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ON FANO MANIFOLDS OF PICARD NUMBER ONE

LAURENT MANIVEL

ABSTRACT. Küchle classified the Fano fourfolds that can be obtained as zero loci of global sections of homogeneous vector bundles on Grassmannians. Surprisingly, his classification exhibits two families of fourfolds with the same discrete invariants. Kuznetsov asked whether these two types of fourfolds are deformation equivalent. We show that the answer is positive in a very strong sense, since the two families are in fact the same! This phenomenon happens in higher dimension as well.

1. INTRODUCTION

The classification of smooth Fano threefolds by Iskhovskih, Mori and Mukai has been one of the highlights of 20th century's algebraic geometry. Many interesting cases from that classification, and especially among the prime Fano manifolds of index one, are obtained by taking suitable sections (mostly linear sections) of certain rational homogeneous spaces, and Mukai wrote a wonderful series of papers about their astonishing geometry (see for example [5], and the more general reference [2]).

It is a general fact that rational homogeneous spaces are a rich source of interesting Fano manifolds, obtained as zero-loci of global sections of vector bundles, and especially homogeneous vector bundles of low rank. In dimension four, O. Küchle [4] began the classification of these Fano manifolds by focusing on Fano fourfolds of index one obtained as subvarieties of Grassmannians, and defined as zero-loci of semisimple homogeneous vector bundles. He obtained a list of fourfolds which, as recently stressed by A. Iliev, are potentially a rich source of nice geometry. In particular some of these varieties have special Hodge structures, that could be relevant in the quest for new hyperkähler manifolds. More precisely, they seem to be good candidates for the ideas of [1] to be implemented successfully.

Recently, A. Kuznetsov obtained nice structural results about the Küchle fourfolds whose Picard number is bigger than one. He also observed that among those whose Picard group is cyclic, there are two families with the very same discrete invariants. He asked whether this coincidence could be explained by the possibility that the two types of Fano fourfolds are deformation equivalent [3, Question 1.1].

The main result of this short note is that this is indeed the case, and that much more is true: the two families are in fact the same one! Moreover, this phenomenon happens in arbitrary dimension: there are two families of prime Fano n -folds of index one, that look different at first sight but in fact coincide. The first one is that of $(n + 2)$ -codimensional linear sections of the Grassmannian $G(2, n + 3)$. The second one is that of zero-loci of sections of the twisted quotient bundle on $G(2, n + 2)$. We prove in Theorem 3.1 that these two types of varieties are the

same up to projective equivalence. Meanwhile we provide a few elements about the geometry of these Fano manifolds, that would probably deserve further study.

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2. TWO FAMILIES OF FANO MANIFOLDS OF INDEX ONE

We will denote by G_n the Grassmannian $G(2, n+2)$ parametrizing planes in a complex vector space V_{n+2} of dimension $n+2$. This is a smooth Fano variety of dimension $2n$, Picard number one and index $n+2$. The very ample generator of the Picard group defines the Plücker embedding of G_n , with respect to which its degree is equal to the Catalan number $c_n = \frac{1}{n+1} \binom{2n}{n}$.

2.1. Linear sections of Grassmannians. Any smooth linear section X of G_{n+1} of codimension $n+2$ is again a Fano variety, of dimension n , Picard number one and index one. This will be our first family of smooth Fano manifolds. Note that for $n=2$ we get a del Pezzo surface of degree five, and for $n=3$ a prime Fano threefold of genus eight.

There is a moduli space for these manifolds, that we can construct as the GIT quotient of an open subset of the Grassmannian $G(n+2, \wedge^2 V_{n+3}^*)$ by the reductive group PGL_{n+3} . The dimension of this moduli space is $N = (n+1)(n+2)^2/2 - (n+2)(n+4) = (n+3)(n^2-4)/2$. Locally around a point in the moduli space corresponding to a given X , deformations are unobstructed: recall this is the case for any Fano manifold, as a consequence of the Kodaira-Akizuki-Nakano vanishing theorem. The tangent space $H^1(TX)$ to the local Kuranishi space can be computed from the normal exact sequence, which yields a long exact sequence of cohomology groups on X :

$$H^0(TX) \rightarrow H^0(TG_{n+1|X}) \rightarrow H^0(O_X(1))^{n+2} \rightarrow H^1(TX) \rightarrow H^1(TG_{n+1|X}).$$

Indeed, using the Koszul resolution of O_X and Bott’s theorem on G_{n+1} , one checks that $H^1(TG_{n+1|X}) = 0$ and $H^0(TG_{n+1|X}) = H^0(TG_{n+1}) = sl_{n+3}$. Moreover $H^0(TX) = 0$, or equivalently:

