theorem. *Let $x \geq x_0$ where x_0 is a sufficiently large positive number. For any integer $k \geq 1$, we have*

$$\frac{1}{k!} \sum_{n \leq x} \frac{n}{\phi(n)} \left(1 - \frac{n}{x}\right)^k = c_1(k)x + c_2(k) \log x + c_3(k) + E_k(x)$$

where $c_1(k)$, $c_2(k)$ and $c_3(k)$ are certain specific constants (depend only on k) and

$$E_k(x) \ll \frac{1}{x^{\frac{1}{2}-\delta}}$$

for any small fixed positive constant δ satisfying $\delta < \frac{1}{100}$ and the implied constant is independent of k .

Remark: One expects the error term of the sum

$$\sum_{n \leq x} \frac{n}{\phi(n)} (x - n)$$

(whose average is the first Riesz mean of $\frac{n}{\phi(n)}$) to behave like the error term of the sum $\sum_{n \leq x} d(n)$ (where $d(n)$ is the number of positive divisors of n) since the generating function related to $\frac{n}{\phi(n)}$ behaves almost like $\zeta(s)\zeta(s+1)$ (see for example Lemma 3.1 in the sequel). This justifies the conjecture of Sitaramachandrarao. We also observe that for every integer $k \geq 1$, we have

$$\frac{1}{k!} \sum_{n \leq x} \frac{n}{\phi(n)} \left(1 - \frac{n}{x}\right)^k = \frac{1}{k!} \sum_{n \leq x} \frac{n}{\phi(n)} \left(1 + O\left(2^k \frac{n}{x}\right)\right)$$

Thus, it is reasonable to expect the error $E_1(x)$ to dominate over all other errors $E_k(x)$ in absolute value. In view of the above main theorem, we propose the following :

Conjecture. For every integer $k \geq 1$,

$$E_k(x) \ll \frac{1}{x^{\frac{3}{4}-\delta}}$$

for any small fixed positive constant δ and the implied constant is independent of k .

Remark: The constants $c_1(k)$, $c_2(k)$ and $c_3(k)$ are determined explicitly in the last section 5. With $k = 1$, we find that

$$c_1(1) = \frac{315\zeta(3)}{2\pi^4}, \quad c_2(1) = -\frac{1}{2}$$

and

$$c_3(1) = \frac{1}{2} - \frac{\gamma_0}{2} - \frac{1}{2} \log(2\pi) - \frac{1}{2} \sum_p \frac{\log p}{p(p-1)} = \frac{1-D}{2}.$$

Thus, the main theorem with $k = 1$ improves Sitaramachandrarao's bound on $|E_1(x)|$ in (9) considerably though his conjecture is still far from being resolved.

2. NOTATIONS AND PRELIMINARIES

Notations: 1. Throughout the paper, $s = \sigma + it$; the parameters T and x are sufficiently large real numbers and k is an integer ≥ 1 .

2. δ , ϵ always denote sufficiently small positive constants.

3. As usual $\zeta(s)$ denotes the Riemann zeta-function and γ_0 is Euler's constant.

3. SOME LEMMAS

Lemma 3.1. *For $\Re s > 1$, we have*

$$F(s) := \sum_{n=1}^{\infty} \frac{n}{\phi(n)n^s} = \zeta(s)\zeta(s+1)g(s)$$

where

$$g(s) := \prod_p \left(1 + \frac{1/p^{s+2} - 1/p^{2s+2}}{(1 - 1/p)} \right)$$

with $g(s)$ is absolutely and uniformly convergent in any compact set in the half-plane $\Re s \geq -\frac{1}{2} + 2\delta$ for any small fixed positive δ satisfying $0 < \delta < \frac{1}{100}$.

