HAL
open science

Alternating Euler sums at the negative integers.

Khristo Boyadzhiev, H. Gopalkrishna Gadiyar, R Padma

To cite this version:

Khristo Boyadzhiev, H. Gopalkrishna Gadiyar, R Padma. Alternating Euler sums at the negative integers.. Hardy-Ramanujan Journal, 2009, Volume 32 - 2009, pp.24-37. 10.46298/hrj.2009.165 . hal-01112352

HAL Id: hal-01112352

https://hal.science/hal-01112352v1

Submitted on 2 Feb 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Hardy-Ramanujan Journal
Vol. 32 (2009)24-37

Alternating Euler Sums at the Negative Integers

Khristo N. Boyadzhiev
Ohio Northern University
Department of Mathematics
Ada, Ohio, 45810, USA
E-mail: k-boyadzhiev@onu.edu

H. Gopalkrishna Gadiyar and R. Padma
AU−KBC Research Centre
M. I. T. Campus of Anna University
Chromepet, Chennai 600 044, INDIA
E-mail: padma@au-kbc.org
gadiyar@au-kbc.org

This paper is dedicated to the memory of
Professor R. Sitaramachandra Rao

Abstract

We study three special Dirichlet series, two of them alternating, related to the Riemann zeta function. These series are shown to have extensions to the entire complex plane and we find their values at the negative integers (or residues at poles). These values are given in terms of Bernoulli and Euler numbers.

Keywords: Dirichlet series, Bernoulli number, Euler Polynomial, Euler-Maclaurin summation formula, Euler-Boole formula, Euler sum, Hankel contour integration, Riemann zeta function

2000 Mathematics Subject Classification: Primary 11M41, Secondary 11B68

1 Introduction

Let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} \tag{1}$$

($n = 1, 2, ...$) be the harmonic numbers. Apostol-Vu [2] and Matsuoka [8] studied the function

$$h(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} H_n \tag{2}$$

and showed that it has a second-order pole at $s = 1$ and simple poles at zero and the negative odd integers. The values at the negative even integers were evaluated in terms of Bernoulli numbers. These results were discussed and specified further in [6]

In this article we study three similar series

$$u(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} H_n \,, \tag{3}$$

$$v(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} H_n^- \,, \text{ and} \tag{4}$$

$$w(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} H_n^- , \tag{5}$$

where

$$H_n^- = 1 - \frac{1}{2} + \frac{1}{3} + ... + \frac{(-1)^{n-1}}{n}. \tag{6}$$

The functions u and v are well defined and holomorphic for $Re\ s > 0$, as $H_n \sim \log n$ when $n \to \infty$. The function w is defined by (5) for $Re\ s\ > 1$. We prove here that $u(s)$ extends to an entire function on the complex plane $\mathbb{C}$, $w(s)$ extends to $\mathbb{C}$ with a simple pole at $s = 1$, while $v(s)$ extends to $\mathbb{C}$ with poles at $0, -1, -3, \cdots$, much like $h(s)$. We also evaluate these functions (when defined) for $s = 0, -1, -2 \cdots$. The values $v(-2n)$ are connected to a certain convolution of Bernoulli numbers in the spirit of [6].

In our work we shall use the Riemann zeta function ($Re\ s > 1$)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \,, \tag{7}$$

and the Euler-Dirichlet eta function ($Re\ s > 0$)

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}\,, \tag{8}$$

with the relationship between them for all complex numbers

$$\eta(s) = (1 - 2^{1-s})\,\zeta(s). \tag{9}$$

The alternating Euler sums above have been studied and evaluated for certain positive integers by Sitaramachandra Rao [10] in terms of zeta values. See also [7].