Proposition 2.1. *For $n \geq 4$, any smooth X has a finite automorphism group.*

Proof. Since X has index one, $H^0(TX) = H^0(\Omega_X^{n-1}(1))$. Taking the $(n-1)$ -th wedge power of the conormal exact sequence, we get that this cohomology group is zero as soon as

$$H^k(X, \Omega_{G_{n+1}|X}^{n-1-k}(1-k)) = 0 \quad \forall 0 \leq k \leq n-1.$$

Using the Koszul resolution of the structure sheaf of X , we get that this vanishing will hold as soon as

$$H^{k+\ell}(G_{n+1}, \Omega_{G_{n+1}}^{n-1-k}(1-k-\ell)) = 0 \quad \forall 0 \leq \ell \leq n+2.$$

If $k+\ell \geq 2$, this follows from the Kodaira-Akizuki-Nakano vanishing theorem, since $(k+\ell) + (n-1-k) = n-1+\ell$ is smaller than the dimension of G_{n+1} . If $k+\ell \geq 1$, we get $H^1(\Omega_{G_{n+1}}^{n-1})$ and $H^1(\Omega_{G_{n+1}}^{n-2})$, which are both zero for $n \geq 4$ since $H^q(\Omega_{G_{n+1}}^p)$

is always zero for $p \neq q$. Finally if $k + \ell \geq 0$, we get $H^0(\Omega_{G_{n+1}}^{n-1}(1)) = 0$ by [6, Theorem 2.3].

Remark. It would be interesting to have closed formulas for the Hodge numbers of linear sections of Grassmannians. In the case we are interested in, the Lefschetz hyperplane theorem gives $h^{p,q}(X) = h^{p,q}(G_{n+1})$ for $p + q < n$. Hence $h^{p,q}(X) = \delta_{p,q} \left\lceil \frac{p+1}{2} \right\rceil$ under this condition.

The case $p + q = n$ is more difficult. It would be enough to compute the holomorphic Euler characteristic of the bundles of p -forms for $p \leq n$, which can be done by standard techniques from Schubert calculus but remains computationally hard. Even the topological Euler characteristic seems difficult to compute. If we try to use the Gauss-Bonnet formula $e(X) = \int_X c_n(TX)$ we can obtain the Chern class of the tangent bundle from the normal exact sequence. We deduce that if

$$P_n(x, y) = \sum_{k=0}^{n+1} p_{n,k} x^{2k} y^{2n+2-2k} = \left[\frac{x^{n+2}}{(1+x)^{n+2}} \frac{(1+x+y^2)^{n+3}}{1-x^2+4y^2} \right]_{2n+2}$$

the degree $2n + 2$ part of the Taylor expansion of this rational function, then

$$e(X) = \sum_{k=0}^{n+1} p_{n,k} c_k.$$

We can deduce the missing Betti number $b_n = b_n(X)$ for X of small dimension n : $b_2 = 5$, $b_3 = 10$, $b_4 = 69$, $b_5 = 380$, $b_6 = 2321$, $b_7 = 9442$.

2.2. Zero loci of the twisted quotient bundle. Recall that the Grassmannian G_n is endowed with two natural vector bundles, the tautological rank two bundle U , and the quotient bundle Q , which has rank n . Moreover $\det(U^*) = \det(Q) = O_{G_n}(1)$, the very ample generator of the Picard group. The quotient bundle Q is generated by global sections, and the zero locus of a non zero section is just a projective space. More interesting are the zero loci of global sections of the twisted quotient bundle $Q(1)$. Since $\det(Q(1)) = O_{G_n}(n+1)$, the zero locus Y of a general global section of $Q(1)$ is a smooth Fano manifold of dimension n and index one. This is our second family of such manifolds.

In classical language, Y defines a congruence of lines in $\mathbb{P}(V_{n+2}) = \mathbb{P}^{n+1}$. Recall that the *order* of such a congruence is defined as the number of lines from Y passing through a general point in \mathbb{P}^{n+1} .

Proposition 2.2. *The congruence of lines defined by Y has order $n + 1$.*

Proof. The set of lines passing through a point in \mathbb{P}^{n+1} is isomorphic to \mathbb{P}^n , and the restriction of $Q(1)$ to this projective space is $R(1)$, if R denotes the tautological quotient bundle on \mathbb{P}^n . The order of the congruence is the number of zeroes of the induced section of $R(1) = T\mathbb{P}^n$. For Y general and a general point in \mathbb{P}^{n+1} we get a general vector field on \mathbb{P}^n , and we deduce that the order is $c_n(T\mathbb{P}^n) = e(\mathbb{P}^n) = n+1$. \square

The space of global sections of $Q(1) = Q \otimes \det(U^*)$ is

$$S_n := S_{10\dots 0-1-1} V_{n+2} = \text{Ker}(V_{n+2} \otimes \wedge^2 V_{n+2}^* \longrightarrow V_{n+2}^*)$$

(the latter morphism being the natural contraction map), as follows from the Borel-Weil theorem. Its dimension is $n(n+2)(n+3)/2$.

