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LOWER BOUNDS FOR THE EIGENVALUES OF THE Spin^c DIRAC OPERATOR ON SUBMANIFOLDS

ROGER NAKAD AND JULIEN ROTH

ABSTRACT. We prove lower bounds for the eigenvalues of the Spin^c Dirac operator on submanifolds. These estimates are expressed in terms of extrinsic and intrinsic quantities. We also give estimates involving the Energy-Momentum tensor as well as conformal bounds. The limiting cases of these estimates give rise to particular spinor fields, called *generalized twisted Killing spinors*, which are also studied.

1. INTRODUCTION AND PRELIMINARIES

The limiting cases of estimates for eigenvalues of the Dirac operator on compact (with or without boundary) manifolds give rise to examples of special geometries. For instance, equality in the classical inequality of Friedrich [3]

$$\lambda_1 \leq \frac{n}{4(n-1)} \inf_M \text{Scal}_M,$$

where n is the dimension of the manifold M and Scal_M its scalar curvature, forces the manifold to be Einstein with positive scalar curvature, due to the fact that the eigenspinor associated with the first eigenvalue is then a real Killing spinor.

This is also the case for the conformal inequality of Hijazi [9]

$$\lambda_1 \leq \frac{n}{4(n-1)} \mu_1,$$

where μ_1 is the first eigenvalue of the Yamabe operator as well for the inequality involving the Energy-momentum tensor [10]

$$\lambda_1 \leq \frac{1}{4} \inf_M (\text{Scal}_M + |Q_\varphi|^2),$$

where Q_φ is the Energy-Momentum tensor associated with a first eigenspinor φ .

On the other hand, in the recent years, many estimates have been proved for the eigenvalues of the Spin^c Dirac operator. A great difference between both cases is that the equality cases for Spin^c lower bounds are less and therefore give larger classes of limiting manifolds (see [8, 13, 14] for instance).

In this note, we prove a new lower bound for the eigenvalues of the Spin^c Dirac operator on submanifolds of Spin^c manifolds (see Theorem 2.1). This generalizes for the Spin^c case, previous estimates by Hijazi-Zhang [11, 12] and Ginoux-Morel [5]. The limiting case of this estimate is characterized by the existence of particular spinor fields called *generalized twisted Killing spinors*. We will study these particular spinor fields and show that under a natural assumption on the dimension and the codimension of the submanifold, they are in fact *twisted Killing spinors* which generalize naturally the usual Killing spinors for twisted spinor bundles.

Finally, we also prove conformal estimates (Theorem 2.3) as well as estimates involving the Energy-Momentum tensor associated with an eigenspinor (Theorem 2.2).

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We recall briefly some basic facts about Spin^c manifolds and their hypersurfaces (the reader can refer to [1, 2, 4, 5]). Let $(\widetilde{M}^{m+n}, \widetilde{g})$ be a Riemannian Spin^c manifold and M^m a submanifold isometrically immersed into \widetilde{M} . Assume that M is also Spin^c and denote by NM the normal bundle of the immersion of M into \widetilde{M} . We denote by $i\widetilde{\Omega}$ (resp. $i\Omega$) the curvature 2-form of the corresponding auxiliary line bundle. Since the manifolds M and \widetilde{M} are Spin^c , there exists a Spin^c structure on the bundle NM . We denote by ΣN the Spin^c bundle of NM and let

$$\Sigma := \begin{cases} \Sigma M \otimes \Sigma N & \text{if } m \text{ or } n \text{ is even,} \\ \Sigma M \otimes \Sigma N \oplus \Sigma M \otimes \Sigma N & \text{otherwise} \end{cases}$$

It is well known that there is a natural isomorphism between Σ and $\Sigma \widetilde{M}|_M$. Moreover, we denote by ∇ the covariant derivative on Σ defined by

$$\nabla := \begin{cases} \nabla^{\Sigma M} \otimes \text{Id} + \text{Id} \otimes \nabla^{\Sigma N} & \text{if } m \text{ or } n \text{ is even,} \\ \nabla^{\Sigma M} \otimes \text{Id} + \text{Id} \otimes \nabla^{\Sigma N} \oplus \nabla^{\Sigma M} \otimes \text{Id} + \text{Id} \otimes \nabla^{\Sigma N} & \text{otherwise.} \end{cases}$$

