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Regularity of a D-module along a submanifold

Yves Laurent

February 4, 2015

Abstract

We study how regularity along a submanifold of a differential or microdifferential system can propagate from a family of submanifolds to another. The first result is that a microdifferential system regular along a lagrangian foliation is regular. However, when restricted to a fixed submanifold the corresponding result is true only under a condition on the characteristic variety.

Introduction

This initial idea of this paper is a question M. Granger relative to a paper with F. Castro-Jimenez [1]: If a holonomic \mathcal{D}_X -module is regular along a family of hyperplanes crossing on a linear subvariety, is it regular along the intersection.

If we consider this problem from the microlocal point of view, it is equivalent to the following: if a holonomic \mathcal{D}_X -module is regular along the points of a submanifold, is it regular along the submanifold itself. From this point of view, the problem is similar to the well-known theorem of Kashiwara-Kawai [5]: if a \mathcal{D}_X -module is regular along the points of an open set, it is regular on this set.

To solve the problem we first prove a microlocal version of the result of Kashiwara-Kawai:

Consider a conic lagrangian foliation of an open set of the cotangent bundle to a complex manifold. Let \mathcal{M} be a holonomic microdifferential module defined on the open set. If \mathcal{M} is regular along each leaf of the foliation then the module \mathcal{M} is regular.

Then we may give an answer to the initial problem. A natural framework to state it is a "maximally degenerated" involutive submanifold of the cotangent bundle. Such a variety carry a canonical conic lagrangian foliation and contains also a canonical conic lagrangian submanifold, its degeneracy locus. Then, the result is that if a holonomic microdifferential module is regular along the leaves of the foliation, it is regular along the degeneracy locus under the condition that its characteristic variety is contained in the maximally degenerated involutive submanifold.

This apply to our initial problem. If a holonomic differential or microdifferential module has its characteristic variety contained in the set $\langle x, \xi \rangle = 0$ of $T^*\mathbb{C}^n$ and regular along the hyperplanes containing the origin, then it is regular along the origin. In the same way, a holonomic module which is regular along the points of a submanifold Y of a complex variety X is regular along Y under the condition that, in a neighborhood of the conormal to Y , the characteristic variety is contained in the inverse image of Y by the projection $T^*X \rightarrow X$.

However, this result is not always true if the condition on the characteristic variety is not satisfied. In the last example, it may be untrue if the singular support of the module has components tangent to the variety Y . We show this by constructing a counterexample.

In sections 1 and 2, we recall the different definitions of regularity and give some classical results that we will use later.

In section 3, we prove a complex microlocal Cauchy theorem that we use in section 4 to prove our main result. In section 5, we show how this applies to maximally degenerated involutive manifolds and give examples.

Section 6, is devoted to the calculation of a counterexample when the condition on the characteristic variety is not fulfilled.

1 Regularity

In dimension 1 regularity of \mathcal{D}_X -modules is equivalent to the notion of differential equation with regular singularity. In higher dimension, there are two different kind of regularity. The first one is global, that is concerns a \mathcal{D}_X -module on an open set while the second is

relative to a subvariety. In both cases, there is an equivalence between growth conditions on the solutions and algebraic conditions on the module itself.

Algebraic conditions uses the so-called V-filtration which is a natural extension of the algebraic conditions in the definition of "regular singular".

Let X be a complex manifold and Y a submanifold of X . Let \mathcal{O}_X be the sheaf of holomorphic functions on X , \mathcal{D}_X the sheaf of differential operators on X with coefficients in \mathcal{O}_X .

The sheaf \mathcal{D}_X is provided with the usual filtration by the order of operators, this filtration will be denoted by $(\mathcal{D}_{X,m})_{m \geq 0}$. Kashiwara defined in [3] an other filtration, the V-filtration, by:

$$V_k \mathcal{D}_X = \{ P \in \mathcal{D}_X \mid \forall \ell \in \mathbb{Z}, P \mathcal{I}_Y^\ell \subset \mathcal{I}_Y^{\ell-k} \} \quad (1.1)$$

where \mathcal{I}_Y is the ideal of definition of Y and $\mathcal{I}_Y^\ell = \mathcal{O}_X$ if $\ell \leq 0$.

If Y is given in local coordinates by $Y = \{(x_1, \dots, x_p, t_1, \dots, t_q) \mid t = 0\}$, then the function x_i and the derivations $D_{x_i} = \frac{\partial}{\partial x_i}$ are of order 0 for the V-filtration while t_j is of order -1 and D_{t_j} of order 1.

Let $\tau : T_Y X \rightarrow Y$ be the normal bundle to Y in X and $\mathcal{O}_{[T_Y X]}$ the sheaf of holomorphic functions on $T_Y X$ which are polynomial in the fibers of τ . Let $\mathcal{O}_{[T_Y X]}[k]$ be the subsheaf of $\mathcal{O}_{[T_Y X]}$ of homogeneous functions of degree k in the fibers of τ . There are canonical isomorphisms between $\mathcal{I}_Y^k / \mathcal{I}_Y^{k-1}$ and $\tau_* \mathcal{O}_{[T_Y X]}[k]$, between $\bigoplus \mathcal{I}_Y^k / \mathcal{I}_Y^{k-1}$ and $\tau_* \mathcal{O}_{[T_Y X]}$. Hence the graded ring $gr^V \mathcal{D}_X$ associated to the V-filtration on \mathcal{D}_X acts naturally on $\mathcal{O}_{[T_Y X]}$. An easy calculation [13] shows that as a subring of $\mathcal{E}nd(\tau_* \mathcal{O}_{[T_Y X]})$ it is identified to $\tau_* \mathcal{D}_{[T_Y X]}$ the sheaf of differential operators on $T_Y X$ with coefficients in $\mathcal{O}_{[T_Y X]}$.