We would be tempted to think of our family of Fano manifolds Y , as the quotient of an open subset of $\mathbb{P}(S_n)$ by PGL_{n+2} . This is not correct. Indeed, an unusual phenomenon happens: $H^0(TG_{n|Y})$ is bigger than the sl_{n+2} we would have expected. In fact,

$$H^0(TG_{n|Y}) = sl_{n+2} \oplus V_{n+2}.$$

This means that there are more linear isomorphisms between these varieties than those coming from PSL_{n+2} . We will explain the appearance of that extra factor in the next section.

Note that $Q(1) = Hom(\wedge^2 U, Q)$, and $S_n \subset Hom(\wedge^2 V_{n+2}, V_{n+2})$. Hence, for any $\omega \in S_n$, the zero locus of the associated section of $Q(1)$ is

$$Y_\omega = \{\langle a, b \rangle \in G(2, V_{n+2}), \quad \omega(a, b) \in \langle a, b \rangle\}.$$

Note that this makes sense for any $\omega \in Hom(\wedge^2 V_{n+2}, V_{n+2}) = S_n \oplus V_{n+2}^*$; but the component on V_{n+2}^* is in fact insignificant, since for any $v^* \in V_{n+2}^*$, $v^*(a, b) = v^*(b)a - v^*(a)b$ always belongs to $\langle a, b \rangle$.

More interestingly, the previous description shows that Y_ω has a natural rational map to $\mathbb{P}(V_{n+2})$, that we denote by Ω . By definition

$$\Omega(\langle a, b \rangle) = [\omega(a, b)].$$

Proposition 2.3. *The rational map Ω is a birational isomorphism with a determinantal hypersurface of degree $n + 1$ in \mathbb{P}^{n+1} .*

Proof. Let $Z = \Omega(Y)$. A point $[c]$ of $\mathbb{P}(V_{n+2})$ belongs to Z if and only if there exists an independant vector d such that $\omega(c, d) = 0$ belongs to c^\perp . Otherwise said, the induced map from $V_{n+2}/\langle c \rangle \rightarrow (c^\perp)^*$ must not be injective. Note that there is a natural duality between c^\perp and $V_{n+2}/\langle c \rangle$. Globally, we can therefore describe Z as the first degeneracy locus of a morphism between vector bundles

$$\bar{\omega} : R(-1) \longrightarrow R.$$

This implies that $[Z] = c_1(Hom(R(-1), R))$ is $n + 1$ times the hyperplane class, so Z is a hypersurface of degree $n + 1$. Moreover the fiber of σ over any point can be identified with the kernel of $\bar{\omega}$, in particular it is always a linear space, and it reduces to a single point for the general point of Z .

Let $\bar{Y} \subset F(1, 2, V_{n+2})$ be the variety parametrizing incident lines and planes $\langle c \rangle \subset \langle c, d \rangle$ such that $\omega(c, d)$ belongs to $\langle c \rangle$. The two projections yield a diagram

$$\begin{array}{ccc} & \bar{Y} & \\ p_1 \swarrow & & \searrow p_2 \\ G(2, V_{n+2}) \supset Y & & Z \subset \mathbb{P}(V_{n+2}) \end{array}$$

It is easy to see that for ω generic, the variety \bar{Y} is the zero-locus of a generic section of a globally generated vector bundle, hence a smooth variety. Moreover its projection to Y is birational. More precisely, this projection is the blow-up of the smooth subvariety S_ω defined as

$$S_\omega = \{\langle a, b \rangle \in G(2, V_{n+2}), \quad \omega(a, b) = 0\}.$$

Note that S_ω is a Calabi-Yau variety, of codimension two in Y .

The projection to Z fails to be an isomorphism over the locus where $\bar{\omega}$ drops rank. For ω generic, this occurs on a codimension three subvariety $C \subset Z$ which is

also the singular locus of Z . The projection from \bar{Y} to Z has fibers over the general points of C which are lines, in particular this projection is a small morphism. \square

Remark. For $n = 2$, the surface Y is a del Pezzo surface of degree five and \bar{Y} is its blow-up at two points. The projection to the cubic surface Z is an isomorphism.

For $n = 3$, the threefold Y is a prime Fano threefold of genus eight, and \bar{Y} is obtained by blowing-up the elliptic curve $E = S_\omega$. The projection to the quartic determinantal threefold Z contracts 25 lines to the 25 singular points of Z .