We have the following identification between the Clifford multiplication

$$(1) \quad X \cdot_M \varphi = (X \cdot \omega_\perp \cdot \psi)|_M,$$

where $\varphi = \psi|_M$, $\psi \in \Gamma(\Sigma \widetilde{M})$, $\omega_\perp := \omega_n$ if n is even and $\omega_\perp = -i\omega_n$ if n is odd, with $\omega_n = i^{\lfloor \frac{n+1}{2} \rfloor} \nu_1 \cdots \nu_n$ the complex volume element of the normal bundle. We also recall the Spin^c Gauss formula

$$(2) \quad \widetilde{\nabla}_X \varphi = \nabla_X \varphi + \frac{1}{2} \sum_{j=1}^m e_j \cdot B(X, e_j) \cdot \varphi,$$

where $\widetilde{\nabla}$ is the spinorial connection on \widetilde{M} and B is the second fundamental form of M in \widetilde{M} . We will denote by H the mean curvature.

Now, let us define the following Dirac operators

$$\mathbf{D} = \sum_{j=1}^m e_j \cdot \nabla_{e_j}, \quad \widetilde{D} := \sum_{j=1}^m e_j \cdot \widetilde{\nabla}_{e_j},$$

and

$$D_H = (-1)^n \omega_\perp \cdot \widetilde{D} = (-1)^n \omega_\perp \cdot \mathbf{D} + \frac{1}{2} H \cdot \omega_\perp \cdot .$$

Clearly, from the spinorial Gauss formula (2), we have $\widetilde{D} = \mathbf{D} - \frac{1}{2} H \cdot$. Moreover (see [1, Lemma 2.1]), \mathbf{D} and D_H are formally self-adjoint and $D_H^2 = \widetilde{D}^* \widetilde{D}$ where \widetilde{D}^* is the formal adjoint of \widetilde{D} with respect to the L^2 -scalar product $\int_M (\cdot, \cdot) dv_g$.

We finish this section of preliminaries by the two following lemmas. The first one generalizes the classical Lichnerowicz formula in the context of twisted Spin^c spinor bundles. Before stating the lemma, we need to introduce the following function associated to a spinor field $\varphi \in \Gamma(\Sigma)$, $R_\varphi^N := 2 \sum_{i,j=1}^n \langle e_i \cdot e_j \cdot \text{Id} \otimes \mathcal{R}_{e_i, e_j}^N \varphi, \frac{\varphi}{|\varphi|^2} \rangle$, on $M_\varphi = \{x \in M \mid \varphi(x) \neq 0\}$. Here, \mathcal{R}_{e_i, e_j}^N stands for the spinorial curvature of the normal bundle NM .

Proposition 1.1 (Twisted Lichnerowicz formula). *For any spinor field $\varphi \in \Gamma(\Sigma)$, point-wise on M_φ , we have*

$$(3) \quad \langle \mathbf{D}^2 \varphi, \varphi \rangle = \langle \nabla^* \nabla \varphi, \varphi \rangle + \frac{1}{4} (\text{Scal}_M + R_\varphi^N) |\varphi|^2 + \frac{i}{2} \langle \Omega \cdot \varphi, \varphi \rangle.$$

Proof: We give the proof in the case where m or n is even. The other case is similar. We compute the square of the Dirac operator \mathbf{D} acting on $\varphi = \alpha \otimes \sigma$. We have

$$(4) \quad \mathbf{D}^2 \varphi = \nabla^* \nabla \varphi + \frac{1}{2} \sum_{i,j=1}^m e_i \cdot e_j \cdot \mathcal{R}_{e_i, e_j} \varphi,$$

where ∇^* is the formal adjoint of ∇ and \mathcal{R} is the spinorial curvature associated with the connection ∇ . From the definition of ∇ , we see easily that

$$\mathcal{R}_{e_i, e_j} \varphi = (\mathcal{R}_{e_i, e_j}^M \sigma) \otimes \alpha + \sigma \otimes (\mathcal{R}_{e_i, e_j}^N \alpha).$$