2 The Euler-Boole summation formula

In order to show that the function $h(s)$ extends to all complex numbers and to investigate its values at the negative integers, Apostol-Vu and Matsuoka used the Euler-Maclaurin summation formula. For the alternating series $u(s)$ and $v(s)$ it is appropriate to use the alternating version of this formula, called in [3] the Euler-Boole formula (see also p. 34, [9]). The alternating version is less known, but equally important. While the ordinary Euler summation formula is based on the Bernoulli numbers and polynomials, the alternating version is naturally based on the Euler polynomials $E_n(x)$ defined by the generating function [1],

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad (|t| < \pi)\,. \tag{10}$$

Thus

$$E_0(x) = 1, \qquad E_1(x) = x - \frac{1}{2}\,, \tag{11}$$

$$E_2(x) = x^2 - x\,, \qquad E_3(x) \ = x^3 - \frac{3}{2}x^2 + \frac{1}{4}\,, \text{ etc.} \tag{12}$$

The following lemma can be found in [3].

Lemma 1 *Let f be a function defined for $t \geq x$ with continuous derivatives of order q and such that $f^{(k)}(t) \to 0 (t \to \infty), k = 0, 1, \cdots, q$. Then*

$$\sum_{k=0}^{\infty}(-1)^k f(x+k) = \frac{1}{2}\sum_{m=0}^{q-1} \frac{E_m(0)}{m!}\, f^{(m)}(x) + R_q\,, \tag{13}$$

where

$$R_q = \frac{1}{2}\int_0^\infty \bar{E}_{q-1}(-t)\,\frac{f^{(q)}(x+t)}{(q-1)!}\,dt\,, \tag{14}$$

and $\bar{E}_q$ *is an antiperiodic function defined as follows:*

$$\bar{E}_q(x) = E_q(x) \text{ for } 0 \le x \le 1\,,\ \bar{E}_q(x+1) = -\bar{E}_q(x) \text{ etc.} \tag{15}$$

($\bar{E}_q$ is periodic with period 2.)
Note that

$$E_n(0) = \frac{2}{n+1}\,(1-2^{n+1})\,B_{n+1}\,, \tag{16}$$

where B_k are the Bernoulli numbers. Thus

$$E_{2n}(0) = 0,\ n = 1, 2, \cdots\,. \tag{17}$$

(The infinite version of (13) without remainder can be seen in [(3.30), [4]].)

Taking $f(x) = x^{-s}$, $Re\ s > 0$, we find

$$f^{(m)}(x) = (-1)^m s(s+1)\cdots(s+m-1)\frac{1}{x^{s+m}}. \tag{18}$$

Set here $x = n$. Then (13) gives (with some $q\ >\ 1 + |Res|$)

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k}{(n+k)^s} &= \frac{1}{2n^s} - \frac{1}{2}\sum_{m=0}^{q-1}\frac{s(s+1)(s+2)...(s+2m)}{n^{s+2m+1}(2m+1)!}\,E_{2m+1}(0)\\ &+ \frac{s(s+1)(s+2)...(s+2q-1)}{(2q-1)!}\int_0^\infty\frac{\bar{E}_{2q-1}(-t)dt}{2(n+t)^{s+2q}}\,.\end{aligned} \tag{19}$$

Theorem 1 *The function* $u(s)$ *admits an extension to the closed left half plane* $Re\ s \le 0$ *as an entire function. Moreover,*

$$u(0) = \frac{1}{2}\log 2\,, \tag{20}$$

and for $n\ \ge\ 1$,

$$2u(-2n) = \eta(1-2n) + nE_{2n-1}(0)\,, \tag{21}$$

or

$$u(-2n) = \frac{(2^{2n}-1)(1-2n)}{4n} B_{2n}\,. \tag{22}$$

Also,

$$u(-1) = \frac{1}{4} - \frac{1}{4}\log 2 \tag{23}$$

and for $n \geq 2$

$$\begin{aligned} 2u(1-2n) &= E_1(0)\,(2n-1)\,\eta(3-2n) \\ &\quad + \frac{1}{3!}\,E_3(0)\,(2n-1)(2n-2)(2n-3)\,\eta(5-2n) \\ &\quad + \frac{1}{5!}\,E_5(0)\,(2n-1)(2n-2)(2n-3) \\ &\qquad (2n-4)(2n-5)\,\eta(7-2n) \\ &\quad + ... + E_{(2n-1)}(0)\,\eta(1)\,. \end{aligned} \tag{24}$$

Note: In the last term here, $(2n-1)!$ appears in both the numerator and the denominator and thus cancels out.