The Euler vector field θ of $T_Y X$ is the vector field which acts on $\mathcal{O}_{[T_Y X]}[k]$ by multiplication by k . Let ϑ be any differential operator in $V_0 \mathcal{D}_X$ whose image in $gr_0^V \mathcal{D}_X$ is θ .

Definition 1.1. The holonomic \mathcal{D}_X -module \mathcal{M} is *regular along Y* if any section u of \mathcal{M} is annihilated by a differential operator of the form $\vartheta^N + P + Q$ where P is in $\mathcal{D}_{X,N-1} \cap V_0 \mathcal{D}_X$ and Q is in $\mathcal{D}_{X,N} \cap V_{-1} \mathcal{D}_X$

Definition 1.2. A polynomial b is a *regular b -function* for u along Y if there exists an operator Q in $V_{-1} \mathcal{D}_X \cap \mathcal{D}_{X,m}$ where m is the degree of b such that $(b(\vartheta) + Q)u = 0$.

It is proved in [7] that \mathcal{M} is regular along Y if and only if all sections of u admit a regular b -function.

Let us denote by $\mathcal{O}_{\widehat{X|Y}}$ the formal completion of \mathcal{O}_X along Y , that is

$$\mathcal{O}_{\widehat{X|Y}} = \text{proj lim } k \mathcal{O}_X / \mathcal{I}_Y^k$$

We proved in [10] that if \mathcal{M} is regular along Y then

$$\forall j \geq 0, \quad \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{O}_{\widehat{X|Y}}) = \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{O}_X) \quad (1.2)$$

We proved in [11] that the converse is true if Y is a hypersurface.

Let d be the codimension of Y . Then $\mathcal{B}_{Y|X}^\infty = \mathcal{H}_Y^d(\mathcal{O}_X)$ is the cohomology of \mathcal{O}_X with support in Y and $\mathcal{B}_{Y|X} = \mathcal{H}_{[Y]}^d(\mathcal{O}_X)$ is the corresponding algebraic cohomology.

We proved also that if \mathcal{M} is regular along Y then

$$\forall j \geq 0, \quad \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{B}_{Y|X}) = \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{B}_{Y|X}^\infty) \quad (1.3)$$

Assume that Y is a hypersurface and let $j : X \setminus Y \hookrightarrow X$ be the canonical injection. We denote by $\mathcal{O}_X[*Y]$ is the sheaf of meromorphic functions with poles on Y and by $j_*j^{-1}\mathcal{O}_X$ the sheaf of holomorphic functions with singularities on Y . Then $\mathcal{B}_{Y|X}^\infty = j_*j^{-1}\mathcal{O}_X/\mathcal{O}_X$ and $\mathcal{B}_{Y|X} = \mathcal{O}_X[*Y]/\mathcal{O}_X$.

In [11], we proved also that \mathcal{M} is regular along Y if equality (1.3) is true.

If Y is not a hypersurface, this is no more true and we have to microlocalize the definition to get an equivalence.

2 Microlocal Regularity

We denote by \mathcal{E}_X the sheaf of microdifferential operators of [12], filtered by the order. We will denote that filtration by $\mathcal{E}_X = \bigcup \mathcal{E}_{X,k}$ and call it the usual filtration.

In [9], we extended the definitions of V-filtrations and b -functions to microdifferential equations and lagrangian subvarieties of the cotangent bundle. These definitions are invariant under quantized canonical transformations.

Let Λ be a lagrangian conic submanifold of the cotangent bundle T^*X and \mathcal{M}_Λ be a simple holonomic \mathcal{E}_X -module supported by Λ . By definition, such a module is generated by a non degenerate section u_Λ , that is such that the ideal of the principal symbols of the microdifferential operators annihilating u_Λ is the ideal of definition of Λ . It always exists locally [12].

Let $\mathcal{M}_{\Lambda,k} = \mathcal{E}_{X,k}u_\Lambda$. Then the V-filtration on \mathcal{E}_X along Λ is defined by:

$$V_k\mathcal{E}_X = \{ P \in \mathcal{E}_X \mid \forall \ell \in \mathbb{Z}, P\mathcal{M}_{\Lambda,\ell} \subset \mathcal{M}_{\Lambda,\ell+k} \} \quad (2.1)$$

This filtration is independent of the choices of \mathcal{M}_Λ and u_Λ , so it is globally defined.

Let $\mathcal{O}_\Lambda[k]$ be the sheaf of holomorphic functions on Λ homogeneous of degree k in the fibers of $\Lambda \rightarrow X$ and $\mathcal{O}_{(\Lambda)} = \bigoplus_{k \in \mathbb{Z}} \mathcal{O}_\Lambda[k]$. Then there is an isomorphism between $\mathcal{M}_{\Lambda,k}/\mathcal{M}_{\Lambda,k-1}$ and $\mathcal{O}_{\Lambda,k}$. By this isomorphism the graded ring $gr^V\mathcal{E}_X$ acts on $\mathcal{O}_{(\Lambda)}$ and may be identified to the sheaf $\mathcal{D}_{(\Lambda)}$ of differential operators on Λ with coefficients in $\mathcal{O}_{(\Lambda)}$.

All these definitions are invariant under quantized canonical transformations [9].

Definition 2.1. The holonomic \mathcal{E}_X -module \mathcal{M} is *regular along* Λ (on an open set of T^*X) if any section u of \mathcal{M} is annihilated by a microdifferential operator of the form $\vartheta^N + P + Q$ where P is in $\mathcal{E}_{X,N-1} \cap V_0\mathcal{E}_X$ and Q is in $\mathcal{E}_{X,N} \cap V_{-1}\mathcal{E}_X$

We have the fundamental result:

Theorem 2.2. (theorem 2.4.2 of [11]) *The holonomic \mathcal{E}_X -module \mathcal{M} is regular along the conormal T_Y^*X to a submanifold Y of X on the open set Ω of T^*X if and only if*

$$\forall j \geq 0, \quad \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{C}_{Y|X})|_\Omega = \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{C}_{Y|X}^\infty)|_\Omega \quad (2.2)$$

where $\mathcal{C}_{Y|X}$ is the sheaf of holomorphic microfunctions of [12].