Then, a classical computation on each factor gives the desired formula. \square

We have this second elementary lemma

Proposition 1.2. *For any spinor field $\varphi \in \Gamma(\Sigma)$, we have*

$$(5) \quad \langle i\Omega \cdot \varphi, \varphi \rangle \geq -\frac{c_m}{2} |\Omega| |\varphi|^2.$$

Proof: If m or n is even, then $\Sigma M \otimes \Sigma N$ and a spinor $\varphi \in \Sigma$ can be written $\varphi = \sigma \otimes \alpha$. Hence, we have

$$\begin{aligned} \langle i\Omega \cdot \varphi, \varphi \rangle &= \langle i\Omega \cdot (\sigma \otimes \alpha), \sigma \otimes \alpha \rangle = \langle i(\Omega \cdot_M \sigma) \otimes \alpha, \sigma \otimes \alpha \rangle \\ &= \langle i(\Omega \cdot_M \sigma), \sigma \rangle |\alpha|^2 \geq -\frac{c_m}{2} |\Omega| |\sigma|^2 |\alpha|^2 \geq -\frac{c_m}{2} |\Omega| |\varphi|^2. \end{aligned}$$

Note that we use the fact that the scalar product on Σ is the product of the scalar products on ΣM and ΣN . We also use the classical estimate on M , that is $\langle i\Omega \cdot \sigma, \sigma \rangle \geq -\frac{c_m}{2} |\Omega| |\sigma|^2$ (see [8]). If m and n are odd, then $\Sigma = \Sigma M \otimes \Sigma N \oplus \Sigma M \otimes \Sigma N$ and a spinor $\varphi \in \Sigma$ is of the form $\varphi = (\alpha \otimes \sigma, \sigma' \otimes \alpha')$. Thus, we have

$$\begin{aligned} \langle i\Omega \cdot \varphi, \varphi \rangle &= \langle i\Omega \cdot (\sigma \otimes \alpha), \sigma \otimes \alpha \rangle + \langle i\Omega \cdot (\sigma' \otimes \alpha'), \sigma' \otimes \alpha' \rangle \\ &\geq -\frac{c_m}{2} |\Omega| (|\sigma|^2 |\alpha|^2 + |\sigma'|^2 |\alpha'|^2) \geq -\frac{c_m}{2} |\Omega| |\varphi|^2. \end{aligned}$$

This concludes the proof. \square

2. EIGENVALUE ESTIMATES FOR SUBMANIFOLDS

Now, we have all the ingredients to state the eigenvalue estimates. We begin by the following basic estimates involving intrinsic terms (scalar curvature, curvature of the line bundle over M) and extrinsic terms (mean curvature and spinorial normal curvature). This result generalizes in the Spin^c setting the estimate of Hijazi-Zhang [11] (extending to any codimension by Ginoux-Morel [5]).

Theorem 2.1. *Let (M^m, g) be a compact Riemannian Spin^c manifold isometrically immersed into a Riemannian Spin^c manifold $(\widetilde{M}^{m+n}, \widetilde{g})$. Consider a non-trivial eigenspinor field $\varphi \in \Gamma(\Sigma)$ for the submanifold Dirac operator D_H , i.e. $D_H \varphi = \lambda \varphi$. Assume that $m \geq 2$ and*

$$\text{Scal}_M + R_\varphi^E - c_m |\Omega^M| > \frac{m-1}{m} \|H\|^2 > 0$$

on M_φ , then, we have

$$\lambda^2 \geq \frac{1}{4} \inf_{M_\varphi} \left(\sqrt{\frac{m}{m-1} (\text{Scal}_M + R_\varphi^N - c_m |\Omega|)} - \|H\| \right)^2.$$

Moreover, if equality holds, then φ is a twisted generalized Killing spinor.

Proof: Let λ be an eigenvalue of the submanifold Dirac operator D_H and q a smooth function, nowhere equal to $\frac{1}{m}$. We consider the following modified connection $\nabla^{\lambda,q}$ defined by

$$\nabla_X^{\lambda,q}\psi = \nabla_X\psi + \frac{1-q}{2(1-mq)}X \cdot H \cdot \psi + q\lambda X \cdot \omega_\perp \cdot \psi,$$

for any spinor field $\psi \in \Gamma(\Sigma)$. Let φ be an eigenspinor for D_H associated with the eigenvalue λ . Using the Twisted Lichnerowicz formula (3), we can easily compute

$$\begin{aligned} \int_M |\nabla^{\lambda,q}\varphi|^2 v_g &= \int_M (1 + mq^2 - 2q) \left[\lambda^2 - \frac{1}{4} \left(\frac{\text{Scal}_M + R_\varphi^N}{1 + mq^2 - 2q} - \frac{(m-1)\|H\|^2}{(1-mq)^2} \right) |\varphi|^2 \right. \\ (6) \quad &\quad \left. - \frac{i}{2(1 + mq^2 - 2q)} \langle \Omega \cdot \varphi, \varphi \rangle \right] v_g \end{aligned}$$