Proof. For $\Re s > 1$, we have

$$\begin{aligned} F(s) &:= \sum_{n=1}^{\infty} \frac{n}{\phi(n)n^s} \\ &= \prod_p \left(1 + \frac{p}{\phi(p)p^s} + \frac{p^2}{\phi(p^2)p^{2s}} + \cdots \right) \\ &= \prod_p \left(1 + \frac{p}{(p-1)p^s} \frac{1}{(1 - 1/p^s)} \right) \\ &= \prod_p \left(\frac{(1 - 1/p)(1 - 1/p^s) + 1/p^s}{(1 - 1/p)(1 - 1/p^s)} \right) \\ &= \zeta(s) \prod_p \left(1 + \frac{1}{p^{s+1}(1 - 1/p)} \right) \\ &= \zeta(s)\zeta(s+1) \prod_p \left(1 + \frac{1}{p^{s+1}(1 - 1/p)} \right) \left(1 - \frac{1}{p^{s+1}} \right) \\ &= \zeta(s)\zeta(s+1) \prod_p \left(1 + \frac{1/p^{s+2} - 1/p^{2s+2}}{(1 - 1/p)} \right) \\ F(s) &= \zeta(s)\zeta(s+1)g(s) \end{aligned}$$

where

$$g(s) := \prod_p \left(1 + \frac{1/p^{s+2} - 1/p^{2s+2}}{(1 - 1/p)} \right).$$

We observe that $g(s)$ is an infinite product of the form $\prod_p (1 + a_p)$ which is absolutely convergent if and only if the sum $\sum |a_p|$ is convergent. Thus, the sum

$$\sum_p \left| \frac{1/p^{s+2} - 1/p^{2s+2}}{(1 - 1/p)} \right| \leq 2 \sum_p \frac{1}{p^{\sigma+2}} + 2 \sum_p \frac{1}{p^{2\sigma+2}}$$

is absolutely and uniformly convergent in any compact set in the half-plane $\sigma > -\frac{1}{2} + 2\delta$ for any fixed δ satisfying $0 < \delta < \frac{1}{100}$. \square

We prove the following lemma 3.2 adapting the arguments of A.E. Ingham (see p.31 Theorem B of [2]) with the dependence of the implied constants on k explicit.

Lemma 3.2. *Let c and y be any positive real numbers and $T \geq T_0$ where T_0 is sufficiently large. Then we have,*

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s}{s(s+1)\dots(s+k)} ds = \begin{cases} \frac{1}{k!} \left(1 - \frac{1}{y}\right)^k + O\left(\frac{4^k y^c}{T^k}\right) & \text{if } y \geq 1, \\ O\left(\frac{1}{T^k}\right) & \text{if } 0 < y \leq 1. \end{cases}$$

Proof. If $y \geq 1$, we move the line of integration to the far left say to the line $\Re s = -R$ (with $R \geq 10k$, sufficiently large). Then, in the rectangular contour formed by the line segments joining the points $c - iT, c + iT, -R + iT, -R - iT$ and $c - iT$ in this anti-clockwise order, we find that $s = 0, -1, \dots, -k$ are the simple poles. The residue at $s = -r$ is $\frac{(-1)^r}{r!(k-r)!} y^{-r}$ and hence the sum of the residues namely

$$(12) \quad \sum_{r=0}^k \frac{(-1)^r}{r!(k-r)!} y^{-r} = \frac{1}{k!} \left(1 - \frac{1}{y}\right)^k.$$

The sum of the horizontal lines contributions in absolute value is

$$(13) \quad \ll \int_{-R}^c \frac{y^\sigma}{T^{k+1}} d\sigma \ll \frac{(R+c)y^c}{T^{k+1}}.$$

The left vertical line contribution in absolute value is

$$\begin{aligned}
&\ll \int_{-T}^T \frac{y^{-R}}{|(-R+it)(-R+1+it)\dots(-R+k+it)|} dt \\
&\ll \int_{|t|\leq R} \dots + \int_{T\geq|t|>R} \dots \\
&\ll \frac{Ry^{-R}}{R(R-1)(R-2)\dots(R-k)} + \frac{y^{-R}}{kR^k} \\
&\ll \frac{2^k y^{-R}}{R^k} + \frac{y^{-R}}{kR^k} \\
(14) \quad &\ll \frac{2^k y^{-R}}{R^k}.
\end{aligned}$$

With $R = \frac{T}{2}$, we obtain the desired asymptotic when $y \geq 1$.