As the restriction to the zero section of T^*X of the sheaf $\mathcal{C}_{Y|X}$ is $\mathcal{B}_{Y|X}$ while the restriction of $\mathcal{C}_{Y|X}^\infty$ is $\mathcal{B}_{Y|X}^\infty$, the equation (1.3) is a special case of (2.2).

If Λ is not the conormal bundle to a submanifold Y , the result is still true if $\mathcal{C}_{Y|X}$ is replaced by a simple holonomic module \mathcal{M}_Λ

$$\forall j \geq 0, \quad \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{M}_\Lambda) = \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M}_\Lambda) \quad (2.3)$$

The regularity property is also equivalent to other properties like the existence of a regular b -function or conditions on the microcharacteristic varieties (see [7]).

If Λ is a smooth part of an irreducible component of the characteristic variety of \mathcal{M} , definition 2.1 is equivalent to what Kashiwara-Kawai call "to have regular singularities along Λ " (definition 1.1.11. of [5]) The equivalence is proved in theorem 3.1.7. of [8].

So definition 1.1.16. of [5] may be reformulated as:

Definition 2.3. The holonomic \mathcal{E}_X -module \mathcal{M} has *regular singularities* (or is *regular*) if for each irreducible component Λ of its characteristic variety, \mathcal{M} is regular along a Zarisky open subset of the regular part of Λ .

A holonomic \mathcal{D}_X -module is regular if $\mathcal{E}_X \otimes_{\mathcal{D}_X} \mathcal{M}$ is regular.

Let us now recall two important results of Kashiwara-Kawai:

Theorem 2.4. (theorem 4.1.1. of [4]) *If \mathcal{M} is a regular holonomic \mathcal{E}_X -module then it is regular along any lagrangian submanifold of T^*X .*

Theorem 2.5. (theorem 6.4.1 of [5]) *A holonomic \mathcal{D}_X -module \mathcal{M} is regular on X if and only if at each point $x \in X$*

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_x \simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \widehat{\mathcal{O}}_x)$$

Here \mathcal{O}_X is the sheaf of holomorphic functions while $\widehat{\mathcal{O}}_x$ is the set of formal power series at x .

3 A complex Cauchy problem for \mathcal{E}_X -modules

Let X be a complex analytic manifold and Y be a smooth hypersurface of X . The inverse image of a \mathcal{E}_X -module on Y has been defined in [12] and we refer to [13] for the details.

Let $\varpi : T^*X \times_X Y \rightarrow T^*X$, $\varrho : T^*X \times_X Y \rightarrow T^*Y$ and $\pi : T^*X \times_X Y \rightarrow Y$ be the canonical maps. The sheaf $\mathcal{E}_{Y \rightarrow X}$ may be defined as

$$\mathcal{E}_{Y \rightarrow X} = \pi^{-1} \mathcal{O}_Y \otimes_{\pi^{-1} \mathcal{O}_X} \varpi^{-1} \mathcal{E}_X$$

If t is an equation of Y we have $\mathcal{E}_{Y \rightarrow X} = \mathcal{E}_X / t \mathcal{E}_X$.

The inverse image of a coherent left \mathcal{E}_X -module \mathcal{M} by $i : Y \hookrightarrow X$ is defined as

$$i^* \mathcal{M} = \mathcal{M}_Y = \varrho_* (\mathcal{E}_{Y \rightarrow X} \otimes_{\varpi^{-1} \mathcal{E}_X} \varpi^{-1} \mathcal{M})$$

The characteristic variety of a \mathcal{E}_X -module is, by definition, its support. A submanifold Y of X is non characteristic for a \mathcal{E}_X -module \mathcal{M} if the map $\varrho : \varpi^{-1} Ch(\mathcal{M}) \rightarrow T^*Y$ is proper and finite.

Let Z be a smooth hypersurface of Y and $\Lambda = T_Z^*X$ be the conormal bundle to Z . Let (r, s) be two rational numbers such that $+\infty \gg r \geq s \geq 1$. The microcharacteristic variety of type (r, s) of \mathcal{M} along Λ has been defined in [8] and [9]. It is a subvariety of $T^*\Lambda$ denoted by $Ch_{\Lambda}^2(r, s)(\mathcal{M})$. Thanks to lemma 3.1, we will not need the definition of this variety.

The map ϖ induces an isomorphism on Λ while ϱ induces a map $\varrho : \Lambda = T_Z^*X \rightarrow \Lambda' = T_Z^*Y$. The submanifold Y is non microcharacteristic of type (r, s) for \mathcal{M} if $Ch_{\Lambda}^2(r, s)(\mathcal{M})$ does not meet the conormal bundle $T_{\Lambda}^*\Lambda'$ outside of the zero section.

Lemma 3.1. *The conormal bundle $T_{\Lambda}^*\Lambda'$ is contained in the zero section hence Z is non non microcharacteristic of type (r, s) for any (r, s) and any coherent \mathcal{E}_X -module.*

Proof. The problem being local, take local coordinates (x, y, t) of X such that $Y = \{(x, y, t) \in X \mid t = 0\}$ and $Z = \{(x, y, t) \in X \mid t = 0, y = 0\}$. Let $(x, y, t, \xi, \eta, \tau)$ be the local coordinates of T^*X defined by (x, y, t) .