Then, using Inequality (5) and by assuming $m(\text{Scal}_M + R_\varphi^N - c_m|\Omega|) > (m-1)\|H\|^2 > 0$, we can choose q so that

$$(7) \quad (1 - mq)^2 = \frac{(m-1)\|H\|}{\sqrt{\frac{m}{m-1}(\text{Scal}_M + R_\varphi^N - c_m|\Omega|) - \|H\|}},$$

on M_φ . Inserting (7) in (6), and since the complement of M_ψ in M is of measure 0, we conclude because the left member of (6) is nonnegative.

If equality occurs, $\nabla_X^{\lambda,q}\varphi = 0$, which implies that $|\varphi|$ is constant on M and

$$\mathbf{D}\varphi = \frac{m(1-q)}{2(1-mq)}H \cdot \varphi + mq\lambda\omega_\perp \cdot \varphi.$$

On the other hand, since φ is an eigenspinor for D_H and the link between D_H and \mathbf{D} , we get

$$\begin{aligned} 0 &= \lambda\omega_\perp \cdot \varphi + \frac{H}{2} \cdot \varphi - \frac{m(1-q)}{2(1-mq)}H \cdot \varphi - mq\lambda\omega_\perp \cdot \varphi \\ &= (1-mq)^2\lambda\omega_\perp \cdot \varphi - (m-1)\frac{H}{2} \cdot \varphi. \end{aligned}$$

Moreover, equality also implies $2|\lambda| = \sqrt{\frac{m}{m-1}(\text{Scal}_M + R_\varphi^N - c_m|\Omega|) - \|H\|}$. From the expression of q and the above relation, we have $\omega_\perp \cdot \varphi = \text{sgn}(\lambda)\frac{H}{\|H\|} \cdot \varphi$ and thus φ satisfies $\nabla_X\varphi = -\frac{f}{m}X \cdot \omega_\perp \cdot \varphi$, with $f = \frac{\text{sgn}(\lambda)}{2}\sqrt{\frac{m}{m-1}(\text{Scal}_M + R_\varphi^N - c_m|\Omega|)}$. That is, ψ is a generalized twisted Killing spinor. Note that here f is *a priori* a function. We will see in Section 3 that under some assumptions on the dimensions m and n , the function f is constant. \square

Now, we define the Energy-Momentum tensor associated with a spinor field $\psi \in \Gamma(\Sigma)$ on M_ψ by

$$Q_{ij}^\psi = \frac{1}{2}(e_i \cdot \omega_\perp \cdot \nabla_{e_j} + e_j \cdot \omega_\perp \cdot \nabla_{e_i}, \frac{\psi}{|\psi|^2})$$

Note that

$$Q_{ij}^\psi = \frac{1}{2}(e_i \cdot_M \nabla_{e_j} + e_j \cdot_M \nabla_{e_i}, \frac{\psi}{|\psi|^2})$$

so it is the intrinsic energy momentum tensor and it is the one appearing in the Einstein Dirac equation. We have the following estimate involving the Energy-Momentum tensor.

Theorem 2.2. *Let (M^m, g) be a compact Riemannian Spin^c manifold isometrically immersed into a Riemannian Spin^c manifold $(\widetilde{M}^{m+n}, \widetilde{g})$. Consider a non-trivial eigenspinor field $\varphi \in \Gamma(\Sigma)$ for the submanifold Dirac operator D_H , i.e. $D_H\varphi = \lambda\varphi$. Assume that $m \geq 2$ and*

$$\text{Scal}_M + R_\varphi^N + 4|Q^\varphi|^2 - c_m|\Omega^M| > \|H\|^2 > 0$$

on M_φ , then, we have

$$\lambda^2 \geq \frac{1}{4} \inf_{M_\varphi} (Scal_M + R_\varphi^N + 4|Q^\varphi|^2 - c_m|\Omega|) - \|H\|^2.$$

Moreover, if equality holds, then φ is a twisted (symmetric) EM-spinor.