If $0 < y \leq 1$, then we move the line of integration to the far-right namely to the line $\Re s = R_1$ (say with R_1 sufficiently large). Since there are no poles in the rectangular contour formed by the line segments joining the points $c - iT, c + iT, R_1 + iT, R_1 - iT$ and $c - iT$ in the anti-clockwise order and the integrand is analytic, by Cauchy's theorem for analytic function of a rectangular contour, we obtain the main term to be zero. However, the horizontal lines together contribute an error which is in absolute value

$$(15) \quad \ll \int_c^{R_1} \frac{y^\sigma}{T^{k+1}} d\sigma \ll \frac{R_1 y^{R_1}}{T^{k+1}} \ll \frac{R_1}{T^{k+1}},$$

and the vertical line contributes an error which is in absolute value

$$(16) \quad \ll \int_{-T}^T \frac{y^{R_1}}{|(R_1+it)(R_1+1+it)\dots(R_1+k+it)|} dt \ll \frac{T y^{R_1}}{R_1^{k+1}} \ll \frac{T}{R_1^{k+1}}.$$

We choose $R_1 = T$. This proves the lemma. \square

Lemma 3.3. *The Riemann zeta-function is extended as a meromorphic function in the whole complex plane \mathbb{C} with a simple pole at $s = 1$ and it satisfies a functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ where*

$$\chi(s) = \frac{\pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right)}{\pi^{-s/2} \Gamma\left(\frac{s}{2}\right)}.$$

Also, in any bounded vertical strip, using Stirling's formula, we have

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-1/2} e^{i(t+\frac{\pi}{4})} (1 + O(t^{-1}))$$

as $|t| \rightarrow \infty$. Thus, in any bounded vertical strip,

$$|\chi(s)| \asymp t^{1/2-\sigma} (1 + O(t^{-1}))$$

as $|t| \rightarrow \infty$.

Proof. See for example p.116 of [8] or p.8-12 of [3]. \square

Lemma 3.4. *For any fixed σ satisfying $\frac{1}{2} < \sigma < 1$, we have*

$$\int_1^T |\zeta(\sigma + it)|^2 dt = \zeta(2\sigma)T + O(T^{2-2\sigma} \log T).$$

Proof. See for example p.151 of [8]. \square

Lemma 3.5. *Let $U \geq U_0$ where U_0 is sufficiently large. Then, unconditionally there exists a point $t^* \in [U, U + U^{1/3}]$ such that the estimate*

$$\max_{1/2 \leq \sigma \leq 2} |\zeta(\sigma + it^*)| \ll \exp(c^*(\log \log U)^2)$$

holds where c^* is an absolute positive constant.

Proof. This is part of theorem 2 of [5]. See for example Lemma 2 in p.73 of [5]. \square

4. PROOF OF THE MAIN THEOREM

We first choose the free large parameter T such that

$$(17) \quad \max_{1/2 \leq \sigma \leq 2} |\zeta(\sigma + iT)| \ll \exp(c^*(\log \log T)^2).$$

The existence of such a T is ensured by Lemma 3.5.

From Lemma 3.2, with $c = 1 + \frac{1}{\log x}$ and writing $F(s) := \zeta(s)\zeta(s+1)g(s)$, we have

$$(18) \quad \begin{aligned} S &:= \frac{1}{k!} \sum_{n \leq x} \frac{n}{\phi(n)} \left(1 - \frac{n}{x}\right)^k \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s(s+1)(s+2)\cdots(s+k)} ds \\ &= \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s(s+1)\cdots(s+k)} ds + O\left(\frac{4^k x^c \log x}{T^k}\right) \end{aligned}$$