Then ϱ is given by $\varrho(x, y, 0, \xi, \eta, \tau) = (x, y, \xi, \eta)$ and $\Lambda = \{(x, y, t, \xi, \eta, \tau) \in T^*X \mid t = 0, y = 0, \xi = 0\}$. So $\Lambda' = \varrho(\Lambda) = \{(x, y, \xi, \eta) \in T^*Y \mid y = 0, \xi = 0\}$ and the map $\varrho : T^*\Lambda \rightarrow T^*\Lambda'$ is a submersion.

Thus the conormal bundle $T_{\Lambda}^*\Lambda'$ is contained in the zero section and Z is non non microcharacteristic for any microcharacteristic variety. \square

Proposition 3.2. *Let \mathcal{M} be holonomic \mathcal{E}_X -module which is defined in a neighborhood Ω of T_Z^*X and assume that Y is non characteristic for \mathcal{M} . Then we have:*

$$\begin{aligned} \varrho_* \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{Z|X}) &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{C}_{Z|Y}) \\ \varrho_* \mathbb{R}\mathcal{H}om_{\mathcal{E}_X} \mathcal{M}, \mathcal{C}_{Z|X}^{\infty} &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{C}_{Z|Y}^{\infty}) \end{aligned}$$

Proof. By lemma 3.1, Y is non microcharacteristic of type $(\infty, 1)$ and $(1, 1)$ for \mathcal{M} . Hence we may apply directly [8, Prop. 3.2.2.] which give the result with $\mathcal{C}_{Z|X}$ in case $(\infty, 1)$ and $\mathcal{C}_{Z|X}^{\infty}$ in case $(1, 1)$. Remark that the latter case had been proved before by Kashiwara-Schapira [6]. \square

The result is completely different if Z is not contained in Y [11].

4 Microlocal regularity

The aim of this section is to prove a microlocal version of theorem 2.5.

Let M be a conic symplectic manifold. Here we assume that M is a complex manifold but proposition 4.1 is true as well in the real differentiable case.

We recall that a structure of conic symplectic manifold on a complex manifold M is given by a 1-form ω_M whose differential $\sigma_M = d\omega_M$ is a symplectic 2-form on M . A *conic lagrangian foliation* of M is a foliation by conic lagrangian submanifolds.

Proposition 4.1. *Let M be conic symplectic manifold with a conic lagrangian foliation. There is locally a homogeneous symplectic map from M to the cotangent bundle T^*X of a complex manifold X which transforms the leaves into the fibers of $\pi : T^*X \rightarrow X$.*

This result has been proved in the non-homogeneous case and when M is a Banach space by Weinstein [14, Cor. 7.2].

Proof. A lagrangian variety is involutive, hence if two functions vanish on a lagrangian variety Λ , their Poisson bracket vanishes on Λ . But the Poisson bracket depend only on the derivative of the functions hence if two functions are constant on Λ , their Poisson bracket vanishes on Λ .

Let \mathcal{O} be the sheaf of ring of holomorphic functions on M which are constant on the leaves of the foliation. The Poisson bracket of two functions of \mathcal{O} vanishes everywhere. Let $2n$ be the dimension of M . As the foliation has codimension n , we can find locally n functions u_1, \dots, u_n in \mathcal{O} whose differentials are linearly independent at each point. The Poisson bracket of two of them is always 0 hence by the proof of Darboux theorem as it is given in [2, theorem 3.5.6.], there are n functions v_1, \dots, v_n on M such that $u_1, \dots, u_n, v_1, \dots, v_n$ is a canonical symplectic (non-homogeneous) system of coordinates for M . By definition, the canonical symplectic 2-form σ_M of M is thus equal to $\sum dv_i \wedge du_i$.

The functions u_1, \dots, u_n , are constant on the leaves of the foliation which are conic varieties, hence they are constant on the fibers of the \mathbb{C} -action. Let θ be Euler vector field associated to this action and for $i = 1, \dots, n$ let $w_i = \theta(v_i)$. As $\theta(u_i) = 0$, θ is equal to $\sum_{i=1}^n w_i(u, v) \frac{\partial}{\partial v_i}$.

By definition the canonical 1-form α of the homogeneous symplectic manifold M is equal to the inner product $\omega \lrcorner \theta$, hence $\alpha = \sum w_i du_i$. As $d\alpha = \omega$, the functions $u_1, \dots, u_n, w_1, \dots, w_n$ define a canonical symplectic homogeneous system of coordinates for M .

In the coordinates (u, w) , the leaves are given by $u = \text{constant}$, which shows the proposition. \square

Theorem 4.2. *Let Ω be a conic open subset of T^*X and (Λ_α) be a conic lagrangian foliation of Ω .*

Let \mathcal{M} be a holonomic \mathcal{E}_X -module defined on Ω . If \mathcal{M} is regular along each leaf Λ_α , then \mathcal{M} is a regular holonomic module on Ω .

Proof. We will prove the proposition by induction on the dimension of X . As definition 2.3 concerns only the part of the characteristic variety outside of the zero section of T^*X , we will work in a neighborhood of a point of \tilde{T}^*X .

If the dimension of X is 1, the characteristic variety of \mathcal{M} is the union of the conormal bundles to isolated points of X . On the other hand, a conic lagrangian foliation is necessarily given by the conormal bundles to the points of X . Hence by the hypothesis, \mathcal{M} is regular along each components of its characteristic variety, hence \mathcal{M} is regular by [5, Def 1.1.16.].

Assume now that the dimension of X is > 1 . The problem being local on T^*X , we may use proposition 4.1 and transform the foliation into the union of the conormals to the points of an open subset U of X . So Ω is a conic open subset of $\pi^{-1}(U)$ and \mathcal{M} is regular along $T_{\{x\}}^*X$ on Ω for $x \in U$.