Proof By changing the order of summation we have

$$\begin{aligned} u(s) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \sum_{k=1}^{n} \frac{1}{k} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=n}^{\infty} \frac{(-1)^{k-1}}{k^s} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)^s}. \end{aligned} \tag{25}$$

The last sum can be evaluated by using (19). Multiplying (19) by $\frac{(-1)^{n-1}}{n}$ and summing from $n=1$ to infinity we obtain

$$\begin{aligned} 2\,u(s) &= \eta(s+1) \\ &\quad - \sum_{m=0}^{q-1} \frac{s(s+1)(s+2)\cdots(s+2m)}{(2m+1)!}\,E_{2m+1}(0)\,\eta(s+2m+2) \\ &\quad + \frac{s(s+1)(s+2)...(s+2q-1)}{(2q-1)!} \int_0^{\infty} \bar{E}_{2q-1}(-t)\,\phi^-(s+2q,t)\;dt\,, \end{aligned} \tag{26}$$

where

$$\phi^-(s,t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,(n+t)^s}\,. \tag{27}$$

This important representation provides a lot of information about the function $u(s)$. It shows that this function extends to $Re\ s \leq 0$. For every complex number s we can take large enough q to show that $u(s)$ becomes well defined. For the evaluation at the negative integers we remind that, according to (8)

$$\eta(-p) = (1-2^{1+p})\,\zeta(-p)\ ,\ p = 0, 1, \cdots\ , \tag{28}$$

and since

$$\zeta(0) = -\frac{1}{2},\ \zeta(-2n) = 0,\ \zeta(1-2n) = -\frac{B_{2n}}{2n},\ n = 1, 2, \cdots\ , \tag{29}$$

we have

$$\eta(0) = \frac{1}{2},\ \eta(-2n) = 0,\ \eta(1-2n) = \frac{2^{2n}-1}{2n} B_{2n},\ n = 1, 2, \cdots\ . \tag{30}$$

Next, setting $s = 0,\ -1,\ -2$, we find

$$u(0) = \frac{1}{2}\eta(1) = \frac{1}{2}\ln 2\,, \tag{31}$$

$$\begin{aligned} 2\,u(-1) &= \eta(0) - E_1(0)\,(-1)\,\eta(1)\,,\ \text{that is,} \\ u(-1) &= \frac{1}{4}(1-\log 2)\,. \end{aligned} \tag{32}$$

$$\begin{aligned} 2\,u(-2) &= \eta(-1) - E_1(0)\,(-2)\,\eta(0)\,,\ \text{that is,} \\ u(-2) &= -\frac{1}{8}\,. \end{aligned} \tag{33}$$

Following this pattern, we obtain from (26) the general formula

$$2\,u(-2n) = \eta(1-2n) + 2\,n\,E_{2n-1}(0)\,\eta(0)\,, \tag{34}$$

which gives (21).

The function $u(s)$ was evaluated for the positive even integers by Sitaramachandra Rao [10]. For instance,

$$u(2) = \frac{5}{8}\,\zeta(3)\,. \tag{35}$$

Remark 1 Instead of the function $u(s)$ one can consider the more general function

$$\begin{aligned} u(s;p) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} (1 + \frac{1}{2^p} + ... + \frac{1}{n^p}) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)^s} , \end{aligned} \tag{36}$$

where $p \geq 1$. The above proof shows that the only change in the representation (26) will be a shift $s \to s + p - 1$ in the functions $\eta(s)$ and $\phi^-(s)$. All other formulas will undergo the same shift. Therefore, the essential case is $p = 1$. This remark applies also to the functions $v(s)$ and $w(s)$.