Proof: For any real function q that never vanishes, consider the modified covariant derivative defined on $\Gamma(\Sigma)$ by

$$\nabla_{e_i}^Q \psi = \nabla_{e_i} \psi - \frac{1}{2mq} e_i \cdot H \cdot \psi + (-1)^{n+1} q \lambda e_i \cdot \omega_\perp \cdot \psi + \sum_j Q_{ij}^\psi e_j \cdot \omega_\perp \cdot \psi$$

Again, we can compute, for an eigenspinor φ

$$\begin{aligned} \int_M |\nabla^Q \varphi|^2 v_g &= \int_M (1 + mq)^2 \left[\lambda^2 - \frac{1}{4} \left(\frac{Scal_M + R_\varphi^N + 4|Q^\varphi|^2}{(1 + mq^2)} - \frac{\|H\|^2}{mq^2} \right) \right] |\varphi|^2 v_g \\ &\quad - \frac{1}{4} \int_M (1 + mq^2) \left[\frac{2}{mq(1 + mq^2)} \left(\|H\|^2 - \frac{\langle H \cdot \varphi, \omega_\perp \cdot \varphi \rangle^2}{|\varphi|^4} \right) \right] |\psi|^2 v_g \\ (8) \quad &\quad - \frac{i}{2} \int_M \langle \Omega \cdot \varphi, \varphi \rangle v_g \end{aligned}$$

Now, we use again (5) and if moreover, $Scal_M + R_\varphi^N - c_m|\Omega| + 4|Q^\varphi|^2 > \|H\|^2 > 0$, we take

$$q = \sqrt{\frac{\|H\|}{m(\sqrt{Scal_M + R_\varphi^N - c_m|\Omega| + 4|Q^\varphi|^2} - \|H\|)}},$$

and then by the Cauchy Schwarz inequality, we have

$$\|H\|^2 - \frac{\langle H \cdot \varphi, \omega_\perp \cdot \varphi \rangle^2}{|\varphi|^4} \geq 0.$$

If equality holds, then $\nabla^Q \varphi = 0$ and equality occurs in the Cauchy-Schwarz inequality, that is, $\|H\|^2 - \frac{\langle H \cdot \varphi, \omega_\perp \cdot \varphi \rangle^2}{|\varphi|^4} = 0$. Thus, proceeding as in the proof of Theorem 2.1, we deduce that $\nabla_X \varphi = -Q(X) \cdot \omega_\perp \cdot \varphi$, that is ψ is a twisted (symmetric) Energy-Momentum spinor (EM-spinor). \square

Note that, by a completely similar computation, we can obtain a lower bound involving both symmetric and skew-symmetric Energy-Momentum tensors as in [7]. We do not write it in this note.

Finally, following the idea of Hijazi [9], we consider a conformal change of the metric $\bar{g} = e^{2u} \tilde{g}$. Let $\Sigma \rightarrow \bar{\Sigma}$, $\psi \rightarrow \bar{\psi}$ be the corresponding isometry between the two spinor bundles. Recall that for 2 spinors ψ and φ on Σ and for any vector field X on \tilde{M} , we have

$$(\varphi, \psi) = (\bar{\varphi}, \bar{\psi})_{\bar{g}} \quad \text{and} \quad \bar{X} \cdot \bar{\psi} = \overline{X \cdot \psi}$$

Note that we have

$$\bar{\mathbf{D}}(e^{-\frac{(m-1)}{2}u} \bar{\psi}) = e^{-\frac{(m+1)}{2}u} \mathbf{D} \bar{\psi}$$

where $\bar{\mathbf{D}}$ denotes the Dirac operator w.r.t the metric \bar{g} . Moreover, the corresponding mean curvature is given by

$$\bar{H} = e^{-2u} (H - m \text{grad}^N u)$$

Now, assume that $\text{grad}^N u = 0$, then D_H is also conformally covariant and we have

$$\bar{D}_H(e^{-\frac{(m-1)}{2}u} \bar{\psi}) = e^{-\frac{(m+1)}{2}u} D_H \bar{\psi}$$

From now on, we will consider regular conformal change of the metric g , i.e., $\text{grad}^N u = 0$.