Let Y be a smooth hypersurface of X which is non characteristic for \mathcal{M} . Let x be a

point of $U \cap Y$. Then by lemma 3.2 applied to $Z = \{x\}$ we have

$$\begin{aligned} \mathbb{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{C}_{Z|Y}) &\xrightarrow{\sim} \varrho_* \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{Z|X}) \\ &\xrightarrow{\sim} \varrho_* \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{Z|X}^\infty) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{C}_{Z|Y}^\infty) \end{aligned}$$

So \mathcal{M}_Y is regular along $T_{\{x\}}^*Y$ on $\Omega \cap T^*Y$ for $x \in U \cap Y$, hence by the hypothesis of induction, \mathcal{M}_Y is regular on $\Omega \cap T^*Y$.

According to definition 2.3, we will now prove that \mathcal{M} is regular by proving that it is regular along a Zarisky open subset of each irreducible component of its characteristic variety. Such an irreducible component is a conic lagrangian subvariety of T^*X hence generically we may assume that it is the conormal to a smooth subvariety Z of X . Let Λ be such a component. If Λ is the conormal to a point of X , then \mathcal{M} is regular along Λ by the hypothesis. So we may assume that Λ is not the conormal to a point and consider a point μ of T^*X where Λ is the conormal to a submanifold Z of X of dimension ≥ 1 .

Locally, there are local coordinates $(x_1, \dots, x_p, t_1, \dots, t_q)$ of X such that μ has coordinates $x = 0, t = 0, \xi = 0, \tau = (1, 0, \dots, 0)$ and $\Lambda = \{(x, t, \xi, \tau) \in T^*X \mid x = 0, \tau = 0\}$. Let $V_a = \{(x, t) \in X \mid t_1 = a\}$. The conormal to the hypersurface V_a does not meet Λ hence V_a is not characteristic for \mathcal{M} and \mathcal{M}_{V_a} is regular if $a \ll 1$. Now we apply theorem 6.4.5. of [5] and get that \mathcal{M} is regular near μ hence regular along Λ . \square

5 Applications and examples

Let Σ be a submanifold of T^*X with a conic lagrangian foliation. This implies that Σ is conic involutive [2, th 3.6.2], We assume that this foliation is the restriction of a conic lagrangian foliation of T^*X .

Let \mathcal{M} be a holonomic \mathcal{E}_X -module whose characteristic variety is contained in Σ and which is regular along each leaf of this foliation. Then by theorem 4.2, \mathcal{M} is regular hence regular along any lagrangian submanifold of T^*X .

A typical example of this situation is given by "maximally degenerated" involutive manifolds, as we will explain now.

The canonical projection $T^*X \rightarrow X$ defines a map $T^*X \times_X T^*X \rightarrow T^*(T^*X)$ which composed with the diagonal map $T^*X \rightarrow T^*X \times_X T^*X$ defines the canonical 1-form of T^*X that is $\omega_X : T^*X \rightarrow T^*(T^*X)$. We now restrict our attention to the complementary \dot{T}^*X of the zero section in T^*X . The set of points of $\Sigma \cap \dot{T}^*X$ where $\omega_X|_\Sigma$ vanishes is isotropic hence of dimension less or equal to the dimension of X . It is called the *degeneracy locus* of Σ .

Definition 5.1. The involutive submanifold Σ of \dot{T}^*X is said to be *maximally degenerated* if the degeneracy locus is of maximal dimension that is the dimension of X .

Then the degeneracy locus is a lagrangian submanifold Λ_0 of \dot{T}^*X . Being involutive, the manifold Σ has a canonical foliation by bicharacteristic leaves.

Lemma 5.2. *Let Σ be a maximally degenerated involutive submanifold of \dot{T}^*X and Λ_0 its degeneracy locus.*

1. Λ_0 is a union of bicharacteristic leaves.

2. For each leaf L of Λ_0 , there is one and only one lagrangian homogeneous submanifold of Σ whose intersection with Λ_0 is exactly L .
3. These lagrangian submanifolds define a foliation of Σ which we will call the "lagrangian foliation".

For a detailed study of maximally degenerated involutive submanifold, we refer to Duistermaat [2].

Locally on Λ_0 , there is a homogeneous symplectic transformation of \dot{T}^*X and local coordinates of X $(x_1, \dots, x_{n-p}, t_1, \dots, t_p)$ which transform Σ into $\{(x, t, \xi, \tau) \in \dot{T}^*X \mid t = 0\}$. Then $\Lambda_0 = \{(x, t, \xi, \tau) \in \dot{T}^*X \mid t = 0, \xi = 0\}$ and the lemma is easy to prove (see example 5.4).

We may now apply theorem 4.2 to these involutive manifold:

Corollary 5.3. *Let Σ be a maximally degenerated submanifold of \dot{T}^*X with degeneracy locus Λ_0 and lagrangian foliation (L_α) .*

Let \mathcal{M} be a holonomic \mathcal{E}_X -module whose characteristic variety is contained in Σ and which is regular along each leaf L_α . Then \mathcal{M} is regular holonomic hence regular along Λ_0 .

Example 5.4. Let Y be a submanifold of X and $\Sigma = \pi^{-1}(Y)$ where $\pi : \dot{T}^*X \rightarrow X$ is the canonical projection. Then the degeneracy locus of Σ is T_Y^*X , the conormal bundle to Y . The lagrangian foliation is given by the conormal bundles to the points of Y .