Remark 2 The numbers

$$G_n = 2(1 - 2^n)\, B_n = n\, E_{n-1}(0) \tag{37}$$

are known as the Genocchi [Jenoki] numbers. They are defined by the generating function

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \quad (|t| < \pi) . \tag{38}$$

3 The function $v(s)$

Theorem 2 *The function $v(s)$ has the representation*

$$\begin{aligned} 2v(s) &= \zeta(s+1) - \sum_{m=0}^{q-1} \frac{s(s+1)(s+2)...(s+2m)}{(2m+1)!} E_{2m+1}(0)\, \zeta(s+2m+2) \\ &\quad + \frac{s(s+1)(s+2)...(s+2q-1)}{(2q-1)!} \int_0^{\infty} \bar{E}_{2q-1}(-t)\, \phi^+(s+2q,t)\ dt , \end{aligned} \tag{39}$$

where $q > 1 + |Res|$ is a positive integer and

$$\phi^+(s,t) = \sum_{n=1}^{\infty} \frac{1}{n\,(n+t)^s} . \tag{40}$$

Therefore, it admits a holomorphic extension to the closed left half plane $Re\, s \leq 0$ with simple poles at $s = 0, -1, -3, \cdots$. The residues at these poles are as follows.

$$Res[s = 0] = \frac{1}{2}\,. \tag{41}$$

$$Res[s = -2n - 1] = \frac{1}{2}\, E_{2n+1}(0)\,, (\ n = 0,\ 1, \cdots)\,. \tag{42}$$

Moreover, for $n = 1,\ 2, \cdots$,

$$2v(-2n) = \zeta(1 - 2n) - n\, E_{2n-1}(0)\,. \tag{43}$$

Proof Changing the order of summation we obtain

$$v(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)^s}\,. \tag{44}$$

The representation (39) then follows from (19).

4 The function $w(s)$

Theorem 3 *The function $w(s)$ defined by (5) for Re $s > 1$ has the representation*

$$\begin{aligned} w(s) &= \frac{\eta(s)}{s-1} + \frac{1}{2}\,\eta(s+1) \\ &\quad + \sum_{m=1}^{q} \frac{s(s+1)...(s+2m-2)}{(2m)!}\, B_{2m}\, \eta(s+2m) + R_q(s)\,, \end{aligned} \tag{45}$$

where

$$R_q(s) = -\frac{s(s+1)...(s+2q)}{(2q+1)!} \int_0^{\infty} \bar{B}_{2q-1}(t)\, \phi^-(s+2q+1, t)\, dt\,, \tag{46}$$

where B_{2m} are the Bernoulli numbers, q is a sufficiently large positive integer and $\bar{B}_k(t)$ are bounded functions. Thus $w(s)$ extends as an analytic function to all complex numbers with a simple pole at $s = 1$ with residue $\eta(1) =$

$\log 2$. *Its values* $w(-p)$, $p = 0,\ 1,\ 2, \cdots$, *can be found from the above representation (note that* $R_q(-p) = 0$ *when* $2q > p$ *). In particular,*

$$w(0) = \frac{1}{2}\eta(1) - \eta(0), \tag{47}$$

$$w(-1) = \frac{-1}{2}\,\eta(-1) + \frac{1}{2}\,\eta(0) - \frac{B_2}{2!}\,\eta(1)\,, \tag{48}$$

$$w(-2) = \frac{1}{2}\,\eta(-1) - B_2\,\eta(0)\,, \tag{49}$$

etc. The general formulas are

$$w(-2n) = \frac{1}{2}\,\eta(1-2n) - B_{2n}\,\eta(0)\,, \tag{50}$$

and

$$\begin{aligned} w(1-2n) &= \frac{-1}{2n}\,\eta(1-2n) \\ &\quad - \sum_{m=1}^{n} \frac{(2n-1)(2n-2)\cdots(2n-2m+1)}{(2m)!} B_{2m}\eta(-2n+2m+1), \end{aligned} \tag{51}$$

for $n = 1,\ 2, \cdots$.