Theorem 2.3. *Let (M^m, g) be a compact Riemannian Spin^c manifold isometrically immersed into a Riemannian Spin^c manifold $(\widetilde{M}^{m+n}, \widetilde{g})$. Consider a non-trivial eigenspinor field $\varphi \in \Gamma(\Sigma)$ for the submanifold Dirac operator D_H , i.e. $D_H\psi = \lambda\psi$. For any regular conformal change of metric $\bar{g} = e^{2u}\widetilde{g}$, assume that $m \geq 3$ and*

$$\overline{\text{Scal}}_M e^{2u} + R_\varphi^N + 4|Q^\psi|^2 - c_m|\Omega^M| > \|H\|^2 > 0$$

on M_φ , then, we have

$$\lambda^2 \geq \frac{1}{4} \inf_{M_\psi} \left(\sqrt{\overline{\text{Scal}}_M e^{2u} + R_\psi^N + 4|Q^\psi|^2 - c_m|\Omega|} - \|H\| \right)^2.$$

Proof: For $\psi \in \Gamma(\Sigma)$ an eigenspinor of D_H with eigenvalue λ , let $\bar{\varphi} := e^{-\frac{(n-1)u}{2}}\bar{\psi}$. then, we have $\overline{D_H}\bar{\varphi} = \lambda e^{-u}\bar{\varphi}$. Recall that

$$\bar{\nabla}_{e_i}\bar{\psi} = \bar{\nabla}_{e_i}\psi - \frac{1}{2}\overline{e_i \cdot du \cdot \psi} - \frac{1}{2}e_i(u)\bar{\psi},$$

and $\bar{e}_i = e^{-u}e_i$. Now, it is straightforward to get $\overline{Q_{i,j}^\psi} = e^{-u}Q_{i,j}^\psi$, hence $|\overline{Q^\psi}|^2 = e^{-2u}|Q^\psi|^2$. Equation (8) is also true on (\widetilde{M}, \bar{g}) . If, we apply it to $\bar{\varphi}$, we get

$$\begin{aligned} \int_M |\bar{\nabla}\bar{Q}\bar{\varphi}|^2_{\mathbf{v}_{\bar{g}}} &= \int_M (1+mq)^2 \left[(\lambda e^{-u})^2 - \frac{1}{4} \left(\frac{\overline{\text{Scal}}_M + \overline{R}_\varphi^N + 4|\overline{Q^\psi}|^2}{(1+mq^2)} - \frac{\|\tilde{H}\|_{\bar{g}}^2}{mq^2} \right) \right] |\bar{\varphi}|^2_{\mathbf{v}_{\bar{g}}} \\ &\quad - \frac{1}{4} \int_M (1+mq^2) \left[\frac{2}{mq(1+mq^2)} \left(\|\tilde{H}\|_{\bar{g}}^2 - \frac{\langle \tilde{H} \cdot \bar{\varphi}, \bar{\omega}_\perp \cdot \bar{\varphi} \rangle_{\bar{g}}}{|\varphi|_{\bar{g}}^4} \right) \right] |\bar{\varphi}|^2_{\mathbf{v}_{\bar{g}}} \\ (9) \quad &\quad - \frac{i}{2} \int_M \langle \Omega \cdot \bar{\varphi}, \bar{\varphi} \rangle_{\bar{g}} \mathbf{v}_{\bar{g}} \end{aligned}$$

since $\tilde{H} = e^{-u}\bar{H}$ and $\overline{R}_\varphi^N = e^{-2u}R_\psi^N$, we have