In local coordinates (x, t) of X where Y is given by $t = 0$, the bicharacteristic leaves of Σ are the sets

$$F_{x_0, \xi_0} = \{(x, t, \xi, \tau) \in \dot{T}^*X \mid t = 0, x = x_0, \xi = \xi_0\}$$

The degeneracy locus is $\Lambda_0 = \{(x, t, \xi, \tau) \in T^*X \mid t = 0, \xi = 0, \tau \neq 0\}$ while the lagrangian foliation is given by the manifolds

$$\Phi_{x_0} = \{(x, t, \xi, \tau) \in T^*X \mid t = 0, x = x_0, \tau \neq 0\}$$

Example 5.5. Let $X = \mathbb{C}^n$ and $\Sigma = \{(x, \xi) \in T^*X \mid \langle x, \xi \rangle = 0, \xi \neq 0\}$.

Then the degeneracy locus of Σ is the conormal bundle to the origin of \mathbb{C}^n and the lagrangian foliation is given by the conormal bundles to the hyperplanes of X which contain the origin.

Example 5.6. More generally, we may consider a linear subvariety Z of X and Σ the union of the conormal bundles to the hyperplanes which contain Z . The lagrangian foliation is given by these conormals and the degeneracy locus is the conormal bundle to Z .

Example 5.7. Let $V = \{(x, t, \xi, \tau) \in T^*\mathbb{C}^2 \mid x\xi + t\tau = \xi^2/\tau\}$. Then Λ_0 is the conormal to the curve $S = \{t + x^2 = 0\}$ while the lagrangian foliation is given by the conormal to the tangent lines to S .

Applied to \mathcal{D}_X -modules, corollary 5.3 gives the following result:

Corollary 5.8. *Let Σ be a maximally degenerated submanifold of \dot{T}^*X with degeneracy locus Λ_0 and lagrangian foliation (L_α) .*

*Let \mathcal{M} be a holonomic \mathcal{D}_X -module. We assume that the characteristic variety of \mathcal{M} is contained in Σ in a conic neighborhood Ω of Λ_0 in \dot{T}^*X . If \mathcal{M} is regular along each leaf L_α in Ω , then \mathcal{M} is regular along Λ_0 .*

Remark that in this corollary, \mathcal{M} is microlocally regular in a neighborhood of $T_Y^*X \cap \dot{T}^*X$ but may be not regular as a \mathcal{D}_X -module.

Example 5.9. The singular support of a \mathcal{D}_X -module is the projection of the intersection of its characteristic variety with \dot{T}^*X . It is also the set of points of X where \mathcal{M} is not locally isomorphic to a power of \mathcal{O}_X . Assume that the singular support of a \mathcal{D}_X -module \mathcal{M} is a normal crossing divisor and that Y is an irreducible component of this divisor. Then, if \mathcal{M} is regular along each point of Y , it is regular along Y .

More generally, if no component of the characteristic variety is the conormal to a variety tangent to Y or singular on Y , then the result is still true because the characteristic variety is contained in $\pi^{-1}(Y)$ in a neighborhood of T_Y^*X .

6 A counterexample

In this section, we give an example which shows that the condition on the characteristic variety in corollary 5.3 is necessary. In particular, if the singular support of a holonomic \mathcal{D}_X -module \mathcal{M} has components tangent to the manifold Y , then \mathcal{M} may be regular along each point of Y but not regular along Y .

Let $X = \mathbb{C}^2$ with coordinates (x, t) , we denote by $D_x = \frac{\partial}{\partial x}$ and $D_t = \frac{\partial}{\partial t}$ the corresponding derivations. We consider two differential operators $P = x^2 D_x + 1$ and $Q = t$.

Let \mathcal{I} be the ideal of \mathcal{D}_X generated by P and Q and \mathcal{M} be the holonomic \mathcal{D}_X -module $\mathcal{D}_X/\mathcal{I}$.

We denote by Y_0 the hypersurface of equation $\{t = 0\}$ and by Y_φ the hypersurface of equation $\{t - \varphi(x) = 0\}$ where φ is a holomorphic function of one variable defined in a neighborhood of 0 in \mathbb{C} .

Proposition 6.1.

1. The module \mathcal{M} is regular along Y_0 .
2. If $\varphi(0) = 0$ and $\varphi(x) \neq 0$ when $x \neq 0$, the module \mathcal{M} is not regular along Y_φ .
3. The module \mathcal{M} is not regular along $\{0\}$. In fact, it is not regular along $T_{\{0\}}^*X$ at any point of $T_{\{0\}}^*X$.

Proof. As the module is supported by Y_0 it is trivially regular along Y_0 .

To prove that \mathcal{M} is not regular along Y_φ , we will prove that

$$\mathcal{E}xt_{\mathcal{D}_X}^1(\mathcal{M}, \mathcal{B}_{Y_\varphi|X}^\infty/\mathcal{B}_{Y_\varphi|X})_0 \neq 0$$

Let $j : X \setminus Y_\varphi \hookrightarrow X$ be the canonical injection. Let $j_*j^{-1}\mathcal{O}_X$ be the sheaf of holomorphic functions with singularities on Y_φ and $\mathcal{O}_X[*Y_\varphi]$ be the sheaf of meromorphic functions with poles on Y_φ . By definition $\mathcal{B}_{Y_\varphi|X}^\infty = j_*j^{-1}\mathcal{O}_X/\mathcal{O}_X$ and $\mathcal{B}_{Y_\varphi|X} = \mathcal{O}_X[*Y_\varphi]/\mathcal{O}_X$ hence we have to calculate the $\mathcal{E}xt^1$ with values in $\mathcal{F} = j_*j^{-1}\mathcal{O}_X/\mathcal{O}_X[*Y_\varphi]$. As P and Q commute the module \mathcal{M} admits a free resolution

$$0 \rightarrow (\mathcal{D}_X)^2 \xrightarrow{\begin{pmatrix} P \\ Q \end{pmatrix}} (\mathcal{D}_X)^2 \xrightarrow{(P, Q)} \mathcal{D}_X \rightarrow \mathcal{M} \rightarrow 0$$

Hence $\mathcal{E}xt_{\mathcal{D}_X}^1(\mathcal{M}, \mathcal{F})$ vanishes if and only if the system

$$\begin{cases} Pf = g \\ Qf = h \end{cases} \quad \text{with} \quad Ph = Qg \quad (6.1)$$

has a solution f in \mathcal{F} for any data (g, h) in $\mathcal{F} \times \mathcal{F}$.