Proof Changing the order of summation (just like in (25) and (44)) we write

$$w(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{k=0}^{\infty} \frac{1}{(n+k)^s}\,. \tag{52}$$

The second sum on the right hand side is not alternating and we need to use the Euler-Maclaurin summation formula (as Apostol and Vu did on p. 88 in [2]),

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(n+k)^s} &= \frac{1}{s-1}\frac{1}{n^{s-1}} + \frac{1}{2n^s} \\ &\quad + \sum_{m=1}^{q} \frac{s(s+1)...(s+2m-2)}{(2m)!} \frac{B_{2m}}{n^{s+2m-1}} \\ &\quad - \frac{s(s+1)...(s+2q)}{(2q+1)!} \int_0^{\infty} \frac{\bar{B}_{2q-1}(t)dt}{(n+t)^{s+2q+1}}\,, \end{aligned} \tag{53}$$

where q is a large enough positive integer and $\bar{B}_k(t)$ is the periodic extension of the Bernoulli polynomial $B_k(t)$ from $[0,1]$ to $\mathbb{R}$ with period 1.

Multiplying (53) by $\frac{(-1)^{n-1}}{n}$ and summing for $n \geq 1$ we obtain the desired representation (45).

5 Equation for $v(s)$ with a Hankel contour integral

Our next theorem shows a connection between the function $v(s)$ defined by (4) and a special Hankel integral. For $s \neq 1,\ 2, \cdots$, define

$$G(s) = \frac{\Gamma(1-s)}{2\pi i} \int_L \frac{z^{s-1}e^z}{1+e^z} \log\left(\frac{e^z-1}{z}\right)\, dz\,, \tag{54}$$

where the contour $L = L_- \cup L_+ \cup L_\epsilon$ consists of three parts: L_- is the "lower side" (i.e. $\arg(z) = -\pi$) of the ray $(-\infty, -\epsilon), \epsilon > 0$, traced left to right, L_+ is the "upper side" ($\arg(z) = \pi$) of this ray traced right to left, and $L_\epsilon = \{z = \epsilon e^{\theta i} : -\pi \leq \theta \leq \pi\}$ is a small circle traced counterclockwise and connecting the two sides of that ray. This contour was used also in [6].

The following theorem is a modification of the main result in [6].

Theorem 4 *For every complex s where both sides are defined we have the equation*

$$G(s) = v(s) - \zeta(s+1) - \psi(s)\,\eta(s) - \eta'(s)\,. \tag{55}$$

(Here $\psi(s)$ is the digamma function and the eta function is as defined in (8).)

Proof For $Re\ s > 1$ set

$$I(s) = \frac{1}{2\pi i} \int_L \frac{z^{s-1}e^z}{1+e^z} \log\left(\frac{e^z-1}{z}\right)\, dz\,. \tag{56}$$

Now let $\epsilon \to 0$. The integral over L_ϵ turns into zero, as the function $\dfrac{e^z}{1+e^z} \log\left(\dfrac{e^z-1}{z}\right)$ is holomorphic in a neighborhood of zero. Noticing that $z = xe^{-\pi i}$ on L_- and $z = xe^{\pi i}$ on L_+ , we find that

$$\begin{aligned} I(s) &= \frac{e^{-\pi i s}}{2\pi i} \int_\infty^0 \frac{x^{s-1}e^{-x}}{1+e^{-x}} \log\left(\frac{1-e^{-x}}{x}\right)\, dx \\ &\quad + \frac{e^{\pi i s}}{2\pi i} \int_0^\infty \frac{x^{s-1}e^{-x}}{1+e^{-x}} \log\left(\frac{1-e^{-x}}{x}\right)\, dx \qquad (57) \\ &= \frac{\sin \pi s}{\pi} \int_0^\infty \frac{x^{s-1}e^{-x}}{1+e^{-x}} \log\left(\frac{1-e^{-x}}{x}\right)\, dx\,. \qquad (58) \end{aligned}$$