Now move the line of integration in the above integral to $\Re s = -1/2 + 2\delta$ (where δ is any fixed positive constant $< \frac{1}{100}$). In the rectangular contour formed by the line segments joining the points $c - iT$, $c + iT$, $-\frac{1}{2} + 2\delta + iT$, $-\frac{1}{2} + 2\delta - iT$ and $c - iT$ in the anticlockwise order, we observe that $s = 1$ is a simple pole and $s = 0$ is a double

pole of the integrand, thus we get the main term from the sum of the residues coming from the poles $s = 1$ and $s = 0$, namely $c_1(k)x + c_2(k) \log x + c_3(k)$. We note that

$$(19) \quad \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s(s+1)\dots(s+k)} ds \\ = \int_{-\frac{1}{2}+2\delta+iT}^{c+iT} \dots + \int_{-\frac{1}{2}+2\delta-iT}^{-\frac{1}{2}+2\delta+iT} \dots + \int_{c-iT}^{-\frac{1}{2}+2\delta-iT} \dots + \text{sum of the residues.}$$

The left vertical line segment contributes the quantity:

$$(20) \quad Q_1 = \frac{1}{2\pi} \int_{-T}^T F\left(-\frac{1}{2} + 2\delta + it\right) \frac{x^{-\frac{1}{2}+2\delta+it}}{\left(-\frac{1}{2} + 2\delta + it\right)\left(\frac{1}{2} + 2\delta + it\right) \dots \left(k - \frac{1}{2} + 2\delta + it\right)} dt \\ = \frac{1}{2\pi} \left(\int_{-1}^1 + \int_{1 < |t| \leq T} \right) \frac{x^{-\frac{1}{2}+2\delta+it} \zeta\left(-\frac{1}{2} + 2\delta + it\right) \zeta\left(\frac{1}{2} + 2\delta + it\right) g\left(-\frac{1}{2} + 2\delta + it\right)}{\left(-\frac{1}{2} + 2\delta + it\right) \left(\frac{1}{2} + 2\delta + it\right) \dots \left(k - \frac{1}{2} + 2\delta + it\right)} dt \\ \ll \frac{x^{-\frac{1}{2}+2\delta}}{(k-1)!} + x^{-\frac{1}{2}+2\delta} \int_{1 < |t| \leq T} t^{\frac{1}{2} - (-\frac{1}{2}+2\delta)} |\zeta(3/2 + 2\delta + it)| |\zeta(1/2 + 2\delta + it)| \frac{dt}{t^k} \\ \ll \frac{x^{-\frac{1}{2}+2\delta}}{(k-1)!} + x^{-\frac{1}{2}+2\delta} \int_{1 < |t| \leq T} \frac{1}{\sqrt{t}} \frac{|\zeta(1/2 + 2\delta + it)|}{t^{k-\frac{1}{2}}} dt \\ \ll \frac{x^{-\frac{1}{2}+2\delta}}{(k-1)!} + x^{-\frac{1}{2}+2\delta} \left(\int_1^T \frac{1}{t} dt \right)^{\frac{1}{2}} \left(\int_1^T \frac{|\zeta(1/2 + 2\delta + it)|^2}{t^{2k-1}} dt \right)^{\frac{1}{2}} \\ \ll \frac{x^{-\frac{1}{2}+2\delta}}{(k-1)!} + x^{-\frac{1}{2}+2\delta} (\log T)$$

since, by Lemma 3.4, letting

$$(21) \quad v(T) := \int_1^T |\zeta(1/2 + 2\delta + it)|^2 dt (\ll_{\delta} T)$$

we have using integration by parts

$$\begin{aligned}
\int_1^T \frac{|\zeta(1/2 + 2\delta + it)|^2}{t^{2k-1}} dt &= \int_1^T \frac{1}{t^{2k-1}} dv(T) \\
&= \frac{v(t)}{t^{2k-1}} \Big|_1^T + (2k-1) \int_1^T \frac{v(t)}{t^{2k}} dt \\
&\ll_{\delta} \frac{1}{T^{2k-2}} + 1 + \max(1, \log T) \\
(22) \quad &\ll \log T
\end{aligned}$$

(by splitting the cases $k = 1$ and $k \geq 2$ separately) where the implied constant in (22) is independent of k and note that δ is any small fixed positive constant.