Remark. As Kuznetsov points it out, $\omega \in \text{Hom}(\wedge^2 V_{n+2}, V_{n+2})$ might be considered as defining a bracket $[a, b] = \omega(a, b)$, although not a Lie bracket in general since the Jacobi identity has no reason to hold. Then Y_ω parametrizes the planes in V_{n+2} on restriction to which the bracket defines a Lie algebra structure: each plane in Y_ω is required to be stable under the bracket, and the Jacobi identity automatically holds for dimensional reasons. Moreover the codimension two subvariety S_ω can be interpreted as parametrizing the two-dimensional abelian subalgebras.

3. AND THEIR COINCIDENCE

Let X be a smooth linear section of G_{n+1} , defined by an $(n+2)$ -dimensional space of linear forms $H_{n+2} \subset \wedge^2 V_{n+3}^*$. We suppose in this section that H_{n+2} is generic.

Fix a decomposition $V_{n+3} = V_{n+2} \oplus \langle v_{n+3} \rangle$, yielding a decomposition

$$\wedge^2 V_{n+3}^* = \wedge^2 V_{n+2}^* \oplus v_{n+3}^* \wedge V_{n+2}^*,$$

where the linear form v_{n+3}^* has kernel V_{n+2} . Generically, the projection on the second factor of this decomposition yields an isomorphism $H_{n+2} \simeq V_{n+2}^*$. The variety X is thus defined by a monomorphism $\omega \in \text{Hom}(V_{n+2}^*, \wedge^2 V_{n+2}^*)$ (we use the same notation for ω and its transpose): it is cut out by the space of linear forms defined as the graph

$$H_{n+2} = \{\omega(u) + v_{n+3}^* \wedge u, \quad u \in V_{n+2}^*\}.$$

Since ω is injective, X does not contain any line passing through $[v_{n+3}]$. Any line in X is of the form $\langle a, b + \chi(b)v_{n+3} \rangle$ for some non zero vectors $a, b \in V_{n+2}$, with the condition that

$$\chi(b)u(a) = \omega(u)(a, b) \quad \forall u \in V_{n+2}^*.$$

This implies that $\omega(u)(a, b) = 0$ for all $u \in a^\perp$, while the remaining equation determines $\chi(b)$. This means in particular that the linear projection from $\wedge^2 V_{n+3}$ to $\wedge^2 V_{n+2}$, which induces a rational map

$$G_{n+1} = G(2, V_{n+3}) \dashrightarrow G_n = G(2, V_{n+2}) \simeq G(2, V_{n+3}/\langle v_{n+3} \rangle),$$

restricts to a well-defined map from X to the subvariety Y_ω of G_n , which is moreover injective.

The inverse mapping can be described as follows. Note that $\omega(a, b)$ is the vector $c \in V_{n+2}$ defined by the identity $u(c) = \omega(u)(a, b)$ for all $u \in V_{n+2}^*$. The equations defining X reduce to the condition that $c = \chi(b)a$. The line $\langle a, b + \chi(b)v_{n+3} \rangle$ is thus represented by

$$[a \wedge (b + \chi(b)v_{n+3})] = [a \wedge b + c \wedge v_{n+3}] = [a \wedge b + \omega(a, b) \wedge v_{n+3}].$$

This concludes the proof of our main result:

Theorem 3.1. $X \subset G_{n+1}$ and $Y_\omega \subset G_n$ are projectively equivalent.

Remark. In order to identify X with the subvariety Y_ω of G_n , we started from a decomposition $V_{n+3} = V_{n+2} \oplus \langle v_{n+3} \rangle$. Note that if we choose another hyperplane V_{n+2} , or equivalently if we change the defining linear form v_{n+3} into $v_{n+3}^* - e^*$ for some $e^* \in V_{n+2}^*$, then ω is changed into the morphism from V_{n+2}^* to $\wedge^2 V_{n+2}^*$ that sends u to $\omega(u) + e^* \wedge u$. In particular the class of ω in S_n is not affected.

On the contrary, changing the line $\langle v_{n+3} \rangle$ has a non trivial effect on the class of ω (that could easily be expressed explicitly), but does not affect $Y = Y_\omega$ up to projective equivalence. This is precisely what explains the extra factor V_{n+2} inside $H^0(TG_n|_Y)$.

Remark. The Calabi-Yau two codimensional subvariety S of Y can be seen directly in X as the intersection of $G(2, V_{n+2}) \subset G(2, V_{n+3})$ with the linear space that defines X . Of course there is a whole family of such Calabi-Yau's in Y , parametrized by an open subset of the projective space of hyperplanes in V_{n+3} . In particular, for $n = 2$ we get a four dimensional family of K3 surfaces covering Y .

Question. As we have seen, our two families of Fano manifolds of index one coincide generically. An intriguing question is to decide whether they coincide stricto sensu: can any smooth member of each family be described as a member of the second family? A negative answer would be particularly interesting, as a new example of the pathological behaviors of the moduli spaces of Fano varieties.

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