At the same time,

$$\int_0^\infty \frac{x^{s-1}e^{-x}}{1+e^{-x}} \log\left(\frac{1-e^{-x}}{x}\right)\, dx = \int_0^\infty \frac{x^{s-1}e^{-x}}{1+e^{-x}} \log(1-e^{-x})\, dx - \int_0^\infty \frac{x^{s-1}}{e^x+1} \log x\, dx\,, \tag{59}$$

and the two integrals on the right hand side are easy to evaluate. First, we use the representation

$$\frac{\log(1-e^{-x})}{1+e^{-x}} = \sum_{n=1}^\infty (-1)^n H_n^- e^{-nx}\,. \tag{60}$$

Multiplying both sides here by $x^{s-1}e^{-x}$ and integrating from zero to infinity we find

$$\begin{aligned}\int_0^\infty \frac{x^{s-1}e^{-x}}{1+e^{-x}} \log(1-e^{-x})\, dx &= \sum_{n=1}^\infty (-1)^n H_n^- \int_0^\infty x^{s-1}e^{-(n+1)x}\, dx \\ &= \Gamma(s) \sum_{n=1}^\infty \frac{(-1)^n H_n^-}{(n+1)^s} \\ &= \Gamma(s)(v(s)-\zeta(s+1))\,. \end{aligned} \tag{61}$$

Second, differentiating for s the well-known representation

$$\Gamma(s)\eta(s) = \int_0^\infty \frac{x^{s-1}}{1+e^x}\, dx\,, \tag{62}$$

we obtain

$$\begin{aligned}\int_0^\infty \frac{x^{s-1}}{1+e^x} \log x\, dx &= \Gamma'(s)\,\eta(s) + \Gamma(s)\,\eta'(s) \\ &= \Gamma(s)\,(\psi(s)\,\eta(s) + \eta'(s))\,. \end{aligned} \tag{63}$$

Collecting together the two evaluations (61) and (63) we obtain

$$\int_0^\infty \frac{x^{s-1}e^{-x}}{1+e^{-x}} \log\left(\frac{1-e^{-x}}{x}\right)\, dx = \Gamma(s)\,(v(s)-\zeta(s+1)-\psi(s)\,\eta(s)-\eta'(s)), \tag{64}$$

that is,

$$I(s) = \frac{1}{\pi}\,\Gamma(s)\,\sin(\pi s)\,(v(s)-\zeta(s+1)-\psi(s)\,\eta(s)-\eta'(s)), \tag{65}$$

and the theorem follows from here in view of the identity

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin \pi s}\,. \tag{66}$$

6 A special identity for Bernoulli numbers

In this section we shall find a relation between the values $v(-2n)$ and the numbers C_n , $n = 1,\ 2, \cdots$, defined as the convolutions.

$$C_n = \frac{1}{2} \sum_{k+j=n} \frac{1}{j!} E_j(1) \frac{B(k)}{k!\,k}\,,\ k = 1,\ 2, \cdots ,;\ j = 0,\ 1,\ \cdots , \tag{67}$$

where $B(n) = B_n$ are the Bernoulli numbers for $n \neq 1$ and $B(1) = -B_1 = 1/2$. Thus for $|z| < \pi$,

$$\frac{ze^z}{e^z - 1} = \frac{-z}{e^{-z} - 1} = \sum_{n=0}^{\infty} \frac{B(n)}{n!} z^n\,. \tag{68}$$

The notation $B(n)$ is used here in order to avoid the negative sign in B_1 . Integrating (68) we find

$$\log\left(\frac{e^z - 1}{z}\right) = \sum_{n=1}^{\infty} \frac{B(n)}{n!\,n} z^n\,. \tag{69}$$

¿From (10)

$$\frac{e^z}{e^z + 1} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} E_n(1)\, z^n\,. \tag{70}$$

The product of the functions in (69) and (70) is the generating function of the numbers C_n ,

$$\frac{e^z}{e^z + 1} \log\left(\frac{e^z - 1}{z}\right) = \sum_{n=1}^{\infty} C_n\, z^n\,. \tag{71}$$

The Euler polynomials given by (10) have the property

$$E_n(t + 1) + E_n(t) = 2\, t^n\,, \tag{72}$$

which shows that

$$E_n(1) = -E_n(0)\,,\ n \geq 1\,, \tag{73}$$

(and also $E_0(1) = E_0(0) = 1$ from (11)).