Now we will estimate the contributions coming from the upper horizontal line (lower horizontal line is similar). Let

$$\begin{aligned}
Q_2 &:= \max_{-1/2 \leq \sigma \leq 1/2} |\zeta(\sigma + iT)| \\
&\ll \max_{-1/2 \leq \sigma \leq 1/2} |T|^{1/2-\sigma} |\zeta(1 - \sigma - iT)| \\
&\ll T \max_{1/2 \leq 1-\sigma \leq 3/2} |\zeta(1 - \sigma - iT)| \\
(23) \quad &\ll T \exp(c^*(\log \log T)^2)
\end{aligned}$$

Therefore with our choice of T , from (17), we have

$$(24) \quad \max_{-1/2 \leq \sigma \leq 2} |\zeta(\sigma + iT)| \ll T \exp(c^*(\log \log T)^2).$$

Thus the horizontal lines in total contribute a quantity which is in absolute value

$$\begin{aligned}
&\ll \int_{-1/2+2\delta}^c \left| \zeta(\sigma + iT) \zeta(\sigma + 1 + iT) \frac{g(\sigma + iT) x^{\sigma+iT}}{(\sigma + iT)(\sigma + 1 + iT) \cdots (\sigma + k + iT)} \right| d\sigma \\
(25) \quad &\ll_{\delta} \frac{(x \exp(2c^*(\log \log T)^2))}{T^k}
\end{aligned}$$

Collecting all the estimates, we get

$$(26) \quad E_k(x) \ll \frac{4^k x^c \log x}{T^k} + x^{-1/2+2\delta} (\log T) + \frac{(x \exp(2c^*(\log \log T)^2))}{T^k}.$$

Note that

$$\frac{(x \exp(2c^*(\log \log T)^2))}{T^k} \ll \frac{x}{T^{k-\delta}}$$

for any fixed small positive constant δ . Now we choose $T := 4x^{\frac{3}{2k}}$ so that from (26), we obtain

$$\begin{aligned}
 E_k(x) &\ll \frac{1}{x^{\frac{1}{2}-3\delta}} + \frac{1}{4^{k-\delta}} \cdot \frac{x}{x^{\frac{3}{2}-\frac{3\delta}{2k}}} \\
 &\ll \frac{1}{x^{\frac{1}{2}-3\delta}} + \frac{x}{x^{\frac{3}{2}-2\delta}} \\
 (27) \quad &\ll \frac{1}{x^{\frac{1}{2}-3\delta}}.
 \end{aligned}$$

Note that the implied constant in (27) is independent of k . This proves the theorem provided $c_1(k)$, $c_2(k)$ and $c_3(k)$ are precisely determined. This is done in the following section.

5. EVALUATION OF THE CONSTANTS $c_1(k)$, $c_2(k)$ AND $c_3(k)$

We recall that

$$g(s) := \prod_p \left(1 + \frac{1/p^{s+2} - 1/p^{2s+2}}{(1 - 1/p)} \right)$$

is absolutely and uniformly convergent in any compact set contained in the half-plane $\Re s \geq -\frac{1}{2} + 2\delta$ for any fixed small positive δ and thus, we observe that, in any compact region in the half plane $\sigma \geq -\frac{1}{2} + 2\delta$ (taking the logarithmic derivative of $g(s)$) we have

$$g'(s) = g(s) \left(\sum_p \frac{1}{1 + \frac{1/p^{s+2}(1-1/p^s)}{1-1/p}} \frac{1}{1-1/p} \left(-\frac{\log p}{p^{s+2}}(1-1/p^s) + \frac{1}{p^{s+2}} \frac{\log p}{p^s} \right) \right).$$

We note that $g(0) = 1$ and

$$g(1) = \prod_p \left(1 + \frac{1/p^3 - 1/p^4}{1 - 1/p} \right) = \prod_p \left(1 + \frac{1}{p^3} \right) = \prod_p \left(\frac{(1 - \frac{1}{p^6})}{(1 - \frac{1}{p^3})} \right) = \frac{\zeta(3)}{\zeta(6)}$$

and $g'(0) = \sum_p \frac{\log p}{p(p-1)}$.