Lemma 6.2. *There exists a function $h(x, t)$ in $j_*j^{-1}\mathcal{O}_X$ such that $h(x, 0) = \exp(1/x)$.*

Proof. There is some integer $n > 0$ and a function ψ such that $\varphi(x) = x^n\psi(x)$ and $\psi(0) \neq 0$. Let $\lambda_0(x) = 1$ and for $j \geq 1$, let $\lambda_j(x) = \sum_{k=0}^{n-1} \frac{x^k}{(nj-k)!}$. We have $\lambda_0(x) + \sum_{j \geq 1} \lambda_j(x) \frac{1}{x^{nj}} = \exp(1/x)$. The function

$$h(x, t) = \sum_{j \geq 0} \psi(x)^j \lambda_j(x) \frac{1}{(\varphi(x) - t)^j}$$

is a solution to the lemma. \square

Proof of proposition continued: Let $h(x, t)$ be the function given by lemma 6.2. We have

$$(P(x, D_x)h(x, t))|_{t=0} = P(x, D_x)h(x, 0) = P(x, D_x)\exp(1/x) = 0$$

hence there exists some function g in $j_*j^{-1}\mathcal{O}_X$ such that $tg(x, t) = P(x, D_x)h(x, t)$.

As $h(x, 0)$ is a function on $\mathbb{C} \setminus \{0\}$ which is not meromorphic the equation $Qf = h$ has no solution in $\mathcal{F} = j_*j^{-1}\mathcal{O}_X/\mathcal{O}_X[*Y_\varphi]$. Hence equation 6.1 has no solution and $\mathcal{E}xt_{\mathcal{D}_X}^1(\mathcal{M}, \mathcal{F})$ does not vanish. This shows point 2) of the proposition.

To prove point 3) we consider the function $f(x, t) = (1/t)\exp(1/x)$ which defines the element $u = \sum_{j \geq 0} \frac{(-1)^j}{j!(j+1)!} \delta^{(j)}(x) \delta(t)$ in $\mathcal{B}_{\{0\}|X}^\infty$. As u is a solution of $Pu = Qu = 0$ which do not belong to $\mathcal{B}_{\{0\}|X}$, the module \mathcal{M} is not regular on $\{0\}$.

Moreover, we may consider the microfunction v in $\mathcal{C}_{\{0\}|X}^\infty$ of symbol $\sum_{j \geq 0} \frac{(-1)^j}{j!(j+1)!} \xi^j$, that is the microfunction associated to u . It is a solution of $Pv = Qv = 0$ which do not belong to $\mathcal{C}_{\{0\}|X}$ at any point of $T_{\{0\}}^*X$. This shows point 3) of the proposition. \square

Let $Z = \mathbb{C}^2$ with coordinates (y, s) . The coordinates will be (x, t, ξ, τ) on T^*X and (y, s, η, σ) on T^*Z . The Legendre transform is defined from T^*X to T^*Z when $\tau \neq 0$ and $\sigma \neq 0$ by the equations

$$\begin{cases} y = \xi\tau^{-1} \\ s = t + x\xi\tau^{-1} \\ \eta = -x\tau \\ \sigma = \tau \end{cases}$$

According to [12, §3.3 ch.II], a quantized canonical transformation associated to it is given by:

$$\begin{cases} x = -D_y D_s^{-1} \\ t = (D_s s + D_y y) D_s^{-1} = s + y D_y D_s^{-1} + 2D_s^{-1} \\ D_x = y D_s \\ D_t = D_s \end{cases}$$

So, if we apply this transformation to the \mathcal{E}_X -module $\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$ with $\pi : T^*X \rightarrow X$, we get the coherent \mathcal{E}_Z -module $\tilde{\mathcal{N}} = \mathcal{E}_Z / (\mathcal{E}_Z \tilde{P} + \mathcal{E}_Z \tilde{Q})$ with

$$\begin{cases} \tilde{P}(x, s, D_x, D_s) = D_y^2 D_s^{-2} y D_s + 1 = y D_y^2 D_s^{-1} + 2D_y D_s^{-1} + 1 \\ \tilde{Q}(x, s, D_x, D_s) = s + y D_y D_s^{-1} + 2D_s^{-1} \end{cases}$$

Consider the following Lagrangian manifolds:

$$\begin{aligned} \Lambda_0 &= T_{Y_0}^* X = \{ (x, t, \xi, \tau) \in T^* X \mid t = 0, \xi = 0 \} \\ \Lambda_\alpha &= T_{Y_\varphi}^* X = \{ (x, t, \xi, \tau) \in T^* X \mid t = \alpha x, \xi = -\alpha \tau \} && \text{with } \varphi(x) = \alpha x \\ \Lambda'_\lambda &= T_{Y_\varphi}^* X = \{ (x, t, \xi, \tau) \in T^* X \mid t = 4\lambda x^2, \xi = -8\lambda x \tau \} && \text{with } \varphi(x) = 4\lambda x^2 \\ \Lambda'_0 &= T_{\{0\}}^* X = \{ (x, t, \xi, \tau) \in T^* X \mid t = 0, x = 0 \} \end{aligned}$$

$$\begin{aligned} \tilde{\Lambda}_0 &= T_{\{0\}}^* Z = \{ (s, y, \eta, \sigma) \in T^* Z \mid s = 0, y = 0 \} \\ \tilde{\Lambda}_\alpha &= T_{\{p_\alpha\}}^* Z = \{ (s, y, \eta, \sigma) \in T^* Z \mid s = 0, y = \alpha \} \\ \tilde{\Lambda}_\lambda &= T_{Y_\varphi}^* Z = \{ (s, y, \eta, \sigma) \in T^* Y \mid s + y^2/\lambda = 0, \eta = 2(y/\lambda)\sigma \} && \text{with } \varphi(y) = -y^2/\lambda \\ \tilde{\Lambda}'_0 &= T_{Y_0}^* Z = \{ (s, y, \eta, \sigma) \in T^* Z \mid s = 0, \eta = 0 \} \end{aligned}$$