Corollary 1 *For $n = 1,\ 2, \cdots$, we have*

$$(2n)!\, C_{2n} = \left(\frac{1}{4n} + 2^{2n-1} - \frac{1}{2} \right) B_{2n}\,. \tag{74}$$

Proof This follows from Theorem 2 (Equation (43)) and Theorem 4. From (55)

$$G(-2n) = v(-2n) - \zeta(1-2n) - (\psi(s)\,\eta(s) + \eta'(s))|_{s=-2n}\,, \tag{75}$$

where the last term is, in fact, zero (cf Corollary 1, [6]). This is so because we can use the equation

$$\psi(s) = \psi(1-s) - \pi \cot \pi s \tag{76}$$

and write

$$\psi(s)\eta(s) = \psi(1-s)\eta(s) - \eta(s)\pi\, \cot \pi s\,. \tag{77}$$

For $s = -2n$ we have $\psi(1+2n)\,\eta(-2n) = 0$ and

$$\eta(s)\,\pi\, \cot \pi s|_{s=-2n} = \eta'(-2n)\,, \tag{78}$$

as follows from the Taylor expansion around $s = -2n$,

$$\eta(s)\,\pi\, \cot \pi s = \eta'(-2n) + \frac{1}{2}\,\eta''(-2n)\,(s+2n) + O((s+2n)^2)\,. \tag{79}$$

Therefore,

$$G(-2n) = v(-2n) - \zeta(1-2n)\,. \tag{80}$$

At the same time, the numbers $G(-2n)$ are proportional to the Taylor coefficients of the series (71), that is,

$$G(-2n) = \frac{(2n)!}{2\pi i} \int_{L_\epsilon} \frac{e^z}{1+e^z}\, \log\left(\frac{e^z - 1}{z} \right) \frac{dz}{z^{2n+1}} = (2n)!\, C_{2n}\,. \tag{81}$$

This representation results from (54). For this special integrand the integrals on L_+ and L_- cancel out and (81) follows from Cauchy's formula. From this we find immediately that

$$(2n)!\, C_{2n} = v(-2n) - \zeta(1-2n)\,, \tag{82}$$

and (74) follows in view of (16), (29) and (43).

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions.* Dover, New York, 1970.

[2] Tom Apostol and T. H. Vu, Dirichlet series related to the Riemann zeta function, *J. Number Theory*, 19 (1984), 85-102.

[3] J. M. Borwein, P. B. Borwein, and K. Dilcher, Pi, Euler Numbers, and Asymptotic Expansions, *Amer. Math. Monthly*, 96 (8), 681-687, (1989).

[4] Khristo N. Boyadzhiev, A series transformation formula and related polynomials, *Int. J. Math. Math. Sci.*. 2005:23 (2005) 3849-3866.

[5] Khristo N. Boyadzhiev, Evaluation of series with Hurwitz and Lerch zeta function coefficients by using Hankel contour integrals, *Applied Math. Comput,*, Vol 186/2 (2007), 1559-1571.

[6] Khristo N. Boyadzhiev, H. Gopalkrishna Gadiyar and R. Padma, The values of an Euler sum at negative integers and a relation to a certain convolution of Bernoulli numbers, *Bull. Korean. Math. Soc.*, 45 (2), 277-283 (2008).

[7] Philippe Flajolet and Bruno Salvy, Euler Sums and Contour Integral Representations, *Experimental Mathematics*, 7, (1) 15-35 (1998).

[8] Yasushi Matsuoka, On the values of a certain Dirichlet series at rational integers, *Tokyo J. Math.*, 5, No.2, 399-403 (1982).

[9] Niels Erik Nörlund, *Vorlesungen Über Differenzenreichnung*, Berlin, 1924.

[10] R. Sitaramachandra Rao, A formula of S. Ramanujan, *J. Number Theory*, 25, 1-19 (1987).

Received and Accepted on April 21, 2009