$$\begin{aligned} \int_M |\bar{\nabla}\bar{Q}\bar{\varphi}|^2_{\mathbf{v}_{\bar{g}}} &= \int_M (1+mq)^2 e^{-2u} \left[(\lambda)^2 - \frac{1}{4} \left(\frac{\overline{\text{Scal}}_M e^{2u} + R_\psi^N + 4|Q^\psi|^2}{(1+mq^2)} - \frac{\|H\|^2}{mq^2} \right) \right] |\bar{\varphi}|^2_{\mathbf{v}_{\bar{g}}} \\ &\quad - \frac{1}{4} \int_M (1+mq^2) e^{-2u} \left[\frac{2}{mq(1+mq^2)} \left(\|H\|^2 - \frac{\langle H \cdot \psi, \omega_\perp \cdot \psi \rangle}{|\psi|^4} \right) \right] |\bar{\varphi}|^2_{\mathbf{v}_{\bar{g}}} \\ (10) \quad &\quad - \frac{i}{2} \int_M \langle \Omega \cdot \bar{\varphi}, \bar{\varphi} \rangle_{\bar{g}} \mathbf{v}_{\bar{g}} \end{aligned}$$

Now, we use again (5) and if moreover, $\overline{\text{Scal}}_M e^{2u} + R_\psi^N - c_M|\Omega| + 4|Q^\psi|^2 > \|H\|^2 > 0$, we take

$$q = \sqrt{\frac{\|H\|}{m(\sqrt{\overline{\text{Scal}}_M e^{2u} + R_\psi^N - c_M|\Omega| + 4|Q^\psi|^2} - \|H\|)}},$$

and then we use the Cauchy Schwarz inequality

$$\|H\|^2 - \frac{\langle H \cdot \psi, \omega_\perp \cdot \psi \rangle}{|\psi|^4} \geq 0,$$

to get the desired result. \square

By taking u as first eigenfunction of the Yamabe operator on M , we get the following corollary, where μ_1 is the first eigenvalue of the Yamabe operator.

Corollary 2.4. *Under the assumptions of Theorem 2.3 and if*

$$\mu_1 + R_{\psi}^N + 4|Q^{\psi}|^2 - c_m|\Omega^M| > \|H\|^2 > 0$$

on M_{φ} , then, we have

$$\lambda^2 \geq \frac{1}{4} \inf_{M_{\psi}} \left(\sqrt{\mu_1 + R_{\psi}^N + 4|Q^{\psi}|^2 - c_m|\Omega|} - \|H\| \right)^2.$$

3. GENERALIZED TWISTED KILLING SPINORS

We have seen in Theorem 2.1 that if equality occurs, then, the eigenspinor φ is in fact a generalized twisted Killing spinor, that is satisfies the equation

$$(11) \quad \nabla_X \varphi = fX \cdot \omega_{\perp} \cdot \varphi = fX \cdot_M \varphi,$$

where f is a real function. We prove the following

Proposition 3.1. *Let (M^m, g) be a compact Riemannian Spin^c manifold isometrically immersed into a Riemannian Spin^c manifold $(\widetilde{M}^{m+n}, \widetilde{g})$. Let $\varphi \in \Gamma(\Sigma)$ be a generalized twisted Killing spinor with real-valued function f . If $m > n + 4$, then f is constant.*

Proof: We define the following forms for $p \in \{1, \dots, m\}$,

$$\omega_p(X_1, \dots, X_p) = \left\langle (X_1 \wedge X_2 \wedge \dots \wedge X_p) \cdot_M \varphi, \varphi \right\rangle,$$

We have the following easy facts (see [6, 8] for instance). For any $k \geq 0$, the forms ω_{4k+1} and ω_{4k+2} are imaginary-valued whereas the forms ω_{4k+3} and ω_{4k} are real-valued. Moreover, we have for any $p \geq 0$

$$d\omega_p = ((-1)^p f - f)\omega_{p+1}.$$

In particular, we have for any $k \geq 1$

$$(12) \quad df \wedge \omega_{2k} = 0.$$

Assume that f is not constant and let $x \in M$ such that $df \neq 0$, on a neighborhood V of x . Hence df^{\perp} is of dimension $m - 1$ and we consider $\{e_1, \dots, e_{m-1}\}$ an orthonormal frame of df^{\perp} . From this, we have

$$\omega_{2k}(e_{i_1}, \dots, e_{i_{2k}}) = 0,$$

for any subset $\{i_1, \dots, i_{2k}\}$ of $\{1, \dots, m - 1\}$. Thus, for $l = \lfloor \frac{m-1}{2} \rfloor$, we deduce that the spinor fields $\varphi, e_{i_1} \cdot_M e_{i_2} \cdot_M \varphi, \dots$ and $e_{i_1} \cdot_M e_{i_2} \cdots e_{i_{2l}} \cdot_M \varphi$ are orthonormal on V .