By the Legendre transform, the manifolds Λ_0 , Λ_α , Λ'_λ and Λ'_0 are respectively transformed into $\tilde{\Lambda}_0$, $\tilde{\Lambda}_\alpha$, $\tilde{\Lambda}'_\lambda$ and $\tilde{\Lambda}'_0$ outside of the set $\{\tau = 0\}$. As the regularity along a lagrangian manifold is invariant under quantized canonical transformation, we get from proposition 6.1 that $\tilde{\mathcal{N}}$ is a holonomic \mathcal{E}_Z -module which is regular along $\tilde{\Lambda}_0$ but not regular along $\tilde{\Lambda}_\alpha$, $\tilde{\Lambda}'_\lambda$ and $\tilde{\Lambda}'_0$.

Let again be $Z = \mathbb{C}^2$ with coordinates (y, s) and let us define the following differential operators:

$$\begin{cases} P_1(x, s, D_x, D_s) = y D_y^2 + 2D_y + D_s \\ Q_1(x, s, D_x, D_s) = s D_s + y D_y + 3 \end{cases}$$

Let \mathcal{J} be the ideal of \mathcal{D}_Z generated by P_1 and Q_1 and \mathcal{N} be the holonomic \mathcal{D}_Z -module $\mathcal{D}_Z/\mathcal{J}$.

Proposition 6.3. *The module \mathcal{N} is regular along all points of $Y = \{(y, s \mid s = y^2)\}$ but is not regular along Y itself.*

Remark 6.4. We have also that \mathcal{N} is irregular along each point of $Y_0 = \{(y, s \mid s = 0)\}$ except 0 and is irregular along Y_0 .

Proof. The operator Q_1 is a b -function for \mathcal{N} at 0 hence \mathcal{N} is regular along this point (locally but also microlocally at each point of T_0^*Z). Moreover, the characteristic variety of \mathcal{N} is contained in $\{(s, y, \eta, \sigma) \in T^*Z \mid s\sigma = 0, y\eta = 0\}$ hence \mathcal{N} is elliptic at each point of Y except 0, so it is regular along these points.

The module $\mathcal{E}_Z \otimes_{\pi^{-1}\mathcal{D}_Z} \pi^{-1}\mathcal{N}$ is by definition isomorphic to $\tilde{\mathcal{N}}$ outside of the set $\{\tau = 0\}$. We have seen that $\tilde{\mathcal{N}}$ is not regular along $\tilde{\Lambda}'_\lambda$. Hence, taking $\lambda = -1$, we get that \mathcal{N} is not regular along Y as well as Y_0 and its non zero points. \square

References

- [1] F. Castro-Jimenez and M. Granger, *Gevrey expansion of hypergeometric integrals I*, arXiv:1212.1410v2 (2013).
- [2] J.J. Duistermaat, *Fourier integral operators*, Courant Institute of Mathematical Sciences, New York University, 1973.
- [3] M. Kashiwara, *Vanishing cycles and holonomic systems of differential equations*, Lect. Notes in Math., vol. 1016, Springer, 1983, pp. 134–142.
- [4] M. Kashiwara and T. Kawai, *Second microlocalization and asymptotic expansions*, Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory, Lect. Notes in Physics, vol. 126, Springer, 1980, pp. 21–76.
- [5] ———, *On the holonomic systems of microdifferential equations III. systems with regular singularities*, Publ. RIMS, Kyoto Univ. **17** (1981), 813–979.
- [6] M. Kashiwara and P. Schapira, *Problème de Cauchy dans le domaine complexe*, Inv. Math. **46** (1978), 17–38.
- [7] Y. Laurent, *Regularity and b-functions for \mathcal{D} -modules*, arXiv:1501.06932. (2015)
- [8] ———, *Théorie de la deuxième microlocalisation dans le domaine complexe*, Progress in Math., vol. 53, Birkhäuser, 1985.
- [9] ———, *Polygone de Newton et b-fonctions pour les modules microdifférentiels*, Ann. Ec. Norm. Sup. 4e série **20** (1987), 391–441.
- [10] ———, *Vanishing cycles of \mathcal{D} -modules*, Inv. Math. **112** (1993), 491–539.
- [11] Y. Laurent and Z. Mebkhout, *Pentes algébriques et pentes analytiques d'un \mathcal{D} -module*, Ann. Ec. Norm. Sup. 4e série **32** (1999), 39–69.
- [12] M. Sato, T. Kawai, and M. Kashiwara, *Hyperfunctions and pseudo-differential equations*, Lect. Notes in Math., vol. 287, Springer, 1980, pp. 265–529.
- [13] P. Schapira, *Microdifferential systems in the complex domain*, Grundlehren der Math., vol. 269, Springer, 1985.
- [14] A. Weinstein, *Symplectic manifolds and their lagrangian submanifolds*, Adv. in Math. **6** (1971), 329–346.

Université de Grenoble
Institut Fourier
UMR 5584 CNRS/UJF
BP 74
38402 St Martin d'Hères Cedex
email: Yves.Laurent@ujf-grenoble.fr