The Bernoulli numbers B_n are defined to be the coefficients of the exponential generating function, precisely by the relation

$$\frac{t}{e^t - 1} := \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

We note that $\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}$ for any integer $n \geq 1$, $B_2 = \frac{1}{6}$ and $B_6 = \frac{1}{42}$ so that $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(6) = \frac{(2\pi)^6}{2 \cdot 6! \cdot 42}$. Since $s = 1$ is a simple pole inside the rectangular

contour, we obtain

$$(28) \quad \text{Res}_{s=1} \left(F(s) \frac{x^s}{s(s+1)(s+2)\cdots(s+k)} \right) = \frac{\zeta(2)\zeta(3)}{k!\zeta(6)} x = \frac{315\zeta(3)}{2\pi^4 k!} x.$$

Hence,

$$(29) \quad c_1(k) = \frac{315\zeta(3)}{2\pi^4 k!}.$$

We observe that in the neighbourhood of $s = 0$, we have

$$\zeta(s+1) - \frac{1}{s} = \gamma_0 + \gamma_1 s + \gamma_2 s^2 + \cdots$$

and hence

$$\lim_{s \rightarrow 0} \left(\zeta(s+1) - \frac{1}{s} \right) = \gamma_0.$$

If we write

$$h(s) := \prod_{j=1}^k \frac{1}{(s+j)},$$

then

$$h'(s) = h(s) \left(- \sum_{j=1}^k \frac{1}{(s+j)} \right)$$

and hence

$$h'(0) = h(0) \left(- \sum_{j=1}^k \frac{1}{j} \right) = \frac{1}{k!} \left(- \sum_{j=1}^k \frac{1}{j} \right).$$

Now $s = 0$ is a double pole in the rectangular contour and hence the residue is

$$\begin{aligned} Q_3 &:= \text{Res}_{s=0} \left[\frac{F(s)x^s}{s(s+1)(s+2)\cdots(s+k)} \right] \\ &= \lim_{s \rightarrow 0} \frac{d}{ds} \left(s^2 \frac{F(s)x^s}{s(s+1)(s+2)\cdots(s+k)} \right) \\ &= \lim_{s \rightarrow 0} \frac{d}{ds} \left(s\zeta(s)\zeta(s+1)g(s) \frac{x^s}{(s+1)(s+2)\cdots(s+k)} \right) \\ &= \lim_{s \rightarrow 0} \frac{d}{ds} \left[s\zeta(s) \left(\zeta(s+1) - \frac{1}{s} \right) g(s) \frac{x^s}{(s+1)(s+2)\cdots(s+k)} \right] \\ &\quad + \lim_{s \rightarrow 0} \frac{d}{ds} \left[\zeta(s)g(s) \frac{x^s}{(s+1)(s+2)\cdots(s+k)} \right] \\ &= \frac{1}{k!} \{ \zeta(0)\gamma_0 g(0) + \zeta'(0)g(0) + \zeta(0)g'(0) + \zeta(0)g(0) \log x \} + \zeta(0)g(0)h'(0) \\ &= \frac{1}{k!} \left\{ -\frac{\gamma_0}{2} - \frac{1}{2} \log(2\pi) - \frac{1}{2} \sum_p \frac{\log p}{p(p-1)} - \frac{1}{2} \log x + \frac{1}{2} \left(\sum_{j=1}^k \frac{1}{j} \right) \right\}. \end{aligned}$$

Therefore,

$$(30) \quad c_2(k) = -\frac{1}{2(k!)}$$

and

$$(31) \quad c_3(k) = \frac{1}{k!} \left\{ \frac{1}{2} \left(\sum_{j=1}^k \frac{1}{j} \right) - \frac{\gamma_0}{2} - \frac{1}{2} \log(2\pi) - \frac{1}{2} \sum_p \frac{\log p}{p(p-1)} \right\}.$$

This completes the proof of the main theorem.

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