Consequently, the space spanned by these spinors is a vector subspace of Σ_x of complex dimension

$$1 + \binom{m-1}{2} + \binom{m-1}{4} + \dots + \binom{m-1}{2l} = 2^{m-2}.$$

Since the complex dimension of Σ is

$$d(m, n) = \begin{cases} 2^{\frac{m+n}{2}} & \text{if } m+n \text{ is even} \\ 2^{\frac{m+n-1}{2}} & \text{if } m+n \text{ is odd} \end{cases}$$

we conclude that $2^{m-2} \leq d(m, n)$. Therefore, f is constant if $2^{m-2} > d(m, n)$ which corresponds to the condition expressed in the statement of the Proposition. Indeed, on one hand, if $m+n$ is even, then $2^{m-2} > d(m, n)$ is equivalent to $m-2 > \frac{n+m}{2}$, that is $m > n+4$. On the other hand, if $m+n$ is odd, then $2^{2m-2} > d(m, n)$ is equivalent to $m-2 > \frac{n+m-1}{2}$, that is $m > n+3$. But since $m+n$ is odd, then m cannot be equal to $n+4$, hence we also have $m > n+4$. \square

Hence, we deduce the following

Corollary 3.2. *Let (M^m, g) be a compact Riemannian Spin^c manifold isometrically immersed into a Riemannian Spin^c manifold $(\widetilde{M}^{m+n}, \widetilde{g})$. if $m > n + 4$ and equality occurs in Theorem 2.1, then M admits a twisted Killing spinor.*

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REFERENCES

- [1] C. Bär, *Extrinsic bounds for eigenvalues of the Dirac operator*, Ann. Glob. Anal. Geom. 16 (1998) 573-596.
- [2] J. P. Bourguignon, O. Hijazi, J. L. Milhorat, A. Moroianu & S. Moroianu, *A spinorial approach to Riemannian and conformal geometry*, Monograph in Mathematics, EMS.
- [3] T. Friedrich, *Der erste Eigenwert des Dirac-operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung*, Math. Nach. 97 (1980), 117-146.
- [4] T. Friedrich, *Dirac operators in Riemannian Geometry*, Graduate studies in mathematics, Volume 25, American Mathematical Society.
- [5] N. Ginoux & B. Morel, *On eigenvalue estimates for the submanifold Dirac operator*, Int. J. Math. 13 (2002), No. 5, 533-548.
- [6] N. Grosse & R. Nakad, *Complex generalized Killing spinors on Riemannian Spin^c manifolds*, arxiv:1311.0969 (to appear in Results in Mathematics).
- [7] G. Habib, *Energy-Momentum tensor on foliations*, J. Geom. Phys. 57 (2007), 2234-2248.
- [8] M. Herzlich et & Moroianu, *Generalized Killing spinors and conformal eigenvalue estimates for Spin^c manifold*, Ann. Glob. Anal. Geom. 17 (1999), 341-370.
- [9] O. Hijazi, *A conformal Lower Bound for the Smallest Eigenvalue of the Dirac Operator and Killing Spinors*, Commun. Math. Phys. 104, (1986) 151–162.
- [10] O. Hijazi, *Lower bounds for the eigenvalues of the Dirac operator*, J. Geom. Phys., 16 (1995) 27-38.
- [11] O. Hijazi & X. Zhang, *Lower bounds for the Eigenvalues of the Dirac Operator, Part I. The Hypersurface Dirac Operator*, Ann. Global Anal. Geom. 19, (2001) 355-376.
- [12] O. Hijazi & X. Zhang, *Lower bounds for the Eigenvalues of the Dirac Operator, Part II. The Submanifold Dirac Operator*, Ann. Global Anal. Geom. 19, (2001) 163-181.
- [13] R. Nakad, *Lower bounds for the eigenvalues of the Dirac operator on Spin^c manifolds*, J. Geom. Phys. 60 (2010), 1634-1642.
- [14] R. Nakad & J. Roth, *The Spin^c Dirac operator on hypersurfaces and applications*, Diff. Geom. Appl., 31 (1), (2013), pp 93-103

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